Spring 2014 Math 214 Final Exam (Prof. Shi)

Name:

Problem	01	02	03	04	05	06	07	08	09	10	11	Total
Max Score	24	30	12	12	12	12	12	12	12	12	12	162
Score												

Instruction:

- There are 11 numbered problems, and the maximum total is 162=150+12. The full score is 150, and there are 12 extra points. While there is no designated extra point problem, I suggest that you try to solve all problems on page 1-4, and choose 2 from the 3 problems on page 5 to solve.
- To get full credit, you have to justify your answer carefully. Partial credit will be given only if your attempt has made significant progress on the problem.
- Use the back of a page or extra paper if you need more space, but clearly indicate in the space following the problem where extra work can be found.
- Be sure to neatly cross out all work that you do not want to be considered. Otherwise it may have the effect of reducing your score.
- Some definitions/axoims/theorems are given in the last two pages as reference.

1. (24 points, 4 points each) **Give brief and complete answers to the following**:

   (a) State the contrapositive of "If n is an odd integer, then n^3 is divisible by 3".

   (b) State the negation of "For every $x \geq 5$, there exists $y < 4$ such that $f(x, y) = 8$."

   (c) Let $S = \{1, 2, 3\}$. Define $A = \{f : S \to S\}$ to be the set of functions from S to itself.
   (i) What is the cardinality $|A|$ of the set A?
   (ii) How many elements in A are bijections?

   (d) A function $f : \mathbb{Z} \to \mathbb{Z}$ is defined by $f(x) = x^3 - 4x$. Is f injective? If it is, prove the injectivity; otherwise show why it is not.

   (e) Prove or disprove: Suppose that $f : \mathbb{R} \to \mathbb{R}$ is an injection and $g : \mathbb{R} \to \mathbb{R}$ is an injection, then $f + g : \mathbb{R} \to \mathbb{R}$ is also an injection.

   (f) Determine the following sets are finite, denumerable or uncountable (no proof necessary):
   (a) $\{x \in \mathbb{R} : x^2 \in \mathbb{Q}\}$;
   (b) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \in \mathbb{Q}\}$.

2. (30 points, 6 points each) **Give brief and complete answers to the following**:

   (a) Let A, B and C be arbitrary sets. Define $P = (A - B) \cup C$ and $Q = (A \cup C) - B$. Give an example of A, B and C so that P and Q have different cardinalities. Which of P and Q has larger cardinality?

   (b) A relation R is defined on $\mathbb{N}$ by $x\ R\ y$ if either $x|y$ or $y|x$. Determine which of the properties reflexive, symmetric, and transitive does the relation R process. Justify your answer by prove or disprove each property.

(c) Define $f : \mathbb{Z}_6 \to \mathbb{Z}_4$ by $f([a]) = [3a]$ for each $a \in \mathbb{Z}$. Is this relation a well-defined function? If yes, prove it is well-defined; if no, show by example why it is not well-defined.

(d) (i) Define a function $f : \mathbb{N} \to \mathbb{N}$ which is an injection but it is not a surjection.
(ii) Define a function $f : \mathbb{N} \to \mathbb{N}$ which is a surjection but it is not an injection.
(You do not need to prove it is injection/surjection.)

(e) Find $x = gcd(2013, 858)$ by using the Euclidean Algorithm, and find a solution (a, b) to $2013a + 858b = x$.

Give complete proof for problem 3-11.

3. (12 points) Suppose that x and y are positive integers. Prove that if $x^2 + y^2$ and xy are both even, then x and y are both even.

4. (12 points) Using mathematical induction, prove that

$$\sum_{i=1}^{n}(3i-1)^2 = 2^2 + 5^2 + 8^2 + \cdots + (3n-1)^2 = \frac{n(6n^2+3n-1)}{2}.$$

5. (12 points) Let $x, y \in \mathbb{R}$, $x > 1$ and $y > 1$. Prove that if $x + \frac{1}{x} \leq y + \frac{1}{y}$, then $x \leq y$.

6. (12 points) Let $A = \mathbb{R} - \{0\}$. Define a relation R on $A \times A$ as $(x, y)\ R\ (z, w)$ if and only if $xw = yz$. Prove R is an equivalence relation on $A \times A$. Describe the geometric profile of its equivalence class $[(2, 3)]$.

7. (12 points) Prove that $5|(3^{3n+1} + 2^{n+1})$ for every nonnegative integer n.

8. (12 points) Define a function $f : (0, \infty) \to (-\infty, \infty)$ to be $f(x) = \frac{x}{2} - \frac{1}{2x}$.

   (a) Prove that f is a bijection, and find the inverse function $f^{-1} : (-\infty, \infty) \to (0, \infty)$.

   (b) Define the composition function $f \circ f$, and find its domain and range.

9. (12 points) Let $A = (0, 1) \cup (2, 3)$ and $B = [1, 2]$. Prove that $|A| = |B|$ by either constructing a bijection between A and B or using Schröder-Bernstein Theorem.

10. (12 points) Let the function $f : \mathbb{N} \to \mathbb{Z}$ be defined by $f(n) = \frac{1 + (-1)^n(2n-1)}{4}$.

    (a) Prove f is a bijection. (Hint: you may use the alternative definition of $f(n)$ with cases n is even or n is odd)

    (b) Find the inverse function $f^{-1} : \mathbb{Z} \to \mathbb{N}$.

11. (12 points) Prove that there exist no positive integers m and n satisfying $m\sqrt{2} = n\sqrt{3}$. (Hint: prove by contradiction, and you may use some results in Chapter 11 included in the Notes)