Vertex Identifying Code in Infinite Hexagonal Grid

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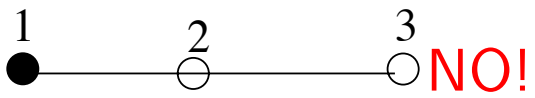
  - $\forall u, v \in V(G)$ if $u \neq v$ then $N[u] \cap D \neq N[v] \cap D$
- Definition: We call such a set D a (vertex identifying) code.

Examples: codes and non-codes



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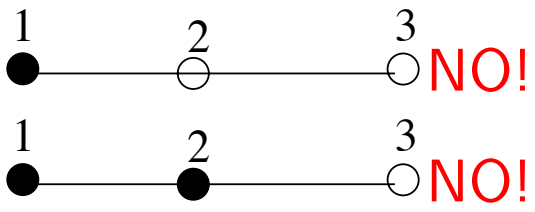

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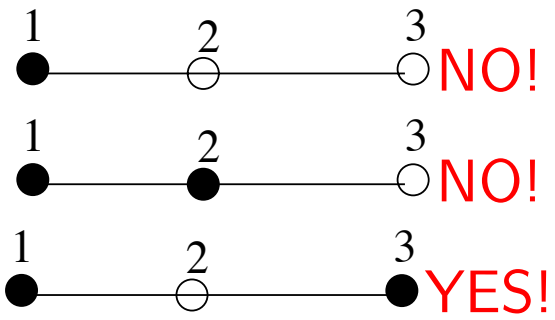

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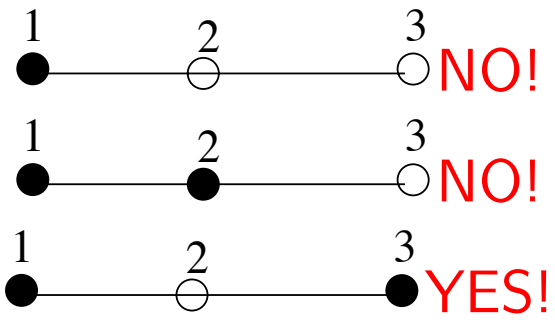

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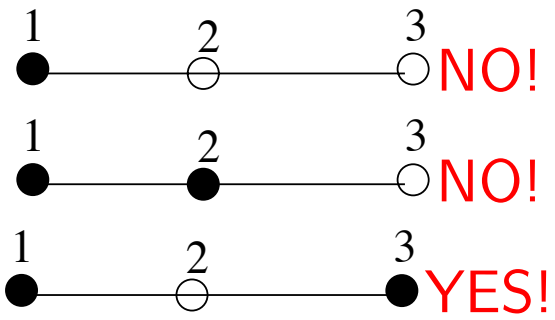

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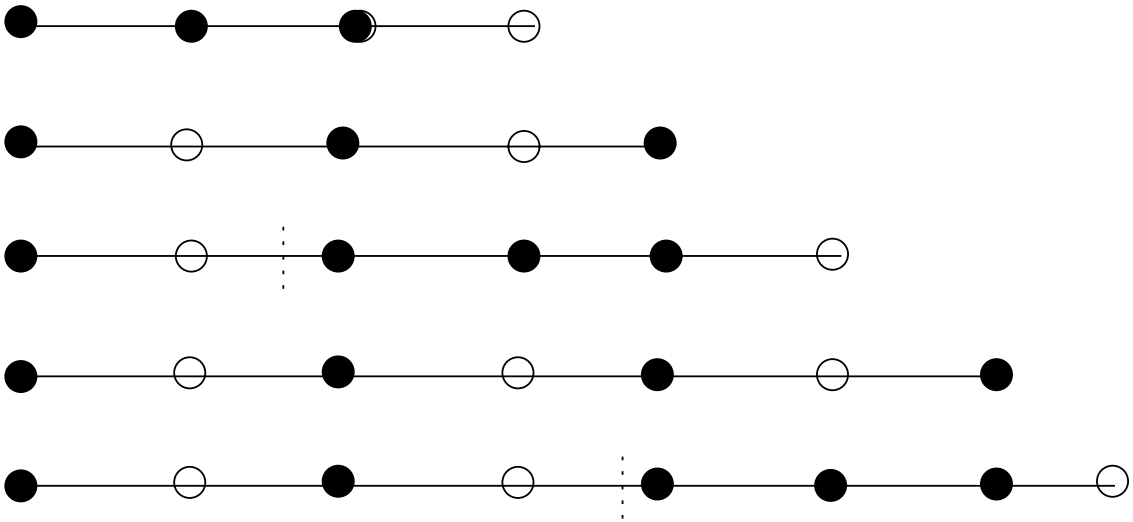

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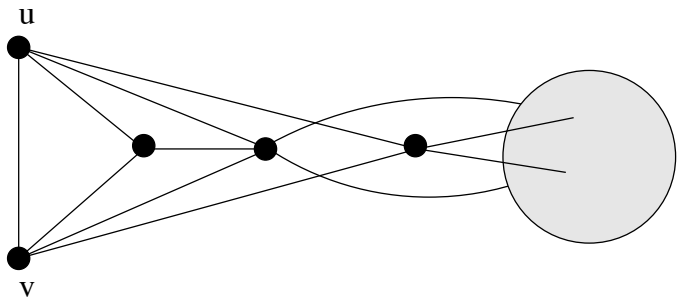

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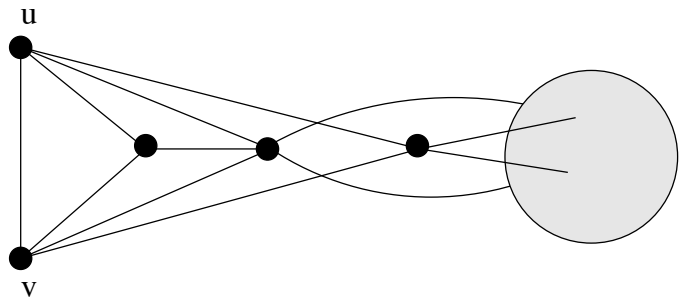

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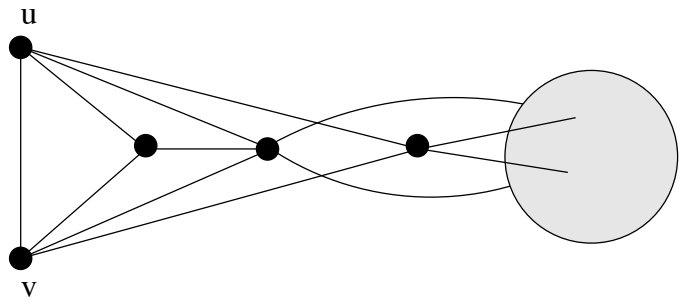

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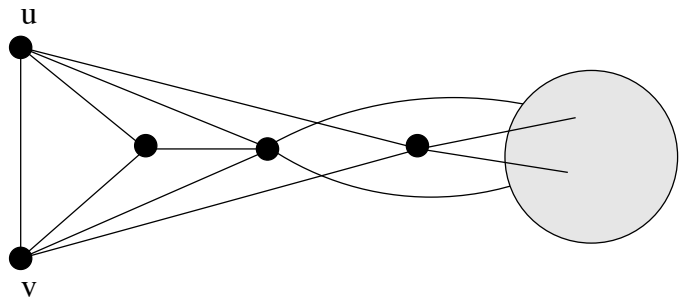

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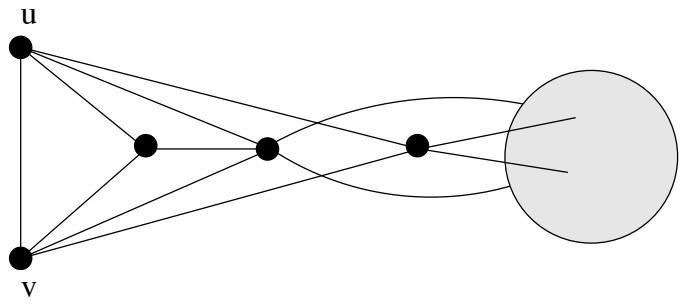

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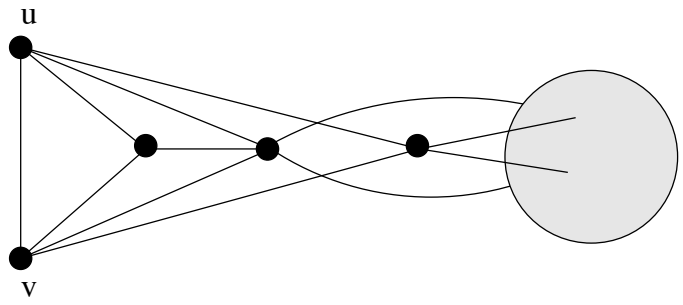

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- We are most interested in infinite grids.

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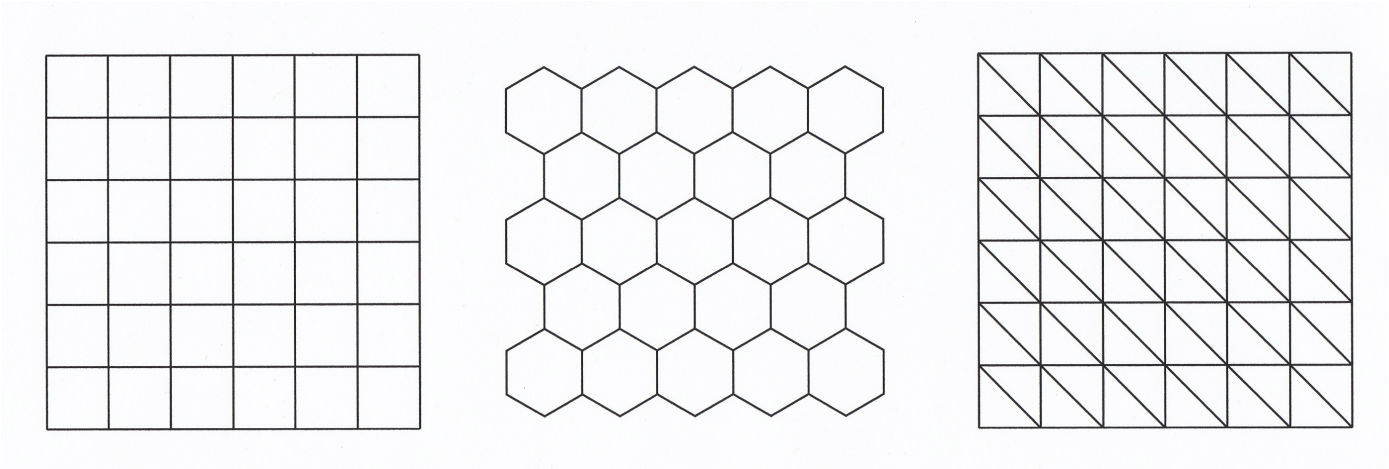

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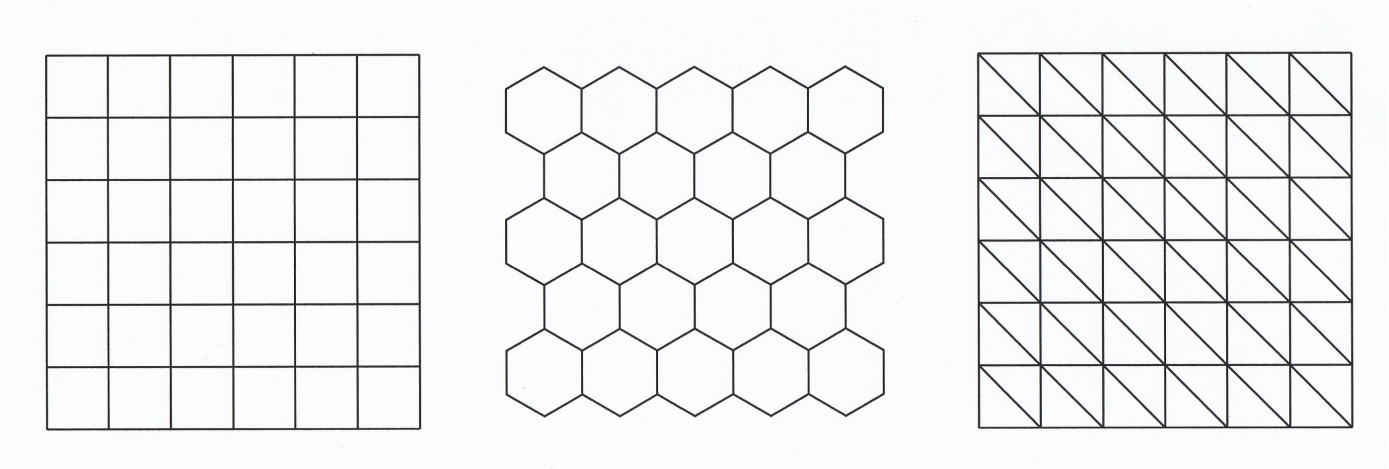

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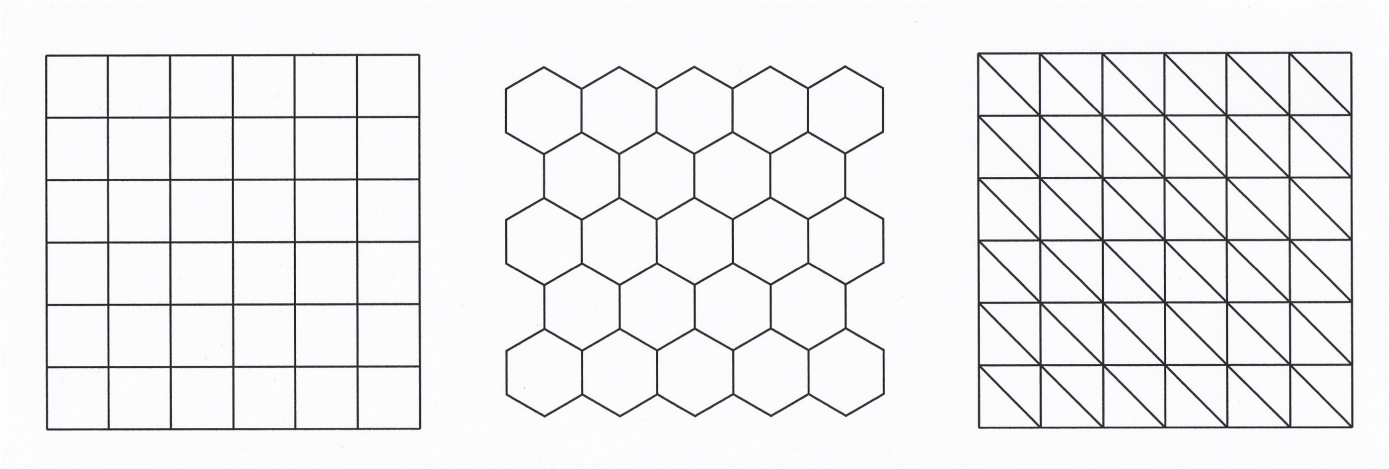

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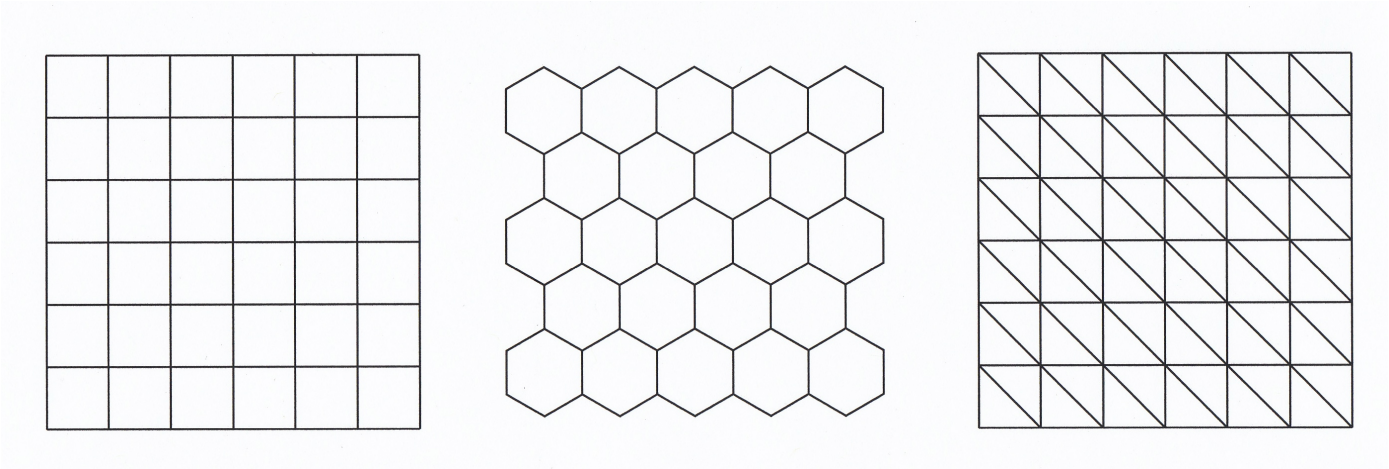

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- Question: what is $\tau(G_Z)$?

Density of Square and Triangular Grids

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- Square Grid: Cohen-Hongala-Lobstein-Zémor (2000) showed that $\tau \leq \frac{7}{20}$; and Ben-Haim-Litsyn (2005) showed that $\tau \geq \frac{7}{20}$.

Code for Hex Grid—Upper bound

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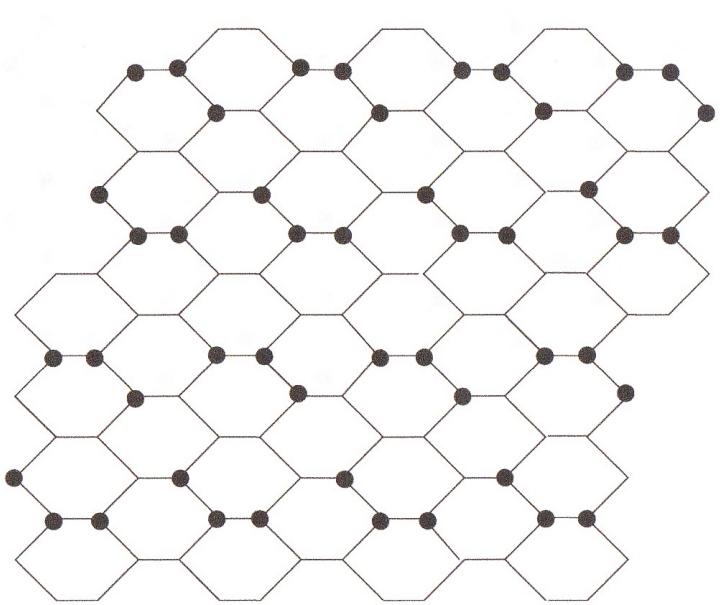

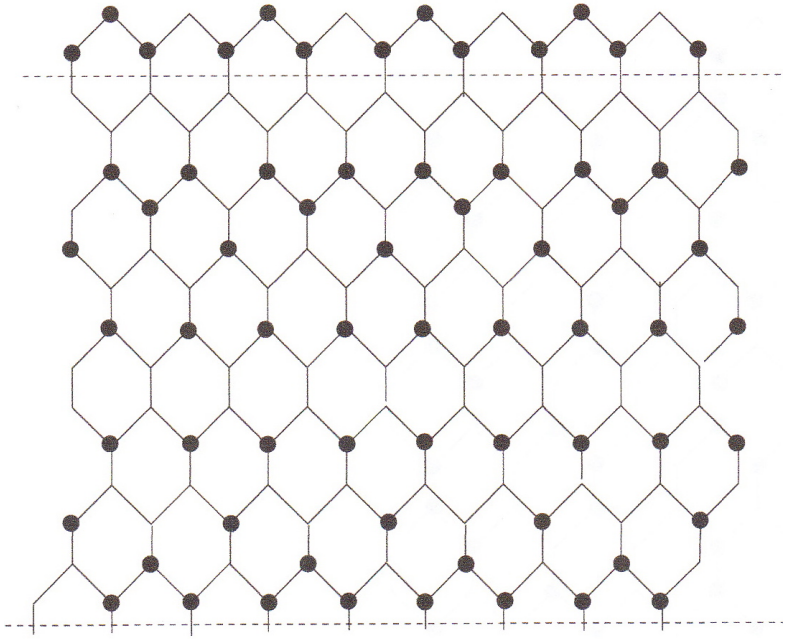

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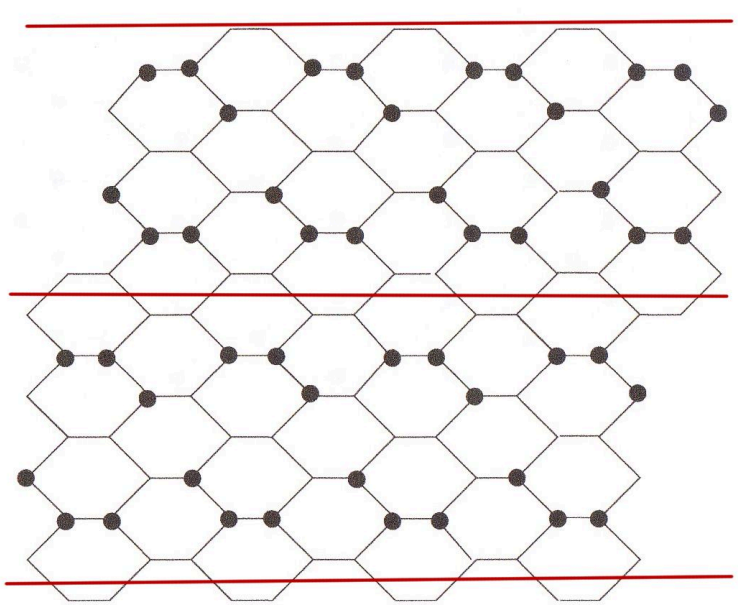

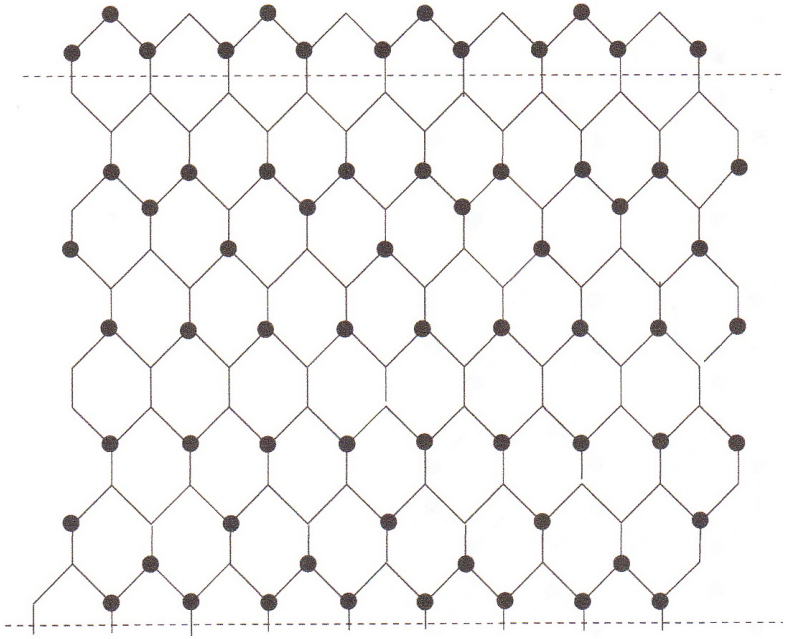

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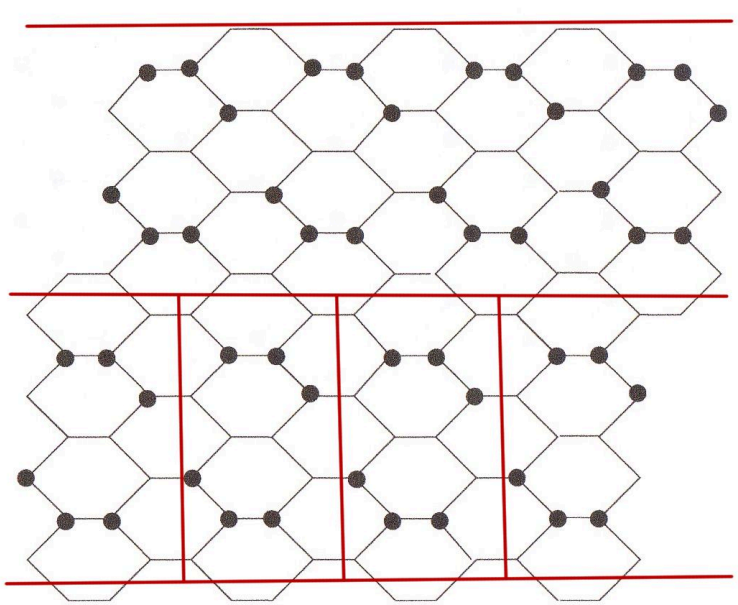

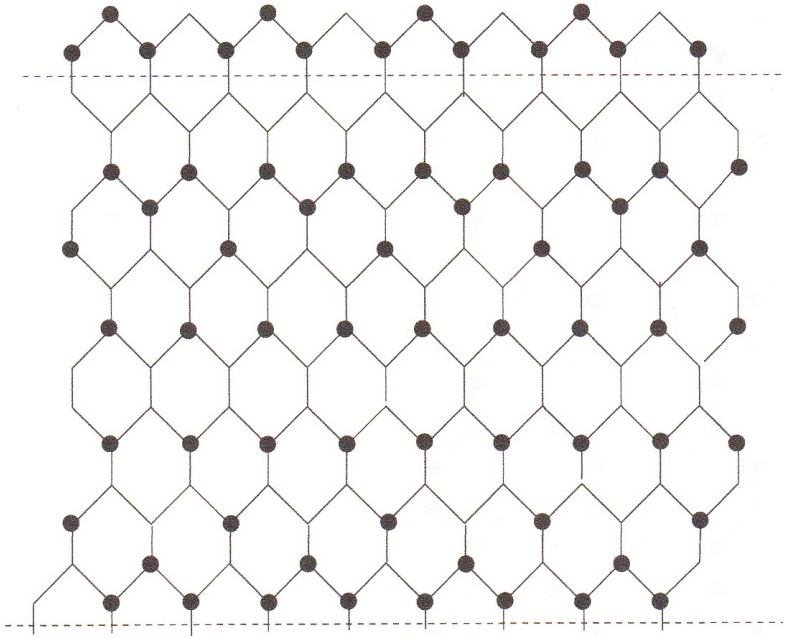

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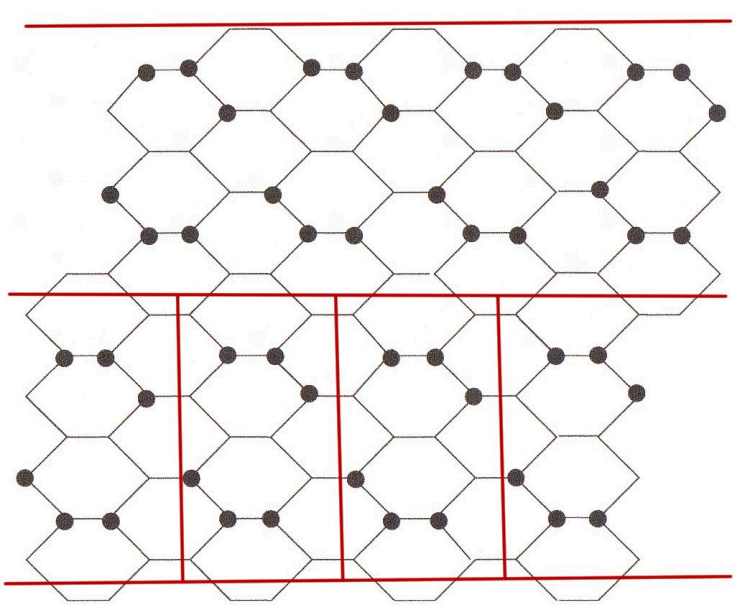

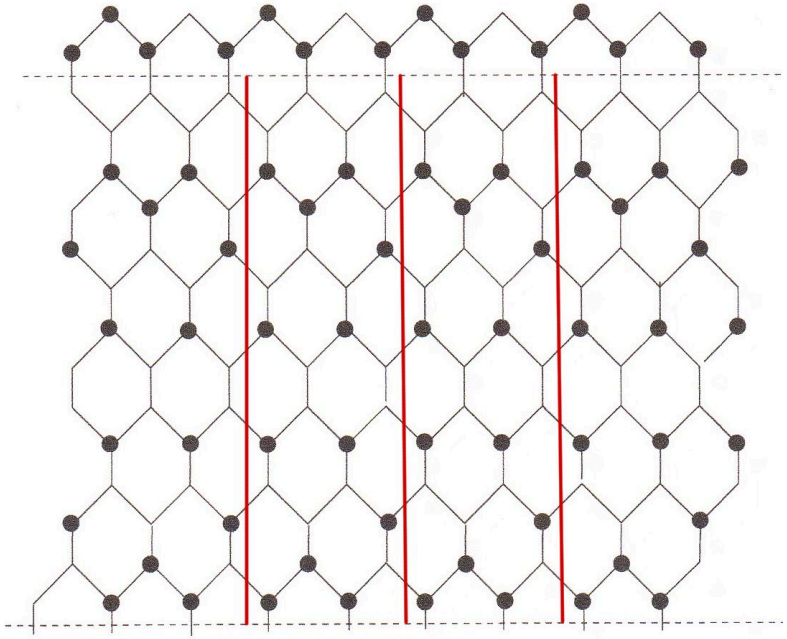

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- Theorem: $\tau \geq \frac{12}{29} \approx 0.41379...$

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- Key: how should we share the cake?

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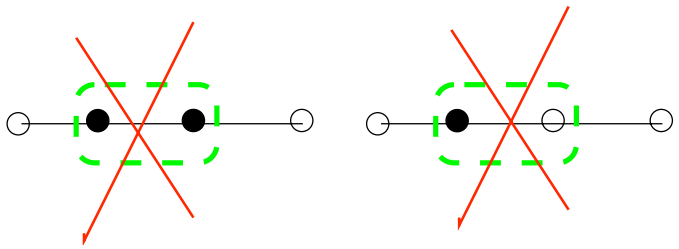

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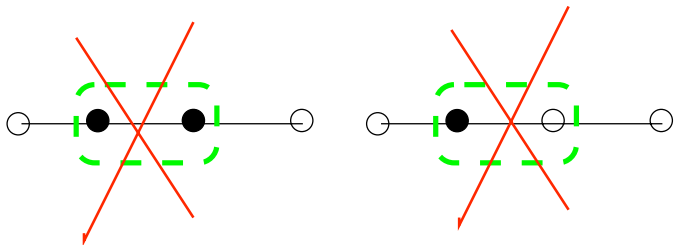

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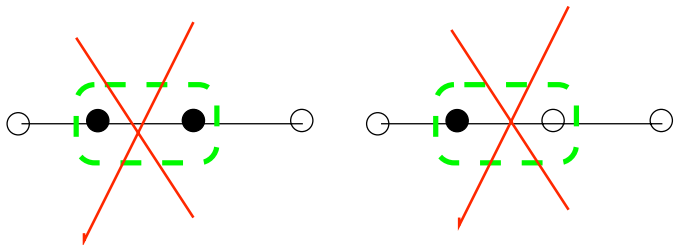

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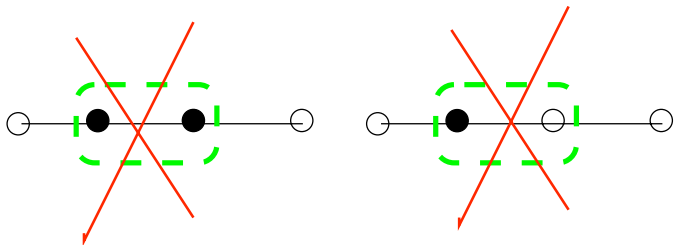

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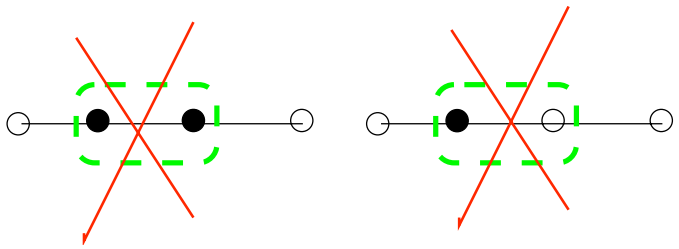

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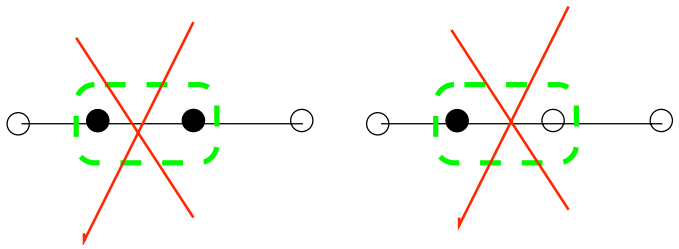

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  - $v \notin D$: $k\frac{2}{5k} = \frac{2}{5}$.
  - v in a 1-cluster: $1 - 3(\frac{2}{5(2)}) = \frac{2}{5}$.
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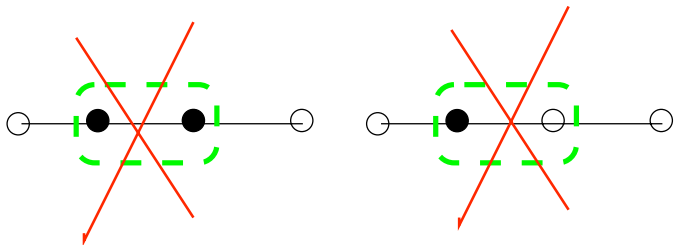
    - v has 3 neighbors in cluster: $1 - 0 = 1$

Proving a Lower Bound (real proof)

- Theorem: For the hex grid, $\tau \geq \frac{2}{5}$.

  Proof. each vertex in D gives $\frac{2}{5k}$ cake to each neighbor not in D that has k neighbors in D.

  We must show that each vertex v has at last $\frac{2}{5}$ of a cake.



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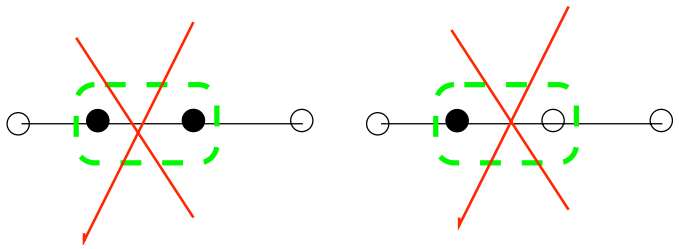

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- How about 3-dimensional grids?
- It is NP-hard to find the minimum ID-code for a given graph, even a given connected planar graph with maximum degree 4 and girth at least $k \geq 3$.

  Can we find any good bounds for such graphs?

Questions?