Reaction-Diffusion Models and Bifurcation Theory
Lecture 8: Global bifurcation

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7. Let L be a linear compact operator from X to X. Then $I - L$ is a linear Fredholm operator with index 0.

Elliptic Operators

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. Consider the Possion's equation

$$-\Delta u(x) = f(x), \quad x \in \Omega, \quad u(x) = 0, \quad x \in \partial\Omega.$$

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4. $-\Delta$ is an invertible linear operator. Let $K = (-\Delta)^{-1}$. Then $K : L^p(\Omega) \to W^{2,p}(\Omega) \bigcap W_0^{1,p}(\Omega)$ or $K : C^\alpha(\overline{\Omega}) \to C_0^{2,\alpha}(\overline{\Omega})$ is a bounded linear operator. So $(-\Delta)^{-1}$ is bounded.

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6. Hence $(-\Delta)^{-1} = i \circ K : L^p(\Omega) \to L^p(\Omega)$ ($C^\alpha(\overline{\Omega}) \to C^\alpha(\overline{\Omega})$) is a linear compact operator.

Nonlinear Mappings

1. Compact mapping: a mapping $F : X \to Y$ is compact if F is continuous, and for every bounded subset B of X, the image $F(B) = \{F(u) : u \in B\}$ is a relatively compact subset of Y.

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Let L be a linear compact operator on X. From Riesz-Schauder theory, the set of eigenvalues of L is at most countably many, and the only possible limit point is $\lambda = 0$. For any eigenvalue λ of L, the subspace

$$X_\lambda = \bigcup_{n=1}^{\infty} \{u \in X : (L - \lambda I)^n u = 0\}$$

is finite dimensional, and $dim(X_\lambda)$ is the algebraic multiplicity of the eigenvalue λ. The geometric multiplicity of λ is defined as $dim\{u \in X : (L - \lambda I)u = 0\}$.

Krasnoselski-Rabinowitz Global Bifurcation Theorem

Consider

$$F(\lambda, u) = u - \lambda L u - H(\lambda, u), \tag{1}$$

where $L : X \to X$ is a linear compact operator, and $H(\lambda, u)$ is compact on $U \subset \mathbb{R} \times X$ such that $||H(\lambda, u)|| = o(||u||)$ near $u = 0$ uniformly on bounded λ intervals. Note the conditions imply that $F_u(\lambda, 0) = I - \lambda L$, and if 0 is an eigenvalue of $F_u(\lambda_0, 0)$, then λ_0^{-1} must be an eigenvalue of the linear operator L. Define

$$S = \{(\lambda, u) \in U : F(\lambda, u) = 0, u \neq 0\}.$$

We say $(\lambda_0, 0)$ is a bifurcation point for the equation (1) if $(\lambda_0, 0) \in \overline{S}$ ($\overline{S}$ is the closure of S).

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Theorem 8.1. (Krasnoselski-Rabinowitz Global Bifurcation Theorem) [Rabinowitz, 1971, JFA] Let X be a Banach space, and let U be an open subset of $\mathbb{R} \times X$ containing $(\lambda_0, 0)$. Suppose that L is a linear compact operator on X, and $H(\lambda, u) : \overline{U} \to X$ is a compact operator such that $||H(\lambda, u)|| = o(||u||)$ as $u \to 0$ uniformly for λ in any bounded interval. If $1/\lambda_0$ is an eigenvalue of L with odd algebraic multiplicity, then $(\lambda_0, 0)$ is a bifurcation point. Moreover if C is the connected component of $\overline{S}$ which contains $(\lambda_0, 0)$, then one of the following holds:

(i) C is unbounded in U;

(ii) $C \bigcap \partial U \neq \emptyset$; or

(iii) C contains $(\lambda_i, 0) \neq (\lambda_0, 0)$, such that λ_i^{-1} is also an eigenvalue of L.

Leray-Schauder degree

Let X be a Banach space, and let U be an open bounded subset of X. Denote by $K(\overline{U})$ the set of compact operators from $\overline{U}$ to X, and define

$$M = \{(I - G, U, y) : U \subset X \text{ open bounded }, G \in K(\overline{U}), \text{ and } y \notin (I - G)(\partial U)\}.$$

Then the Leray-Schauder degree $d : M \to \mathbf{Z}$ is a well-defined function, which satisfies the following properties:

1. $d(I, U, y) = 1$ if $y \in U$, and $d(I, U, y) = 0$ if $y \notin \overline{U}$;
2. (Additivity) $d(I - G, U, y) = d(I - G, U_1, y) + d(I - G, U_2, y)$ if U_1 and U_2 are disjoint open subsets of U so that $y \notin (I - G)(\overline{U} \backslash (U_1 \bigcup U_2))$;
3. (Homotopy invariance) Suppose that $h : [0, 1] \times \overline{U} \to X$ is compact and $y : [0, 1] \to X$ is continuous, and $y(t) \notin (I - G)(\partial U)$, then $D(t) = d(I - h(t, \cdot), U, y(t))$ is a constant independent of $t \in [0, 1]$.
4. (Existence) If $d(I - G, U, y) \neq 0$, then there exists $u \in U$ such that $u - G(u) = y$;
5. If for $G_1, G_2 \in K(\overline{U})$, $G_1(u) = G_2(u)$ for any $u \in \partial U$, then $d(I - G_1, U, y) = d(I - G_2, U, y)$.

Degree in finite dimensional spaces

For continuous $f : U \equiv [-1,1] \to \mathbb{R}$, $d(I-f, U, 0)$ can be defined by $d(I-f, U, 0) = sgn(f(-1)) \cdot sgn(f(1))$, where $sgn(y) = y/|y|$. Another definition is $d(I-f, U, 0) = \sum_{x \in U, x=f(x)} f'(x)$ for function f satisfying $f'(x) \neq 0$ whenever $f(x) = x$. Here we must have $x \neq f(x)$ for $x = \pm 1$.

For continuous $f : U(\subset \mathbb{R}^n) \to \mathbb{R}^n$ $(n \geq 2)$, $d(I-f, U, 0) = \sum_{x \in U, x=f(x)} sgn(Det(f'(x)))$ for function f satisfying $Det(f'(x)) \neq 0$ whenever $f(x) = x$. Here we must have $x \neq f(x)$ for $x \in \partial U$.

Toy proof: Let $f(x,y) = (x^2, y^2)$. Prove that $f(x,y) = (3,4)$ has a positive solution (x_1, y_1).

1. Let $U = \{(x,y) \in \mathbb{R}^2 : x > 0, y > 0, x^2 + y^2 = 6^2\}$.
2. Prove that $G_t(x,y) = I(x,y) - tf(x,y) = (x - tx^2, y - ty^2)$ has no solution $(x_t, y_t) \in \partial U$ satisfying $G_t(x_t, y_t) = (3,4)$.
3. $G_0(x,y) = I(x,y) = (x,y)$ has a solution $(x_0, y_0) = (3,4) \in U$ such that $G_0(x,y) = (3,4)$ since $(3,4) \in U$.
4. $d(G_0, U, (3,4)) = 1$ since $Det(I) = 1$ and $(3,4)$ is the unique solution of $G_0(x,y) = (3,4)$.
5. $d(G_0, U, (3,4)) = d(G_1, U, (3,4))$ from the homotopy invariance property of Leray-Schauder degree, then $d(G_1, U, (3,4)) = 1$.
6. Then $G_1(x,y) = (3,4)$ has a solution in U.

Properties of Leray-Schauder degree

1. If L is a linear compact operator on X, then $d(I - \lambda L, B_R(0), 0) = (-1)^{\beta}$, where $B_R(v)$ is a ball centered at v with radius R, and β is the sum of algebraic multiplicity of eigenvalues μ of L satisfying $\lambda\mu > 1$.
2. Suppose that $G \in K(\overline{U})$, $u_0 \in U$ and $R > 0$ such that u_0 is the unique solution satisfies $u - G(u) = 0$ in $B_R(u_0)$, then the derivative $G'(u_0) : X \to X$ is a linear compact operator; if $\lambda = 1$ is not an eigenvalue of $G'(u_0)$, then $d(I - G, B_R(u_0), 0) = d(I - G'(u_0), B_R(0), 0)$ for some sufficiently small $R > 0$ (this number is also called *fixed point index* of u_0 with respect to G).

Example: $\Delta u + \lambda f(u) = 0,\ x \in \Omega, \quad u = 0,\ x \in \partial\Omega.$

Suppose that $f(0) = 0$. What is the fixed point index of $u = 0$?

1. $K : (-\Delta)^{-1} : C^{\alpha}(\overline{\Omega}) \to C^{\alpha}(\overline{\Omega})$ is well defined as $K(f) = u$ that $u \in C_0^{2,\alpha}(\overline{\Omega})$ such that $-\Delta u = f$ for any $f \in C^{\alpha}(\overline{\Omega})$.
2. For $H(\lambda, u) = u - \lambda K(f(u)) = 0$, $H_u(\lambda, u)[w] = w - \lambda K(f'(u))[w]$ and $H_u(\lambda, 0)[w] = w - \lambda K(f'(0))[w]$.
3. For $L[w] = K(f'(0))[w] = \mu w$, we have $\mu\lambda\Delta w + \lambda f'(0)w = 0$ or $\Delta w + (\mu\lambda)^{-1}\lambda f'(0)w = 0$.
4. So for $\mu\lambda > 1$, let $\rho_k < \lambda f'(0) < \rho_{k+1}$ then the fixed point index of 0 is $\sum_{j=1}^{k} M(\rho_k)$, whereρ_k is the k-th eigenvalue of $\Delta\phi + \rho\phi = 0$, and $M(\rho_k)$ is the algebraic multiplicity of ρ_k.

Krasnoselski Bifurcation Theorem

[Krasnoselski, 1964] Let X be a Banach space, and let U be an open subset of $\mathbb{R} \times X$ containing $(\lambda_0, 0)$. Suppose that L is a linear compact operator on X, and $H(\lambda, u) : \overline{U} \to X$ is a compact operator such that $||H(\lambda, u)|| = o(||u||)$ as $u \to 0$ uniformly for λ in any bounded interval. If $1/\lambda_0$ is an eigenvalue of L with odd algebraic multiplicity, then $(\lambda_0, 0)$ is a bifurcation point.

Proof. Suppose not, then there exists a $R > 0$ such that in the region $O = \{(\lambda, u) : |\lambda - \lambda_0| \leq R, |u| \leq R\}$, the only solutions of $F(\lambda, u) = 0$ are $\{(\lambda, 0) : |\lambda - \lambda_0| \leq R\}$. We choose λ_-, λ_+ so that $\lambda_0 - R < \lambda_- < \lambda_0 < \lambda_+ < \lambda_0 + R$. From the homotopy invariance of the Leray-Schauder degree,

$$d(F(\lambda_-, \cdot), B_\rho(0), 0) = d(F(\lambda_+, \cdot), B_\rho(0), 0),$$

for any $\rho \in (0, R)$. For ρ small enough, $d(F(\lambda_\pm, \cdot), B_\rho(0), 0) = d(I - \lambda_\pm L, B_\rho(0), 0)$. But on the other hand,

$$|d(I - \lambda_+ L, B_\rho(0), 0) - d(I - \lambda_- L, B_\rho(0), 0)| = 1,$$

since λ_0^{-1} is the only eigenvalue of L in between λ_-^{-1} and λ_+^{-1}, and the algebraic multiplicity of λ_0^{-1} is odd. That is a contradiction. Thus $(\lambda_0, 0)$ is a bifurcation point.

Rabinowitz Global Bifurcation Theorem

If C is the connected component of $\overline{S}$ which contains $(\lambda_0, 0)$, then one of the following holds:

(i) C is unbounded in U;

(ii) $C \bigcap \partial U \neq \emptyset$; or

(iii) C contains $(\lambda_i, 0) \neq (\lambda_0, 0)$, such that λ_i^{-1} is also an eigenvalue of L.

Separation Lemma. Let (M, d) be a compact metric space, and let A and B be close subsets of M such that $A \bigcap B = \emptyset$. Then there exist compact subsets M_A and M_B of M such that $M_A \bigcup M_B = M$, $M_A \bigcap M_B = \emptyset$, $M_A \supset A$, and $M_B \supset B$.

Proof (1)

Proof. we assume the stated alternatives do not hold, then C is bounded in U, $C \bigcap \partial U = \emptyset$, and $C \bigcap \{(\lambda, 0) \in U\} = \{(\lambda_0, 0)\}$. From the compactness of L and H, C is compact since it is bounded. Let $C_\varepsilon = \{(\lambda, u) \in U : dist((\lambda, u), C) < \varepsilon\}$. Let $A = C$ and $B = S \bigcap \partial C_\varepsilon$. From Separation Lemma, there exists compact M_A and M_B such that $M_A \bigcap M_B = \emptyset$, $M_A \bigcup M_B = S \bigcap \overline{C_\varepsilon}$, $M_A \supset C$ and $M_B \supset S \bigcap \partial C_\varepsilon$. Hence there exists an open bounded $U_0 = M_A$ such that

$$C \subset U_0 \subset \overline{U_0} \subset U, \quad \text{and } \overline{S} \bigcap \partial U_0 = \emptyset. \tag{2}$$

Define $U_0(\lambda) = \{u \in X : (\lambda, u) \in U_0\}$ for $\lambda \in I$ where $I = \{\lambda \in \mathbb{R} : (\{\lambda\} \times X) \bigcap U_0 \neq \emptyset\}$. Then $D(\lambda) = d(F(\lambda, \cdot), U_0(\lambda), 0)$ is constant for $\lambda \in I$ since $\overline{S} \bigcap \partial U_0 = \emptyset$ and the homotopy invariance of $d(F, \Omega, 0)$, where $d(F(\lambda, \cdot), \Omega, 0)$ is the Leray-Schauder degree.

Since $(\lambda_0, 0)$ is the only intersection of C with the line $\{(\lambda, 0)\}$, U_0 can be chosen so that $U_0 \bigcap \{(\lambda, 0) \in U\} = [\lambda_0 - \delta, \lambda_0 + \delta] \times \{0\}$, and no any point λ in $[\lambda_0 - 2\delta, \lambda_0 + 2\delta]$ satisfies that λ^{-1} is an eigenvalue of L. We choose $\lambda_\pm$ which satisfy $\lambda_0 - \delta < \lambda_- < \lambda_0 < \lambda_+ < \lambda_0 + \delta$. We choose $\rho > 0$ small enough so that $F(\lambda, u) \neq 0$ for $\lambda \in [\lambda_+, \lambda_0 + 2\delta]$ and $u \in B_\rho(0) \backslash \{0\}$, and we also choose $\lambda^* > \lambda_0 + 2\delta$ such that $U_0(\lambda^*) = \emptyset$.

Proof (2)

From the homotopy invariance of the Leray-Schauder degree on $U_0\backslash([\lambda_+,\lambda^*]\times\overline{B}_\rho(0))$, we have

$$d(F(\lambda_+,\cdot),U_0(\lambda_+)\backslash\overline{B}_\rho(0),0)=d(F(\lambda^*,\cdot),U_0(\lambda^*),0)=0. \tag{3}$$

For the same argument,

$$d(F(\lambda_-,\cdot),U_0(\lambda_-)\backslash\overline{B}_\rho(0),0)=0. \tag{4}$$

On the other hand, from the additivity of the Leray-Schauder degree,

$$D(\lambda_\pm)=d(F(\lambda_\pm,\cdot),U_0(\lambda_\pm)\backslash\overline{B}_\rho(0),0)+d(F(\lambda_\pm,\cdot),B_\rho(0),0). \tag{5}$$

Hence we obtain

$$d(F(\lambda_+,\cdot),B_\rho(0),0)=d(F(\lambda_-,\cdot),B_\rho(0),0). \tag{6}$$

For $\rho>0$ small enough,

$$d(F(\lambda_\pm,\cdot),B_\rho(0),0)=d(I-\lambda_\pm L,B_\rho(0),0). \tag{7}$$

From the formula of Leray-Schauder degree of $I-\lambda L$, we have

$$|d(I-\lambda_+L,B_\rho(0),0)-d(I-\lambda_-L,B_\rho(0),0)|=1, \tag{8}$$

But (8) is a contradiction with (7). Hence the alternatives in the theorem hold.

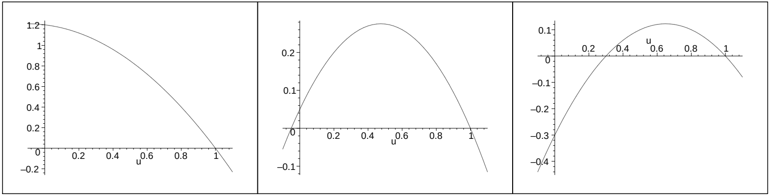
Unilateral bifurcation theorem

[Rabinowitz, 1971, JFA], [Dancer, 1974, Indiana Math J], [Lopez-Gomez, 2000, book]
Theorem 8.2. Let X be a Banach space, and let U be an open subset of $\mathbb{R} \times X$ containing $(\lambda_0, 0)$. Suppose that L is a linear compact operator on X, and $H(\lambda, u) : \overline{U} \to X$ is a compact operator such that $||H(\lambda, u)|| = o(||u||)$ as $u \to 0$ uniformly for λ in any bounded interval. Suppose that $1/\lambda_0$ is an eigenvalue of L with algebraic multiplicity 1. We define $\Gamma_+ = \{(\lambda(s), u(s)) : s \in (0, \epsilon)\}$ and $\Gamma_- = \{(\lambda(s), u(s)) : s \in (-\epsilon, 0)\}$. Let $\mathcal{C}$ be a connected component of $\overline{S}$ where $S = \{(\lambda, u) \in V : H(\lambda, u) = 0, u \neq 0\}$ containing $(\lambda_0, 0)$. Let $\mathcal{C}^+$ (resp. $\mathcal{C}^-$) be the connected component of $\mathcal{C}\backslash\Gamma_-$ which contains Γ_+ (resp. the connected component of $\mathcal{C}\backslash\Gamma_+$ which contains Γ_-). Then each of the sets $\mathcal{C}^+$ and $\mathcal{C}^-$ satisfies one of the following:
(i) it is unbounded;
(ii) it contains a point $(\lambda_*, 0)$ with $\lambda_* \neq \lambda_0$; or
(iii) it contains a point (λ, z), where $z \neq 0$ and $z \in Z$ which any complement of $span\{w_0\} = \mathcal{N}(H_u(\lambda_0, 0))$ in X.

Application 1: Reaction-diffusion population model

$$\begin{cases} \dfrac{\partial u}{\partial t} = \mathcal{D}\Delta u + uf(x,u), & x \in \Omega, \quad t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \Omega. \end{cases}$$

$u(x,t)$: population density at position x and time t
Ω: a bounded habitat, $u = 0$ on boundary $\partial\Omega$: hostile exterior environment
$f(x,u)$: heterogeneous growth rate per capita



$f(x,u)$: (a) logistic; (b) weak Allee effect; (c) strong Allee effect.

Bifurcation problem

$$\Delta u + \lambda u f(x,u) = 0, \quad x \in \Omega, \quad u = 0, \quad x \in \partial\Omega.$$

$u = 0$ is always a solution for any $\lambda > 0$, $\lambda_1(f,\Omega)$ (minimal patch size) is the principal eigenvalue of $\Delta\psi + \lambda f(x,0)\psi = 0$, $x \in \Omega$, $\psi = 0$, $x \in \partial\Omega$. We consider positive solutions only.

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Logistic case: a supercritical transcritical bifurcation occurs at $\lambda_1(f, \Omega)$; for $\lambda > \lambda_1$, there is a unique steady state which is globally stable. [Cantrell-Cosner, 2003]

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Weak Allee effect case:
(A) a subcritical (backward) transcritical bifurcation occurs at $\lambda_1(f,\Omega) > 0$;
(B) for $\lambda \in (\lambda_*, \lambda_1)$, there are at least two steady state solutions (bistability);
(C) a saddle-node bifurcation occurs at λ_* (at least when Ω is a ball);
(D) for λ large, it is similar to logistic case. [Shi-Shivaji, JMB, 2006]

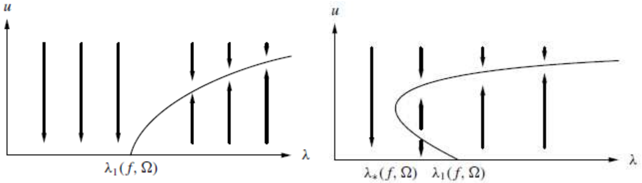
The solution curve near $(\lambda, u) = (\lambda_1, 0)$ is $\{(\lambda(s), u(s)) : 0 < s < \delta\}$, where $\delta > 0$ is a constant, $\lambda(s) = \lambda_1(f,\Omega) + \eta(s)$, $u(s) = s\varphi_1 + sv(s)$, $0 < s < \delta$, $\eta(0) = 0$ and $v(0) = 0$, and

$$\eta'(0) = -2[\lambda_1(f,\Omega)]^2 \frac{\int_\Omega f_u(x,0)\varphi_1^3(x)dx}{\int_\Omega |\nabla\varphi_1(x)|^2 dx}.$$

Allee effect caused by diffusion (ODE with weak Allee effect is similar to logistic case); danger of hysteresis. [Jiang-Shi, book chapter, 2009]

Bifurcation Diagrams

$$\Delta u + \lambda u f(x, u) = 0, \quad x \in \Omega, \quad u = 0, \quad x \in \partial\Omega,$$



(Left) logistic; (Right) weak Allee effect;

Proposition: the bifurcating branch belongs to a global continuum in $C^{\alpha}(\overline{\Omega})$.

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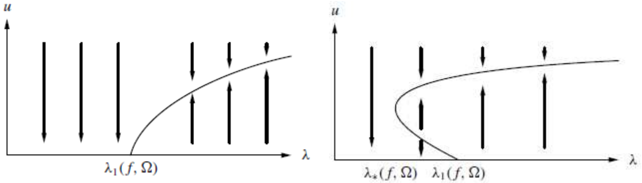

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1. $K = (-\Delta)^{-1} : C^{\alpha}(\overline{\Omega}) \to C^{\alpha}(\overline{\Omega})$ is well defined as $K(f) = u$ that $u \in C_0^{2,\alpha}(\overline{\Omega})$ such that $-\Delta u = f$ for any $f \in C^{\alpha}(\overline{\Omega})$.

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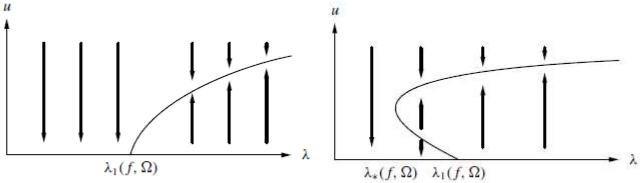

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2. We apply K to the equation, and we consider

$$G(\lambda, u) \equiv u - \lambda K u f(x,u) = 0,$$

where $u \in C^\alpha(\overline{\Omega})$.

Global Branch

1. Let $E = C^{\alpha}(\overline{\Omega})$, and let $S = \{(\lambda, u) \in \mathbb{R}^{+} \times E : G(\lambda, u) = 0, u \neq 0\}$. Then there exists a a connected component $\mathcal{C}$ of $\overline{S}$ such that $(\lambda_1, 0) \in \mathcal{C}$.

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2. Let $E^+ = \{u \in E : u(x) \geq 0 \text{ in } \Omega\}$. Then $\mathcal{C}_+ = \mathcal{C} \bigcap (\mathbb{R}^+ \times E^+)$ is unbounded.
3. If $f(x, u)$ satisfies $f(x, u) < 0$ for $u > 1$, then $0 < u(x) < 1$ for $x \in \Omega$, where u is any positive solution of

$$\Delta u + \lambda u f(x, u) = 0, \quad x \in \Omega, \quad u = 0, \quad x \in \partial\Omega.$$

Therefore the projection of $\mathcal{C}_+$ to λ-axis is unbounded in $\mathbb{R}^+$, and hence it contains (λ_1, ∞). This proves the existence of a positive solution for any $\lambda > \lambda_1$.

Uniqueness in logistic case

Assume $f_u(x,u) \le 0$ (logistic case). A positive solution to

$$\Delta u + \lambda u f(x,u) = 0, \quad x \in \Omega, \quad u = 0, \quad x \in \partial\Omega,$$

is stable if all eigenvalues μ_i of

$$-\Delta\phi - \lambda f(x,u)\phi - \lambda u f_u(x,u)\phi = \mu\phi, \quad x \in \Omega, \quad \phi = 0, \quad x \in \partial\Omega,$$

are positive.

From u's equation, u is the principal eigenfunction of the eigenvalue problem (with $\eta = 0$):

$$-\Delta\psi - \lambda f(x,u)\psi = \eta\psi, \quad x \in \Omega, \quad \psi = 0, \quad x \in \partial\Omega,$$

But

$$\begin{aligned}\mu_1 &= \min_{\phi \in W_0^{1,2}(\Omega),\ \phi \neq 0} \frac{\int_\Omega \left(|\nabla\phi|^2 - \lambda f(x,u)\phi^2 - \lambda u f_u(x,u)\phi^2\right) dx}{\int_\Omega \phi^2 dx} \\ &> \min_{\psi \in W_0^{1,2}(\Omega),\ \psi \neq 0} \frac{\int_\Omega \left(|\nabla\psi|^2 - \lambda f(x,u)\psi^2\right) dx}{\int_\Omega \psi^2 dx} = \eta_1 = 0,\end{aligned}$$

So any positive solution u is stable, hence non-degenerate.

Application 2: 1-D scalar problem

[Rabinowitz, 1972, book chapter]
Consider

$$-(p(x)u')' + q(x)u = \lambda a(x)u + \lambda F(x, u, u'), \quad 0 < x < \pi,$$
$$a_0 u(0) + b_0 u'(0) = 0, \quad a_1 u(1) + b_1 u'(1) = 0,$$

where a_0, b_0, a_1, b_1 satisfy $(a_0^2 + b_0^2)(a_1^2 + b_1^2) \neq 0$. Let E be the set of functions in $C^1[0, \pi]$ satisfying boundary condition. Let S_k^+ be the set of $\phi \in E$ such that ϕ has exactly $k - 1$ simple zeros in $(0, \pi)$, all zeros of ϕ in $[0, \pi]$ are simple, and ϕ is positive near $x = 0$. Set $S_k^- = -S_k^+$ and $S_k = S_k^+ \bigcup S_k^-$. Let S be the closure in $\mathbb{R} \times E$ of the set of nontrivial solutions of the equation. It is known that the eigenvalue problem

$$-(p(x)\phi')' + q(x)\phi = \lambda a(x)\phi, \quad 0 < x < \pi,$$
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has a sequence of eigenvalues $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \to \infty$.

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has a sequence of eigenvalues $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \to \infty$.

Theorem. If $F(x, u, u') = o(|u| + |u'|)$ near $(u, u') = (0, 0)$, then for each positive integer $k > 0$, S contains a connected component C_k, which meets $(\lambda_k, 0)$ and is unbounded in $\mathbb{R} \times S_k$.

In particular, if the equation has only trivial solutions when $\lambda > 0$, and all solutions are uniformly bounded, then for $\lambda > \lambda_k$, the problem has at least k distinct solutions, with one each in S_k.

Application 3: Global Turing Bifurcation

Theorem: Suppose that $f(u_0, v_0) = g(u_0, v_0) = 0$, and at (u_0, v_0),
(A) $f_u < 0$ (inhibitor), $g_v > 0$ (activator);
(B) $D_1 = f_u g_v - f_v g_u > 0$ and $f_u + g_v < 0$.

For fixed $\lambda > 0$, if $d_k(\lambda) \equiv \dfrac{\lambda[g_v k^2 - \lambda D_1]}{k^2(k^2 - \lambda f_u)} \neq d_j(\lambda)$ for any $j \neq k$,

(i) $d = d_k$ is a bifurcation point where a continuum Σ of non-trivial solutions of

$$\begin{cases} u_{xx} + \lambda f(u, v) = 0, \ \ dv_{xx} + \lambda g(u, v) = 0, & x \in (0, \ell\pi), \\ u_x(0) = u_x(\ell\pi) = v_x(0) = v_x(\ell\pi) = 0, \end{cases}$$

bifurcates from the line of trivial solutions (d, u_0, v_0);
(ii) The continuum Σ is either unbounded in the space of (d, u, v), or it connects to another $(d_j(\lambda), u_0, v_0)$;
(iii) Σ is locally a curve near $(d_k(\lambda), u_0, v_0)$ in form of
$(d, u, v) = (d(s), u_0 + sA\cos(kx) + o(s), v_0 + sB\cos(kx) + o(s))$, $|s| < \delta$, and
$d'(0) = 0$ thus the bifurcation is of pitchfork type ($d''(0)$ can be computed in term of $D^3(f, g)$).

[Rabinowitz, 1971, JFA],

[Shi-Wang, 2009, JDE] [Shi, 2009, Frontier Math. China]

Turing bifurcation in CIMA model

$$\begin{cases} u_t = u_{xx} + 5\alpha - u - \dfrac{4uv}{1+u^2}, & x \in (0, \ell\pi),\ t > 0, \\ v_t = m\left(dv_{xx} + u - \dfrac{uv}{1+u^2}\right), & x \in (0, \ell\pi),\ t > 0, \\ u_x(x,t) = v_x(x,t) = 0, & x = 0, \ell\pi,\ t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in (0, \ell\pi), \end{cases} \tag{9}$$

Constant equilibrium: $(u_*, v_*) = (\alpha, 1+\alpha^2)$

Jacobian at (u_*, v_*): $J = \dfrac{1}{1+\alpha^2}\begin{pmatrix} 3\alpha^2 - 5 & -4\alpha \\ 2\alpha^2 & -\alpha \end{pmatrix}$.

Assume $0 < 3\alpha^2 - 5 < \alpha$ (or $1.291 < \alpha < 1.468$)

$f_u > 0$, $g_v < 0$, $D_1 = f_u g_v - f_v g_u > 0$ and $f_u + g_v < 0$.

Bifurcation points: $d_j = \dfrac{\alpha}{1+\alpha^2} \cdot \dfrac{5+\lambda_j}{\lambda_j(f_0 - \lambda_j)}$,

where $f_0 = \dfrac{3\alpha^2 - 5}{1+\alpha^2}$, and $\lambda_j = j^2/\ell^2$.

[Ni-Tang, 2005] also true for higher dimensions

Global Turing Bifurcation for CIMA reaction

[Ni-Tang, 2005] *Trans. Amer. Math. Soc.*:
(A) For $d > 0$ small, (u_*, v_*) is the only steady state solution;
(B) All non-negative steady state solution satisfies $0 < u(x) < 5\alpha$, $0 < v(x) < 1 + 25\alpha^2$.

[Jang-Ni-Tang, 2004] *J. Dynam. Diff. Equa.*:
(C) Each connected component bifurcated from (d_j, u_*, v_*) is unbounded in the space of (d, u, v), and its projection over d-axis covers (d_j, ∞).
(D) For each $d > \min\{d_j\}$ and $d \neq d_k$, there exists a non-constant steady state solution.

[Jin-Shi-Wei-Yi, to appear] Bifurcation with α as parameter.

Bifurcation in CIMA system

$$\begin{cases} u_{xx} + 5\alpha - u - \dfrac{4uv}{1+u^2} = 0, & x \in (0, \ell\pi), \\ dv_{xx} + u - \dfrac{uv}{1+u^2} = 0, & x \in (0, \ell\pi), \\ u_x(0) = v_x(0) = u_x(\ell\pi) = v_x(\ell\pi) = 0. \end{cases}$$

$$D(\alpha, p) := d(1+\alpha^2)p^2 - p[d(3\alpha^2 - 5) - \alpha] + 5\alpha = 0$$

Bifurcation points: $(p = j^2/\ell^2)$, $\alpha(p) = \dfrac{p + 5 + \sqrt{(p+5)^2 + 4d^2p^2(3-p)(p+5)}}{2dp(3-p)}$.

For $0 < j^2/\ell^2 < 3$, there are finitely many j points $\alpha_j^S = \alpha_j^S(d, \ell)$, $1 \le j \le n$, satisfying

$$\alpha_* < \alpha_1^S < \alpha_2^S < \cdots < \alpha_n^S < \infty,$$

with $\alpha_j^S = \alpha_2(j^2/\ell^2)$, and $\alpha = \alpha_n^S$ is a possible bifurcation point.

1. There exists a C^∞ smooth curve Γ_j of solutions bifurcating from $(\alpha, u, v) = (\alpha_j^S, u_{\alpha_j^S}, v_{\alpha_j^S})$, with Γ_j contained in a global branch $\mathcal{C}_j$ of solutions;
2. Near $(\alpha, u, v) = (\alpha_j^S, u_{\alpha_j^S}, v_{\alpha_j^S})$, $\Gamma_j = \{(\alpha_j(s), u_j(s), v_j(s)) : s \in (-\epsilon, \epsilon)\}$, where $u_j(s) = \alpha_j^S + sa_j \cos(jx/\ell) + s\psi_{1,j}(s)$, $v_j(s) = (1 + \alpha_j^S)^2 + sb_j \cos(jx/\ell) + s\psi_{2,j}(s)$ for $s \in (-\epsilon, \epsilon)$ for some C^∞ smooth functions $\alpha_j, \psi_{1,j}, \psi_{2,j}$ such that $\alpha_j(0) = \alpha_j^S$ and $\psi_{1,j}(0) = \psi_{2,j}(0) = 0$;
3. Each $\mathcal{C}_j$ is unbounded: the projection of $\mathcal{C}_j$ on the α-axis contains (α_j^S, ∞).

Unbounded branches

Since $0 < u(x) < 5\alpha$ and $0 < v(x) < 1 + 25\alpha^2$, then $\mathcal{C}_j$ must remain bounded for finite α. Suppose that the projection of $\mathcal{C}_j$ in α-axis is bounded. Then $\mathcal{C}_j$ must contain another bifurcation point $(\alpha_i^S, \alpha_i^S, 1 + (\alpha_i^S)^2)$ for some $i \neq j$. Indeed $\mathcal{C}_j$ contains finitely many bifurcation points in form of $(\alpha_i^S, \alpha_i^S, 1 + (\alpha_i^S)^2)$, since there are only finitely $i \in \mathbb{N}$ such that $i^2/\ell^2 < 3$ for fixed $\ell > 0$. Among these finitely many α_i^S, there is one with largest index i_M. Notice that the equation is also well-defined for the interval $(0, \ell\pi/i_M)$, and the bifurcation points (depending on the length) have the relation

$$\alpha_{i_M}^S(\ell\pi) = \alpha_2(i_M^2/\ell^2) = \alpha_1^S(\ell\pi/i_M) := \alpha^M.$$

Hence α^M is also a bifurcation point for the equation with interval $(0, \ell\pi/i_M)$. From global bifurcation theorem, the global branch $\mathcal{C}_1^M$ bifurcating from $\alpha = \alpha^M$ for the equation with interval $(0, \ell\pi/i_M)$ is also unbounded or contains another bifurcation point. But any solution with interval $(0, \ell\pi/i_M)$ can be extended to $(0, \ell\pi)$ by reflection, hence that $\mathcal{C}_1^M$ is unbounded implies that $\mathcal{C}_j$ is unbounded, which contradicts our assumption. Or $\mathcal{C}_1^M$ contains another bifurcation point $\alpha_k^S(\ell\pi/i_M)$, but that will imply $\alpha_{ki_M}^S(\ell\pi) = \alpha_k^S(\ell\pi/i_M)$ is on the branch $\mathcal{C}_j$, and clearly $ki_M > i_M$ since $k > 1$, which contradicts with the maximality of i_M. Therefore the projection of $\mathcal{C}_j$ in α-axis is not bounded.

The weakness of Rabinowitz Theorem

It requires strong compactness. For applications in PDEs, it usually requires to take inverse of Δ operators or more general elliptic operators. For some applications with cross-diffusion or nonlinear boundary conditions, taking inverse operators are not easy.

Cross-diffusion equation: [Shigesada-Kawasaki-Teramoto, 1979]

$$\begin{cases} \Delta[(1+\alpha v)u] + u(\lambda - u - bv) = 0, & x \in \Omega, \\ \Delta[(1+\beta u)v] + v(\mu + cu - v) = 0, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega. \end{cases}$$

Problem with nonlinear boundary condition

$$\begin{cases} u'' - f(u)v = 0, & x \in (0,1), \\ \lambda v'' - \chi(v\psi'(u)u')' + (kf(u) - \theta - \beta v)v = 0, & x \in (0,1), \\ u'(0) = 0, \quad u'(1) = h(1 - u(1)), & \\ \lambda v' - \chi v\psi'(u)u' = 0, & x = 0, 1. \end{cases}$$

Global Bifurcation From Simple Eigenvalue

Theorem [Crandall-Rabinowitz, 1971, JFA]

Let $F : \mathbb{R} \times X \to Y$ be continuously differentiable. Suppose that $F(\lambda, u_0) = 0$ for $\lambda \in \mathbb{R}$, the partial derivative $F_{\lambda u}$ exists and is continuous. At (λ_0, u_0), F satisfies

(F1) $dimN(F_u(\lambda_0, u_0)) = codimR(F_u(\lambda_0, u_0)) = 1$, and

(F3) $F_{\lambda u}(\lambda_0, u_0)[w_0] \notin R(F_u(\lambda_0, u_0))$, where $w_0 \in N(F_u(\lambda_0, u_0))$,

Then the solutions of $F(\lambda, u) = 0$ near (λ_0, u_0) consists precisely of the curves $u = u_0$ and $(\lambda(s), u(s))$, $s \in I = (-\delta, \delta)$, where $(\lambda(s), u(s))$ are C^1 functions such that $\lambda(0) = \lambda_0$, $u(0) = u_0$, $u'(0) = w_0$.

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Then the solutions of $F(\lambda, u) = 0$ near (λ_0, u_0) consists precisely of the curves $u = u_0$ and $(\lambda(s), u(s))$, $s \in I = (-\delta, \delta)$, where $(\lambda(s), u(s))$ are C^1 functions such that $\lambda(0) = \lambda_0$, $u(0) = u_0$, $u'(0) = w_0$.

[Pejsachowicz-Rabier, 1998] [Shi-Wang, 2009, JDE]
If in addition, $F_u(\lambda, u)$ is a Fredholm operator for all $(\lambda, u) \in \mathbb{R} \times X$, then the curve $\{(\lambda(s), u(s)) : s \in I\}$ is contained in $\mathcal{C}$, which is a connected component of $S = \{(\lambda, u) \in \mathbb{R} \times X : F(\lambda, u) = 0, u \neq u_0\}$; and either $\mathcal{C}$ is not compact, or $\mathcal{C}$ contains a point $(\lambda_*, 0)$ with $\lambda_* \neq \lambda_0$.

Unilateral Theorem

Suppose that all conditions above are satisfied. Let $\mathcal{C}$ be defined as above. We define $\Gamma_+ = \{(\lambda(s), u(s)) : s \in (0, \epsilon)\}$ and $\Gamma_- = \{(\lambda(s), u(s)) : s \in (-\epsilon, 0)\}$. In addition we assume that

1. $F_u(\lambda, u_0)$ is continuously differentiable in λ for $(\lambda, u_0) \in V$;
2. The norm function $u \mapsto ||u||$ in X is continuously differentiable for any $u \neq 0$;
3. For $k \in (0, 1)$, if (λ, u_0) and (λ, u) are both in V, then $(1-k)F_u(\lambda, u_0) + kF_u(\lambda, u)$ is a Fredholm operator.

Let $\mathcal{C}^+$ (resp. $\mathcal{C}^-$) be the connected component of $\mathcal{C} \backslash \Gamma_-$ which contains Γ_+ (resp. the connected component of $\mathcal{C} \backslash \Gamma_+$ which contains Γ_-). Then each of the sets $\mathcal{C}^+$ and $\mathcal{C}^-$ satisfies one of the following: (i) it is not compact; (ii) it contains a point (λ_*, u_0) with $\lambda_* \neq \lambda_0$; or (iii) it contains a point $(\lambda, u_0 + z)$, where $z \neq 0$ and $z \in Z$.

Fredholm Operators of Index Zero

Quasilinear elliptic systems with nonlinear boundary conditions are Fredholm operators of index zero

Theorem [Shi-Wang, 2009, JDE]

Suppose that $p > n$, $\partial\Omega \in C^3$, and the regularity assumption above holds. Let U be an open connected set of $\mathbb{R} \times (W^{2,p}(\Omega))^N$. Assume that for each fixed $(\lambda, u) \in U$, $D_u T(\lambda, u) = (D_u A(\lambda, u), D_u B(\lambda, u))$ is elliptic on $\overline{\Omega}$, and that for a particular $(\lambda_0, u_0) \in U$, $D_u T(\lambda_0, u_0)$ satisfies Agmon's condition at a θ_0, then the Fredholm index of $D_u T(\lambda, u)$ is 0 for all $(\lambda, u) \in U$.

It will have many applications in reaction-diffusion systems in mathematical biology, physics, and chemistry.

Application

Cross-diffusion system:

$$\begin{cases} \Delta[(1+\alpha_1 u+\alpha_2 v)u]+u(\lambda-u-bv)=0, & x\in\Omega, \\ \Delta[(1+\beta_1 u+\beta_2 v)v]+v(\mu+cu-v)=0, & x\in\Omega, \\ u=v=0, & x\in\partial\Omega. \end{cases}$$

Competing species with passive diffusion, self-diffusion, cross-diffusion.
[Shigesada, Kawasaki and Teramoto, 1979]
[Nakashima, Yamada, 1996] [Kuto, Yamada, 2004]: $\alpha_1=\beta_2=0$
Their idea: $U=(1+\alpha_2 v)u$, $V=(1+\beta u)v$, then the system becomes semilinear but with messy nonlinearities.

We prove the existence of a bounded branch of coexistence solutions which connecting the two semi-trivial solution branches via our new global bifurcation theorem. Our method is definitely more direct.