Some previous honors projects

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Preserver Problems

General problem

Let $\mathcal{M}$ be a matrix space. Give a complete description of those functions $f : \mathcal{M} \to \mathcal{M}$ with some special properties (related to topics on computational mathematics).

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- Let $\sim$ be a relation on $\mathcal{M}$. Study $f : \mathcal{M} \to \mathcal{M}$ such that $f(A) \sim f(B)$ whenever (if and only if) $A \sim B$ in $\mathcal{M}$.

Examples

Theorem [Frobenius, 1897]

A linear function $f : M_n \to M_n$ satisifes

$$\det(f(A)) = \det(A) \qquad \text{for all } A \in M_n$$

if and only if there are $M, N \in M_n$ with $\det(MN) = 1$ such that

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- Preservers of the rank function, or the set of rank-k matrices.
- Preservers of the eigenvalues, spectrum, or spectral radius.
- Preservers of the numerical range of $A \in M_n$ defined by

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n,\ x^*x = 1\}.$$

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Also, we studied the problem of partitioning a set of $(0, 1)$ matrices into different subsets with special structures.

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Suppose $f : M_{m,n} \to M_{m,n}$ is Schur multiplicative and maps rank k matrices to rank k matrices for $k \leq 1$. Then there are permutation matrices $P \in M_m$, $Q \in M_n$, and a multiplicative function $\tau : \mathbb{C} \to \mathbb{C}$ with $\tau(\mathbb{C}^*) \subseteq \mathbb{C}^*$ such that

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- $m = n$ and $f(A) = P(\tau(a_{ij}))^t Q$ for all $A = (a_{ij}) \in M_{m,n}$.

Recall that the spectral radius of a matrix $A \in M_n$ is

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For $1 \leq k < n$, the rank-k numerical range of $A \in M_n$ is defined by

$$\Lambda_k(A) = \{\gamma \in \mathbb{C} : \exists\, X \in M_{n,k} \text{ with } X^*X = I_k \text{ and } X^*AX = \gamma I_k\}.$$

Theorem [Clark, Li, Mahle and Rodman, 2009]

A linear function $f : M_n \to M_n$ satisfies

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We also solved the problem for Hermitian matrices in the same paper, and characterized multiplicative functions which preserve the rank k-numerical range in a paper with Sze.

Current and future projects

- With Herman, Forstall, Yannello, we are studying functions preserving different eigenvalue containment regions $R(A)$ such as the Gershgorin region, the Brauer set, the Ostrowski set. In particular, we characterize f such that

$$R(f(A) - f(B)) = R(A - B) \qquad \text{for all } A, B \in M_n,$$

and

$$R(f(A)f(B)) = R(AB) \qquad \text{for all } A, B \in M_n.$$

- I also plan to study preservers of the pseudo spectrum of matrices.
- What do you want to preserve?

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