A NONLINEAR STURM-LIOUVILLE THEOREM

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A well known theorem of Sturm-Liouville in the theory of ordinary differential equations concerns the eigenvalue problem:

(1) $$\mathcal{L}u \equiv -(p(x)u')' + q(x)u = \lambda a(x)u \ , \quad 0 < x < \pi$$

together with the separated boundary conditions (henceforth denoted by B.C.)

(2) $$a_0 u(0) + b_0 u'(0) = 0 \ , \quad a_1 u(\pi) + b_1 u'(\pi) = 0$$

where $(a_0^2 + b_0^2)(a_1^2 + b_1^2) \neq 0$ and p, q, a are respectively continuously differentiable and positive, continuous, and continuous and positive on the interval $[0,\pi]$. Under the above assumptions, the eigenvalues μ_n of (1) - (2) are simple and form an increasing sequence with $\mu_n \to \infty$ as $n \to \infty$. In addition, any eigenvector v_n corresponding to μ_n has exactly $(n - 1)$ simple zeroes in $(0,\pi)$. We are interested in obtaining a nonlinear version of this result together with some applications.

First the result will be stated in a somewhat different fashion. As a technical convenience for what follows suppose that 0 is not an eigenvalue for $\mathcal{L}$ under the B.C. (2). Then (1) - (2) can be converted to an equivalent integral equation:

(3) $$u(x) = \lambda \int_0^\pi g(x, y)\, a(y)\, u(y)\, dy$$

where g is the Greens function for $\mathcal{L}$ together with the B. C. Let $E = C^1[0,\pi] \cap$ B. C. under the usual maximum norm:

$$\|u\|_1 = \max_{x\in[0,\pi]} |u(x)| + \max_{x\in[0,\pi]} |u'(x)| \ .$$

By a solution of (3) we mean a pair $(\lambda, u) \in \mathbb{R} \times E$. Let S_k^+ denote the set of $\varphi \in E$ such that φ has exactly $k - 1$ simple zeroes in $(0,\pi)$, all zeroes of φ in $[0,\pi]$ are simple, and φ is positive in a deleted neighborhood of $x = 0$. Set $S_k^- = -S_k^+$ and $S_k = S_k^+ \cup S_k^-$. Then S_k^+, S_k^-, and S_k are open subsets of E and any eigenfunction v_k corresponding to μ_k belongs to S_k. We make v_k unique by requiring that $\|v_k\|_1 = 1$ and $v_k \in S_k^+$.

Using the terminology just introduced, the Sturm-Liouville theorem can be reformulated as follows: (3) possesses a line of trivial solutions $\{(\lambda, 0) | \lambda \in \mathbb{R}\}$ and in addition for each integer $k > 0$, a line of nontrivial solutions given by $\{(\mu_k, \alpha v_k) | \alpha \in \mathbb{R}\}$. We will obtain a nonlinear analogue of this result. Consider

(4) $$\mathcal{L}u = \lambda F(x, u, u') , \quad 0 < x < \pi$$

together with the B.C. (2). It is assumed that F is a continuous function of its arguments in $[0, \pi] \times \mathbb{R}^2$ and $F(x, \xi, \eta) = a(x)\xi + 0((\xi^2 + \eta^2)^{1/2})$ near $(\xi, \eta) = (0, 0)$. As above, (4), (2) can be converted to the equivalent integral equation:

(5) $$u(x) = \lambda \int_0^{\pi} g(x, y) F(y, u(y), u'(y)) dy .$$

Because of the form of F, (5) also possesses the line of trivial solutions $\{(\lambda, 0) | \lambda \in \mathbb{R}\}$. Let $\mathcal{S}$ denote the closure in $\mathbb{R} \times E$ of the set of nontrivial solutions of (5). By a theorem of Krasnoselski ([4]), the only possible trivial solutions belonging to $\mathcal{S}$ are the points $(\mu_k, 0)$, $k \in \mathbb{N}$, i.e. the possible bifurcation points.

Concerning the structure of $\mathcal{S}$, we have:

<u>Theorem 6</u>: For each integer $k > 0$, $\mathcal{S}$ contains a component, $\mathcal{C}_k$, which meets $(\mu_k, 0)$ and is unbounded in $\mathbb{R} \times S_k$.

(By a component of $\mathcal{S}$ we mean a maximal closed connected subset). Thus the statement of Theorem 6 contains in particular the linear case (3) where $\mathcal{C}_k$ is a line. To prove Theorem 6, a general theorem from nonlinear functional analysis which will be stated below and two lemmas are employed.

Let $\hat{E}$ be a real Banach space and $G : \mathbb{R} \times \hat{E} \to \hat{E}$ be compact, i.e. be continuous and map bounded sets into relatively compact sets. Suppose further that $G(\lambda, u) = \lambda Lu + H(\lambda, u)$ where L is a compact linear map and $H(\lambda, u) = 0(\|u\|)$ near $u = 0$ uniformly on bounded λ intervals. Consider the equation

(7) $$u = G(\lambda, u) .$$

A solution of (7) is a pair $(\lambda, u) \in \mathbb{R} \times \hat{E}$. Then (7) possesses the line of trivial solutions. Let $\hat{\mathcal{S}}$ denote the closure of the set of nontrivial solutions of (7) and let μ be a real characteristic value of L. Then we have:

<u>Theorem 8</u>: If μ is a real characteristic value of L of odd multiplicity, $\hat{\mathcal{S}}$ contains a component, $\mathcal{C}$, containing $(\mu, 0)$ and which is either unbounded or meets $(\hat{\mu}, 0)$

where $\mu \neq \hat{\mu}$ is a real characteristic value of L.

For a proof of Theorem 8, see Rabinowitz [7] or [8].

Note that (5) has the form (7) and μ_k being a simple eigenvalue of $\mathfrak{L}$ is a simple characteristic value of the corresponding linear integral operator. Hence the hypotheses of Theorem 8 are satisfied here and $\mathcal{C}_k$, the component of $\mathcal{S}$ containing $(\mu_k, 0)$ is either unbounded in $\mathbb{R}\times E$ or meets $(\mu_j, 0)$, $j \neq k$. Actually $\mathcal{C}_k$ is unbounded in $\mathbb{R}\times S_k$ as we shall see via the following two lemmas.

Lemma 9: There exists a neighborhood $\mathcal{N}_j$ of $(\mu_j, 0)$ such that $(\lambda, u) \in \mathcal{N}_j \cap \mathcal{S}$ implies $u \in S_j$ or $u \equiv 0$.

Proof: If not, there exists a sequence $(\lambda_n, u_n) \in \mathcal{S}$ such that $(\lambda_n, u_n) \to (\mu_j, 0)$ as $n \to \infty$ and $u_n \notin S_j$. From (5) or equivalently (7),

$$(10) \qquad \frac{u_n}{\|u_n\|_1} = \lambda_n L \frac{u_n}{\|u_n\|_1} + \frac{H(\lambda_n, u_n)}{\|u_n\|_1} .$$

The $0(\|u\|_1)$ condition on H implies $\|u_n\|_1^{-1} H(\lambda_n, u_n) \to 0$ as $n \to \infty$. Moreover the compactness of L and boundedness of $\{u_n / \|u_n\|_1\}$ implies that $\{Lu_n / \|u_n\|_1\}$ possesses a convergent subsequence $\{Lu_{n_p} / \|u_{n_p}\|_1\}$. From (10) this subsequence converges in E to v with $\|v\|_1 = 1$ and satisfying

$$(11) \qquad v = \mu_j L v .$$

Hence $v = v_j$ or $v = -v_j$. In either event $v \in S_j$ and since S_j is open, $u_{n_p} / \|u_{n_p}\|_1 \in S_j$ for all j large. But this implies $u_{n_p} \in S_j$, a contradiction. Thus the lemma is established.

Lemma 12: Suppose (λ, u) is a solution of (4) and u has a double zero, i.e. there exists $\tau \in [0, \pi]$ such that $u(\tau) = 0 = u'(\tau)$. Then $u \equiv 0$.

Proof: If $F(x, \xi, \eta)$ is locally Lipschitz continuous with respect to (ξ, η), the result follows immediately from the uniqueness theorem for the initial value problem for (4). The more general case amounts to reproving the uniqueness result for the special initial data (0, 0) using the "0-condition" on $F - a\xi$. See [8] for a proof.

Proof of Theorem 6: Fix $k > 0$. Suppose $\mathcal{C}_k \subset (\mathbb{R}\times S_k) \cup \{(\mu_k, 0)\}$. Then by Theorem 8, $\mathcal{C}_k$ must be unbounded in this set and we are through. Thus suppose

$\mathcal{C}_k \not\subset (\mathbb{R}\times S_k) \cup \{(\mu_k, 0)\}$. By Lemma 9, $\mathcal{C}_k \cap \mathcal{M}_k \subset (\mathbb{R}\times S_k) \cup \{(\mu_k, 0)\}$. Since $\mathcal{C}_k$ is connected, there exists $(\bar{\lambda}, \bar{u}) \in \mathcal{C}_k \cap (\mathbb{R}\times \partial S_k)$, $(\bar{\lambda}, \bar{u}) \neq (\mu_k, 0)$, and $(\bar{\lambda}, \bar{u}) = \lim_{n\to\infty} (\lambda_n, u_n)$ with $(\lambda_n, u_n) \in \mathcal{C}_k \cap (\mathbb{R}\times S_k)$. This implies $\bar{u}$ has a double zero and by Lemma 12, $\bar{u} \equiv 0$. Hence $(\bar{\lambda}, \bar{u}) = (\mu_j, 0)$ for some $j \neq k$. (Recall that the only trivial solutions belonging to $\mathcal{S}$ correspond to bifurcation points). But this implies $(\lambda_n, u_n) \in \mathcal{M}_j$ and therefore $u_n \in S_j$ for n large. Since this is impossible, the theorem is proved.

Remarks: If 0 is an eigenvalue of $\mathcal{L}$, Theorem 6 still obtains with the aid of an approximation argument. See Rabinowitz [6] or [8]. A sharper version of Theorem 8 for μ a simple characteristic value (see [7]) shows that $\mathcal{C}_k = \mathcal{C}_k^+ \cup \mathcal{C}_k^-$ where $\mathcal{C}_k^+ \cap \mathcal{C}_k^- = \{(\mu_k, 0)\}$ and $\mathcal{C}_k^+$, $\mathcal{C}_k^-$ are unbounded in $\mathbb{R}\times S_k^+$, $\mathbb{R}\times S_k^-$ respectively. Thus we get an even nicer correspondence with the linear case (3). If F is smooth near $(\xi, \eta) = (0, 0)$, then a general theorem on bifurcation from simple eigenvalues (see e.g. [2] or [3]) implies that $\mathcal{C}_k$ near $(\mu_k, 0)$ is a smooth curve of the form $(\lambda, u) = (\mu_k + 0(1), \alpha v_k + 0(|\alpha|))$ for α near 0. Lastly we note that Theorem 6 can readily be generalized to permit $\mathcal{L}$ to be nonlinear and a more general dependence of $\mathcal{L}$ and F on λ (see [6]).

To give some idea of what the sets $\mathcal{C}_k$ may look like, consider the following example:

$$(13)\qquad \begin{cases} -u'' = \lambda(1 + f(u^2 + (u')^2))u & 0 < x < \pi \\ u(0) = 0 = u(\pi) \end{cases}$$

where f is continuous and $f(0) = 0$. The linearization of (13) about $u = 0$ is:

$$(14)\qquad -v'' = \mu v \;, \quad 0 < x < \pi \;, \quad v(0) = 0 = v(\pi)$$

which possesses eigenvalues $\mu_n = n^2$ and eigenfunctions $v_n = \alpha_n \sin nx$. We will study $\mathcal{C}_1$ for (14) and in particular try for solutions of the form $(\lambda, u) = (\lambda, c \sin x)$. This leads to the equation:

$$(15)\qquad 1 = \lambda(1 + f(c^2))$$

relating λ and c. The freedom we have in choosing f leads to a wide range of possible behavior for $\mathcal{C}_1$.

Next some qualitative consequences of Theorem 6 will be studied.

Corollary 16: Suppose in (4) $F(x, \xi, \eta) = a(x)\xi + f(x, \xi, \eta)\xi$ where f is continuous in its arguments and $f(x, 0, 0) \equiv 0$. If $\mu_k > 0$ and $f \geq 0$, then $\mathcal{C}_k$ lies in $[0, \mu_k] \times S_k$ while if $\mu_k > 0$ and $f \leq 0$, $\mathcal{C}_k$ lies in $[\mu_k, \infty) \times S_k$.

Proof: The result is an immediate consequence of Theorem 6 together with a comparison argument which shows if e.g. $f \geq 0$ and $(\lambda, u) \in (\mathbb{R} \times S_k) \cap \mathcal{S}$, then $\lambda \in [0, \mu_k]$. See [6] for details.

The effect of a priori bounds on the sets $\mathcal{C}_k$ will be studied next.

Corollary 17: Suppose there exists a continuous real valued function $M(\lambda)$ for $\lambda \in \mathbb{R}^+$ such that (λ, u) a solution of (5) with $\lambda \geq 0$ implies $\|u\|_1 \leq M(\lambda)$. If $\mu_k > 0$, then for all $\lambda \in (\mu_k, \infty)$ there exists $u \in S_k$ such that $(\lambda, u) \in \mathcal{C}_k$.

Proof: By Theorem 6, $\mathcal{C}_k$ is unbounded in $\mathbb{R} \times S_k$. Note that $(0, u)$ cannot be a solution of (5). Hence $\mathcal{C}_k$ lies in $\mathbb{R}^+ \times S_k$. The existence of $M(\lambda)$ implies the projection of $\mathcal{C}_k$ on $\mathbb{R}^+$ cannot be bounded. Hence the result follows from the connectedness of $\mathcal{C}_k$.

Conditions under which such a priori bounds may be obtained can be found in Crandall and Rabinowitz [2] and Wolkowiskey [8]. As another application of Theorem 6 involving a different kind of a priori bound, we will prove a generalized version of a theorem of Nehari [5]. Consider

$$(18) \qquad -u'' = f(x, u)u, \quad 0 < x < \pi, \quad u(0) = 0 = u(\pi)$$

where f is continuous on $[0, \pi] \times \mathbb{R}$, $f(x, 0) = 0$, $f(x, u) > 0$ if $u \neq 0$, and there exists a continuous function $\rho : \mathbb{R} \to \mathbb{R}^+$ with $\rho(s) \to \infty$ as $|s| \to \infty$ and such that $|u| > s$ implies $f(x, u) > \rho(s)$.

Note that (18) differs from the equations treated earlier in that its right hand side has no linear part at $u = 0$.

Theorem 19: Under the above hypotheses on f, for each integer $k > 0$ there exists $u_k \in S_k$ such that u_k satisfies (18).

To prove Theorem 19, we require the following lemma which will be proved in the Appendix.

Lemma 21: Consider the equation

(22) $\quad -u'' = \lambda(b(x) + f(x,u))u \;,\quad 0 < x < \pi \;,\quad u(0) = 0 = u(\pi)$

where f is as above and $b \geq 0$ is continuous in $[0,\pi]$. Then for each integer $k > 0$ there exists a continuous function $M_k(\lambda) : (0,\infty) \to \mathbb{R}^+$ such that if (λ, u) is a solution of (22) with $u \in S_k$, then $\|u\|_1 \leq M_k(\lambda)$.

Proof of Theorem 19: An approximation argument is used. Let $\theta \in (0,1)$. Consider

$(23)_\theta$ $\quad -u'' = \lambda(\theta + f(x,u))u \;,\quad 0 < x < \pi \;;\quad u(0) = 0 = u(\pi)\;.$

The eigenvalues $\omega_k(\theta)$ of

(24) $\quad -w'' = \omega\theta w \;,\quad 0 < x < \pi \;;\quad w(0) = 0 = w(\pi)$

are $\omega_k(\theta) = k^2/\theta > 1$ for all $k \geq 1$. By Corollary 16 $(23)_\theta$ possesses a component $\mathcal{C}_k(\theta)$ of solutions which is unbounded in $[0, \omega_k(\theta)] \times S_k$. By Lemma 21, the projection of $\mathcal{C}_k(\theta)$ on $\mathbb{R}$ contains $(0, \omega_k(\theta))$. In particular there exists $u_k(\theta) \in S_k$ such that $(1, u_k(\theta)) \in \mathcal{C}_k(\theta)$ and $\|u_k(\theta)\|_1 \leq M_k(1,\theta)$. The proof of Lemma 21 shows that M_k can be chosen independent of θ. From $(23)_\theta$, $\max\limits_{x\in[0,\pi]} |u_k''(\theta)|$ can be bounded independently of θ. These bounds, the Arzela Ascoli Theorem, and $(23)_\theta$ imply there is a sequence $\theta_n \to 0$ as $n \to \infty$ such that $u_k(\theta_n)$ converges in $C^2[0,\pi]$ to a solution u_k of (18) with $u_k \in \overline{S_k}$. It only remains to show that $u_k \not\equiv 0$. But this follows by the argument of Lemma 9. The theorem is proved.

Remark 25: By using the ideas contained in the above proof, a version of Theorem 19 can be obtained for (22) with $\lambda = 1$.

Many people have studied nonlinear eigenvalue problems such as (4), (2) by examining a corresponding initial value problem and using shooting techniques. It seems unlikely that such methods can be used to obtain Theorem 6. On the other hand, Theorem 6 can be employed to shed some light on the corresponding initial value problem. For convenience we replace (2) by the B.C.

(26) $$u(0) = 0 = u(\pi)\;.$$

Moreover we assume F in (4) is Lipschitz continuous in ξ and η and therefore the initial value problem for (4) possesses a unique solution.

Consider the map $\Psi : \mathcal{S} \to \mathbb{R}^2$, $\Psi(\lambda,u) = (\lambda, u'(0))$. The map Ψ is 1-1 via uniqueness of solutions to the initial value problem and is continuous. Therefore $\Psi(\mathcal{C}_k) \equiv \mathcal{J}_k$ is a connected subset of $\mathbb{R}^2$ and $\mathcal{J}_k \cap \mathcal{J}_j = \emptyset$ if $k \neq j$. Note that even though $\mathcal{C}_k$ is unbounded in $\mathbb{R} \times E$, $\mathcal{S}_k$ may be bounded in $\mathbb{R}^2$. By the remarks following the proof of Theorem 6, $\mathcal{S}_k = \mathcal{S}_k^+ \cup \mathcal{S}_k^- \equiv \Psi(\mathcal{C}_k^+) \cup \Psi(\mathcal{C}_k^-)$. As an interesting consequence of Corollary 16 we get

Corollary 27: Suppose in addition to the hypotheses of Corollary 17, F is Lipschitz continuous in ξ and η and (26) obtains. Let μ_r be the smallest positive eigenvalue of (3), (2). If $\lambda > \mu_k > \mu_r$, then there exist constants $C_r^+ > \ldots > C_k^+ > 0 > C_k^- > \ldots > C_r^-$ such that $(\lambda, C_j^\pm) \in \mathcal{S}_j^\pm$, $r \leq j \leq k$.

Proof: The result follows immediately from Corollary 16 and the properties of the sets $\mathcal{S}_j^\pm$.

Since C_r^+, $-C_r^- \leq M(\lambda)$, one could use a shooting technique in the interval of initial derivatives $[-M(\lambda), M(\lambda)]$ to find the solutions whose existence is given by Corollary 27.

We conclude with some remarks on periodic B.C. Suppose all functions involved in (1) and (4) are π periodic in x and (2) is replaced by

$$u(0) = u(\pi) \ , \quad u'(0) = u'(\pi) \ . \tag{28}$$

This case is interesting because some of the important structure obtained for the separated B.C. case is lost. The linear theory here again gives an increasing sequence of eigenvalues ζ_n with $\zeta_n \to \infty$ as $n \to \infty$. However the eigenvalues need not be simple although they are of multiplicity at most 2 and then have two corresponding linearly independent eigenvectors. More precisely (see Coddington-Levinson [1]) $\zeta_1 < \zeta_2 \leq \zeta_3 < \zeta_4 \leq \zeta_5 < \ldots$ etc. Any eigenfunction corresponding to ζ_1 has no zeroes in $[0,\pi]$; any eigenfunctions corresponding to ζ_{2k}, ζ_{2k+1}, $k \geq 1$ have exactly 2k simple zeroes in $[0,\pi]$. Thus in particular, ζ_1 is a simple eigenvalue. We again set up a family of open sets to take advantage of the nodal properties. Let $\hat{E}$ denote the subset of $C^1[0,\pi]$ of π periodic functions. Let $T_0^+ = \{\varphi \in \hat{E} \mid \varphi > 0\}$, $T_0^- = -T_0^+$, $T_0 = T_0^+ \cup T_0^-$, and $T_k = \{\varphi \in \hat{E} \mid \varphi$ has exactly 2k simple zeroes in $[0,\pi)\}$.

Consider now (4), (28). Again as a convenience we assume 0 is not an eigenvalue of $\mathcal{L}$. Hence (4), (28) can be converted to an operator equation of the form (7). Let $\hat{\mathcal{S}}$ denote the closure of the set of nontrivial solutions of this equation. With a small modification, the proof of Lemma 9 gives us:

Lemma 29: There exists a neighborhood $\mathfrak{m}_j$ of $(\zeta_{2j}, 0)$ (and $(\zeta_{2j+1}, 0)$ if $j \neq 0$)

such that $(\lambda, u) \in \mathfrak{m}_j \cap \hat{\mathcal{S}}$ implies $u \in T_j$ or $u \equiv 0$.

Since ζ_1 is a simple eigenvalue of (1), (28), a combination of Lemmas 29 and 12, and Theorem 8 yields:

Theorem 30: $\hat{\mathcal{S}}$ contains a component $\hat{\mathcal{C}}_1$ which meets $(\zeta_1, 0)$ and is unbounded in $\mathbb{R} \times T_0$.

Whenever ζ_{2k}, $k \geq 1$ is simple, the above lemmas and Theorem 8 implies that $\hat{\mathcal{S}}$ contains a component $\hat{\mathcal{C}}_k$ in $(\mathbb{R} \times T_k) \cup \{(\zeta_{2k}, 0)\}$ meeting $(\zeta_{2k}, 0)$. However $\hat{\mathcal{C}}_k$ need not be unbounded in $\mathbb{R} \times T_k$ but may also meet $(\zeta_{2k+1}, 0)$. Moreover if $\zeta_{2k} = \zeta_{2k+1}$, i.e. we have an eigenvalue of multiplicity 2, bifurcation need not occur at all. A simple such example is given by:

$$(31) \qquad -u'' + u = \lambda(u + (u')^3) \qquad 0 < x < \pi$$

with u satisfying (28). Multiplying (31) by u' and integrating over a period yields:

$$\lambda \int_0^{2\pi} (u')^4 \, dx = 0 \quad .$$

Since the equation possesses no solutions when $\lambda = 0$, (31), (28) possesses only the trivial solutions and the line of solutions $\{(1,\alpha) \mid \alpha \in \mathbb{R}\}$ in $\mathbb{R} \times T_0$.

Thus in general other than for $(\zeta_1, 0)$ results analogous to Theorem 6 do not obtain for (4), (19) and even to obtain bifurcation more hypotheses will have to be made. One way to guarantee bifurcation and even some sort of global result is to impose variational structure on the problem. More precisely suppose that F in (4) is independent of u', $q \geq 0$, and $F(x,u) = \frac{\partial}{\partial u} \hat{F}(x,u)$ with $\hat{F}(x,0) \equiv 0$. Then (4), (19) is the Euler equation of the variational problem:

$$\text{Extremize} \int_0^{\pi} \hat{F}(x, \varphi)\, dx$$

over the class of $\varphi \in \hat{E}$ with

$$\int_0^{\pi} (p(\varphi')^2 + q\varphi^2)\, dx = R \ , \qquad R \text{ a constant} \ .$$

By a theorem of Krasnoselski [4], each point $(\zeta_k, 0)$ will be a bifurcation point for (4), (28). Moreover if $\hat{F}$ is odd in u and appropriate technical conditions are satisfied, it follows from a theorem of Ljusternik [4] that for each $R > 0$, there exist infinitely many distinct solutions $(\lambda_n(R), u_n(R))$ of (4), (28) with

$$\int_0^{\pi} (p|u_n'|^2 + q u_n^2)\, dx = R \ .$$

An interesting open question is whether for all integers $k \geq 0$, there exists a solution $(\lambda_k(R), u_k(R))$ with $u_k(R) \in T_k$.

APPENDIX

Proof of Lemma 21: The proof consists of two steps. First we show there exists $M_k(\lambda, c)$ such that if (λ, u) is as in the statement of the lemma with $|u'(0)| \leq c$, then $\|u\|_1 \leq M_k(\lambda, c)$. Then we show $M_k(\lambda, c)$ can be chosen independently of c.

Suppose $u'(0) > 0$. (The argument for $u'(0) < 0$ is the same.) Let y_1 be the first zero of u'. Then u is a monotone increasing function and from (18), u'' is a monotone decreasing function in $[0, y_1]$. Hence u' is monotone decreasing in $[0, y_1]$ and $u'(0) = \max\limits_{[0,y_1]} u'(x) \leq c$, $\max\limits_{[0,y_1]} u \leq c\, y_1$. Let z_1 be the first zero of u in $(0, \pi)$. Then from (18), u and u' are monotone decreasing in $[y_1, z_1]$ so $\max\limits_{[y_1,z_1]} u \leq c\, y_1$. Moreover integrating (18):

$$(32) \qquad -u'(z_1) = \lambda \int_{y_1}^{z_1} (b + f) u \, dx \ .$$

The bounds obtained for u in $[y_1, z_1]$ and (32) give a bound for $|u'(x)|$ in $[y_1, z_1]$. Continuing in this fashion leads to an estimate $\|u\|_1 \leq M_k(\lambda, c)$ where M_k is continuous in λ and c. Note also that if u' is known at any zero z_j of u, an estimate of the same form for $\|u\|_1$ obtains with c replaced by $|u'(z_j)|$.

It remains to show that $M_k(\lambda, c)$ can be chosen independently of c. If not, there exists a sequence (λ, u_n) satisfying (18) with $u_n \in S_k$ and $|u_n'(0)| \to \infty$. Let $\sigma_{j,n}$ denote the j^{th} zero of u_n in $[0, \pi]$, $0 \leq j \leq k$. By our above remarks, $|u_n'(\sigma_{j,n})| \to \infty$ as $n \to \infty$. Let $I_{j,n} = [\sigma_{j,n}, \sigma_{j+1,n}]$. Since u_n' has a zero in $I_{j,n}$, the Mean Value Theorem implies that $\max\limits_{I_{j,n}} |u_n''(x)| \to \infty$ as $n \to \infty$. Hence from (18), $\max\limits_{I_{j,n}} |u_n(x)| \to \infty$ as $n \to \infty$. Therefore for any $s > 0$, if $n = n(s)$ is sufficiently large, there exists $x_{j,n} \in I_{j,n}$ such that $u_n(x_{j,n}) > s$ and $f(x_{j,n}, u_n(x_{j,n})) > \rho(s)$.

Consider the subinterval of $I_{j,n}$ in which $s = s(\lambda, k)$ is so large that $\rho(s) > \frac{4k^2}{\lambda}$ and therefore $\lambda(b + f) > 4k^2$. By the Sturm Comparison Theorem the length of this subinterval is less than $\frac{\pi}{2k}$. At least one of the intervals $I_{j,n}$, say $I_{p,n}$ has length $\geq \frac{\pi}{k}$. Consequently we can find a subinterval $[\gamma_n, \delta_n]$ of length $\geq \frac{\pi}{4k}$ such that $|f| \leq \frac{k^2}{\lambda} = \rho(\bar{s})$, $|u| \leq \bar{s}$ with $|u| = \bar{s}$ at one end of the subinterval and $u = 0$ at the other end. For convenience suppose $u(\gamma_n) = 0$, $u(\delta_n) = \bar{s}$. From (18),

$$|u_n'(\gamma_n) - u_n'(\delta_n)| = \left|\int_{\gamma_n}^{\delta_n} \lambda(b + f)u\,dx\right| \leq (\lambda\ \|b\| + k^2)\bar{s}\ . \tag{33}$$

This implies:

$$u(\delta_n) = \bar{s} = \int_{\gamma_n}^{\delta_n} u'(x)dx \geq \frac{\pi}{4k}\ [|u_n'(\gamma_n)| - (\lambda\|b\| + k^2)\bar{s}]\ . \tag{34}$$

But the right hand side of (34) is unbounded as $n \to \infty$ while the left hand side is bounded. Hence we have a contradiction. The uniformity of the argument in λ on bounded λ intervals gives the continuity of M_k in λ and the lemma is proved.

Remark: Note that if b is replaced by θb, $\theta \in [0, y]$, $M_k(\lambda)$ can be chosen independently of θ. Note also that as $\lambda \to 0$, $M_k(\lambda) \to \infty$.

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