Reaction-Diffusion Models and Bifurcation Theory Lecture 5: Stability in reaction-diffusion equations

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Banach spaces

1. **Metric space**: a pair (M, d) where M is a set and $d : M \times M \to \mathbb{R}$ is a metric which satisfies for any $x, y, z \in M$: (i) $d(x, y) \geq 0$; (ii) $d(x, y) = 0$ if and only if $x = y$; (iii) $d(x, y) = d(y, x)$; and (iv) (triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$.
2. **Complete**: A metric space (M, d) is *complete* if any Cauchy sequence $\{x_n\} \subset M$ has a limit in M.
3. **Normed vector space**: a pair $(V, ||\cdot||)$ where V is a linear vector space over real numbers and the norm $||\cdot|| : V \to \mathbb{R}$ is a function which satisfies for any $a \in \mathbb{R}$ and $x, y \in V$: (i) $||ax|| = |a| \cdot ||x||$; (ii) $||x|| \geq 0$, and $||x|| = 0$ if and only if $x = 0$ (the zero vector); and (iii) (triangle inequality) $||x + y|| \leq ||x|| + ||y||$. A normed vector space is a metric space with the metric $d(x, y) = ||x - y||$.
4. **Banach space**: a complete normed vector space.
5. **Linear operator**: Let X and Y be Banach spaces. A mapping $F : X \to Y$ is linear if (i) $F(ax) = aF(x)$ and (ii) $F(x + y) = F(x) + F(y)$ for any $a \in \mathbb{R}$ and $x, y \in X$.
6. **Nonlinear operator**: A mapping $F : X \to Y$ which is not linear.
7. **Dimension**: For $n \in \mathbb{N}$, a Banach space X is n-dimensional if there is a subset of X with n non-zero linear independent elements, but there is no such a set with $n + 1$ element.
8. Example: $\mathbb{R}^n$, $C(\overline{\Omega})$ (the set of continuous functions defined on $\overline{\Omega}$).

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7. Let L be a linear compact operator from X to X. Then $I - L$ is a linear Fredholm operator with index 0.

Function spaces

Let Ω be a bounded domain in $\mathbb{R}^n$.

1. Sobolev spaces: $L^p(\Omega)$, $W^{k,p}(\Omega)$, $W_0^{k,p}(\Omega)$ and for $k \in \mathbb{N}$ and $1 \leq p \leq \infty$ (Banach spaces).
2. $W^{k,2}(\Omega)$ and $L^2(\Omega)$ (Hilbert spaces).
3. Hölder spaces $C^{k,\alpha}(\overline{\Omega})$, $C_0^{k,\alpha}(\overline{\Omega})$, $C^k(\overline{\Omega})$ and $C_0^k(\overline{\Omega})$ for $k \in \mathbb{N} \cup \{0\}$ and $\alpha \in (0,1]$ (Banach spaces).
4. $W_0^{k,p}(\Omega)$, $C_0^{k,\alpha}(\overline{\Omega})$: $u = 0$ on $\partial\Omega$.
5. $W_N^{k,p}(\Omega)$ $(k \geq 2)$, $C_N^{k,\alpha}(\overline{\Omega})$ $(k \geq 1)$: $\nabla u \cdot n = 0$ on $\partial\Omega$.
6. (Sobolev embedding)
   (i) If $p < n$, then $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $1 \leq q \leq \dfrac{np}{n-p}$
   (ii) If $p > n$, then $W^{k,p}(\Omega) \hookrightarrow C^{k-1,\alpha}(\overline{\Omega})$ for $\alpha = 1 - n/p$.

Eigenvalues of Elliptic Operators

Let Ω be a bounded domain with smooth boundary. For any $q(x)$ in $C\left(\bar{\Omega}\right)$ and $d > 0$, consider the linear eigenvalue problem

$$\begin{cases} -d\Delta\phi + q(x)\phi = \rho\phi, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega. \end{cases}$$

1. Any eigenvalue is real-valued, and the eigenvalue problem has an infinite sequence of eigenvalues, $\rho_1 < \rho_2 \leq \rho_3 \leq \cdots \to \infty$.
2. The principal eigenvalue $\rho_1 = \rho_1\left(-d\Delta + q(x)\right)$ is simple, and associated eigenfunction $\phi_1(x)$ can be chosen as positive.
3. An eigenfunction $\phi_i(x)$ corresponding to ρ_i with $i \geq 2$ are sign-changing since $\int_\Omega \phi_i(x)\phi_1(x)dx = 0$.
4. ρ_1 is strictly increasing in the sense that $q_1(x), q_2(x) \in C\left(\bar{\Omega}\right)$, $q_1(x) \leq q_2(x)$ and $q_1(x) \not\equiv q_2(x)$ implies that $\rho_1\left(-d\Delta + q_1(x)\right) < \rho_1\left(-d\Delta + q_2(x)\right)$.
5. The variational characterization for ρ_1:

$$\rho_1\left(-d\Delta + q(x)\right) = \inf_{\varphi \in W_0^{1,2}(\Omega), \varphi \neq 0} \frac{\int_\Omega \left(d|\nabla\varphi|^2 + q\varphi^2\right) dx}{\int_\Omega \varphi^2 dx}.$$

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$n = 1$: $-\phi''(x) = \lambda\phi(x)$, $x \in (0, \pi)$, and $\phi(0) = \phi(\pi) = 0$.
$\lambda_i = i^2$ $(i = 1, 2, 3, \cdots)$, and $\phi_i(x) = \sin(i\pi x)$. (all eigenvalues are simple ones.)

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$n = 2$ (rectangle):
$-\phi_{xx} - \phi_{yy} = \lambda\phi$, $(x, y) \in (0, a\pi) \times (0, b\pi)$, and $\phi(x, y) = 0$ on the boundary.
$\lambda_{m,n} = \dfrac{m^2}{a^2} + \dfrac{n^2}{b^2}$ $(m, n = 1, 2, 3, \cdots)$, and $\phi_{m,n}(x, y) = \sin(m\pi x/a)\sin(n\pi y/b)$.
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Positive or negative eigenvalues?

$-d\Delta\phi + q(x)\phi = \rho\phi,\ x \in \Omega,\ \ \phi = 0,\ x \in \partial\Omega$
has an infinite sequence of eigenvalues, $\rho_1 < \rho_2 \leq \rho_3 \leq \cdots \to \infty$, and the principal eigenvalue ρ_1 is the <u>smallest</u> eigenvalue. This is mostly used by people in elliptic equations.
$d\Delta\phi - q(x)\phi = \rho\phi,\ x \in \Omega,\ \ \phi = 0,\ x \in \partial\Omega$
has an infinite sequence of eigenvalues, $\rho_1 > \rho_2 \geq \rho_3 \geq \cdots \to -\infty$, and the principal eigenvalue ρ_1 is the <u>largest</u> eigenvalue. This is mostly used by people in dynamical systems (as linearization of nonlinear equation like $u_t = d\Delta u + f(u)$).

Elliptic Operators

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. Consider the Possion's equation

$$-\Delta u(x) = f(x), \quad x \in \Omega, \quad u(x) = 0, \quad x \in \partial\Omega.$$

1. $-\Delta$ is a linear operator, but it is NOT bounded since it has a sequence of eigenvalues ρ_i which tend to ∞.

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2. For every $f \in L^p(\Omega)$, $p > 1$, Possion's equation has a unique solution $u \in W^{2,p}(\Omega) \bigcap W_0^{1,p}(\Omega)$, and $||u||_{W^{2,p}} \le c||f||_{L^p}$. ($L^p$ estimate)

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3. For every $f \in C^\alpha(\overline{\Omega})$, $\alpha \in (0,1)$, Possion's equation has a unique solution $u \in C_0^{2,\alpha}(\overline{\Omega})$, and $||u||_{C^{2,\alpha}} \leq c||f||_{C^\alpha}$. (Schauder estimate)

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4. $-\Delta$ is an invertible linear operator. Let $K = (-\Delta)^{-1}$. Then $K : L^p(\Omega) \to W^{2,p}(\Omega) \bigcap W_0^{1,p}(\Omega)$ or $K : C^\alpha(\overline{\Omega}) \to C_0^{2,\alpha}(\overline{\Omega})$ is a bounded linear operator. So $\underline{(-\Delta)^{-1} \text{ is bounded}}$.

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5. The inclusion mapping $i : W^{2,p}(\Omega) \to L^p(\Omega)$ or $i : C_0^{2,\alpha}(\overline{\Omega}) \to C^\alpha(\overline{\Omega})$ (defined by $i(x) = x$) is a linear compact mapping.

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6. Hence $(-\Delta)^{-1} = i \circ K : L^p(\Omega) \to L^p(\Omega)$ ($C^\alpha(\overline{\Omega}) \to C^\alpha(\overline{\Omega})$) is a linear compact operator.
7. $(-\Delta)^{-1} : C^{1,\alpha}(\overline{\Omega}) \to C^{1,\alpha}(\overline{\Omega})$ or $(-\Delta)^{-1} : C(\overline{\Omega}) \to W^{2,p}(\overline{\Omega})$ $(p > n)$ is also compact.

Other boundary conditions and systems

1. Neumann boundary:

$$\begin{cases} -d\Delta\phi + q(x)\phi = \rho\phi, & x \in \Omega, \\ \nabla\phi \cdot n = 0, & x \in \partial\Omega. \end{cases}$$

   All are same except

$$\rho_1\left(-d\Delta + q(x)\right) = \inf_{\varphi \in W^{1,2}(\Omega), \varphi \neq 0} \frac{\int_\Omega \left(d|\nabla\varphi|^2 + q\varphi^2\right) dx}{\int_\Omega \varphi^2 dx}.$$

2. The Neumann problem

$$-\Delta u(x) = f(x), \;\; x \in \Omega, \quad \nabla u(x) \cdot n = 0, \;\; x \in \partial\Omega.$$

   is solvable if $\displaystyle\int_\Omega f(x)dx = 0$. Let $X_0 = \{f \in X : \int_\Omega f(x)dx = 0\}$. Then $-\Delta : Y_0 \to X_0$ is invertible where X, Y are appropriate function spaces.
3. All these properties are still valid if the scalar equation is replaced by a system of equations with similar boundary conditions. (at least when there is no cross-diffusion)

Maximum Principle

(General version) Define $L_c = -\Delta + c(x)$. Suppose that Ω is a bounded connected domain in $\mathbb{R}^n$ with $n \geq 2$, $c \in L^\infty(\Omega)$, and $c(x) \geq 0$ for $x \in \overline{\Omega}$. If $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, $L_c u \geq 0$ in Ω, $u(x) \geq 0$ on $\partial\Omega$, then

1. (weak maximum principle) $u(x) \geq 0$ for $x \in \Omega$;
2. (strong maximum principle) If there exists $x_* \in \Omega$ such that $u(x_*) = 0$, then $u(x) \equiv 0$ for $x \in \overline{\Omega}$;
3. (Hopf boundary lemma) If there is a ball $B_r \subset \Omega$ such that $x_* \in \partial B_r \cap \partial\Omega$, and $u(x) > u(x_*)$ for all $x \in B_r$, then for any outward direction ν at x_* with respect to ∂B_r (i.e., $\nu \cdot n(x_*) > 0$ for the outer normal vector $n(x_*)$ of ∂B_r), $\limsup\limits_{t \to 0} \dfrac{u(x_*) - u(x_* - t\nu)}{t} < 0$; if $u \in C^1(\overline{\Omega})$, then $\dfrac{\partial u}{\partial n}(x_*) < 0$.

(Special version). If $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, $L_c u \geq 0$ in Ω, $u(x) \geq 0$ on $\partial\Omega$. If $u(x) \geq 0$ for all $x \in \Omega$, then either $u(x) > 0$ for all $x \in \Omega$, or $u(x) \equiv 0$. If $u(x) > 0$ in Ω, $\partial\Omega$ is of class $C^{2,\alpha}$, $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$, and there exists $x_* \in \partial\Omega$ such that $u(x_*) = 0$, then $\dfrac{\partial u}{\partial n}(x_*) < 0$. there is no assumption on the sign of $c(x)$!

Optimal Maximum Principle

We say that the *maximum principle* holds for L_c in Ω if $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, $L_c u \geq 0$ in Ω, $u \geq 0$ on $\partial\Omega$, then $u \geq 0$ in Ω.

1. The maximum principle holds in Ω if the principal eigenvalue $\rho_1(c)$ is positive.
2. If $\rho_1(c) = 0$, and these exists $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, $L_c u \geq 0$ in Ω, $u \geq 0$ on $\partial\Omega$, then $u = c\phi_1$, where ϕ_1 is an eigenfunction corresponding to $\rho_1(c)$.
3. Consider the equation $-L_c u + \lambda u = f$, where $f \in L^p(\Omega)$ $(p > n)$ and $f > 0$. Then
   (a) If $\lambda < \rho_1(c)$, then there is a unique solution $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that $u > 0$ in Ω and $\nabla u \cdot n < 0$ on $\partial\Omega$.
   (b) If $\lambda = \rho_1(c) = 0$, then there is no solution.
   (c) (anti-maximum principle) If $\rho_1(c) < \lambda < \rho_1(c) + \delta$, then there is a unique solution $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that $u < 0$ in Ω and $\nabla u \cdot n > 0$ on $\partial\Omega$.
   [Clement-Peletier, 1979, JDE] [Hess, 1981, JDE]

A priori bound

1. (Dirichlet Boundary) Suppose that $f \in C(\overline{\Omega} \times \mathbb{R}^+)$ and Ω is a bounded connected domain in $\mathbb{R}^n$ $(n \geq 2)$ with $C^{2,\alpha}$ boundary. Suppose that $u \in C^2(\overline{\Omega}) \cap C^1(\overline{\Omega})$ is a solution of

$$\begin{cases} \Delta u(x) + f(x, u(x)) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

satisfying $u(x) \geq 0$ for $x \in \Omega$.
(a) If $f(x, 0) \geq 0$ for all $x \in \overline{\Omega}$, then either $u(x) > 0$ for all $x \in \Omega$ and $\nabla u(x) \cdot n(x) < 0$ for all $x \in \partial\Omega$, or $u(x) \equiv 0$ for $x \in \Omega$.
(b) If $u(x) \not\equiv 0$, and $u_M = u(x_0) = \max_{x \in \overline{\Omega}} u(x)$, then $f(x, u_M) > 0$.

2. (Neumann Boundary) Suppose that $f \in C(\overline{\Omega} \times \mathbb{R}^+)$ and Ω is a bounded connected domain in $\mathbb{R}^n$ $(n \geq 2)$ with $C^{2,\alpha}$ boundary. Suppose that $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies

$$\begin{cases} \Delta u(x) + f(x, u(x)) \geq 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} \leq 0 & \text{on } \partial\Omega, \end{cases}$$

and $u(x_0) = \max_{x \in \overline{\Omega}} u(x)$, then $f(x_0, u(x_0)) \geq 0$.

Smoothness of solutions to elliptic equations

Example: Fisher-KPP equation

$$d\Delta u + u(1-u) = 0, \quad x \in \Omega, \quad u = 0, \quad x \in \partial\Omega.$$

Prove: if $u(x)$ is a nonnegative solution, then either $u(x) \equiv 0$ or $0 < u(x) < 1$ for $x \in \Omega$.

$$d\Delta u + u(1-u) = 0, \quad x \in \Omega, \quad \nabla u \cdot n = 0, \quad x \in \partial\Omega.$$

Prove: if $u(x)$ is a nonnegative solution, then either $u(x) \equiv 0$ or $u(x) \equiv 1$ for $x \in \Omega$. That is: there is no non-constant solution.

$$\Delta u + f(x,u) = 0, \quad x \in \Omega, \quad u = 0, \quad x \in \partial\Omega.$$

$u : \overline{\Omega} \to \mathbb{R}$ is a classical solution if $u \in C^2(\Omega) \cap C^(\overline{\Omega})$, satisfying $\Delta u(x) + f(x, u(x)) = 0$ for any $x \in \Omega$, and $u(x) = 0$ for any $x \in \partial\Omega$.

Often we obtain a weak solution $u \in W^{2,p}(\Omega)$ for $p \geq 2$. Then we can prove the solution is indeed a classical one by using Schauder estimates, L^p estimates and Sobolev embedding theorems, if f is a Hölder continuous function.

Parabolic equations

$$\begin{cases} u_t = d\Delta u - V\cdot\nabla u + f(t,x,u), & x\in\Omega, \quad t>0,\\ u(x,t)=0, & x\in\partial\Omega, \quad t>0,\\ u(x,0)=u_0(x)\geq 0, & x\in\Omega. \end{cases}$$

$u:\overline{\Omega}\times(0,T)\to\mathbb{R}$ is a <u>classical solution</u> if $u\in C^{2,1}(\Omega\times[0,T])\cap C^{(}\overline{\Omega}\times(0,T])$, satisfying $u_t(x,t)=d\Delta u(x,t)+f(t,x,u(x,t))=0$ for any $(x,t)\in\Omega\times(0,T]$, $u(x,t)=0$ for any $(x,t)\in\partial\Omega\times(0,T]$, and $u(x,0)=u_0(x)$ for $x\in\Omega$.

Existence Theorem: Suppose that f is locally Lipschitz continuous, $u_0\in X^\alpha$ where X^α is a proper subspace of $X=W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)$, then the parabolic equation above has a unique classical solution $u(x,t)$ for $t\in[0,T]$.

1. Usually the initial value function $u_0(x)$ is required to be at least continuous, although the equation may still have a solution even if u_0 is not continuous, or u_0 is a generalized function like Delta function.
2. The solution $u(x,t)$ exists <u>globally</u> if a classical solution $u(x,t)$ exists for $(x,t)\in\overline{\Omega}\times(0,\infty)$, and $u(x,t)$ is <u>uniformly bounded</u> if there exists $M>0$ such that $|u(x,t)|\leq M$ for $(x,t)\in\overline{\Omega}\times(0,T]$.
3. The solution map $S_t: u_0\mapsto u(\cdot,t)$ generates a <u>semiflow</u> in X^α, which is useful in the dynamical system approach. A semiflow $S: X^\alpha\times(0,T)\to X^\alpha$ satisfies (i) $S(x,0)=x$; (ii) $S(S(x,t_1),t_2)=S(x,t_1+t_2)$; (iii) $S_t=S(\cdot,t): X^\alpha\to X^\alpha$ is continuous; and (iv) for each $x\in X^\alpha$, $t\mapsto S(x,t)$ is continuous.

Comparison Principle

$$\begin{cases} u_t = d\Delta u - V \cdot \nabla u + f(t,x,u), & x \in \Omega, \quad t > 0, \\ u(x,t) = 0, \quad \text{or} \quad \nabla u \cdot n = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \Omega. \end{cases}$$

Comparison Principle: Suppose that f is C^1, and $u, v \in C^{2,1}(\Omega \times [0,T]) \cap C^(\overline{\Omega} \times (0,T])$ satisfy

$$\begin{cases} u_t - d\Delta u + V \cdot \nabla u - f(t,x,u) \geq v_t - d\Delta v + V \cdot \nabla v - f(t,x,v), & x \in \Omega, \quad t \in (0,T], \\ u(x,t) \geq v(x,t), \quad \text{or} \quad \nabla u \cdot n \geq \nabla v \cdot n, & x \in \partial\Omega, \quad t \in (0,T], \\ u(x,0) \geq v(x,0), & x \in \Omega, \end{cases}$$

then $u(x,t) \geq v(x,t)$ for $(x,t) \in \Omega \times (0,T]$; and if $u(x,0) \not\equiv v(x,0)$, then $u(x,t) > v(x,t)$ for $(x,t) \in \Omega \times (0,T]$.

Example: Fisher-KPP equation

$$\begin{cases} u_t = d\Delta u - V \cdot \nabla u + u(1-u), & x \in \Omega, \quad t > 0, \\ u(x,t) = 0, \quad \text{or} \quad \nabla u \cdot n = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \Omega. \end{cases}$$

Prove: If $u_0(x) \geq 0$ is continuous for $x \in \Omega$, then a solution $u(x,t)$ exists globally for $t \in (0,\infty)$. If the boundary condition is Neumann, prove that $\lim_{t\to\infty} u(x,t) = 1$ uniformly for $x \in \overline{\Omega}$.

Steady state solution

$$\begin{cases} u_t = d\Delta u - V \cdot \nabla u + f(t,x,u), & x \in \Omega, \quad t > 0, \\ u(x,t) = 0, \quad \text{or} \quad \nabla u \cdot n = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \Omega. \end{cases}$$

Steady state solution: if $f(t,x,u) = f(x,u)$, then a solution $u(x,t)$ satisfies $u(x,t) \equiv u(x)$, and it satisfies an elliptic equation

$$\begin{cases} d\Delta u - V \cdot \nabla u + f(x,u) = 0, & x \in \Omega, \\ u(x) = 0, \quad \text{or} \quad \nabla u \cdot n = 0, & x \in \partial\Omega. \end{cases}$$

Periodic solution: a solution $u(x,t)$ satisfies $u(x,t+T) = u(x,t)$ for all $(x,t) \in \Omega \times (0,\infty)$ and a $T > 0$. We assume that there is a smallest such $T > 0$ exists. Notice that if $u(x,t)$ is periodic, so is $u(x,t+t_1)$ for $0 < t_1 < T$. But they have the same orbit in the phase space: $\{u(x,t) : t > 0\}$. Hence a periodic solution corresponds to a periodic orbit.

Stability

$$\begin{cases} u_t = d\Delta u - V \cdot \nabla u + f(x,u), & x \in \Omega, \quad t > 0, \\ u(x,t) = 0, \quad \text{or} \quad \nabla u \cdot n = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \Omega. \end{cases}$$

Suppose that $v(x)$ is a steady state solution. Let X be an open set of Banach space such that $u_0, v \in X$, and the solution $u(x,t;u_0)$ exists.

1. A steady state solution v is <u>stable</u> (Lyapunov stable) in X if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that when $||u_0 - v||_X < \delta$, then $||u(\cdot,t;u_0) - v(\cdot)||_X < \varepsilon$ for all $t > 0$; v is <u>unstable</u> if it is not stable.
2. A steady state solution v is <u>(locally) asymptotically stable</u> (attractive) in X if v is stable, and there exists $\eta > 0$, such that when $||u_0 - v||_X < \eta$, then $\lim\limits_{t\to\infty} ||u(\cdot,t;u_0) - v(\cdot)||_X = 0$.
3. If $v(x)$ is locally asymptotically stable, then the set $X_v = \left\{ u_0 \in X : \lim\limits_{t\to\infty} ||u(\cdot,t;u_0) - v(\cdot)||_X = 0 \right\}$ is the basin of attraction of v. A steady state solution v is <u>globally asymptotically stable</u> if $X_v = X$.

Examples of X: $C(\overline{\Omega})$, $W^{1,p}(\Omega)$, $C^{2,\alpha}(\overline{\Omega})$, $C_+(\overline{\Omega}) = \{u_0 \in C(\overline{\Omega}) : u_0(x) \geq 0,\ x \in \overline{\Omega}\}$.
[Brown-Dunne-Gardner, 1981, JDE] these stabilities are same (under some conditions).

Principle of Linearized Stability

$$\begin{cases} u_t = d\Delta u - V \cdot \nabla u + f(x,u), & x \in \Omega, \quad t > 0, \\ u(x,t) = 0, \quad \text{or} \quad \nabla u \cdot n = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \Omega. \end{cases}$$

Principle of Linearized Stability: Suppose that $v(x)$ is a steady state solution. Consider the eigenvalue problem:

$$\begin{cases} L\phi \equiv d\Delta\phi - V \cdot \nabla\phi + f_u(x, v(x))\phi = \mu\phi, & x \in \Omega, \\ \phi(x) = 0, \quad \text{or} \quad \nabla\phi \cdot n = 0, & x \in \partial\Omega. \end{cases}$$

If all the eigenvalues of L have negative real part, then v is locally asymptotically stable; and if at least one of eigenvalues of L has positive real part, then v is unstable.
Corollary.
1. Let ρ_1 be the principal eigenvalue of L, then v is locally asymptotically stable if $\rho_1 < 0$, and v is unstable if $\rho_1 > 0$.
2. Define $I(\phi) = \int_\Omega [|\nabla\phi(x)|^2 - f_u(x, v(x))\phi^2(x)]dx$. If $V = 0$, then v is locally asymptotically stable if $I(\phi) > 0$ for any $\phi \in W^{1,2}(\Omega)$, and v is unstable if there exists a ϕ ($\in W^{1,2}(\Omega)$ for Neumann or $\in W_0^{1,2}(\Omega)$ for Dirichlet) such that $I(\phi) < 0$.

[Smoller, 1982, book, Chapter 11], [Webb, 1985, book]

Autonomous Neumann Problem

$$\begin{cases} u_t = d\Delta u + f(u), & x \in \Omega, \quad t > 0, \\ \nabla u \cdot n = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \Omega. \end{cases}$$

Linearized equation at a steady state solution v:

$$\begin{cases} d\Delta\phi + f'(v)\phi = \mu\phi, & x \in \Omega, \\ \nabla\phi \cdot n = 0, & x \in \partial\Omega. \end{cases}$$

Theorem. Suppose that $v(x) = c$ is a constant steady state solution (so $f(c) = 0$). Then $v(x) = c$ is locally asymptotically stable if $f'(c) < 0$, and $v(x) = c$ is unstable if $f'(c) < 0$.

Theorem. [Casten-Holland, 1978, JDE], [Matano, 1979, Pub RIMS] Suppose that Ω is a bounded convex domain. Let $v(x)$ be a locally asymptotically stable steady state solution on Ω. Then $v(x)$ is a constant function.

[Matano, 1979, Pub RIMS] If Ω is not convex, then there may exist non-constant locally asymptotically stable steady state solutions.

Non-autonomous sublinear problem

$$\begin{cases} u_t = d\Delta u + f(x,u), & x \in \Omega, \quad t > 0, \\ u = 0, \quad \text{or} \quad \nabla u \cdot n = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \Omega. \end{cases}$$

$$\begin{cases} d\Delta\phi_1 + f_u(x,u)\phi_1 = \mu_1\phi_1, & x \in \Omega, \\ \phi_1 = 0, \quad \text{or} \quad \nabla\phi_1 \cdot n = 0, & x \in \partial\Omega. \end{cases}$$

Sublinear: $f(x,u) \geq uf_u(x,u)$ for $u \geq 0$.

Theorem. If f is sublinear, then any positive steady state solution $u(x)$ is locally asymptotically stable.

Proof. By using Green's Theorem

$$d\int_\Omega [u\Delta\phi_1 - \phi_1\Delta u] + \int_\Omega \phi_1[uf_u(x,u) - f(x,u)] = \mu\int_\Omega u\phi_1.$$

Example: Non-homogeneous Fisher-KPP
$d\Delta u + (m(x) - k(x)u)u = 0, x \in \Omega, u = 0, \quad \text{or } \nabla u \cdot n = 0, \quad x \in \partial\Omega.$

If f is superlinear ($f(x,u) \leq uf_u(x,u)$ for $u \geq 0$), then any positive steady state solution $u(x)$ is unstable.

Upper-Lower solution method

$$\begin{cases} d\Delta u + f(x,u) = 0, & x \in \Omega, \\ u(x) = 0, \quad \text{or} \quad \nabla u \cdot n = 0, & x \in \partial\Omega. \end{cases}$$

$v(x) \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is an upper (lower) solution if

$$\begin{cases} d\Delta v + f(x,v) \leq (\geq) 0, & x \in \Omega, \\ v(x) \geq (\leq) 0, \quad \text{or} \quad \nabla v \cdot n \geq (\leq) 0, & x \in \partial\Omega. \end{cases}$$

Theorem. Suppose that f is Lipschitz continuous in u and Hölder continuous in x. If there is a upper solution $\overline{u}$ and a lower solution $\underline{u}$ satisfying $\overline{u}(x) \geq \underline{u}(x)$ for all $x \in \overline{\Omega}$, then there exist steady state solutions $u_M(x)$ and $u_m(x)$ satisfying $\overline{u}(x) \geq u_M(x) \geq u_m(x) \geq \underline{u}(x)$. Moreover u_M is the maximum solution and u_m is the minimum solution which satisfy $\overline{u}(x) \geq u(x) \geq \underline{u}(x)$.

[Amann, 1976, SIAM-Rev], [Du, 2006, book]

Theorem. [Sattinger, Indiana J Math, 1971] Let $u_M(x)$ and $u_m(x)$ be the maximum solution and minimum solution above. Consider the linearized eigenvalue problem (for $* = M, m$)

$$\begin{cases} d\Delta\phi_1 + f_u(x,u_*)\phi_1 = \mu\phi_1, & x \in \Omega, \\ \phi_1 = 0, \quad \text{or} \quad \nabla\phi_1 \cdot n = 0, & x \in \partial\Omega. \end{cases}$$

Then $\mu_1(u_*) \leq 0$ for $* = M, m$.

Fisher-KPP equation

Example: Non-homogeneous Fisher-KPP equation
$d\Delta u + (m(x) - k(x)u)u = 0, x \in \Omega, \quad u = 0, \quad \text{or } \nabla u \cdot n = 0, \quad x \in \partial\Omega.$

Suppose $k(x) \geq \delta > 0$ for $x \in \overline{\Omega}$, and $M \geq m(x)$ for $x \in \overline{\Omega}$ and there exists a positive measure set $\Omega_0 \subset \Omega$ such that $m(x) > 0$ on Ω_0.

Upper solution $\overline{u} = M/\delta$.

lower solution (Dirichlet): $\underline{u} = \varepsilon\phi_1(x)$, where ϕ_1 satisfies
$d\Delta\phi_1 + m(x)\phi_1 = \mu_1\phi_1, x \in \Omega, \quad \phi_1 = 0, \ x \in \partial\Omega.$
if $0 < d < 1/\rho_1$ or $\mu_1 > 0$

lower solution (Neumann): $\underline{u} = \varepsilon\phi_1(x)$, where ϕ_1 satisfies
$d\Delta\phi_1 + m(x)\phi_1 = \mu_1\phi_1, x \in \Omega, \quad \nabla\phi_1 \cdot n = 0, \ x \in \partial\Omega.$
if $\displaystyle\int_\Omega m(x)dx > 0$ and for any $d > 0$.
or if $\displaystyle\int_\Omega m(x)dx < 0$ and for any $0 < d < 1/\rho_1$.

We need $\mu_1 > 0$.
[Lou, 2006, JDE], [Ni, 2011, book]

Two eigenvalue problems

$$\begin{cases} -d\Delta\phi + q(x)\phi = \mu\phi, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega. \end{cases}$$

Eigenvalues: $\mu_1 < \mu_2 \leq \mu_3 \leq \cdots \to \infty$

$$\mu_1 = \inf_{\varphi \in H_0^1(\Omega), \varphi \neq 0} \frac{\int_\Omega \left(d|\nabla\varphi|^2 + q\varphi^2\right) dx}{\int_\Omega \varphi^2 dx}.$$

$$\begin{cases} -\Delta\phi = \rho q(x)\phi, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega. \end{cases}$$

(Possible) Eigenvalues: $\rho_1 < \rho_2 \leq \rho_3 \leq \cdots \to \infty$ (exist if $\{\varphi : \int_\Omega q(x)\varphi^2 dx > 0\} \neq \emptyset$)
and $\rho_{-1} > \rho_2 \geq \rho_3 \geq \cdots \to -\infty$ (exist if $\{\varphi : \int_\Omega q(x)\varphi^2 dx < 0\} \neq \emptyset$)

$$\rho_1 = \inf_{\varphi \in H_0^1(\Omega), \varphi \neq 0, \int_\Omega q(x)\varphi^2 > 0} \frac{\int_\Omega |\nabla\varphi|^2 dx}{\int_\Omega q(x)\varphi^2 dx}.$$

$$\rho_{-1} = -\inf_{\varphi \in H_0^1(\Omega), \varphi \neq 0, \int_\Omega q(x)\varphi^2 < 0} \frac{\int_\Omega |\nabla\varphi|^2 dx}{\int_\Omega q(x)\varphi^2 dx}.$$

$0 < d < \rho_1^{-1}$ is equivalent to $\mu_1 > 0$.

[Brown-Lin, JMAA, 1980], [Hess-Kato, CPDE, 1980]

Uniqueness

Example: Non-homogeneous Fisher-KPP equation
$d\Delta u + (m(x) - k(x)u)u = 0, x \in \Omega, \quad u = 0, \quad \text{or } \nabla u \cdot n = 0, \quad x \in \partial\Omega.$

Let $u_M(x)$ be the maximum solution, and let $u(x)$ be another solution. Then $u_M(x) \geq u(x)$ for all $x \in \overline{\Omega}$.

$$\begin{aligned}0 =& d\int_\Omega (u\Delta u_M - u_M \Delta u)dx = d\int_\Omega (u\Delta u_M - u_M\Delta u)dx \\ =& \int_\Omega [(m(x) - k(x)u)uu_M - (m(x) - k(x)u_M)u_M u]dx = \int_\Omega k(x)u_M u(u_M - u)dx.\end{aligned}$$

Then $u_M \equiv u$ since $k(x) \geq \delta > 0$.

Theorem. Suppose that $f(x, u) \geq uf_u(x, u)$ for $u \geq 0$.

$$d\Delta u + f(x, u) = 0, x \in \Omega, \quad u = 0, \quad \text{or } \nabla u \cdot n = 0, \quad x \in \partial\Omega,$$

has at most one positive solution, and it is locally asymptotically stable if it exists.

LaSalle's invariance principle

[LaSalle, 1960] [Henry, 1981, Lecture notes, Theorem 4.3.4]

1. Let Y be a complete metric space. A semiflow $S : Y \times (0, T) \to Y$ satisfies (i) $S(x, 0) = x$; (ii) $S(S(x, t_1), t_2) = S(x, t_1 + t_2)$; (iii) $S(t) = S(\cdot, t) : Y \to Y$ is continuous; and (iv) for each $x \in Y$, $t \mapsto S(x, t)$ is continuous.
2. Suppose that $S(t)Y \to Y$ is a semiflow. Then $V : Y \to \mathbb{R}$ is a Lyapunov function, if $\dot{V}(x) = \overline{\lim}_{t\to 0^+} \dfrac{V(S(t)x) - V(x)}{t} \le 0$ for all $x \in Y$.
3. Let $x_0 \in Y$. Then the orbit through x_0 is $\gamma(x_0) = \{S(t)x_0 : t \ge 0\}$.
4. A set $K \subset Y$ is invariant if for any $x_0 \in K$, there exists a continuous curve $x : \mathbb{R} \to K$ such that $x(0) = x_0$ and $S(t)x(\tau) = x(t + \tau)$, for $\tau \in \mathbb{R}$ and $t \ge 0$. That is, $\gamma(x_0) \subset K$.
5. The ω-limit set of x_0 is $\omega(x_0) = \omega(\gamma(x_0)) = \{x \in Y : \text{ there exists } \quad t_n \to \infty \quad \text{ such that } S(t)x_0 \to x\}$.
6. Theorem. Suppose that $x_0 \in Y$, and the orbit $\gamma(x_0)$ is compact. Then $\omega(x_0)$ is nonempty, invariant, connected and compact, and $dist(S(t)x_0, \omega(x_0)) \to 0$ as $t \to \infty$.
7. LaSalle's invariance principle Let V be a Lyapunov function on Y. Define $E = \{x \in Y : \dot{V}(x) = 0\}$, and M is the maximum invariance subset of E. If the orbit $\gamma(x_0)$ is compact, then $S(t)x_0 \to M$ as $t \to \infty$.

Global Stability

$$\begin{cases} u_t = d\Delta u + f(x,u), & x \in \Omega, \quad t > 0, \\ u = 0, \quad \text{or} \quad \nabla u \cdot n = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \Omega. \end{cases}$$

1. It generates a semiflow S_t on X^α if f is Lipchitz continuous.
2. There is a Lyapunov function $V(u) = \displaystyle\int_\Omega [|\nabla u(x)|^2 - 2F(x,u(x))]dx$, where $F(x,u) = \int_0^u f(x,s)ds$.
3. A solution orbit is compact if it is bounded.
4. If $\dot{V}(u) = 0$, then u is a steady state solution.

Example: Non-homogeneous Fisher-KPP equation

$$\begin{cases} u_t = d\Delta u + u(m(x) - k(x)u), & x \in \Omega, \quad t > 0, \\ u = 0, \quad \text{or} \quad \nabla u \cdot n = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \Omega. \end{cases}$$

1. All solutions are uniformly bounded.
2. There is a unique positive steady state solution or $u = 0$ is the only non-negative steady state solution.
3. Theorem. The unique positive steady state solution is globally asymptotically stable when it exists, and otherwise $u = 0$ is globally asymptotically stable.

Grand Finale

Theorem Consider

$$\begin{cases} u_t = d\Delta u + u(m(x) - k(x)u), & x \in \Omega, \quad t > 0, \\ u = 0, \quad \text{or} \quad \nabla u \cdot n = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \Omega. \end{cases}$$

Here Ω is bounded smooth domain, $u_0(\not\equiv 0)$ is continuous, $m(x), k(x)$ are continuous functions on $\overline{\Omega}$ such that $k(x) \geq \delta > 0$ for $x \in \overline{\Omega}$, and $M \geq m(x)$ for $x \in \overline{\Omega}$ and there exists a positive measure set $\Omega_0 \subset \Omega$ such that $m(x) > 0$ on Ω_0.

1. For the Dirichlet boundary case $u = 0$ on $\partial\Omega$, let ρ_1 be the principal eigenvalue of $\Delta\phi_1 + \rho_1 m(x)\phi_1 = 0, x \in \Omega, \quad \phi_1 = 0, \ x \in \partial\Omega$. Then when $d \geq \rho_1^{-1}$, $u = 0$ is globally asymptotically stable; and when $0 < d < \rho_1^{-1}$, there is a unique positive steady state solution u_d which is globally asymptotically stable.
2. For the Neumann boundary case $\nabla u \cdot n = 0$ on $\partial\Omega$, if $\int_\Omega m(x)dx > 0$, then for $d > 0$, there is a unique positive steady state solution u_d which is globally asymptotically stable; if $\int_\Omega m(x)dx < 0$, then for $0 < d < \rho_1^{-1}$, there is a unique positive steady state solution u_d which is globally asymptotically stable, while for $d \geq \rho_1^{-1}$, $u = 0$ is globally asymptotically stable.

Bistability

$$\begin{cases} u_t = d\Delta u + w(1-w)(w-b), & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \Omega. \end{cases}$$

1. For any $d > 0$ and $0 < b < 1$, the constant steady state $u = 0$ is locally asymptotically stable.
2. If $1/2 \leq b < 1$, then for any $d > 0$, the only nonnegative steady state solution is $u = 0$.
3. If $0 < b < 1/2$, then the only nonnegative steady state solution is $u = 0$ if $d > \dfrac{(1-b)^2}{4\lambda_1^D(\Omega)}$, and there exists a constant $D_0 = D_0(\Omega) > 0$ such that for $0 < d < D_0$, it has at least two positive steady state solutions. Moreover, for $0 < d_1 < D_0$, it has a maximal steady state solution $\widetilde{u}(x)$ such that for any steady state solution $u(x)$, $\widetilde{u}(x) > u(x)$ for $x \in \Omega$.
4. If $0 < b < 1/2$, and Ω is a ball of $\mathbb{R}^n$ for $n \geq 1$, then there exists $D_0 > 0$ such that it has exactly two positive steady state solutions for $0 < d < D_0$, has exactly one positive steady state solution for $d = D_0$, and has no positive steady state solution for $d > D_0$. [Ouyang-Shi, JDE, 1998]
5. In the last case with $0 < d < D_0$, there are three non-negative steady state solutions $0 < \hat{u}(x) < \widetilde{u}(x)$. For any $u_*(\geq 0) \in X^\alpha$, there exists $\beta_0(u_*) > 0$ such that if $u_0 = \beta u_*$, then as $t \to \infty$, (i) $u(\cdot,t) \to 0$ if $0 < \beta < \beta_0$, (ii) $u(\cdot,t) \to \hat{u}(\cdot)$ if $\beta = \beta_0$, and (iii) $u(\cdot,t) \to \widetilde{u}(\cdot)$ if $\beta > \beta_0$. [Jiang-Shi, DCDS-A, 2008]

Cooperative system

$$\begin{cases} \Delta u + \lambda f(u,v) = 0, & x \in \Omega, \\ \Delta v + \lambda g(u,v) = 0, & x \in \Omega, \\ u(x) = v(x) = 0, & x \in \partial\Omega, \end{cases}$$

Linearized equation:

$$\begin{cases} \Delta \xi + \lambda f_u(u,v)\xi + \lambda f_v(u,v)\eta = -\mu\xi, & x \in \Omega, \\ \Delta \eta + \lambda g_u(u,v)\xi + \lambda g_v(u,v)\eta = -\mu\eta, & x \in \Omega, \\ \xi(x) = \eta(x) = 0, & x \in \partial\Omega, \end{cases}$$

Cooperative: $f_v(u,v) \geq 0$, $g_u(u,v) \geq 0$ for $(u,v) \in \mathbb{R}^+ \times \mathbb{R}^+$

If the system is cooperative, then

1. A positive solution with $\Omega = B^n$ (ball in $\mathbb{R}^n$) is radially symmetric. [Troy, JDE, 1981]
2. For Neumann boundary problem, if (u,v) is a stable solution, and the domain Ω is convex, then (u,v) must be constant. [Kishimoto-Weinberger, JDE, 1985]
3. The eigenvalue with smallest real part is real-valued, and it is a principal eigenvalue μ_1 with a positive eigenfunction. [Sweers, Math Z, 1992]

Sublinear system

[Chern-Tang-Lin-Shi, PRSE, 2011], [Cui-Li-Shi-Wang, TMNA, 2013 to appear]

$$\begin{cases} \Delta u + \lambda f(u,v) = 0, & x \in \Omega, \\ \Delta v + \lambda g(u,v) = 0, & x \in \Omega, \\ u(x) = v(x) = 0, & x \in \partial\Omega, \end{cases}$$

Suppose that (u,v) is a positive solution. Then (u,v) is stable ($\mu_1 > 0$) if (f,g) satisfies one of the following conditions: for any $(u,v) \in \mathbb{R}^+ \times \mathbb{R}^+$,

(A_1) $f(u,v) > f_u(u,v)u + f_v(u,v)v$, $g(u,v) > g_u(u,v)u + g_v(u,v)v$, or
(A_2) $f(u,v) > f_u(u,v)u + g_u(u,v)v$, $g(u,v) > g_v(u,v)v + f_v(u,v)u$, or
(A_3) $f(u,v) > f_v(u,v)v + g_v(u,v)u$, $g(u,v) > g_u(u,v)u + f_u(u,v)v$, or
(A_4) $f(u,v) > g_v(u,v)u + g_u(u,v)v$, $g(u,v) > f_v(u,v)u + f_u(u,v)v$.

Proof. The conjugate linearized operator has the same principal eigenvalue μ_1 with positive eigenfunction (ξ^*, η^*).

$$\mu_1 \int_\Omega (u\xi^* + v\eta^*)dx = \lambda \int_\Omega [f - f_u u - f_v v]\eta^* dx + \lambda \int_\Omega [g - g_u u - g_v v]\xi^* dx.$$

Example: existence, stability and uniqueness of positive solution of cooperative logistic system (here $a,b,c,d > 0$ and $p,q > 1$)

$$\begin{cases} \Delta u + \lambda(au - u^p + cv) = 0, & x \in \Omega, \\ \Delta v + \lambda(bv - v^q + du) = 0, & x \in \Omega, \\ u(x) = v(x) = 0, & x \in \partial\Omega. \end{cases}$$

Competition model: Sublinear but not cooperative

$$\begin{cases} u_t = d_1 \Delta u + f_1(z(x) - au - bv)u = 0, & x \in \Omega, \\ v_t = d_2 \Delta v + f_2(z(x) - cu - dv)v = 0, & x \in \Omega, \\ Bu(x) = Bv(x) = 0, & x \in \partial\Omega. \end{cases}$$

where f_i is a non-increasing function, the resource function $z(x) > 0$, $a, b, c, d > 0$.

Lotka-Volterra model: $f_i(u) = m_i u$.
[Dockery et.al., JMB, 1998], [Hutson et.al., JDE, 2002, 2005]
[Lou, JDE, 2006], [Lam-Ni, SIAM-MA, 2012], [He-Ni, JDE, 2013]
Chemostat model: $f_i = \dfrac{m_i u}{a_i + u}$, $a = b = c = d = 1$. [Hsu-Waltman, SIAM-AM, 1993]

1. The coexistence steady state solution $(u > 0,\ v > 0)$ may not be stable.
2. The coexistence steady state solution $(u > 0,\ v > 0)$ may not be unique (non-convex domain).
3. When the boundary steady state solutions are unstable, then there is a coexistence steady state solution.

Question: for convex domain, is the coexistence steady state solution unique and globally stable?

Route to global stability

General reaction-diffusion systems: $u_t = d_1\Delta u + f(x,u,v),\ v_t = d_2\Delta v + f(x,u,v)$
General reaction-diffusion equation: $u_t = d\Delta u - V\cdot\nabla u + f(x,u)$
Fisher-KPP equation: $u_t = d\Delta u + u(m(x) - k(x)u)$.

	General system	General equation	Fisher-KPP
Local existence	Yes (usually)	Yes	Yes
Global existence	Can be proved	Can be proved	Yes
Existence of steady state	Harder	Many methods	Yes
Maximum principle	No in general	Yes	Yes
Upper-lower solution method	No in general	Yes	Yes
Lyapunov function	No in general	Yes	Yes
Uniqueness of steady state	No in general	No in general	Yes
Globally stable steady state	No in general	No in general	Yes
Periodic orbit	Maybe	No	No