Spring 2016 Math 214 Final Exam (Prof. Shi)

Name:

Problem	01	02	03	04	05	06	07	08	09	10	11	Total
Max Score	24	30	12	12	12	12	12	12	12	12	12	162
Score												

Instruction:

- There are 11 numbered problems, and the maximum total is 162=150+12. The full score is 150, and there are 12 extra points. While there is no designated extra point problem, I suggest that you try to solve all problems on page 1-5, and choose 3 from the 4 problems on page 6-7 to solve.
- To get full credit, you have to justify your answer carefully. Partial credit will be given only if your attempt has made significant progress on the problem.
- Use the back of a page or extra paper if you need more space, but clearly indicate in the space following the problem where extra work can be found. Be sure to neatly cross out all work that you do not want to be considered. Otherwise it may have the effect of reducing your score.
- Some definitions/axoims/theorems are given in the last two pages as reference.

1. (24 points, 4 points each) **Give brief and complete answers to the following**:

   (a) Consider a statement: "If Donald Trump wins presidential election, then Chicago Cubs wins World Series or Detroit Lions wins Super Bowl".
   (i) State its contrapositive.
   (ii) State its negation.

   (b) Let $P(A)$ be the power set of A. Prove or disprove: for any two sets A and B, $P(A \cup B) = P(A) \cup P(B)$.

   (c) Let $S = \{1, 2\}$. Define R to be the set of all relations from S to itself, and define F to be the set of all functions from S to itself.
   (i) What is the cardinality $|R|$ of the set R?
   (ii) What is the cardinality $|F|$ of the set F?
   (You do not need to list all elements of R or F)

(d) Suppose that $A \subseteq B$. Choose "True" or "False" (no proof needed).
(i) If B is denumerable, then A is denumerable. **T** **F**
(ii) If A is denumerable, then B is denumerable. **T** **F**
(iii) If B is uncountable, then A is uncountable. **T** **F**
(iv) If A is uncountable, then B is uncountable. **T** **F**

(e) Prove or disprove: for any non-empty set A, $|A| < |A \times A|$. (That is: the cardinality of A is smaller than the cardinality of $A \times A$.)

(f) What is the last digit of 2017^{2017}?

2. (30 points, 6 points each) **Give brief and complete answers to the following**:

(a) Define an explicit bijection from $\mathbb{R} - \{0\}$ to $\mathbb{R}$.
(You do not need to prove it is a bijection.)

(b) A relation R is defined on $\mathbb{R}$ by $x\ R\ y$ if $|x - y| \leq 5$. Determine which of the properties reflexive, symmetric, and transitive does the relation R process. Justify your answer by prove or disprove each property.

(c) (i) Define a function $f : \mathbb{Z} \to \mathbb{Z}$ which is an injection but it is not a surjection.
(ii) Define a function $f : \mathbb{Z} \to \mathbb{Z}$ which is a surjection but it is not an injection.

(d) Determine the following sets are empty, finite, denumerable or uncountable (no proof necessary).

(i) $\{(x, y) \in \mathbb{Q} \times \mathbb{Q} : x^2 + y^2 < 25\}$;
(ii) $\{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x^2 + y^2 < 25\}$;
(iii) $\{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 < 25\}$.

(e) Find $x = gcd(2016, 234)$ by using the Euclidean Algorithm, and find a solution (a, b) to $2016a+234b = x$.

Give complete proof for problem 3-11.

3. (12 points) Suppose that x and y are both integers. Prove that if $x^2 - y^2$ and xy are both even, then both x and y are even.

4. (12 points) Using mathematical induction to prove that for any $n \in \mathbb{N}$,

$$\sum_{i=1}^{n}(-1)^i i^2 = -1^2 + 2^2 - 3^2 + 4^2 - \cdots + (-1)^n n^2 = \frac{(-1)^n n(n+1)}{2}.$$

5. (12 points) Prove that $2n+1$ and $9n+4$ are relatively prime for every positive integer n.

6. (12 points) Prove that $36|(7^n - 6n - 1)$ for every positive integer n.

7. (12 points) Define a relation R on $\mathbb{Z}$ by $x\ R\ y$ if $3|(2x - 5y)$. Prove R is an equivalence relation on $\mathbb{Z}$. How many distinct equivalent classes are there for this equivalence relation? List all distinct equivalent classes.

8. (12 points) Let $f : A \to B$ and $g : B \to C$ be two functions. Suppose that $g \circ f : A \to C$ is injective.

   (a) (6 points) Prove that f is injective.

   (b) (3 points) Give an example that g is not injective but $g \circ f : A \to C$ is injective.

   (c) (3 points) Prove that if in addition f is surjective, then g is injective.

9. (12 points) Prove that $\sqrt{n-2} + \sqrt{n+2}$ is an irrational number for every $n \in \mathbb{N}$ satisfying $n \geq 3$.
   (Hint: (i) from 11.20, $\sqrt{k}$ is rational if and only if $k = m^2$ for some $m \in \mathbb{N}$; (ii) from the axioms for rational numbers, if x is rational, then x^2 is also rational.)

10. (12 points) Let $A = (0,1) = \{x \in \mathbb{R} : 0 < x < 1\}$ and $B = [0,1] = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$. Prove that $|A| = |B|$ by using Schröder-Bernstein Theorem: constructing an injection from A to B, another injection from B to A, and prove the functions which you define are injections.

11. (12 points) Define a function $f : (0,1) \to \mathbb{R}$ to be $f(x) = \dfrac{2x-1}{x(1-x)}$.

   (a) (10 points) Prove that f is a bijection, and find the inverse function $f^{-1} : \mathbb{R} \to (0,1)$.

   (b) (2 points) Define the composition function $f \circ f : A \to \mathbb{R}$, where A is a subset of $(0,1)$. For which set A, $f \circ f$ is also a bijection?