Reaction-Diffusion Models and Bifurcation Theory Lecture 9: More applications of local and global bifurcation

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Krasnoselski-Rabinowitz Global Bifurcation Theorem

Consider

$$F(\lambda, u) = u - \lambda L u - H(\lambda, u), \tag{1}$$

where $L : X \to X$ is a linear compact operator, and $H(\lambda, u)$ is compact on $U \subset \mathbb{R} \times X$ such that $||H(\lambda, u)|| = o(||u||)$ near $u = 0$ uniformly on bounded λ intervals. Note the conditions imply that $F_u(\lambda, 0) = I - \lambda L$, and if 0 is an eigenvalue of $F_u(\lambda_0, 0)$, then λ_0^{-1} must be an eigenvalue of the linear operator L. Define

$$S = \{(\lambda, u) \in U : F(\lambda, u) = 0, u \neq 0\}.$$

We say $(\lambda_0, 0)$ is a bifurcation point for the equation (1) if $(\lambda_0, 0) \in \overline{S}$ ($\overline{S}$ is the closure of S).

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Theorem 8.1. (Krasnoselski-Rabinowitz Global Bifurcation Theorem) [Rabinowitz, 1971, JFA] Let X be a Banach space, and let U be an open subset of $\mathbb{R} \times X$ containing $(\lambda_0, 0)$. Suppose that L is a linear compact operator on X, and $H(\lambda, u) : \overline{U} \to X$ is a compact operator such that $||H(\lambda, u)|| = o(||u||)$ as $u \to 0$ uniformly for λ in any bounded interval. If $1/\lambda_0$ is an eigenvalue of L with odd algebraic multiplicity, then $(\lambda_0, 0)$ is a bifurcation point. Moreover if C is the connected component of $\overline{S}$ which contains $(\lambda_0, 0)$, then one of the following holds:

(i) C is unbounded in U;

(ii) $C \bigcap \partial U \neq \emptyset$; or

(iii) C contains $(\lambda_i, 0) \neq (\lambda_0, 0)$, such that λ_i^{-1} is also an eigenvalue of L.

Unilateral bifurcation theorem

[Rabinowitz, 1971, JFA], [Dancer, 1974, Indiana Math J], [Lopez-Gomez, 2000, book]
Theorem 8.2. Let X be a Banach space, and let U be an open subset of $\mathbb{R} \times X$ containing $(\lambda_0, 0)$. Suppose that L is a linear compact operator on X, and $H(\lambda, u) : \overline{U} \to X$ is a compact operator such that $||H(\lambda, u)|| = o(||u||)$ as $u \to 0$ uniformly for λ in any bounded interval. Suppose that $1/\lambda_0$ is an eigenvalue of L with algebraic multiplicity 1. We define $\Gamma_+ = \{(\lambda(s), u(s)) : s \in (0, \epsilon)\}$ and $\Gamma_- = \{(\lambda(s), u(s)) : s \in (-\epsilon, 0)\}$. Let $\mathcal{C}$ be a connected component of $\overline{S}$ where $S = \{(\lambda, u) \in V : u - \lambda Lu - H(\lambda, u) = 0, u \neq 0\}$ containing $(\lambda_0, 0)$. Let $\mathcal{C}^+$ (resp. $\mathcal{C}^-$) be the connected component of $\mathcal{C} \backslash \Gamma_-$ which contains Γ_+ (resp. the connected component of $\mathcal{C} \backslash \Gamma_+$ which contains Γ_-). Then each of the sets $\mathcal{C}^+$ and $\mathcal{C}^-$ satisfies one of the following:
(i) it is unbounded;
(ii) it contains a point $(\lambda_*, 0)$ with $\lambda_* \neq \lambda_0$ such that $1/\lambda_*$ is also an eigenvalue of L; or
(iii) it contains a point (λ, z), where $z \neq 0$ and $z \in Z$ which any complement of $span\{w_0\} = \mathcal{N}(I - \lambda_0 L)$ in X.

Global Bifurcation From Simple Eigenvalue

Theorem 8.3 [Crandall-Rabinowitz, 1971, JFA]
Let $F : \mathbb{R} \times X \to Y$ be continuously differentiable. Suppose that $F(\lambda, u_0) = 0$ for $\lambda \in \mathbb{R}$, the partial derivative $F_{\lambda u}$ exists and is continuous. At (λ_0, u_0), F satisfies

(F1) $dimN(F_u(\lambda_0, u_0)) = codimR(F_u(\lambda_0, u_0)) = 1$, and

(F3) $F_{\lambda u}(\lambda_0, u_0)[w_0] \notin R(F_u(\lambda_0, u_0))$, where $w_0 \in N(F_u(\lambda_0, u_0))$,

Then the solutions of $F(\lambda, u) = 0$ near (λ_0, u_0) consists precisely of the curves $u = u_0$ and $(\lambda(s), u(s))$, $s \in I = (-\delta, \delta)$, where $(\lambda(s), u(s))$ are continuous functions such that $\lambda(0) = \lambda_0$, $u(0) = u_0$. If F is C^2 near (λ_0, u_0), then $u'(0) = w_0$, and

$$\lambda'(0) = -\frac{\langle l, F_{uu}(\lambda_0, u_0)[w_0, w_0]\rangle}{2\langle l, F_{\lambda u}(\lambda_0, u_0)[w_0]\rangle}.$$

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[Pejsachowicz-Rabier, 1998, J D'Anal Math] [Shi-Wang, 2009, JDE]
If in addition, $F_u(\lambda, u)$ is a Fredholm operator for all $(\lambda, u) \in \mathbb{R} \times X$, then the curve $\{(\lambda(s), u(s)) : s \in I\}$ is contained in $\mathcal{C}$, which is a connected component of $S = \{(\lambda, u) \in \mathbb{R} \times X : F(\lambda, u) = 0, u \neq u_0\}$; and either $\mathcal{C}$ is not compact, or $\mathcal{C}$ contains a point $(\lambda_*, 0)$ with $\lambda_* \neq \lambda_0$.

Unilateral Theorem

[Shi-Wang, 2009, JDE]

Theorem 8.4 Suppose that all conditions above are satisfied. Let $\mathcal{C}$ be defined as above. We define $\Gamma_+ = \{(\lambda(s), u(s)) : s \in (0, \epsilon)\}$ and $\Gamma_- = \{(\lambda(s), u(s)) : s \in (-\epsilon, 0)\}$. In addition we assume that

1. $F_u(\lambda, u_0)$ is continuously differentiable in λ for $(\lambda, u_0) \in V$;
2. The norm function $u \mapsto ||u||$ in X is continuously differentiable for any $u \neq 0$;
3. For $k \in (0, 1)$, if (λ, u_0) and (λ, u) are both in V, then $(1 - k)F_u(\lambda, u_0) + kF_u(\lambda, u)$ is a Fredholm operator.

Let $\mathcal{C}^+$ (resp. $\mathcal{C}^-$) be the connected component of $\mathcal{C}\backslash\Gamma_-$ which contains Γ_+ (resp. the connected component of $\mathcal{C}\backslash\Gamma_+$ which contains Γ_-). Then each of the sets $\mathcal{C}^+$ and $\mathcal{C}^-$ satisfies one of the following: (i) it is not compact; (ii) it contains a point (λ_*, u_0) with $\lambda_* \neq \lambda_0$; or (iii) it contains a point $(\lambda, u_0 + z)$, where $z \neq 0$ and $z \in Z$.

SIR model

[Kermack-Mckendrick, 1927]
Susceptible population $S(t)$: who are not yet infected
Infective population $I(t)$: who are infected and are able to spread the disease by contact with susceptible
Removed population $R(t)$: who have been infected and then removed from the possibility of being infected again or spreading (Methods of removal: isolation or immunization or recovery or death)

$$S' = -\beta' SI, \quad I' = \beta' SI - \alpha I, \quad R' = \alpha I,$$

1. Total population is a constant N (except death from the disease)
2. A average infective makes contact to transmit infection with $\beta = \beta' N$ others per unit time (β: contact rate)
3. A fraction α of infectives leave the infective class per unit time ($1/\alpha$: infectious period)

(1) If $S(0) < \alpha/\beta'$, then $I(t)$ is a decreasing function which tends to 0, and $S(t)$ is also decreasing and tends to a constant level greater than 0.
(2) If $S(0) > \alpha/\beta'$, then the behavior of $S(t)$ is same, but $I(t)$ will first increase in a time period $(0, T_0)$, then decrease and tends to 0 after T_0.

Define a dimensionless quantity $R_0 = \dfrac{\beta' S(0)}{\alpha}$. This is a threshold quantity. If we introduce a small number of infectives $I(0)$ into the a susceptible population, then an epidemic will occur if $R_0 > 1$.

SIR endemic (including birth and death)

$$\frac{dS}{d\tau} = bN - \beta' SI - dS, \quad \frac{dI}{d\tau} = \beta SI - \alpha I - dI - cI, \quad \frac{dR}{d\tau} = \alpha I - dR.$$

b: birth rate; d: disease-unrelated death rate, c: disease-related death rate

Nondimensionalized version: (assuming $b = d$ and $c = 0$)
$u = \dfrac{S}{N}$, $v = \dfrac{I}{N}$, $w = \dfrac{R}{N}$, $\beta = \beta' N$

$$\frac{du}{dt} = d - \beta uv - du, \ \frac{dv}{dt} = \beta uv - dv - \alpha v, \ \frac{dw}{dt} = \alpha v - dw$$

disease free equilibrium: $(u, v, w) = (1, 0, 0)$,
endemic equilibrium: $(u, v, w) = \left(\dfrac{b+\alpha}{\beta}, \dfrac{b(\beta - b - \alpha)}{\beta(b+\alpha)}, \dfrac{\alpha(\beta - b - \alpha)}{\beta(b+\alpha)} \right)$.

Basic reproductive number: $R_0 = \dfrac{\beta}{\alpha + b}$
When $R_0 < 1$, the disease will die, and the disease free equilibrium is stable; when $R_0 > 1$, the disease will stay in the population, the disease-free equilibrium is unstable, and the endemic equilibrium is stable.

A general framework

[van den Driessche, Watmough, 2002, Math. Biosci.]
"Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission" (cited 1164 times on Google Scholar, and 264 times on MathSciNet)

1. Consider a heterogeneous population whose individuals are distinguishable by age, behaviour, spatial position and/or stage of disease, but can be grouped into n homogeneous compartments. Let $x = (x_1, \cdots, x_n) \in X = \{x \geq 0\}$, with each $x_i \geq 0$, be the number of individuals in each compartment.
2. The first m compartments correspond to infected individuals. Let X_s be the set of all disease free states. That is $X_s = \{x \geq 0 : x_j = 0,\ 1 \leq j \leq m\}$. $(1 \leq m < n)$
3. Let $\mathcal{F}_i(x)$ be the rate of appearance of new infections in compartment i, $\mathcal{V}_i^+(x)$ be the rate of transfer of individuals into compartment i by all other means, and $\mathcal{V}_i^-(x)$ be the rate of transfer of individuals out of compartment i.

Model: $x_i' = \mathcal{F}_i(x) - \mathcal{V}_i(x),\ \ 1 \leq i \leq n$, where $\mathcal{V}_i(x) = \mathcal{V}_i^+(x) - \mathcal{V}_i^-(x)$.

A general framework

Model: $x_i' = \mathcal{F}_i(x) - \mathcal{V}_i(x)$, $1 \le i \le n$, where $\mathcal{V}_i(x) = \mathcal{V}_i^+(x) - \mathcal{V}_i^-(x)$.

(A1) if $x \ge 0$, then $\mathcal{F}_i, \mathcal{V}_i^+, \mathcal{V}_i^- \ge 0$ for $1 \le i \le n$;

(A2) if $x_i = 0$, then $\mathcal{V}_i^-(x) = 0$. In particular, if $x \in X_s$, then $\mathcal{V}_i^-(x) = 0$ for $1 \le i \le m$;

(A3) for $m+1 \le i \le n$, $\mathcal{F}_i(x) = 0$ for all $x \in X$; (the incidence of infection for uninfected compartments is zero)

(A4) for $1 \le i \le m$, $\mathcal{F}_i(x) = 0$ and $\mathcal{V}_i^+(x) = 0$ for all if $x \in X_s$; (if the population is free of disease then the population will remain free of disease)

(A5) If $x_0 \in X_s$ is a DFE (disease free equilibrium), then all eigenvalues of $D\mathcal{F}(x_0)$ have negative real parts. (the DFE is stable in the absence of new infection)

If x_0 is a DFE, and $\mathcal{F}_i$, $\mathcal{V}_i$ satisfies (A1)-(A5), then the derivatives $D\mathcal{F}(x_0)$ and $D\mathcal{V}(x_0)$ are partitioned as

$$D\mathcal{F}(x_0) = \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix}, \quad D\mathcal{V}(x_0) = \begin{pmatrix} V & 0 \\ J_3 & J_4 \end{pmatrix}.$$

Basic Reproduction Number

The basic reproduction number, denoted $\mathcal{R}_0$, is the expected number of secondary cases produced, in a completely susceptible population, by a typical infective individual. If $\mathcal{R}_0 < 1$, then on average an infected individual produces less than one new infected individual over the course of its infectious period, and the infection cannot grow. Conversely, if $\mathcal{R}_0 > 1$, then each infected individual produces, on average, more than one new infection, and the disease can invade the population.

SIR model: $\dfrac{du}{dt} = d - \beta uv - du,\ \dfrac{dv}{dt} = \beta uv - dv - \alpha v,\ \dfrac{dw}{dt} = \alpha v - dw$
disease free equilibrium: $(u, v, w) = (1, 0, 0)$,
endemic equilibrium: $(u, v, w) = \left(\dfrac{b+\alpha}{\beta}, \dfrac{b(\beta - b - \alpha)}{\beta(b+\alpha)}, \dfrac{\alpha(\beta - b - \alpha)}{\beta(b+\alpha)}\right)$.
Basic reproductive number: $R_0 = \dfrac{\beta}{\alpha + b}$

General Model: $x_i' = \mathcal{F}_i(x) - \mathcal{V}_i(x),\ \ 1 \le i \le n$, where $\mathcal{V}_i(x) = \mathcal{V}_i^+(x) - \mathcal{V}_i^-(x)$.

If x_0 is a DFE, and $\mathcal{F}_i$, $\mathcal{V}_i$ satisfies (A1)-(A5), then the derivatives $D\mathcal{F}(x_0)$ and $D\mathcal{V}(x_0)$ are partitioned as $D\mathcal{F}(x_0) = \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix}$, $D\mathcal{V}(x_0) = \begin{pmatrix} V & 0 \\ J_3 & J_4 \end{pmatrix}$.
The matrix FV^{-1} is called the next generation matrix for the model and $\mathcal{R}_0 = \rho(FV^{-1})$, where $\rho(A)$ denotes the spectral radius of a matrix A.
The DFE x_0 is locally asymptotically stable if $\mathcal{R}_0 < 1$, but unstable if $\mathcal{R}_0 > 1$.

Bifurcation of Endemic Equilibria

$x_i' = \mathcal{F}_i(\mu, x) - \mathcal{V}_i(\mu, x)$, $1 \leq i \leq n$, where $\mathcal{V}_i(\mu, x) = \mathcal{V}_i^+(\mu, x) - \mathcal{V}_i^-(\mu, x)$.
Suppose that $x = x_0$ is a DFE for $\mu > 0$. We define $F(\mu, x) = \mathcal{F}(\mu, x) - \mathcal{V}(\mu, x)$.

[van den Driessche, Watmough, 2002, Math. Biosci.]

Theorem 9.1. Suppose that x_0 is a DFE for $\mu > 0$, $\mathcal{F}_i$, $\mathcal{V}_i$ satisfies (A1)-(A5) for $\mu > 0$, and $\mathcal{F}_i$, $\mathcal{V}_i$ are at least C^2 near (μ_0, x_0). Suppose that $\mathcal{R}_0(\mu_0) = \rho(F(\mu_0)V^{-1}(\mu_0)) = 1$ for some $\mu_0 > 0$. Suppose that 0 is a simple eigenvalue of $F_x(\mu_0, x_0)$. Then there exists a family of endemic equilibria $\{(\mu(s), x(s) : 0 < s < \delta\}$ satisfying $\mu(s) = \mu_0 + \mu'(0)s + o(s)$ and $x(s) = x_0 + sw_0 + o(s)$, where w_0 satisfies $F_x(\mu_0, x_0)[w_0] = 0$,

$$\mu'(0) = -\frac{v_0 \cdot F_{xx}(\mu_0, x_0)[w_0, w_0]}{2v_0 \cdot F_{\mu x}(\mu_0, x_0)[w_0]},$$

and v_0 is the left (row) eigenvector of $F_x(\mu_0, x_0)$, that is $v \cdot F_x(\mu_0, x_0) = 0$ or $F_x^T(\mu_0, x_0)[v_0^T] = 0$ (T is the matrix transpose). We can assume that $v_0 \cdot w_0 = 1$.

$\mathcal{R}(F_x(\mu_0, x_0)) = \{y \in \mathbb{R}^n : v_0 \cdot y = 0\}$.

Global bifurcation: there is an unbounded continuum $\Sigma \subset \mathbb{R}^+ \times X^0$ of solutions of $F(\mu, x) = 0$ such that $(\mu_0, x_0) \in \Sigma$, and $proj_\mu(\Sigma) = (\mu_0, \infty)$ (assuming that there is no endemic equilibria for small μ).

Bifurcation and stability of Endemic Equilibria

Remarks:

1. If the function $\mu \mapsto \mathcal{R}_0(\mu)$ is one-to-one, then one can use $\mathcal{R}_0$ instead of μ as bifurcation parameter, then the projection of the unbounded continuum onto $\mathcal{R}_0$ covers the interval $(1, \infty)$.
2. If the bifurcating endemic equilibria exist for $\mathcal{R}_0 < 1$, then they are unstable and the bifurcation is called backward; if the bifurcating endemic equilibria exist for $\mathcal{R}_0 > 1$, then they are stable and the bifurcation is called forward.

Generalization to PDE (reaction-diffusion system):
[Wendi Wang and Xiao-Qiang Zhao, 2012, SIAM-ADS]
Basic Reproduction Numbers for Reaction-Diffusion Epidemic Models

$$(x_i)_t = \nabla(d_i(x)\nabla x_i) + \mathcal{F}_i(\mu, x) - \mathcal{V}_i(\mu, x), \;\; 1 \le i \le n, \; d_i \ge 0.$$

A model of misteltoe and bird

[Wang-Liu-Shi-del Rio, 2013, JMB to appear]

$$\begin{cases} \dfrac{\partial M}{\partial t} = \alpha e^{-d_i\tau} K[f(M(t-\tau,\cdot))B(t-\tau,\cdot)] - d_m M & x\in\overline{\Omega}, t>0,\\ \dfrac{\partial B}{\partial t} = D\Delta B - \nabla\left(\beta B\nabla M\right) + g(B) + cK[f(M(t,\cdot))B(t,\cdot)], & x\in\Omega, t>0,\\ M(t,x) = M_0(t,x), B(t,x) = B_0(t,x), & x\in\Omega, -\tau\le t\le 0,\\ [D\nabla B(t,x) - \beta B(t,x)\nabla M(t,x)]\cdot n(x) = 0, & x\in\partial\Omega, \end{cases} \tag{2}$$

We assume that $K : C(\overline{\Omega}) \to C(\overline{\Omega})$ is a linear mapping satisfying

(K1) $||K[u]||_{C(\overline{\Omega})} \le A_1||u||_{C(\overline{\Omega})}$ for some $A_1 > 0$;

(K2) If $u(x) \ge 0$ for all $x \in \overline{\Omega}$, then for $0 \le C_1 < C_2$, $K[C_1 u](x) \le K[C_2 u](x)$ for $x \in \overline{\Omega}$, and

$$K[u](x) \le A_2 \max\left\{u(x), \int_\Omega u(x)dx\right\}, \tag{3}$$

for some $A_2 > 0$.

The function f satisfies a Holling type growth rate:

(f) $f \in C^1(\mathbb{R}^+)$, $f(0) = 0$, $f'(M) > 0$ for $M \ge 0$, and $\lim_{M\to\infty} f(M) = f_\infty$.

Without mistletoes, the bird population has a logistic growth rate $g(B)$ which satisfies

(g) $g \in C^1(\mathbb{R}^+)$, $g(0) = g(K_B) = 0$, $g(B) > 0$ in $(0, K_B)$, and $g(B) < 0$ for $B > K_B$.

Equilibrium problem

$$\begin{cases} \alpha e^{-d_i\tau} K\left[\dfrac{MB}{M+w}\right](x) - d_m M(x) = 0, & x \in \overline{\Omega}, \\ D\Delta B(x) - \beta\nabla(B(x)\nabla M(x)) + B(x)(1-B(x)) + cK\left[\dfrac{MB}{M+w}\right](x) = 0, & x \in \Omega, \\ [D\nabla B(x) - \beta B(x)\nabla M(x)]\cdot n(x) = 0, & x \in \partial\Omega. \end{cases} \tag{4}$$

Using d_m as a bifurcation parameter, the equilibrium problem (4) can be written in the following abstract form:

$$F(d_m, M, B) = 0, \tag{5}$$

where $F : \mathbb{R} \times W^{2,p}(\Omega) \times W^{2,p}(\Omega) \to W^{2,p}(\Omega) \times L^p(\Omega) \times W^{1,p}(\partial\Omega)$ is defined by

$$F(d_m, M, B) = \begin{pmatrix} \alpha e^{-d_i\tau} K\left[\dfrac{MB}{M+w}\right] - d_m M \\ D\Delta B - \beta\nabla(B\nabla M) + B(1-B) + cK\left[\dfrac{MB}{M+w}\right] \\ (D\nabla B - \beta B\nabla M)\cdot n \end{pmatrix}. \tag{6}$$

The model (4) has two trivial solutions $E_0 = (0,0)$ and $E_1 = (0,1)$ for any $d_m > 0$. We consider the bifurcation of nontrivial solutions to (5) from the line of trivial solutions $\{(d_m, 0, 1) : d_m > 0\}$.

Fredholm integral operator

Besides $(K1)$ and $(K2)$, we also assume that the dispersal operator $K : C(\overline{\Omega}) \to C(\overline{\Omega})$ satisfies

$(K3)$ $K : C(\overline{\Omega}) \to C(\overline{\Omega})$ is compact, and K is strongly positive, that is, for any $u \in C(\overline{\Omega})$ and $u \geq 0$, $K[u](x) > 0$ for $x \in \overline{\Omega}$.

We notice that the identity mapping $K[u] = u$ considered in Section 4 does not satisfy $(K3)$, but the integral operator defined in $(H2)$ satisfies $(K3)$ if the kernel function $k(x, y) > 0$ for $(x, y) \in \overline{\Omega} \times \overline{\Omega}$. The main consequence of the assumption $(K3)$ is the renown Krein-Rutman Theorem which asserts the existence of a principal eigenvalue with a positive eigenvector.

From the compactness assumption in $(K3)$, it follows from well-known results for compact operators, $K : C(\overline{\Omega}) \to C(\overline{\Omega})$ possesses a sequence of eigenvalues $\{\lambda_i\}$ such that $\lambda_i \in \mathbb{R}$,

$$0 \leq \cdots \leq |\lambda_3| \leq |\lambda_2| \leq |\lambda_1|, \tag{7}$$

and the only possible limit point of $\{\lambda_i\}$ is zero. Moreover, since K is strongly positive, then from Krein-Rutman theorem, we have $\lambda_1 > 0$ with its corresponding function $\phi_1(x) > 0$. In the following we normalize ϕ_1 so that $\max_{x \in \overline{\Omega}} \phi_1(x) = 1$, and we also assume that

$(K4)$ $\phi_1 \in W^{2,p}(\Omega)$ for any $p > n$.

Linearized problem

The linearization of F at the boundary equilibrium $E_1 = (0, 1)$ is

$$F_{(M,B)}(d_m, 0, 1)[\phi, \psi] = \begin{pmatrix} \dfrac{\alpha e^{-d_i \tau}}{w} K[\phi] - d_m \phi \\ D\Delta\psi - \psi + \dfrac{c}{w} K[\phi] - \beta\Delta\phi \\ (D\nabla\psi - \beta\nabla\phi) \cdot n \end{pmatrix}.$$

Therefore, 0 is a simple eigenvalue of $F_{(M,B)}(d_m, 0, 1)$ if and only if

$$\begin{cases} K[\phi] = \dfrac{d_m w}{\alpha e^{-d_i \tau}} \phi, & x \in \overline{\Omega}, \\ -D\Delta\psi + \psi = \dfrac{c}{w} K[\phi] - \beta\Delta\phi, & x \in \Omega, \\ \dfrac{\partial\psi}{\partial n} = \dfrac{\beta}{D} \dfrac{\partial\phi}{\partial n}, & x \in \partial\Omega. \end{cases} \tag{8}$$

has a unique nonzero solution up to a constant multiple. Define

$$\widetilde{d}^k_{m,\tau} := \widetilde{d}_{m,\tau} \lambda_1 = \frac{\alpha}{w} e^{-d_i \tau} \lambda_1, \tag{9}$$

and let ψ_1 be the unique solution of

$$-D\Delta\psi + \psi = \frac{c\widetilde{d}^k_{m,\tau}}{\alpha e^{-d_i \tau}} \phi_1 - \beta\Delta\phi_1, \;\; x \in \Omega, \;\; \frac{\partial\psi}{\partial n} = \frac{\beta}{D} \frac{\partial\phi_1}{\partial n}, x \in \partial\Omega. \tag{10}$$

Then when $d_m = \widetilde{d}^k_{m,\tau}$, (8) is solvable thus a bifurcation occurs at $d_m = \widetilde{d}^k_{m,\tau}$.

Global bifurcation

$$\begin{cases} \alpha e^{-d_i\tau} K\left[\dfrac{MB}{M+w}\right](x) - d_m M(x) = 0, & x \in \overline{\Omega}, \\ D\Delta B(x) - \beta\nabla(B(x)\nabla M(x)) + B(x)(1-B(x)) + cK\left[\dfrac{MB}{M+w}\right](x) = 0, & x \in \Omega, \\ [D\nabla B(x) - \beta B(x)\nabla M(x)]\cdot n(x) = 0, & x \in \partial\Omega. \end{cases}$$

Theorem. Assume that $\beta \geq 0$, and the dispersal mapping K satisfies $(K1)-(K4)$. Then there is a smooth curve Γ_τ^k of positive equilibrium solutions of (2) bifurcating from the line of trivial solutions $\{(d_m, 0, 1) : d_m > 0\}$ at $d_m = \widetilde{d}_{m,\tau}^k$, and Γ_τ^k is contained in a global branch $\mathcal{C}_\tau^k$ of positive equilibrium solutions of (2). Moreover

1. Near $(d_m, M, B) = (\widetilde{d}_{m,\tau}^k, 0, 1)$, $\Gamma_\tau^k = \{(d_m(s), M(s,x), B(s,x)) : s \in (0,\epsilon)\}$, where $M(s,x) = s\phi_1(x) + s\Psi_1(s,x)$, $B(s,x) = 1 + s\psi_1(x) + s\Psi_2(s,x)$, ϕ_1 is the principal eigenfunction of K, and ψ_1 is defined as in (10); $d_m(s)$, $\Psi_1(s,\cdot)$ and $\Psi_2(s,\cdot)$ are smooth functions defined for $s \in (0,\epsilon)$ such that $\Psi_1(0,\cdot) = \Psi_2(0,\cdot) = 0$, $d_m(0) = \widetilde{d}_{m,\tau}^k$, and
$$d_m'(0) = \frac{\alpha e^{-d_i\tau}\displaystyle\int_\Omega K[-\phi_1^2(\cdot) + w\phi_1(\cdot)\psi_1(\cdot)](x)\phi_1(x)dx}{w^2\displaystyle\int_\Omega \phi_1^2(x)dx}. \tag{11}$$
2. For $s \in (0,\epsilon)$, the bifurcating solution $(d_m(s), M(s,\cdot), B(s,\cdot))$ is locally asymptotically stable if $d_m'(0) < 0$, and it is unstable if $d_m'(0) > 0$.

Fredholm operator

To apply the global bifurcation theorem (Theorem 8.3), we first show that the linearized operator $F_{(M,B)}$ is a Fredholm operator for any $(d_m, M, B) \in \mathbb{R}^+ \times W^{2,p}(\Omega) \times W^{2,p}(\Omega)$. For that purpose we write $F(d_m, M, B) = F_1(d_m, M, B) + F_2(M, B)$, where

$$F_1(d_m, M, B) = \begin{pmatrix} -d_m M \\ D\Delta B - \beta\nabla(B\nabla M) + B(1-B) \\ (D\nabla B - \beta B\nabla M)\cdot n \end{pmatrix},$$

and

$$F_2(M, B) = \begin{pmatrix} \alpha e^{-d_i \tau} K\left[\dfrac{MB}{M+w}\right] \\ cK\left[\dfrac{MB}{M+w}\right] \\ 0 \end{pmatrix}.$$

It is standard to verify that the linearization $(F_1)_{(M,B)}$ of F_1 at any (d_m, M, B) is Fredholm as $N((F_1)_{(M,B)})$ is finite dimensional, and $R((F_1)_{(M,B)})$ has a finite codimension. And the linearization $(F_2)_{(M,B)}$ of F_2 at any (d_m, M, B) is compact from $(K3)$. Therefore $F_{(M,B)}$ is Fredholm as it is a compact perturbation of a Fredholm operator (see [Kato, 1980, book] page 238 Theorem 5.26). Consequently the existence of a global branch $\mathcal{C}_\tau^k$ containing Γ_τ^k follows from Theorem 8.3.

Predator-prey system with cross-diffusion

Cross-diffusion system:

$$\begin{cases} \Delta[(1+\alpha_1 u+\alpha_2 v)u] + u(\lambda - u - bv) = 0, & x \in \Omega, \\ \Delta[(1+\beta_1 u+\beta_2 v)v] + v(\mu + cu - v) = 0, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega. \end{cases}$$

Competing species with passive diffusion, self-diffusion, cross-diffusion.
[Shigesada, Kawasaki and Teramoto, 1979, JTB]
[Nakashima, Yamada, 1996, ADE] [Kuto, Yamada, 2004, JDE]: $\alpha_1 = \beta_2 = 0$
Their idea: $U = (1+\alpha_2 v)u$, $V = (1+\beta u)v$, then the system becomes semilinear but with messy nonlinearities.

We prove the existence of a bounded branch of coexistence solutions which connecting the two semi-trivial solution branches via our new global bifurcation theorem. Our method is definitely more direct.

Setup

$$\begin{cases} (1+\alpha v)\Delta u + \alpha u \Delta v + 2\alpha \nabla u \cdot \nabla v + u(\lambda - u - bv) = 0, & x \in \Omega, \\ \beta v \Delta u + (1+\beta u)\Delta v + 2\beta \nabla u \cdot \nabla v + v(\mu + cu - v) = 0, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega. \end{cases} \tag{12}$$

Define 2×2 matrix:

$$A_1(u,v) = \begin{pmatrix} 1+\alpha v & \alpha u \\ \beta v & 1+\beta u \end{pmatrix},$$

and for $1 \le i, j \le n$, $\mathbf{u} = (u,v)^T$,

$$f(\mu, \mathbf{u}, \nabla \mathbf{u}) = -\begin{pmatrix} 2\alpha \nabla u \cdot \nabla v + u(\lambda - u - bv) \\ 2\beta \nabla u \cdot \nabla v + v(\mu + cu - v) \end{pmatrix},$$

Then (12) is equivalent to

$$A(\mu, \mathbf{u}) \equiv -A_1(\mathbf{u})\Delta \mathbf{u} + f(\mu, \mathbf{u}, \nabla \mathbf{u}) = 0, \quad \text{where} \quad \mathbf{u} \in X \equiv (W_B^{2,p}(\Omega))^2 = (W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega))^2. \tag{13}$$

Linearization

The linearization of $A(\mu, \mathbf{u})$ at $\mathbf{u}$ is given by ($\mathbf{w} = (w_1, w_2) \in X$)

$$D_{\mathbf{u}}A(\mu, \mathbf{u})[\mathbf{w}] = -A_1(\mathbf{u})\Delta\mathbf{w} - A_2(\mathbf{w})\Delta\mathbf{u} - A_3(\nabla\mathbf{u}) \cdot \nabla\mathbf{w} - J(\mathbf{u})\mathbf{w},$$

where

$$A_2(\mathbf{w}) = \begin{pmatrix} \alpha w_2 & \alpha w_1 \\ \beta w_2 & \beta w_1 \end{pmatrix}, \quad A_3(\nabla\mathbf{u}) = \begin{pmatrix} 2\alpha\nabla v & 2\alpha\nabla u \\ 2\beta\nabla v & 2\beta\nabla u \end{pmatrix},$$

and J is the Jacobian

$$J = \begin{pmatrix} \lambda - 2u - bv & -bu \\ cv & \mu + cu - 2v \end{pmatrix}.$$

For a small $\varepsilon > 0$, we define

$$X_\varepsilon = \{(u, v) \in X : u(x) > -\varepsilon, \ v(x) > -\varepsilon\}.$$

Then X_ε is an open connected subset of X. Clearly for $\mathbf{u} \in X$, $Trace(A_1(\mathbf{u})) > 0$ and $Det(A_1(\mathbf{u})) > 0$. So $A_1(\mathbf{u})$ is an elliptic operator, and $D_u A(\mu, \mathbf{u}) : X \to Y \equiv (L^p(\Omega))^2$ is Fredholm with index 0; Moreover, $A : \mathbb{R} \times X_\varepsilon \to Y$ is C^1 smooth. (see details in [Shi-Wang, 2009, JDE]).

Semitrivial steady states

Denote by $\lambda_1(q)$ the principal eigenvalue of

$$-\Delta\phi + q(x)\phi = \gamma\phi, \quad x \in \Omega, \quad \phi = 0, \quad x \in \partial\Omega,$$

where $q(x)$ is a continuous function in $\overline{\Omega}$. And we also use the notation $\lambda_1 = \lambda_1(0)$. Notice that $\lambda_1(q)$ is an increasing function in q in the sense: if $q_1(x) \geq q_2(x)$ and $q_1(x) \not\equiv q_2(x)$, then $\lambda_1(q_1) > \lambda_1(q_2)$. It is well-known that for the scalar equation

$$\Delta u + u(\lambda - u) = 0, \quad x \in \Omega; \quad u = 0, \quad x \in \partial\Omega,$$

there exists a unique positive solution θ_λ if $\lambda > \lambda_1$. Moreover $\{(\lambda, \theta_\lambda) : \lambda > \lambda_1\}$ is a smooth curve in $\mathbb{R} \times W_B^{2,P}(\Omega)$; θ_λ is stable in the sense that the linearized eigenvalue problem

$$-\Delta\phi - \lambda\phi + 2\theta_\lambda\phi = \eta\phi, \quad x \in \Omega; \quad \phi = 0, \quad x \in \partial\Omega,$$

has a positive principal eigenvalue $\lambda_1(-\lambda + 2\theta_\lambda)$. Thus $-\Delta - \lambda + 2\theta_\lambda$ is invertible and $(-\Delta - \lambda + 2\theta_\lambda)^{-1}\phi$ is positive if ϕ is positive.

A priori estimates

$$\begin{cases} (1+\alpha v)\Delta u + \alpha u \Delta v + 2\alpha \nabla u \cdot \nabla v + u(\lambda - u - bv) = 0, & x \in \Omega, \\ \beta v \Delta u + (1+\beta u)\Delta v + 2\beta \nabla u \cdot \nabla v + v(\mu + cu - v) = 0, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega. \end{cases}$$

We fix $\lambda > \lambda_1$. Then the system has trivial solution $(0,0)$ and semi-trivial solution $(\theta_\lambda, 0)$ for any $\mu \in \mathbb{R}$, and semi-trivial solution $(0, \theta_\mu)$ for $\mu > \lambda_1$.

1. If $\lambda \le \lambda_1$, then there exist no positive solutions.
2. If (u, v) is a positive solution, then

$$0 \le u(x) \le U(x) \le M_1 \equiv \begin{cases} \lambda & \text{if } \lambda\alpha \le b, \\ (\lambda\alpha + b)^2/4\alpha b & \text{if } \lambda\alpha > b, \end{cases}$$
$$0 \le v(x) \le V(x) \le M_2 \equiv (1+\beta M_1)(1+cM_1),$$

   where $U(x) = (1+\alpha v(x))u(x)$ and $V(x) = (1+\beta u(x))v(x)$.
3. There exists $\mu_0 = -cM_1$, and $\mu^0 > \mu_0$ such that there is no positive solution if $\mu < \mu_0$ or $\mu > \mu^0$.

[Nakashima, Yamada, 1996, ADE] [Kuto, Yamada, 2004, JDE]

Bifurcation points

Two semi-trivial solution branches:

$$\Gamma_u = \{(\theta_\lambda, 0) : \mu \in \mathbb{R}\}, \quad \Gamma_v = \{(0, \theta_\mu) : \mu > \lambda_1\}.$$

First we let $\mathbf{u} = (\theta_\lambda, 0)$. Simplifying the equations, we obtain

$$D_{\mathbf{u}} A(\mu, (\theta_\lambda, 0))[\mathbf{w}] = -\begin{pmatrix} \Delta w_1 + (\lambda - 2\theta_\lambda) w_1 + \alpha \Delta(\theta_\lambda w_2) - b\theta_\lambda w_2 \\ \Delta[(1+\beta\theta_\lambda) w_2] + (\mu + c\theta_\lambda) w_2 \end{pmatrix}.$$

If we set $D_{\mathbf{u}} A(\mu, (\theta_\lambda, 0))[\mathbf{w}] = 0$, then the equation of w_2 is equivalent to

$$\Delta W_2 + \frac{\mu + c\theta_\lambda}{1+\beta\theta_\lambda} W_2 = 0, \;\; x \in \Omega; \;\; W_2 = 0, \quad x \in \partial\Omega, \tag{14}$$

where $W_2(x) = (1+\beta\theta_\lambda) w_2(x)$. Thus the possible bifurcation point μ_1 is the one such that

$$\lambda_1\left(\frac{-\mu_1 - c\theta_\lambda}{1+\beta\theta_\lambda}\right) = 0. \tag{15}$$

Bifurcation Points

Similar analysis can be done on the other semi-trivial branch, but $(0, \theta_\mu)$ is not a fixed point in X so we consider the operator $A'(\mu, \mathbf{u}) = A(\mu, \mathbf{u} + (0, \theta_\mu))$ for which $\mathbf{u} = (0, 0)$ is always a solution of $A'(\mu, \mathbf{u}) = 0$ for all μ. The corresponding linearized equation is

$$D_{\mathbf{u}} A'(\mu, 0)[\mathbf{w}] = -\begin{pmatrix} \Delta[(1 + \alpha\theta_\mu) w_1] + (\lambda - b\theta_\mu) w_1 \\ \Delta w_2 + (\mu - 2\theta_\mu) w_2 + \beta\Delta(\theta_\mu w_1) + cb\theta_\mu w_1 \end{pmatrix}.$$

Thus the possible bifurcation point is μ_2 such that

$$\lambda_1 \left(\frac{-\lambda + b\theta_{\mu_2}}{1 + \alpha\theta_{\mu_2}} \right) = 0. \tag{16}$$

Lemma. There exists a unique $\mu_1 \in (-\infty, \infty)$ so that (15) holds, and there exists a unique $\mu_2 \in (\lambda_1, \infty)$ so that (16) holds. Moreover the corresponding null spaces $\mathcal{N}(D_u A(\mu_1, (\theta_\lambda, 0)))$ and $\mathcal{N}(D_u A'(\mu_2, (0, 0)))$ are one-dimensional.

Proof of lemma

Lemma. There exists a unique $\mu_1 \in (-\infty, \infty)$ so that (15) holds, and there exists a unique $\mu_2 \in (\lambda_1, \infty)$ so that (16) holds. Moreover the corresponding null spaces $\mathcal{N}(D_u A(\mu_1, (\theta_\lambda, 0)))$ and $\mathcal{N}(D_u A'(\mu_2, (0,0)))$ are one-dimensional.
Proof: Define

$$f_1(\mu) = \lambda_1 \left(\frac{-\mu - c\theta_\lambda}{1 + \beta\theta_\lambda} \right), \quad \text{and } q_1(\mu) = \frac{-\mu - c\theta_\lambda}{1 + \beta\theta_\lambda}.$$

Then $q_1(\mu)$ is decreasing in μ. From the properties of $\lambda(q)$, we deduce that $f_1(\mu) \to \pm\infty$ as $\mu \to \mp\infty$ and f_1 is decreasing. Hence μ_1 exists and it is unique. With $\mu = \mu_1$, (14) has a positive solution W_2. Then $w_2 = (1 + \beta\theta_\lambda)^{-1} W_2$, and $w_1 = (-\Delta - \lambda + 2\theta_\lambda)^{-1}(\alpha\Delta(\theta_\lambda w_2) - b\theta_\lambda w_2)$ give rise to the unique solution of $D_{\mathbf{u}} A(\mu, (\theta_\lambda, 0))[\mathbf{w}] = 0$ up to a constant multiplier.
Similarly we define

$$f_2(\mu) = \lambda_1 \left(\frac{-\lambda + b\theta_\mu}{1 + \alpha\theta_\mu} \right), \quad \text{and } q_2(\mu) = \frac{-\lambda + b\theta_\mu}{1 + \alpha\theta_\mu}.$$

Since θ_μ is increasing in μ (pointwisely for $x \in \Omega$), then q_2 and f_2 are increasing in μ. One can show that $f_2(\mu) \to \lambda_1 + b/\alpha > 0$ as $\mu \to \infty$, and $f_2(\mu) \to \lambda_1 - \lambda < 0$ as $\mu \to \lambda_1 + 0$. Hence μ_2 exists and is unique. Similarly to the above case, the null space is one-dimensional with $w_1 > 0$.

Global bifurcation

[Shi-Wang, 2009, JDE]

$$\begin{cases} (1+\alpha v)\Delta u + \alpha u \Delta v + 2\alpha \nabla u \cdot \nabla v + u(\lambda - u - bv) = 0, & x \in \Omega, \\ \beta v \Delta u + (1+\beta u)\Delta v + 2\beta \nabla u \cdot \nabla v + v(\mu + cu - v) = 0, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega. \end{cases}$$

Theorem. Suppose that $\alpha, \beta, b, c > 0$ and $\lambda > \lambda_1$. Let $\mathcal{S}^+$ be the set of positive solutions to the equation above. Then there exists a connected component $\mathcal{C}^*$ of $\mathcal{S}^+$ such that the closure of $\mathcal{C}^*$ includes the bifurcation points $(\mu, u, v) = (\mu_1, \theta_\lambda, 0)$ and $(\mu, u, v) = (\mu_2, 0, \theta_{\mu_2})$. In other words, bifurcations occur at both $(\mu, u, v) = (\mu_1, \theta_\lambda, 0)$ and $(\mu, u, v) = (\mu_2, 0, \theta_{\mu_2})$, and the bifurcating continua from the two points are connected to each other.

Proof. We apply the abstract theorem at $(\mu, u, v) = (\mu_1, \theta_\lambda, 0)$ with $V = \mathbb{R} \times X_\varepsilon$. We have already observed that $A : V \to Y$ is C^1 smooth, and $D_u A(\lambda, u)$ is Fredholm with zero index for any $(\lambda, u) \in V$. We have also shown in Lemma that $\mathcal{N}(D_{\mathbf{u}} A(\mu_1, (\theta_\lambda, 0))) = span\{(w_1, w_2)\}$ with $w_2 > 0$. For the transversality condition,

$$D_{\mu\mathbf{u}} A(\mu_1, (\theta_\lambda, 0)) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -w_2 \end{pmatrix} \notin \mathcal{R}(D_{\mathbf{u}} A(\mu_1, (\theta_\lambda, 0))),$$

because the equation $\Delta[(1+\beta\theta_\lambda)\psi] + (\mu_1 + c\theta_\lambda)\psi = w_2$ is not solvable since $\int_\Omega (1+\beta\theta_\lambda) w_2^2 dx \neq 0$.

Proof

Now we can apply Theorem 8.3 to obtain a connected component $\mathcal{C}$ of the set $\mathcal{S}$ of all solutions emanating from $(\mu, u, v) = (\mu_1, \theta_\lambda, 0)$. Similarly we can show the existence of a connected component of $\mathcal{S}$ emanating from $(\mu, u, v) = (\mu_2, 0, \theta_{\mu_2})$. Moreover near the bifurcation point, $\mathcal{C}$ has the form $(\mu(s), \theta_\lambda + o(s), sw_2 + o(s))$ for s small. Then the solution is positive for $s > 0$ since $w_2 > 0$ and $\theta_\lambda > 0$. Let $P = \{(u, v) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) : u > 0, v > 0 \text{ for } x \in \Omega, \partial u/\partial\nu < 0, \partial v/\partial\nu < 0 \text{ for } x \in \partial\Omega\}$, where ν is the unit outer normal vector field of $\partial\Omega$. Then $\mathcal{C} \cap (\mathbb{R} \times P) \neq \emptyset$.

Let $\mathcal{C}^* = \mathcal{C} \cap (\mathbb{R} \times P)$. Let $\mathcal{C}^+$ and $\mathcal{C}^-$ be the sub-continua in Theorem 8.4 (Conditions 1-3 in that theorem can be easily verified). By definition, $\mathcal{C}^* \subset \mathcal{C}^+$. By the elliptic regularity theory, the first alternative in Theorem 8.4 for $\mathcal{C}^+$ is equivalent to "the closure of $\mathcal{C}^+$ intersects ∂V or is unbounded in the norm of $\mathbb{R} \times X$". On the other hand, by the a priori estimates, the positive solutions (u, v) are bounded in L^∞ norm, and the range of μ for existence of such solutions is also bounded. Thus by the elliptic regularity theory again, $\mathcal{C}^*$ cannot be unbounded in $\mathbb{R} \times X$ norm. Now we see that if the first alternative in Theorem 8.4 occurs, then $\overline{\mathcal{C}^*} \cap (\mathbb{R} \times \partial P)$ contains a point (μ^*, u^*, v^*) other than $(\mu_1, \theta_\lambda, 0)$. This is obviously true if the other alternatives occur.

Proof

By continuity, (μ^*, u^*, v^*) is a solution of the equation, and $u^* \geq 0$, $v^* \geq 0$. By the maximum principle, $u^* \equiv 0$ or $u^* > 0$, and the same for v^*. If $(u^*, v^*) = (0, 0)$, then we can show that $D_u A(\mu^*, 0) = -(\Delta + \lambda, \Delta + \mu^*)$ is degenerate and its null space contains a $(w_1, w_2) \geq 0, \neq 0$. Since $\lambda > \lambda_1$, $w_1 = 0$; hence $w_2 > 0$ and $\mu^* = \lambda_1$. Applying Theorem 8.3 to the trivial solution branch $\{(\mu, 0, 0) : \mu \in \mathbb{R}\}$ at $(\lambda_1, 0, 0)$, we have that all the nontrivial solutions of near $(\lambda_1, 0, 0)$ are the semitrivial ones $(\mu, 0, \theta_\mu)$, contradicting the definition of (μ^*, u^*, v^*). Thus $(u^*, v^*) \neq (0, 0)$. Note that $(\mu^*, u^*, v^*) \notin \Gamma_u$ since $\mu = \mu_1$ is the only point on Γ_u where positive solutions bifurcate. We conclude $(\mu^*, u^*, v^*) = (\mu_2, 0, \theta_{\mu_2})$, the only possible point on Γ_v where positive solutions bifurcate. □

Bifurcation branch from one semitrivial solution to another one: [Blat-Brown, 1986, SIAM-MA]

The result implies the existence of positive solutions for $\mu \in (\mu_1, \mu_2)$ or $\mu \in (\mu_2, \mu_1)$ if $\mu_1 \neq \mu_2$. Indeed $\mu \in (\mu_1, \mu_2)$ is equivalent to

$$\lambda_1 \left(\frac{-\mu - c\theta_\lambda}{1 + \beta\theta_\lambda} \right) < 0, \quad \lambda_1 \left(\frac{-\lambda + b\theta_\mu}{1 + \alpha\theta_\mu} \right) < 0; \tag{17}$$

and $\mu \in (\mu_2, \mu_1)$ is equivalent to

$$\lambda_1 \left(\frac{-\mu - c\theta_\lambda}{1 + \beta\theta_\lambda} \right) > 0, \quad \lambda_1 \left(\frac{-\lambda + b\theta_\mu}{1 + \alpha\theta_\mu} \right) > 0. \tag{18}$$

Even when $\mu_1 = \mu_2$, a solution branch still connects the two bifurcation points.

More properties

$$\begin{cases} (1+\alpha v)\Delta u + \alpha u \Delta v + 2\alpha \nabla u \cdot \nabla v + u(\lambda - u - bv) = 0, & x \in \Omega, \\ \beta v \Delta u + (1+\beta u)\Delta v + 2\beta \nabla u \cdot \nabla v + v(\mu + cu - v) = 0, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega. \end{cases}$$

Theorem. Suppose that $\alpha, \beta, b, c > 0$ and $\lambda > \lambda_1$. Let $\mathcal{S}^+$ be the set of positive solutions to the equation above. Then there exists a connected component $\mathcal{C}^*$ of $\mathcal{S}^+$ such that the closure of $\mathcal{C}^*$ includes the bifurcation points $(\mu, u, v) = (\mu_1, \theta_\lambda, 0)$ and $(\mu, u, v) = (\mu_2, 0, \theta_{\mu_2})$. In other words, bifurcations occur at both $(\mu, u, v) = (\mu_1, \theta_\lambda, 0)$ and $(\mu, u, v) = (\mu_2, 0, \theta_{\mu_2})$, and the bifurcating continua from the two points are connected to each other.

Remark:
1. The bifurcation direction. When $\alpha = \beta = 0$, then there is no backward bifurcation. When $\alpha, \beta > 0$, ? (homework)
2. Uniqueness of coexistence steady state. When $\alpha = \beta = 0$ and $n = 1$ ($\Omega = (0, L)$), the uniqueness was proved in [Lopez-Gomez and Pardo, 1993, DIE]. (This can be generalized to the case of $\alpha, beta > 0$?) The higher dimensional case is open.
3. Stability of coexistence steady state. Open even in the case $\alpha = \beta = 0$ and $n = 1$ ($\Omega = (0, L)$).

1-D problem

[Lopez-Gomez and Pardo, 1993, DIE]

$$\begin{cases} u'' + u(\lambda - u - bv) = 0, & x \in (0, L), \\ v'' + v(\mu + cu - v) = 0, & x \in (0, L), \\ u(0) = u(L) = v(0) = v(L) = 0. \end{cases}$$

Fix $\lambda > \lambda_1 = \pi^2/L^2$, we have proved that there exists a positive solution (u, v) if $\mu \in (\mu_1, \mu_2)$, where μ_1 and μ_2 satisfy

$$-\mu_1 + \lambda_1(-c\theta\lambda) = \lambda_1(-\mu_1 - c\theta_\lambda) = 0, \quad -\lambda + \lambda_1(b\theta_{\mu_2}) = \lambda_1(-\lambda + b\theta_{\mu_2}) = 0.$$

Hence $\mu_1 = \lambda_1(-c\theta\lambda) < \lambda_1(0) = \lambda_1 < \mu_2$.

We can prove that when $\mu \leq \mu_1$ or $\mu \geq \mu_2$, then there is no positive solutions.

We only need to show that if (u, v) is a positive solution, then (u, v) is non-degenerate.

Linearization

Suppose that (u, v) is degenerate, then the linearized equation

$$\begin{cases} \phi'' + (\lambda - 2u - bv)\phi - bu\psi = 0, & x \in (0, L), \\ \psi'' + cv\phi + (\mu + cu - 2v)\psi = 0, & x \in (0, L), \\ \phi(0) = \phi(L) = \psi(0) = \psi(L) = 0, \end{cases}$$

has a non-trivial solution (ϕ, ψ).
Define $L_1[\phi] = \phi'' + (\lambda - 2u - bv)\phi$ and $L_2[\psi] = \psi'' + (\mu + cu - 2v)\psi$. Then $\lambda_1(L_1) = \lambda_1(-\lambda + 2u + bv) > \lambda_1(-\lambda + u + bv) = 0$ and $\lambda_1(L_2) = \lambda_1(-\mu - cu + 2v) > \lambda_1(-\mu - cu + v) = 0$. Hence L_1 and L_2 are both invertible, and $(L_i)^{-1}$ is a negative operator on $C[0, L]$ in the sense that if $L_i[g] = f$ and $f \geq 0$, then $g > 0$.

Then the linearized equation becomes $L_1[\phi] = bu\psi$ and $L_2[\psi] = -cv\phi$.

Both ϕ and ψ have to change sign. Suppose $\phi > 0$, then $\psi = L_2^{-1}[-cv\phi] > 0$; and $\phi = L_1^{-1}[bu\psi] < 0$, which is a contradiction.

Nondegeneracy

Suppose that ϕ has $m(\geq 3)$ zeros

$$0 = x_0 < x_1 < x_2 < \cdots < x_{m-1} < x_m = L$$

such that

$$\phi(x) < 0,\ x \in (x_{2j}, x_{2j+1}),\ j \geq 0,\ 2j+1 \leq m,$$
$$\phi(x) > 0,\ x \in (x_{2j-1}, x_{2j}),\ j \geq 1,\ 2j \leq m.$$

Then

$$\psi(x_{2j}) > 0,\ \psi(x_{2j+1}) < 0 \text{ for } x_{2j}, x_{2j+1} \in 0 < 2j < 2j+1 < m.$$

This contradicts with $\psi(x_m) = \psi(L) = 0$.

Remark: The same proof works for radially symmetric positive solutions on n-dimensional balls.

[Dancer, Lopez-Gomez, Ortega, 1995, DIE], [Du, 2005, book chapter]