Reaction-Diffusion Models and Bifurcation Theory
Lecture 7: Abstract Bifurcation Theory

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Nonlinear Functions and Derivatives in Banach Spaces

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Partial derivatives: (Frechét derivative or Gâteaux derivative)

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If $F(\lambda, u) = \Delta u + \lambda f(u)$, then
$F_u(\lambda, u)[\phi] = \Delta \phi + \lambda f'(u)\phi$, $F_\lambda(\lambda, u) = f(u)$, $F_{\lambda u}(\lambda, u) = f'(u)$.
$F_{\lambda\lambda}(\lambda, u) = 0$, $F_{uu}(\lambda, u)[\phi, \psi] = \lambda f'(u)\phi\psi$.

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$F_u(\lambda, u)$ is a linear mapping from X to Y
$\mathcal{N}(F_u(\lambda, u)) \subset X$ is the null space (the space of solutions of $F_u(\lambda, u)[\phi] = 0$)
$\mathcal{R}(F_u(\lambda, u)) \subset Y$ is the range space

Derivatives as linear mappings

$F_u(\lambda, u) : X \to Y$ is a linear mapping.
$F_\lambda(\lambda, u) : \mathbb{R} \to Y$ is a linear mapping. But we use $F_\lambda(\lambda, u)[1]$ as $F_\lambda(\lambda, u)$ since $F_\lambda(\lambda, u)[\tau]$ is $\tau F_\lambda(\lambda, u)[1]$. Hence we can understand that $F_\lambda(\lambda, u) \in Y$.

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$F_{uu}(\lambda, u) : X \times X \to Y$ is a bilinear mapping (linear in both variables). That is,
(i) $F_{uu}(\lambda, u)[\phi_1 + \phi_2, \psi] = F_{uu}(\lambda, u)[\phi_1, \psi] + F_{uu}(\lambda, u)[\phi_2, \psi]$;
$F_{uu}(\lambda, u)[\phi, \psi_1 + \psi_2] = F_{uu}(\lambda, u)[\phi, \psi_1 + \psi_2] + F_{uu}(\lambda, u)[\phi, \psi_1 + \psi_2]$.
(ii) $F_{uu}(\lambda, u)[a\phi, \psi] = aF_{uu}(\lambda, u)[\phi, \psi]$; $F_{uu}(\lambda, u)[\phi, b\psi] = bF_{uu}(\lambda, u)[\phi, \psi]$.

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Similarly, $F_{uuu}(\lambda, u) : X \times X \times X \to Y$ is a trilinear mapping.

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Finite dimensional case:
$F = F(\lambda, u) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 1$.
$F_u = (J_{ij} = \partial_j F_i)$, $n \times n$ Jacobian matrix
$F_{\lambda u} = (M_{ij} = \partial_{\lambda j} F_i)$, $n \times n$ matrix
$F_{uu} = (K_{ijk} = \partial_{jk} F_i)$, $n \times n \times n$ Hessian matroid

Linear operator: $F_u[(x_1, \cdots, x_n)] = \left(\sum_j J_{1j} x_j, \cdots, \sum_j J_{nj} x_j \right)$

Bilinear operator:

$$F_{uu}[(x_1, \cdots, x_n), (y_1, \cdots, y_n)] = \left(\sum_{j,k} K_{1jk} x_j y_k, \cdots, \sum_{j,k} K_{njk} x_j y_k \right).$$

Contraction mapping principle

Theorem 0. [Banach, 1922] Let (M, d) be a non-empty completed metric space. Assume that $T : M \to M$ is a contraction mapping, that is, there exists $k \in (0, 1)$ such that for any $x, y \in M$,

$$d(Tx, Ty) \leq k \cdot d(x, y).$$

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Proof.
1. Choose any $x_0 \in M$, and define $x_n = Tx_{n-1}$ for $n \geq 1$. Then $d(x_{n+1}, x_n) \leq k^n d(x_1, x_0)$.
2. $\{x_n\}$ is a Cauchy sequence in M since $k < 1$, and thus $\{x_n\}$ has a limit $x_* \in M$ from the completeness of (M, d).
3. Since $d(x_{n+1}, Tx_*) \to 0$ as $n \to \infty$, then $Tx_* = x_*$.
4. If there exists $y_* \in M$ such that $Ty_* = y_*$, then $0 \leq d(x_*, y_*) = d(Tx_*, Ty_*) \leq k \cdot d(x_*, y_*)$. But $k < 1$ so $d(x_*, y_*) = 0$.
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Counterexamples:
1. If M is not complete? $f(x) = x/2,\ x \in M = (0, 1)$
2. If $k = 1$? $f(x) = x + 1,\ x \in \mathbb{R}$ or $f(\theta) = \theta + \pi,\ \theta \in S^1 = (\mathbb{R}\ mod\ 2\pi)$
3. If $d(Tx, Ty) < d(x, y)$? $f(x) = \pi + x - \arctan x,\ x \in \mathbb{R}$

Solving from implicit function

Implicit function: $f(x_1, x_2, x_3, \cdots, x_n) = 0$.
Can one solve x_i from an implicit function?

1. $f_1(\lambda, y) = 3\lambda + 4y - 5$. Then $3\lambda + 4y - 5 = 0$, $y = (5 - 3\lambda)/4$. (globally uniquely solvable)
2. $f_2(\lambda, y) = \lambda^2 + y^2 - 1$. Then $\lambda^2 + y^2 - 1 = 0$, $y = \pm\sqrt{1 - \lambda^2}$. (locally uniquely solvable if $\lambda \in (-1, 1)$, not uniquely solvable if near $\lambda = \pm 1$)
3. $f_3(\lambda, y) = \lambda - \sin ye^y$. Then $\lambda - \sin ye^y = 0$, linearization at $(\lambda, y) = (0, 0)$: $\lambda - y = 0$, then $y(\lambda) \approx \lambda$ near $\lambda = 0$. (uniquely solvable? cannot be explicitly solved)

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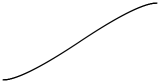
Suppose that $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is smooth, and $f(\lambda_0, y_0) = 0$.
Implicit function theorem: If $f_y(\lambda_0, y_0) \neq 0$, then for λ near λ_0, $f(\lambda, y) = 0$ has a unique solution $y(\lambda)$ so that $f(\lambda, y(\lambda)) = 0$.
Submersion theorem (preimage theorem): If
$\nabla f(\lambda_0, y_0) = (f_\lambda(\lambda_0, y_0), f_y(\lambda_0, y_0)) \neq (0, 0)$, then the set $\{(\lambda, y) : f(\lambda, y) = 0\}$ near (λ_0, y_0) is a curve.

Implicit Function Theorem: No Bifurcation

Theorem 1. Assume that X, Y are Banach spaces. Let $(\lambda_0, u_0) \in \mathbb{R} \times X$ and let F be a continuously differentiable mapping of an open neighborhood V of (λ_0, u_0) into Y. Let $F(\lambda_0, u_0) = 0$ and $F_u(\lambda_0, u_0)$ is invertible ($F_u(\lambda_0, u_0)[\phi] = 0$ only has zero solution). Then the solutions of $F(\lambda, u) = 0$ near (λ_0, u_0) form a curve $(\lambda, u(\lambda))$, $u(\lambda) = u_0 + (\lambda - \lambda_0)w_0 + z(\lambda)$, where $w_0 = -[F_u(\lambda_0, u_0)]^{-1}(F_\lambda(\lambda_0, u_0))$ and $\lambda \mapsto z(\lambda) \in X$ is a continuously differentiable function near $s = 0$ with $z(0) = z'(0) = 0$.



Theorem 1a. (Finite dimensional) Suppose that $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ is a C^k smooth mapping such that $f(\lambda_0, u_0) = 0$. Suppose that at $(\lambda_0, u_0) \in \mathbb{R}^m \times \mathbb{R}^n$, the matrix $f_u(\lambda_0, u_0) : \mathbb{R}^n \to \mathbb{R}^n$ is invertible. Then there exists a neighborhood A of λ_0 and a neighborhood B of u_0, such that for every $\lambda \in A$, there exists a unique $u = u(\lambda) \in B$ such that $f(\lambda, u) = 0$. The map $\lambda \mapsto u(\lambda)$ is C^k.

Implicit Function Theorem: more general version

Theorem 2. Let X, Y and Z be Banach spaces, and let $U \subset X \times Y$ be a neighborhood of (λ_0, u_0). Let $F : U \to Z$ be a continuously differentiable mapping. Suppose that $F(\lambda_0, u_0) = 0$ and $F_u(\lambda_0, u_0)$ is an isomorphism, *i.e.* $F_u(\lambda_0, u_0)$ is one-to-one and onto, and $F_u^{-1}(\lambda_0, u_0) : Z \to Y$ is a linear bounded operator. Then there exists a neighborhood A of λ_0 in X, and a neighborhood B of u_0 in Y, such that for any $\lambda \in A$, there exists a unique $u(\lambda) \in B$ satisfying $F(\lambda, u(\lambda)) = 0$. Moreover $u(\cdot) : A \to B$ is continuously differentiable, and $u'(\lambda_0) : X \to Y$ is defined as $u'(\lambda_0)[\psi] = -[F_u(\lambda_0, u_0)]^{-1} \circ F_\lambda(\lambda_0, u_0)[\psi]$. If F is C^k, then $u : A \to B$ is also C^k.

History:
Implicit function:
[Descartes, 1637] [Newton, 1669] [Leibnitz, 1676] [J. Bernoulli, 1695]
Implicit function theorem in $\mathbb{R}^n$: [Dini, 1878]

Equivalent forms:
Inverse function theorem, Submersion theorem,
Constant rank theorem, Preimage theorem

Inverse function theorem

Theorem 3. Let X, Y be Banach spaces, and let $U \subset X$ be a neighborhood of u_0. Let $F : U \to Y$ be a continuously differentiable mapping. Suppose that $F(u_0) = v_0$ and the Frechét derivative $F'(u_0)$ is an isomorphism, *i.e.* $F'(u_0)$ is one-to-one and onto, and $[F'(u_0)]^{-1} : Y \to X$ is a linear bounded operator. Then there exists a neighborhood A of u_0 in U and a neighborhood B of v_0 in Y, such that $F : A \to B$ is an isomorphism, and for every $v \in B$, there exists a unique $u \in A$ such that $F(u) = v$. Hence $F^{-1} : B \to A$ is well-defined, and $(F^{-1})'(F(u)) = [F'(u)]^{-1}$ for $u \in A$. Moreover F^{-1} is as smooth as F.

Theorem 3a. (Finite dimensional) Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a C^k smooth mapping such that $f(u_0) = 0$. Suppose that the matrix $f'(u_0) : \mathbb{R}^n \to \mathbb{R}^n$ is invertible. Then there exists a neighborhood A of u_0 and a neighborhood B of $f(u_0)$, such that for every $v \in B$, there exists a unique $u = u \in A$ such that $f(u) = v$. The inverse function $f^{-1} : B \to A$ is also C^k.

Proof. Define $G : U \times Y \to Y$ by $G(u, v) = F(u) - v$. Then $G(u_0, v_0) = 0$ and $G_u(u_0, v_0) = F'(u_0)$ is invertible. Then $G(u, v) = 0$ is solvable for $u = u(v)$ so that $G(u(v), v) = 0$.

Preimage Theorem

Theorem 4a. (finite dimensional) Suppose that $f : \mathbb{R}^{m+n} \to \mathbb{R}^n$ is a C^k smooth mapping such that $f(u_0) = 0$. Suppose that the matrix $f'(u_0) : \mathbb{R}^{m+n} \to \mathbb{R}^n$ is rank n (full rank). Then there exists a neighborhood A of u_0, such that the set $S = \{u \in A : f(u) = 0\}$ is a m-dimensional submanifold of A. (a special case is when $m = 1$, then S is a curve)

Let X, Y be Banach spaces.
1. A bounded linear mapping L from X to Y is Fredholm if the dimension of its kernel $\mathcal{N}(L)$ and the co-dimension of its range $\mathcal{R}(L)$ are both finite, and $\mathcal{R}(L)$ is closed. The Fredholm index of L is defined to be $ind(L) = dim\mathcal{N}(L) - codim\mathcal{R}(L)$.
2. A nonlinear mapping $F : X \to Y$ is Fredholm if its derivative $F' : X \to Y$ is Fredholm.
3. For any linear mapping $L : \mathbb{R}^n \to \mathbb{R}^m$, $ind(L) = dim\mathcal{N}(L) - codim\mathcal{R}(L) = n - m$. (rank-nullity theorem in linear algebra)

Theorem 4. [Smale, 1964, AMJ] Let X, Y be Banach spaces, and let $U \subset X$ be a neighborhood of u_0. Let $F : U \to Y$ be a continuously differentiable mapping. Suppose that $F(u_0) = v_0$ and the Frechét derivative $F'(u_0)$ is Fredholm and surjective, then there exists a neighborhood A of u_0 in U, such that the set $S = \{u \in A : F(u) = 0\}$ is a manifold of dimension $ind(F'(u_0)) = dim\mathcal{N}(L)$.

Immersion and submersion

Let X, Y be Banach spaces, and let $U \subset X$ be a subset. Suppose that $f : U \to Y$ is a continuously differentiable mapping, and its derivative at $u_0 \in X$ is a Fredholm operator $f'(u_0) : X \to Y$.

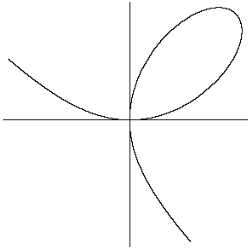
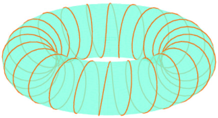
1. f is an submersion at u_0 if $f'(u_0)$ is surjective, *i.e.* $\mathcal{R}(f'(u_0)) = Y$. f is a submersion on U if it is a submersion at each $u_0 \in U$.
2. f is an immersion at u_0 if $f'(u_0)$ is injective, *i.e.* $\mathcal{N}(f'(u_0)) = \{0\}$. f is an immersion on U if it is an immersion at each $u_0 \in U$.

Submersion manifold: $f^{-1}(f(u_0))$ is locally a manifold of dimension $\dim \mathcal{N}(f'(u_0))$ containing u_0.

Immersion manifold: the image $f(U)$ is locally a Banach manifold of codimension $\text{codim}(\mathcal{R}(f'(u_0)))$.

In general $f(U)$ may not be a true manifold. If $f : U \to Y$ is an immersion and injective, and U is compact, then $f(U)$ is a manifold. This is an embedding.

Examples

1. $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = x^3 + y^3 - 3xy$. Then $f(x, y) = 0$ is the Folium of Descrates. It is locally a curve (except $(x, y) = (0, 0)$ since ∇f is surjective except at $(0, 0)$.
2. $f : \mathbb{R} \to \mathbb{R}^2$ defined by $f(t) = (\cos t, \sin t)$. It is an immersion since $f'(t)$ is injective, and $f(\mathbb{R})$ is compact. In this case $f(\mathbb{R})$ is a manifold although f is not injective.
3. (irrational winding of a torus) The curve $y = \sqrt{2}x$ on $\mathbb{R}^2$ induces a curve on a torus $T = \mathbb{R}^2/\mathbb{Z}^2$. This curve is not closed on torus, and it is an immersion, but it is not an embedded manifold. Indeed it is dense on the torus. This is another example of the image of immersion may not be a manifold. This is does not have self-intersecting points as the first one.

Saddle-Node Bifurcation

Theorem 5 [Crandall-Rabinowitz, ARMA, 1973]
Let U be a neighborhood of (λ_0, u_0) in $\mathbb{R} \times X$, and let $F : U \to Y$ be a continuously differentiable mapping. Assume that $F(\lambda_0, u_0) = 0$, F satisfies

(F1) $dim\mathcal{N}(F_u(\lambda_0, u_0)) = codim\mathcal{R}(F_u(\lambda_0, u_0)) = 1$, and

(F2) $F_\lambda(\lambda_0, u_0) \notin \mathcal{R}(F_u(\lambda_0, u_0))$.

If Z is a complement of $\mathcal{N}(F_u(\lambda_0, u_0)) = span\{w_0\}$ in X, then the solutions of $F(\lambda, u) = 0$ near (λ_0, u_0) form a curve $\{(\lambda(s), u(s)) = (\lambda(s), u_0 + sw_0 + z(s)) : |s| < \delta\}$, where $s \mapsto (\lambda(s), z(s)) \in \mathbf{R} \times Z$ is a continuously differentiable function, $\lambda(0) = \lambda'(0) = 0$, and $z(0) = z'(0) = 0$.

Saddle-Node Bifurcation

Theorem 5 [Crandall-Rabinowitz, ARMA, 1973]
Let U be a neighborhood of (λ_0, u_0) in $\mathbb{R} \times X$, and let $F : U \to Y$ be a continuously differentiable mapping. Assume that $F(\lambda_0, u_0) = 0$, F satisfies

(F1) $dim\mathcal{N}(F_u(\lambda_0, u_0)) = codim\mathcal{R}(F_u(\lambda_0, u_0)) = 1$, and

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If Z is a complement of $\mathcal{N}(F_u(\lambda_0, u_0)) = span\{w_0\}$ in X, then the solutions of $F(\lambda, u) = 0$ near (λ_0, u_0) form a curve $\{(\lambda(s), u(s)) = (\lambda(s), u_0 + sw_0 + z(s)) : |s| < \delta\}$, where $s \mapsto (\lambda(s), z(s)) \in \mathbf{R} \times Z$ is a continuously differentiable function, $\lambda(0) = \lambda'(0) = 0$, and $z(0) = z'(0) = 0$.

Proof 1: Define $G(s, \lambda, z) = F(\lambda, u_0 + sw_0 + z)$ and prove that $G_{(\lambda,z)}(0, \lambda_0, 0) : \mathbb{R} \times Z \to Y$ is an isomorphism. Then apply the implicit function theorem.

Proof 2: Apply preimage theorem to $F(\lambda, u)$. Then $F'(\lambda_0, u_0)[\tau, w] = \tau F_\lambda(\lambda_0, u_0) + F_u(\lambda_0, u_0)[w]$ which is surjective. Also $F'(\lambda_0, u_0)[\tau, w] = 0$ has a 1-dimensional kernel $\{(0, w_0)\}$.

We will later consider the case

(F2') $F_\lambda(\lambda_0, u_0) \in \mathcal{R}(F_u(\lambda_0, u_0))$.

Turning direction

Condition **(F1)**: $\dim \mathcal{N}(F_u(\lambda_0, u_0)) = \operatorname{codim} \mathcal{R}(F_u(\lambda_0, u_0)) = 1$

$X = \mathcal{N}(F_u(\lambda_0, u_0)) \oplus Z$
$Y = \mathcal{R}(F_u(\lambda_0, u_0)) \oplus Y_1$
$w_0(\neq 0) \in \mathcal{N}(F_u(\lambda_0, u_0))$
$F_u(\lambda_0, u_0)|_Z : Z \to \mathcal{R}(F_u(\lambda_0, u_0))$ is an isomorphism
there exists $l \in Y^*$ such that $\mathcal{R}(F_u(\lambda_0, u_0)) = \{v \in Y : \langle l, v \rangle = 0\}$
(Y^* is the dual space of Y, and $\langle l, v \rangle$ is the dual pair between Y^* and Y)

Turning direction

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The solution curve $\{(\lambda(s), u(s)) = (\lambda(s), u_0 + sw_0 + z(s)) : |s| < \delta\}$, where $s \mapsto (\lambda(s), z(s)) \in \mathbb{R} \times Z$ is C^1, $\lambda(0) = u_0$, $\lambda'(0) = 0$, $z(0) = z'(0) = 0$.

Turning direction

Condition **(F1)**: $\dim \mathcal{N}(F_u(\lambda_0, u_0)) = \operatorname{codim} \mathcal{R}(F_u(\lambda_0, u_0)) = 1$

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If F is C^2 in u near (λ_0, u_0), then $\lambda(s)$ is C^2 at $\lambda = \lambda_0$, and

$$\lambda''(0) = -\frac{\langle l, F_{uu}(\lambda_0, u_0)[w_0, w_0] \rangle}{\langle l, F_\lambda(\lambda_0, u_0) \rangle}.$$

$F(\lambda, u) = \Delta u + \lambda f(u)$, $F_u(\lambda, u)[w] = \Delta w + \lambda f'(u) w$,
$F_\lambda(\lambda, u) = f(u)$, $F_{uu}(\lambda, u)[\phi, \psi] = \lambda f'(u) \phi \psi$.
If $g \in \mathcal{R}(F_u(\lambda_0, u_0))$, $\Delta w + \lambda f'(u) w = g$ and $\int_\Omega g(x) w_0(x) = 0$
So $l : Y \to \mathbb{R}$ is defined by $l(g) = \int_\Omega g(x) w_0(x) 0$.

Transcritical-Pitchfork Bifurcation

Theorem 6. [Crandall-Rabinowitz, 1971, JFA]
Let U be a neighborhood of (λ_0, u_0) in $\mathbb{R} \times X$, and let $F : U \to Y$ be a continuously differentiable mapping such that $F_{\lambda u}$ exists and continuous in U. Assume that $F(\lambda, u_0) = 0$ for $(\lambda, u_0) \in U$. At (λ_0, u_0), F satisfies

(F1) $dim\mathcal{N}(F_u(\lambda_0, u_0)) = codim\mathcal{R}(F_u(\lambda_0, u_0)) = 1$, and

(F3) $F_{\lambda u}(\lambda_0, u_0)[w_0] \notin \mathcal{R}(F_u(\lambda_0, u_0))$, where $w_0 \in \mathcal{N}(F_u(\lambda_0, u_0))$,

Let Z be any complement of $\mathcal{N}(F_u(\lambda_0, u_0)) = span\{w_0\}$ in X. Then the solution set of $F(\lambda, u) = 0$ near (λ_0, u_0) consists precisely of the curves $u = u_0$ and $\{(\lambda(s), u(s)) : s \in I = (-\epsilon, \epsilon)\}$, where $\lambda : I \to \mathbb{R}$, $z : I \to Z$ are continuous functions such that $u(s) = u_0 + sw_0 + sz(s)$, $\lambda(0) = \lambda_0$, $z(0) = 0$.

known as bifurcation from simple eigenvalue
[Chow and Hale, 1982, book] [Deimling, 1985, book]

Applications: (cited 346 times in MathSciNet)
Existence of steady periodic water waves
[Constantin-Strauss, 2004, CPAM], [Strauss, 2010, BAMS]
Free boundary problem in tumor models [Friedman-Hu, 2007, SIAM-MA]
Nonconstant stationary patterns in spatial ecological models
[Du-Hsu, 2010, SIAM-MA], [Shi-Shivaji, 2006, JMB]

Turning direction

Condition **(F1)**: $dim\mathcal{N}(F_u(\lambda_0, u_0)) = codim\mathcal{R}(F_u(\lambda_0, u_0)) = 1$

$X = \mathcal{N}(F_u(\lambda_0, u_0)) \oplus Z$
$Y = \mathcal{R}(F_u(\lambda_0, u_0)) \oplus Y_1$
$w_0(\neq 0) \in \mathcal{N}(F_u(\lambda_0, u_0))$
$F_u(\lambda_0, u_0)|_Z : Z \to \mathcal{R}(F_u(\lambda_0, u_0))$ is an isomorphism
there exists $l \in Y^*$ such that $R(F_u(\lambda_0, u_0)) = \{v \in Y : \langle l, v \rangle = 0\}$

Turning direction

Condition **(F1)**: $dim\mathcal{N}(F_u(\lambda_0, u_0)) = codim\mathcal{R}(F_u(\lambda_0, u_0)) = 1$

$X = \mathcal{N}(F_u(\lambda_0, u_0)) \oplus Z$
$Y = \mathcal{R}(F_u(\lambda_0, u_0)) \oplus Y_1$
$w_0(\neq 0) \in \mathcal{N}(F_u(\lambda_0, u_0))$
$F_u(\lambda_0, u_0)|_Z : Z \to \mathcal{R}(F_u(\lambda_0, u_0))$ is an isomorphism
there exists $l \in Y^*$ such that $R(F_u(\lambda_0, u_0)) = \{v \in Y : \langle l, v\rangle = 0\}$

The nontrivial solution curve $\{(\lambda(s), u(s)) = (\lambda(s), u_0 + sw_0 + sz(s)) : |s| < \delta\}$, where $s \mapsto (\lambda(s), z(s)) \in \mathbb{R} \times Z$ is C^0, $\lambda(0) = u_0$, $z(0) = 0$, and if F is C^2 in u

$$\lambda'(0) = -\frac{\langle l, F_{uu}(\lambda_0, u_0)[w_0, w_0]\rangle}{2\langle l, F_{\lambda u}(\lambda_0, u_0)[w_0]\rangle}.$$

So if
(F4) $F_{uu}(\lambda_0, u_0)[w_0, w_0] \notin \mathcal{R}(F_u(\lambda_0, u_0))$,
is satisfied, then a transcritical bifurcation occurs.

Turning direction

Condition **(F1)**: $dim\mathcal{N}(F_u(\lambda_0, u_0)) = codim\mathcal{R}(F_u(\lambda_0, u_0)) = 1$

$X = \mathcal{N}(F_u(\lambda_0, u_0)) \oplus Z$
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$w_0(\neq 0) \in \mathcal{N}(F_u(\lambda_0, u_0))$
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The nontrivial solution curve $\{(\lambda(s), u(s)) = (\lambda(s), u_0 + sw_0 + sz(s)) : |s| < \delta\}$, where $s \mapsto (\lambda(s), z(s)) \in \mathbb{R} \times Z$ is C^0, $\lambda(0) = u_0$, $z(0) = 0$, and if F is C^2 in u

$$\lambda'(0) = -\frac{\langle l, F_{uu}(\lambda_0, u_0)[w_0, w_0] \rangle}{2\langle l, F_{\lambda u}(\lambda_0, u_0)[w_0] \rangle}.$$

So if
(F4) $F_{uu}(\lambda_0, u_0)[w_0, w_0] \notin \mathcal{R}(F_u(\lambda_0, u_0))$,
is satisfied, then a transcritical bifurcation occurs.

If **(F4)** is not satisfied, and F is C^3, then (θ satisfies $F_{uu}(\lambda_0, u_0)[w_0, w_0] + F_u(\lambda_0, u_0)[\theta] = 0$)

$$\lambda''(0) = -\frac{\langle l, F_{uuu}(\lambda_0, u_0)[w_0, w_0, w_0] \rangle + 3\langle l, F_{uu}(\lambda_0, u_0)[w_0, \theta] \rangle}{3\langle l, F_{\lambda u}(\lambda_0, u_0)[w_0] \rangle},$$

and if $\lambda''(0) \neq 0$, then a pitchfork bifurcation occurs.

Crossing Curve Bifurcation

Theorem 7. [Liu, Shi and Wang, 2007, JFA]
Let $F : \mathbb{R} \times X \to Y$ be a C^2 mapping. Suppose that $F(\lambda_0, u_0) = 0$, F satisfies

(F1) $dim\mathcal{N}(F_u(\lambda_0, u_0)) = codim\mathcal{R}(F_u(\lambda_0, u_0)) = 1$, and

(F2') $F_\lambda(\lambda_0, u_0) \in \mathcal{R}(F_u(\lambda_0, u_0))$.

Let $X = \mathcal{N}(F_u(\lambda_0, u_0)) \oplus Z$ be a fixed splitting of X, let $v_1 \in Z$ be the unique solution of $F_\lambda + F_u[v] = 0$, and let $l \in Y^*$ such that $\mathcal{R}(F_u(\lambda_0, u_0)) = \{v \in Y : \langle l, v\rangle = 0\}$. We assume that the matrix (all derivatives are evaluated at (λ_0, u_0))

$$H_0 = H_0(\lambda_0, u_0) \equiv \begin{pmatrix} \langle l, F_{\lambda\lambda} + 2F_{\lambda u}[v_1] + F_{uu}[v_1]^2\rangle & \langle l, F_{\lambda u}[w_0] + F_{uu}[w_0, v_1]\rangle \\ \langle l, F_{\lambda u}[w_0] + F_{uu}[w_0, v_1]\rangle & \langle l, F_{uu}[w_0]^2\rangle \end{pmatrix}$$

is non-degenerate, i.e., $\det(H_0) \neq 0$.

1. If H_0 is definite, *i.e.* $\det(H_0) > 0$, then the solution set of $F(\lambda, u) = 0$ near $(\lambda, u) = (\lambda_0, u_0)$ is $\{(\lambda_0, u_0)\}$.
2. If H_0 is indefinite, *i.e.* $\det(H_0) < 0$, then the solution set of $F(\lambda, u) = 0$ near $(\lambda, u) = (\lambda_0, u_0)$ is the union of two intersecting C^1 curves, and the two curves are in form of $(\lambda_i(s), u_i(s)) = (\lambda_0 + \mu_i s + s\theta_i(s), u_0 + \eta_i s w_0 + s y_i(s))$, $i = 1, 2$, where $s \in (-\delta, \delta)$ for some $\delta > 0$, (μ_1, η_1) and (μ_2, η_2) are non-zero linear independent solutions of the equation

$$\begin{aligned}&\langle l, F_{\lambda\lambda} + 2F_{\lambda u}[v_1] + F_{uu}[v_1]^2\rangle\mu^2 + 2\langle l, F_{\lambda u}[w_0] + F_{uu}[w_0, v_1]\rangle\eta\mu \\ &+ \langle l, F_{uu}[w_0]^2\rangle\eta^2 = 0,\end{aligned}$$

   where $\theta_i(s), y_i(s)$ are some functions defined on $s \in (-\delta, \delta)$ which satisfy $\theta_i(0) = \theta_i'(0) = 0$, $y_i(s) \in Z$, and $y_i(0) = y_i'(0) = 0$, $i = 1, 2$.

Remarks

1. When $F_\lambda(\lambda_0, u_0) = 0$, we have $v_1 = 0$.
   $H_1 \equiv \begin{pmatrix} \langle l, F_{\lambda\lambda}(\lambda_0, u_0)\rangle & \langle l, F_{\lambda u}(\lambda_0, u_0)[w_0]\rangle \\ \langle l, F_{\lambda u}(\lambda_0, u_0)[w_0]\rangle & \langle l, F_{uu}(\lambda_0, u_0)[w_0, w_0]\rangle \end{pmatrix}$
   and the equation of tangents of curves become

$$\langle l, F_{\lambda\lambda}(\lambda_0, u_0)\rangle\mu^2 + 2\langle l, F_{\lambda u}(\lambda_0, u_0)[w_0]\rangle\mu\eta + \langle l, F_{uu}(\lambda_0, u_0)[w_0, w_0]\rangle\eta^2 = 0.$$

2. If $F(\lambda, u_0) \equiv 0$, then Theorem 6 (classical transcritical and pitchfork bifurcation theorem) follows from Theorem 7.
3. Theorem 7 is a natural complement Crandall-Rabinowitz saddle-node bifurcation theorem (Theorem 5), where $(F2)$ is imposed. Our result is based on condition the opposite $(F2')$ and a generic second order non-degeneracy condition $\det(H_0) \neq 0$.
4. An example for single point solution set is $F(\lambda, u) = \lambda^2 + u^2$. Then $(\lambda, u) = (0, 0)$ is the only solution (also a degenerate solution)

Secondary Bifurcation

Theorem 8. [Liu-Shi-Wang, 2007, JFA] [Crandall-Rabinowitz, 1971, JFA]
Let X and Y be Banach spaces, let U be an open subset of X and let $F : U \to Y$ be twice differentiable. Suppose

1. $F(u_0) = 0$,
2. $\dim \mathcal{N}(F'(u_0)) = 2$, $\operatorname{codim} \mathcal{R}(F'(u_0)) = 1$.

Then

1. if for any $\phi(\neq 0) \in \mathcal{N}(F'(u_0))$, $F''(u_0)[\phi, \phi] \notin \mathcal{R}(F'(w_0))$, then the set of solutions to $F(u) = 0$ near $u = u_0$ is the singleton $\{u_0\}$.
2. if there exists $\phi_1(\neq 0) \in \mathcal{N}(F'(u_0))$ such that $F''(u_0)[\phi_1]^2 \in \mathcal{R}(F'(u_0))$, and there exists $\phi_2 \in \mathcal{N}(F'(u_0))$ such that $F''(u_0)[\phi_1, \phi_2] \notin \mathcal{R}(F'(u_0))$, then u_0 is a bifurcation point of $F(u) = 0$ and in some neighborhood of u_0, the totality of solutions of $F(u) = 0$ form two differentiable curves intersecting only at u_0. Moreover the solution curves are in form of $u_0 + s\psi_i + s\theta_i(s)$, $s \in (-\delta, \delta)$, $\theta_i(0) = \theta_i'(0) = 0$, where ψ_i $(i = 1, 2)$ are the two linear independent solutions of the equation $\langle l, F''(u_0)[\psi, \psi]\rangle = 0$ and $l \in Y^*$ satisfies $\mathcal{N}(l) = \mathcal{R}(F'(u_0))$.

Summary of bifurcation results

Bifurcation problem: $F(\lambda, u) = 0$, assuming $F(\lambda_0, u_0) = 0$, in all case $\dim \mathcal{N}(F_u(\lambda_0, u_0)) = \text{codim}(F_u(\lambda_0, u_0))$ (Fredholm index 0)

Name	$\dim \mathcal{N}$	solution set	Trans-cond	theorem
implicit function	0	1 monotone curve	none	Theorem 1
saddle-node	1	1 turning curve	$F_\lambda \notin \mathcal{R}$	Theorem 5
transcritical, pitchfork	1	2 crossing curves	$F_{\lambda u}[w_0] \notin \mathcal{R}$	Theorem 6
crossing-curve	1	2 crossing curves	$detH < 0$	Theorem 7
degenerate simple	1	3 crossing curves	$F_{\lambda u}[w_0] \in \mathcal{R}$, more	Theorem 9
double saddle-node	2	2 tangent curves	$F_\lambda \notin \mathcal{R}$	Theorem 10

Mapping (no parameter): $F(u) = 0$, assuming $F(u_0) = 0$

Name	$\dim \mathcal{N}$	$\text{codim}\, \mathcal{R}$	solution set	Theorem
inverse function	0	0	one point	Theorem 3
preimage	1	0	a curve	Theorem 4
preimage	k	0	a k-d manifold	Theorem 4
secondary	2	1	two crossing curves	Theorem 8

Degenerate bifurcation from simple eigenvalue

Theorem 9. [Liu-Shi-Wang, 2013, JFA]
Let U be a neighborhood of (λ_0, u_0) in $\mathbb{R} \times X$, and let $F \in C^3(U, Y)$. Assume that $F(\lambda, u_0) = 0$ for $(\lambda, u_0) \in U$. At (λ_0, u_0), F satisfies **(F1)**,

(F3′) $F_{\lambda u}(\lambda_0, u_0)[w_0] \in \mathcal{R}(F_u(\lambda_0, u_0))$; and

(F4′) $F_{uu}(\lambda_0, u_0)[w_0]^2 \in \mathcal{R}(F_u(\lambda_0, u_0))$.

Let $X = N(F_u(\lambda_0, u_0)) \oplus Z$ be a fixed splitting of X, and let $l \in Y^*$ such that $R(F_u(\lambda_0, u_0)) = \{v \in Y : \langle l, v\rangle = 0\}$. Denote by $v_1 \in Z$ the unique solution of $F_{\lambda u}(\lambda_0, u_0)[w_0] + F_u(\lambda_0, u_0)[v] = 0$, and $v_2 \in Z$ the unique solution of $F_{uu}(\lambda_0, u_0)[w_0]^2 + F_u(\lambda_0, u_0)[v] = 0$. We assume that the matrix (all derivatives are evaluated at (λ_0, u_0))

$$H = H(\lambda_0, u_0) = \begin{pmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{pmatrix}$$

is non-degenerate, *i.e.*, $\det(H) \neq 0$, where H_{ij} is given by

$$H_{11} = \langle l, F_{\lambda\lambda u}[w_0] + 2F_{\lambda u}[v_1]\rangle, \; H_{22} = \frac{1}{3}\langle l, F_{uuu}[w_0]^3 + 3F_{uu}[w_0, v_2]\rangle,$$

$$H_{12} = \frac{1}{2}\langle l, F_{\lambda uu}[w_0]^2 + F_{\lambda u}[v_2] + 2F_{uu}[w_0, v_1]\rangle$$

Degenerate bifurcation from simple eigenvalue

1. If H is definite, *i.e.* $\det(H) > 0$, then the solution set of $F(\lambda, u) = 0$ near $(\lambda, u) = (\lambda_0, u_0)$ is the line $\{(\lambda, u_0)\}$.
2. If H is indefinite, *i.e.* $\det(H) < 0$, then the solution set of $F(\lambda, u) = 0$ near $(\lambda, u) = (\lambda_0, u_0)$ is the union of C^1 curves intersecting at (λ_0, u_0), including the line of trivial solutions $\Gamma_0 = \{(\lambda, u_0)\}$ and two other curves $\Gamma_i = \{(\lambda_i(s), u_i(s)) : |s| < \delta\}$ $(i = 1, 2)$ for some $\delta > 0$, with

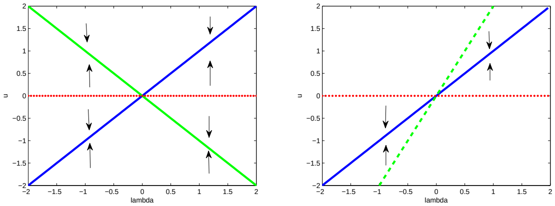
$$\lambda_i(s) = \lambda_0 + \mu_i s + s\theta_i(s), \quad u_i(s) = u_0 + \eta_i s w_0 + s v_i(s),$$

where (μ_1, η_1) and (μ_2, η_2) are non-zero linear independent solutions of the equation

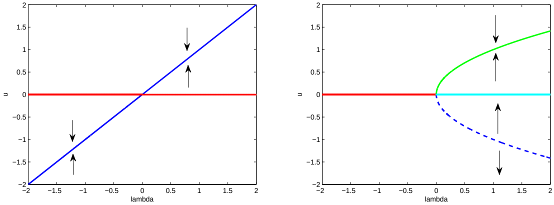
$$H_{11}\mu^2 + 2H_{12}\mu\eta + H_{22}\eta^2 = 0,$$

$\theta_i(0) = \theta_i'(0) = 0$, $v_i(s) \in Z$, and $v_i(0) = v_i'(0) = 0$, $i = 1, 2$.

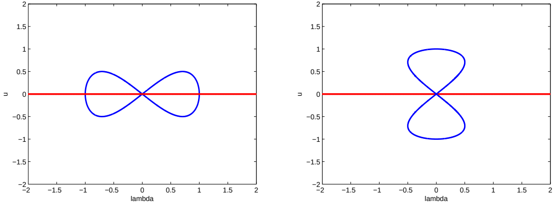
Example: (Left):$F(\lambda, u) = u(\lambda^2 - u^2)$; (Right):$F(\lambda, u) = u(u - \lambda)(u - 2\lambda)$.



More examples



(Left):$F(\lambda, u) = \lambda u^2 - u^3$; (Right):$F(\lambda, u) = \lambda u^2 - u^4$.



(Left): $F(\lambda, u) = u(\lambda^2 - u^2 + u^4)$; (Right): $F(\lambda, u) = u(\lambda^4 - \lambda^2 + u^2)$.

What if Kernel is 2-dimensional?

Let $F : \mathbb{R} \times X \to Y$ be continuously differentiable. $F(\lambda_0, u_0) = 0$, F satisfies

(F1-2) $dimN(F_u(\lambda_0, u_0)) = codimR(F_u(\lambda_0, u_0)) = 2$, and

(F2) $F_\lambda(\lambda_0, u_0) \notin R(F_u(\lambda_0, u_0))$.

and additional non-degeneracy condition on D^2F

Then a saddle-node bifurcation of two curves occurs
[Liu-Shi-Wang, 2013 to appear, CPAA]

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and additional non-degeneracy condition on D^2F

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[Liu-Shi-Wang, 2013 to appear, CPAA]

If F satisfies **(F1-2)**,

(F2') $F_\lambda(\lambda_0, u_0) \in R(F_u(\lambda_0, u_0))$,

more complicated, depending on the symmetry of the problem
(example: two-dimensional surface, four curves)

Double Saddle-Node Bifurcation

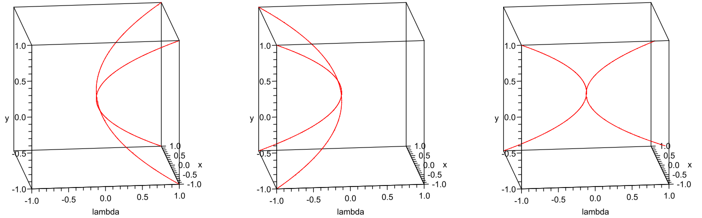
Theorem 10 [Liu-Shi-Wang, 2013 to appear, CPAA]
Let $F : \mathbb{R} \times X \to Y$ be a C^p mapping, and assume (F1-2) and (F2). Let $X = N(F_u(\lambda_0, u_0)) \oplus Z$ be a fixed splitting of X, and let $v_1, v_2 \in Y^*$ such that $R(F_u) = \{h \in Y : \langle v_1, h\rangle = 0 \text{ and } \langle v_2, h\rangle = 0\}$ so that $\langle v_1, F_\lambda\rangle \neq 0$ and $\langle v_2, F_\lambda\rangle = 0$. We assume that the matrix (all derivatives are evaluated at (λ_0, u_0))

$$H_2 = H_2(\lambda_0, u_0) \equiv \begin{pmatrix} \langle v_2, F_{uu}[w_1, w_1]\rangle & \langle v_2, F_{uu}[w_1, w_2]\rangle \\ \langle v_2, F_{uu}[w_1, w_2]\rangle & \langle v_2, F_{uu}[w_2, w_2]\rangle \end{pmatrix}$$

is non-degenerate, i.e., $\det(H_2) \neq 0$. Let S be the solution set of $F(\lambda, u) = 0$ near $(\lambda, u) = (\lambda_0, u_0)$.

1. If H_2 is definite, *i.e.* $\det(H_2) > 0$, then S is $\{(\lambda_0, u_0)\}$.
2. If H_2 is indefinite, *i.e.* $\det(H_2) < 0$, then S is the union of two C^{p-1} tangent curves.

An Example



Three types of double saddle-node bifurcations: (left) $\lambda - x^2 = 0$ and $\lambda - y^2 = 0$, supercritical; (middle) $\lambda - x^2 + 2y^2 = 0$ and $\lambda - y^2 + 2x^2 = 0$, subcritical; (right) $\lambda - x^2 - 2xy = 0$ and $\lambda - y^2 - 2xy = 0$, transcritical.

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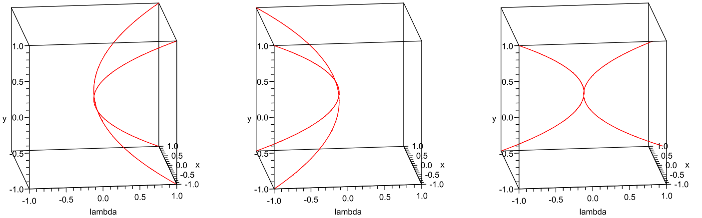

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Question: Is there a realistic example of double saddle-node bifurcation?

Lyapunov-Schmidt reduction

Suppose that $F : \mathbb{R} \times X \to Y$ is a C^p map such that $F(\lambda_0, u_0) = y_0$, and F satisfies $(F1)$ at (λ_0, u_0). Then $F(\lambda, u) = y_0$ for (λ, u) near (λ_0, u_0) can be reduced to $\langle l, F(\lambda, u_0 + tw_0 + g(\lambda, t))\rangle = 0$, where $t \in (-\delta, \delta)$, $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$ where δ is a small constant,, $l \in Y^*$ such that $\langle l, u\rangle = 0$ if and only if $u \in \mathcal{R}(F_u(\lambda_0, u_0))$, and g is a C^p function into Z such that $g(\lambda_0, 0) = 0$ and Z is a complement of $N(F_u(\lambda_0, u_0))$ in X.

Let the projection from Y into $R(F_u(\lambda_0, u_0))$ be Q. Then $F(\lambda, u) = 0$ is equivalent to

$$Q \circ F(\lambda, u) = 0, \quad \text{and } (I - Q) \circ F(\lambda, u) = 0.$$

We rewrite the first equation in form

$$G(\lambda, t, g) \equiv Q \circ F(\lambda, u_0 + tw_0 + g) = 0$$

where $t \in \mathbb{R}$ and $g \in Z$. Calculation shows that $G_g(\lambda_0, 0, 0) = Q \circ F_u(\lambda_0, u_0)$ is an isomorphism from Z to $R(F_u(\lambda_0, u_0))$. Then $g = g(\lambda, t)$ is uniquely solvable from the implicit function theorem for (λ, t) near $(\lambda_0, 0)$, and g is C^2. Hence $u = u_0 + tw_0 + g(\lambda, t)$ is a solution to $F(\lambda, u) = 0$ if and only if $(I - Q) \circ F(\lambda, u_0 + tw_0 + g(\lambda, t)) = 0$. Since $R(F_u(\lambda_0, u_0))$ is co-dimensional one, hence it becomes the scalar equation $\langle l, F(\lambda, u_0 + tw_0 + g(\lambda, t))\rangle = 0$.

Proof of Main Results

1. Lyapunov-Schmidt reduction: reduce $F(\lambda, u) = 0$ to

$G(\lambda, t) \equiv \langle l, F(\lambda, u_0 + tw_0 + g(\lambda, t)) \rangle = 0$

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2. A finite dimensional result (improving result based on Morse lemma)

Morse lemma: [Nirenberg, 1974, book] Suppose that $f : \mathbb{R}^k \to \mathbb{R}$ is a C^p function, $k \geq 2$. If $f(0) = 0$, $f_x(0) = 0$, and the Hessian $f_{xx}(0)$ is a non-degenerate $k \times k$ matrix. Then there exists a local C^{p-2} coordinate change $y(x)$ defined in a neighborhood of the origin with $y(0) = 0$, $y_x(0) = I$ such that

$$f(x) = \frac{1}{2} y(x)^T f_{xx}(0) y(x),$$

where $y(x)^T$ is the transpose of $y(x)$, and $y(x)$ is assumed to be column vector in $\mathbb{R}^k$. In particular if $k = 2$ and $f_{xx}(0)$ is indefinite, then the set of solutions of $f(x) = 0$ near the origin consists of two C^{p-2} curves intersecting only at the origin.

A Finite Dimensional Theorem

[Liu-Wang-Shi, 2007, JFA]
Suppose that $(x_0, y_0) \in \mathbb{R}^2$ and U is a neighborhood of (x_0, y_0). Assume that $f : U \to \mathbb{R}$ is a C^p function for $p \geq 2$, $f(x_0, y_0) = 0$, $\nabla f(x_0, y_0) = 0$, and the Hessian $H = H(x_0, y_0)$ is non-degenerate. Then

1. If H is definite, then (x_0, y_0) is the unique zero point of $f(x, y) = 0$ near (x_0, y_0);
2. If H is indefinite, then there exist two C^{p-1} curves $(x_i(t), y_i(t))$, $i = 1, 2$, $t \in (-\delta, \delta)$, such that the solution set of $f(x, y) = 0$ consists of exactly the two curves near (x_0, y_0), $(x_i(0), y_i(0)) = (x_0, y_0)$. Moreover t can be rescaled and indices can be rearranged so that $(x_1'(0), y_1'(0))$ and $(x_2'(0), y_2'(0))$ are the two linear independent solutions of

$$f_{xx}(x_0, y_0)\eta^2 + 2f_{xy}(x_0, y_0)\eta\tau + f_{yy}(x_0, y_0)\tau^2 = 0.$$

Proof of the "Calculus Problem"

Consider

$$x' = \frac{\partial f(x,y)}{\partial y}, \;\; y' = -\frac{\partial f(x,y)}{\partial x}, \;\; (x(0), y(0)) \in U.$$

Then it is a Hamiltonian system with potential function $f(x, y)$, (x_0, y_0) is the only equilibrium point in U, and (x_0, y_0) is a saddle point. From the invariant manifold theory of differential equations, the set $\{f(x, y) = 0\}$ near (x_0, y_0) is consisted of the 1-dimensional stable and unstable manifolds at (x_0, y_0), which are C^{p-1} since f is C^p.

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Question: is there a proof without using invariant manifold theory but only elementary calculus?

another more general theorem:

splitting lemma [Kuiper, 1972] [Chang, 1993, book] [Li-Li-Liu, 2005, JFA]

Complete the Proof

Proof of Theorem 7 (crossing-curves)

Apply the 2-D lemma to the bifurcation equation $G(\lambda, t) \equiv \langle l, F(\lambda, u_0 + tw_0 + g(\lambda, t))\rangle = 0$.

Proof of Theorem 9 (degenerate simple eigenvalue)

$G(\lambda, t) \equiv \langle l, F(\lambda, u_0 + tw_0 + g(\lambda, t))\rangle = 0$ has a line of trivial solutions: $t = 0$ for all λ. So we apply the 2-D lemma to the function $h(\lambda, t) = \begin{cases} \frac{1}{t} G(\lambda, t), & \text{if } t \neq 0, \\ G_t(\lambda, 0), & \text{if } t = 0. \end{cases}$

Proof of theorem 10 (double saddle-node)

Since the kernel is 2-D, then the Lyapunov-Schmidt reduction reduces it to

$$G_i(\lambda, s_1, s_2) = \langle v_i, F(\lambda, u_0 + s_1 w_1 + s_2 w_2 + g(\lambda, s_1, s_2))\rangle, \ i = 1, 2.$$

But λ can be solved by (s_1, s_2) by implicit function theorem (since $F_\lambda \notin \mathcal{R}(F_u(\lambda_0, u_0))$). Then we can apply the 2-D lemma to $f_2(s_1, s_2) \equiv G_1(\lambda(s_1, s_2), s_1, s_2)$.

Stability

In the bifurcation from simple eigenvalue described in Theorem 6, an exchange of stability occurs at the bifurcation point between the known trivial solutions and the bifurcating nontrivial solutions.

Let $T, K \in B(X, Y)$ (the set of bounded linear maps from X into Y). We say that $\mu \in \mathbb{R}$ is a K-simple eigenvalue of T, if

$$\dim N(T - \mu K) = \operatorname{codim} R(T - \mu K) = 1, \quad N(T - \mu K) = \operatorname{span}\{w_0\},$$

and

$$K[w_0] \notin R(T - \mu K).$$

Lemma[Crandall and Rabinowitz, 1973]: Suppose that $T_0, K \in B(X, Y)$ and μ_0 is a K-simple eigenvalue of T_0. Then there exists $\delta > 0$ such that if $T \in B(X, Y)$ and $\|T - T_0\| < \delta$, then there exists a unique $\mu(T) \in \mathbb{R}$ satisfying $\|\mu(T) - \mu_0\| < \delta$ such that $N(T - \mu(T)K) \neq \emptyset$ and $\mu(T)$ is a K-simple eigenvalue of T. Moreover if $N(T_0 - \mu_0 K) = \operatorname{span}\{w_0\}$ and Z is a complement of $\operatorname{span}\{w_0\}$ in X, then there exists a unique $w(T) \in X$ such that $N(T - \mu(T)K) = \operatorname{span}\{w(T)\}$, $w(T) - w_0 \in Z$ and the map $T \mapsto (\mu(T), w(T))$ is analytic.

Exchange of stability

Theorem [Crandall and Rabinowitz, 1973] Let X, Y, U, F, Z, λ_0, and w_0 be the same as in Theorem 2, and let all assumptions in Theorem 2 on F be satisfied. Let $(\lambda(t), u(t))$ be the curve of nontrivial in Theorem 2, there exist C^2 functions $m(\lambda) : (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \to \mathbb{R}$, $z : (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \to X$, $\mu : (-\delta, \delta) \to \mathbb{R}$, and $w : (-\delta, \delta) \to X$, such that

$$F_u(\lambda, u_0)z(\lambda) = m(\lambda)K(z(\lambda)), \quad \text{for} \quad \lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon), \tag{1}$$

$$F_u(\lambda(t), u(t))\phi(t) = \mu(t)K(\phi(t)), \quad \text{for} \quad t \in (-\delta, \delta). \tag{2}$$

where $m(\lambda_0) = \mu(0) = 0$, $z(\lambda_0) = w(0) = w_0$. Moreover, near $t = 0$ the functions $\mu(t)$ and $-t\lambda'(t)m'(\lambda_0)$ have the same zeroes and, whenever $\mu(t) \neq 0$ the same sign and satisfy

$$\lim_{t \to 0} \frac{-t\lambda'(t)m'(\lambda_0)}{\mu(t)} = 1. \tag{3}$$

Direction of bifurcations

A saddle-node bifurcation is supercritical (forward) if $\lambda''(0) > 0$, and it is subcritical (backward) if $\lambda''(0) < 0$.

A pitchfork bifurcation is supercritical (forward) if $\lambda''(0) > 0$, and it is subcritical (backward) if $\lambda''(0) < 0$.

In a transcritical bifurcation, it is often that only the positive portion (the part with $s \in (0, \delta)$) of curve is concerned). If the solutions with $s \in (0, \delta)$ are stable, then it is forward transcritical bifurcation; and if the solutions with $s \in (0, \delta)$ are unstable, then it is backward transcritical bifurcation.

An ODE system of epidemic model

Adapted from [Wang Wending, 2006, Math. Bios.]

$$S' = A - dS - \lambda SI, \quad I' = \lambda SI - \gamma I - \frac{kI}{1+I}$$

parameters A: recruitment of population, d, γ: mortality rates, λ: infection rate, $T(I)$: treatment

Trivial equilibrium: $(S, I) = (A/d, 0)$. Define $F(\lambda, S, I) = \begin{pmatrix} A - dS - \lambda SI \\ \lambda SI - \gamma I - \dfrac{kI}{1+I} \end{pmatrix}$.

$$F_{(S,I)} = \begin{pmatrix} -d - \lambda I & -\lambda S \\ \lambda I & \lambda S - \gamma - \frac{ka}{(a+I)^2} \end{pmatrix},$$

$$F_{(S,I)}(\lambda, A/d, 0) = \begin{pmatrix} -d & -\lambda A/d \\ 0 & \lambda A/d - \gamma - k/a \end{pmatrix}$$

So bifurcation point $\lambda = \lambda_* = \dfrac{\gamma + k/a}{A/d} = \dfrac{d(a\gamma + k)}{Aa}$.

Eigenvector $w_0 = [\phi, \psi] = (-\lambda_* A, d^2)$

$\mathcal{R}F_{(S,I)}(\lambda, A/d, 0) = \{(a, b) \in \mathbb{R}^2 : b = 0\} = \{(a, b) \in \mathbb{R}^2 : (a, b) \cdot (0, 1) = 0\}$.

An ODE system of epidemic model

$$F_{(S,I)(S,I)}[(\phi,\psi]^2 = \begin{pmatrix} -2\lambda\phi\psi \\ 2\lambda\phi\psi + 2k\psi^2/a \end{pmatrix}$$
$$F_{\lambda(S,I)}[(\phi,\psi] = \begin{pmatrix} -A\psi_1/d \\ A\psi_1/d \end{pmatrix}$$

$$\lambda'(0) = -\frac{\langle l, F_{(S,I)(S,I)}(\lambda_0, A/d, 0)[w_0, w_0]\rangle}{2\langle l, F_{\lambda(S,I)}(\lambda_0, A/d, 0)[w_0]\rangle} = \frac{d^3[(a\gamma+k)^2 - kAa]}{a^2A^2}.$$

So a backward bifurcation ($\lambda'(0) < 0$) occurs if A is large.

Oyster Model

[Jordan-Cooley, Lipcius, Shaw, Shen, Shi, 2011, JTB]

$$\frac{dO}{dt} = rOf(d)\left(1 - \frac{O}{k}\right) - \mu f(d)O - \epsilon(1 - f(d))O, \tag{4}$$

$$\frac{dB}{dt} = \mu f(d)O + \epsilon(1 - f(d))O - \gamma B, \tag{5}$$

$$\frac{dS}{dt} = -\beta S + Cge^{-\frac{FO}{Cg}}. \tag{6}$$

Here $d = \dfrac{O}{2} + B - S$, and $f = f(d)$ satisfies

$$f'(d) > 0, \quad f(0) = \frac{1}{2}, \quad \lim_{d \to -\infty} f(d) = 0, \quad \text{and} \quad \lim_{d \to \infty} f(d) = 1;$$

the function $g(x) = g(O + B)$ satisfies

$$g(0) = 1, \quad g'(x) \leq 0 \quad \text{for} \quad x \geq 0, \quad \text{and} \quad \lim_{x \to \infty} g(x) = 0;$$

and $F = F(y) = F(Cg)$ satisfies

$$F(0) = 0; \quad \lim_{y \to \infty} F(y) = 0; \quad \text{and there exists } y_0 > 0 \text{ such that}$$

$$F'(y) > 0 \text{ for } 0 < y < y_0, \quad F'(y) < 0 \text{ for } y > y_0, \quad \text{and} \quad F(y_0) = F_0.$$

Bifurcation Point

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$$G(C, O, B, z) = \begin{pmatrix} rOf(d)\left(1 - \dfrac{O}{k}\right) - \mu f(d)O - \epsilon(1 - f(d))O \\ rf(d)\dfrac{O^2}{k} + \mu f(d)O - \gamma B + \epsilon(1 - f(d))O \\ Cge^{-\frac{FO}{Cg}} - \beta z - C \end{pmatrix}. \tag{7}$$

Then for $u = (O, B, z)$, the derivative $G_u(C, 0, 0, 0)$ is given by

$$J(0, 0, C/\beta) = \begin{pmatrix} f(-C/\beta)(r - \mu + \epsilon) - \epsilon & 0 & 0 \\ f(-C/\beta)(\mu - \epsilon) + \epsilon & -\gamma & 0 \\ Cg'(0) - F(C) & Cg'(0) & -\beta \end{pmatrix}.$$

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A bifurcation point C_* can be uniquely determined by $f(-C/\beta) = \epsilon/(r - \mu + \epsilon)$.

Transcritical Bifurcation

At $C = C_*$, $G_u(C_*, 0, 0, 0)$ can be written as

$$L \equiv G_u(C_*, 0, 0, 0) = \begin{pmatrix} 0 & 0 & 0 \\ \frac{\epsilon r}{r - \mu + \epsilon} & -\gamma & 0 \\ C_* g'(0) - F(C_*) & C_* g'(0) & -\beta \end{pmatrix}. \quad (8)$$

We take the eigenvector of L to be $w_0 = (1, w_{02}, w_{03})$ where

$$w_{02} = \frac{\epsilon r}{\gamma(r - \mu + \epsilon)},$$
$$w_{03} = \frac{C_* g'(0) - F(C_*)}{\beta} + \frac{C_* g'(0)\epsilon r}{\beta\gamma(r - \mu + \epsilon)},$$

one can see that the range of L is $\{(0, y, z) \in \mathbb{R}^3\}$ which is two-dimensional, so we can take the vector l to be $(1, 0, 0)$.

Turning direction

$\langle l, G_{\lambda u}(C_*, 0, 0, 0)[w_0]\rangle = -f'(-C_*/\beta)(r - \mu + \epsilon)/\beta < 0$, and

$$\langle l, G_{uu}(C_*, 0, 0, 0)[w_0, w_0]\rangle = \frac{2r}{k} f'(-C_*/\beta) I,$$

where

$$I = \frac{\lambda(r - \mu + \epsilon)K}{2r} - \frac{f(-C_*/\beta)}{f'(-C_*/\beta)} + \frac{\lambda \epsilon K}{\gamma} - \frac{\epsilon K}{\gamma \beta}\left(C_* g'(0) - F(C_*) + \frac{C_* g'(0)\epsilon r}{\gamma(r - \mu + \epsilon)}\right). \tag{9}$$

So

$$C'(0) = -\frac{\langle l, G_{uu}(C_*, 0, 0, 0)[w_0, w_0]\rangle}{2\langle l, G_{\lambda u}(C_*, 0, 0, 0)[w_0]\rangle} = \frac{rI}{k(r - \mu + \epsilon)},$$

If $I < 0$ then the bifurcation is forward; if $I > 0$ then it is backward which implies bistable parameter ranges.