Reaction-Diffusion Models and Bifurcation Theory
Lecture 12: Bifurcation in delay reaction-diffusion equations

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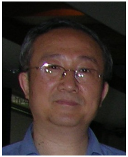
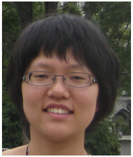
Collaborators/Support





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When $\tau > \dfrac{\pi}{2r}$, $u = K$ is unstable, and $\tau = \pi/(2r)$ is a Hopf bifurcation point.

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[Green-Stech, 1981, book chap], [Busenberg-Huang, 1996, JDE],
[Su-Wei-Shi, 2009, JDE] more general case [Yan-Li, 2010, Nonlinearity]

Diffusive Hutchinson Model Simulation

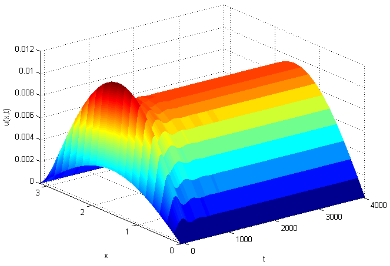

Figure : Numerical simulation of Dirichlet problem with $\lambda = 1.01$ and $c = 0.5$. Top: $\tau = 80$, the solution approaches to the positive steady state. Bottom: $\tau = 120$, the solution still approaches to the positive steady state but with noticeable oscillations.

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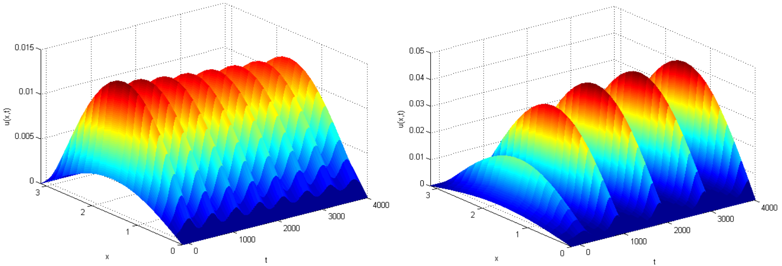

Figure : Numerical simulation of Dirichlet problem with $\lambda = 1.01$ and $c = 0.5$. (A) Left: $\tau = 130$, the solution converges to a time-periodic solution with small oscillations; (B) Right: $\tau = 200$, the solution converges to a time-periodic solution with larger amplitude.

Diffusive Hutchinson Model with Partial Delay

Assume that $a, b \geq 0$ and $a + b = 1$
no-flux boundary condition (and also non-spatial model):

$$u_t = d\Delta u + ru(1 - au - bu(t - \tau)), \quad x \in \Omega, \quad \frac{\partial u}{\partial n} = 0, \ x \in \partial\Omega.$$

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When $a \geq b$, then $u = 1$ is globally stable for any $\tau \geq 0$;

When $a < b$, There exists $\tau_0(r) = \dfrac{1}{r\sqrt{b^2 - a^2}} \arccos\left(-\dfrac{a}{b}\right)$ such that if $\tau < \tau_0(r)$, $u = 1$ is locally stable, and if $\tau > \tau_0(r)$, $u = 1$ is unstable. $\tau = \tau_0(r)$ is a Hopf bifurcation point.

[Yamada, 1982, JMAA], [Kuang-Smith, 1993, J-Aust-MS], [Pao, 1996, JMAA]

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[Pao, 1996, JMAA], [Huang, 1998, JDE],
[Su-Wei-Shi, 2012, JDDE] global continuation of periodic orbits

Global Bifurcation

[Su-Wei-Shi, 2012, JDDE]
$u_t = du_{xx} + ru(1 - au - bu(t-\tau)),\ \ x \in (0,\pi),\ \ u(0) = u(\pi) = 0.$
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1. There exists infinitely many Hopf bifurcation points $\tau = \tau_n$ $(n = 0, 1, 2, \cdots)$ such that $\tau_{n+1} > \tau_n$ so that periodic orbits bifurcate from steady state u_r.
2. The connected component $\mathfrak{C}_n$ of the set of nontrivial periodic orbits bifurcating from $\tau = \tau_n$ is unbounded so that
$$\sup\left\{\max_{t\in\mathbb{R}} |z(t)| + |\tau| + \omega + \omega^{-1} : (z,\tau,\omega) \in \mathfrak{C}_n\right\} = \infty,$$
where $z(t)$ is the orbit and $2\pi/\omega$ is the period.
3. If $(z,\tau,\omega) \in \mathfrak{C}_n$, then $1/(n+1) < \omega < 1/n$ if $n \geq 1$, and $\omega > 1$ if $n = 0$.
4. For $n \neq m$, $\mathfrak{C}_n \bigcap \mathfrak{C}_m = \emptyset$; the projection of $\mathfrak{C}_n$ to τ component contains (τ_n, ∞).

[Wu, 1996, book], [Wu, 1998, Tran-AMS]

Diffusive Hutchinson Model with nonlocal effect

zero boundary condition:

$$u_t = d\Delta u + \lambda u\left(1 - \int_\Omega K(x,y)u(y,t-\tau)dy\right), \quad x \in \Omega, \quad u = 0, \quad x \in \partial\Omega.$$

assume $\lambda > d\lambda_1$ but $\lambda - d\lambda_1$ is small: there exists a $\tau_0(\lambda) > 0$ such that u_λ is locally asymptotically stable when $\tau \in [0, \tau_0(\lambda))$ and it is unstable when $\tau > \tau_0(\lambda)$. And $\tau = \tau_0(\lambda)$ is a Hopf bifurcation point.

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$$u_t = d\Delta u + \lambda u \left(1 - \int_0^\pi u(y,t-\tau)dy\right), \quad x \in (0,\pi), \quad u = 0, \quad x = 0, \pi.$$

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[Chen-Shi, 2012, JDE]

Simulation (1)

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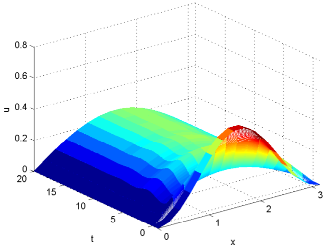
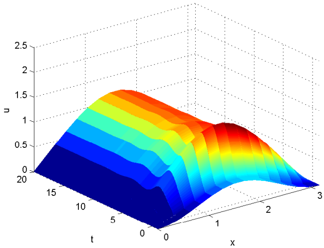


Figure : Spatially homogeneous kernel $K(x,y) = 1$. (Left): $\tau = 1$; (Right): $\tau = 1.6$.

Simulation (2)

$$u_t = d\Delta u + \lambda u \left(1 - \int_0^{\pi} K(x,y)u(y,t-\tau)dy \right), \quad x \in (0,\pi), \quad u = 0, \quad x = 0, \pi.$$



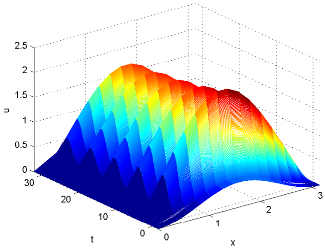


Figure : Spatially nonhomogeneous kernel $K(x,y) = \dfrac{|x-y|}{\pi}$. (Left): $\tau = 1$; (Right): $\tau = 1.6$.

Setting

[Su-Wei-Shi, 2009, JDE]

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} &= d\frac{\partial u^2(x,t)}{\partial x^2} + \lambda u(x,t) f(u(x,t-\tau)), \qquad x \in (0,l),\ t>0, \\ u(0,t) &= u(l,t) = 0, \qquad t \geq 0, \end{aligned} \tag{1}$$

where $d > 0$ is the diffusion coefficient, $\tau > 0$ is the time delay, and $\lambda > 0$ is a scaling constant; the spatial domain is the interval $(0, l)$, and Dirichlet boundary condition is imposed so the exterior environment is hostile. We consider Eq. (1) with the following initial value:

$$u(x,s) = \eta(x,s), \qquad x \in [0,l],\ s \in [-\tau, 0], \tag{2}$$

where $\eta \in \mathcal{C} \stackrel{def}{=} C([-\tau,0], Y)$ and $Y = L^2\big((0,l)\big)$.

The following assumptions are always satisfied:

(A1) There exists $\delta > 0$ such that f is a C^4 function on $[0,\delta]$;

(A2) $f(0) = 1$, and $f'(u) < 0$ for $u \in [0,\delta]$.

Steady State

$$\frac{d^2u(x)}{dx^2} + \lambda u(x)f(u(x)) = 0, \quad x \in (0, l), \qquad (3)$$
$$u(0) = u(l) = 0.$$

It is well known that $Y = \mathcal{N}(dD^2 + \lambda_*) \oplus \mathcal{R}(dD^2 + \lambda_*)$, where

$$D^2 = \frac{\partial^2}{\partial x^2}, \qquad \mathcal{N}(dD^2 + \lambda_*) = \mathrm{Span}\{\sin(\frac{\pi}{l}(\cdot))\}$$

and

$$\mathcal{R}(dD^2 + \lambda_*) = \left\{ y \in Y : \langle \sin(\frac{\pi}{l}(\cdot)), y\rangle = \int_0^l \sin(\frac{\pi}{l}x)y(x)dx = 0 \right\}.$$

Theorem 1 There exist $\lambda^* > \lambda_*$ and a continuously differentiable mapping $\lambda \mapsto (\xi_\lambda, \alpha_\lambda)$ from $[\lambda_*, \lambda^*]$ to $(X \cap \mathcal{R}(dD^2 + \lambda_*)) \times \mathbb{R}^+$ such that Eq.(1) has a positive steady state solution given by

$$u_\lambda = \alpha_\lambda(\lambda - \lambda_*)[\sin(\frac{\pi}{l}(\cdot)) + (\lambda - \lambda_*)\xi_\lambda], \qquad \lambda \in [\lambda_*, \lambda^*]. \qquad (4)$$

Moreover, $\alpha_{\lambda_*} = \dfrac{-\int_0^l \sin^2(\frac{\pi}{l}x)dx}{\lambda_* f'(0)\int_0^l \sin^3(\frac{\pi}{l}x)dx}$ and $\xi_{\lambda_*} \in X$ is the unique solution of the equation $(dD^2 + \lambda_*)\xi + [1 + \lambda_*\alpha_{\lambda_*} f'(0)\sin(\frac{\pi}{l}(\cdot))]\sin(\frac{\pi}{l}(\cdot)) = 0, \quad \langle \sin(\frac{\pi}{l}(\cdot)), \xi\rangle = 0.$

Linearization

$$
\begin{aligned}
&\frac{\partial v(x,t)}{\partial t} = d\frac{\partial^2 v(x,t)}{\partial x^2} + \lambda f(u_\lambda)v(x,t) + \lambda u_\lambda f'(u_\lambda)v(x,t-\tau), \qquad t>0,\\
&\quad v(0,t) = v(l,t) = 0, \qquad t \geq 0,\\
&\quad v(x,t) = \eta(x,t), \qquad (x,t) \in [0,l]\times[-\tau,0],
\end{aligned} \tag{5}
$$

where $\eta \in \mathcal{C}$. We introduce the operator $A(\lambda) : \mathscr{D}(A(\lambda)) \to Y_{\mathbb{C}}$ defined by $A(\lambda) = dD^2 + \lambda f(u_\lambda)$, with domain

$$
\mathscr{D}(A(\lambda)) = \{y \in Y_{\mathbb{C}} : \ \dot{y},\ \ddot{y} \in Y_{\mathbb{C}},\ y(0) = y(l) = 0\} = X_{\mathbb{C}},
$$

and set $v(t) = v(\cdot, t),\ \eta(t) = \eta(\cdot, t)$. Then Eq.(5) can be rewritten as

$$
\begin{aligned}
&\frac{dv(t)}{dt} = A(\lambda)v(t) + \lambda u_\lambda f'(u_\lambda)v(t-\tau), \qquad t>0,\\
&v(t) = \eta(t), \qquad t\in[-\tau,0], \quad \eta \in \mathcal{C},
\end{aligned} \tag{6}
$$

with $A(\lambda)$ an infinitesimal generator of a compact C_0-semigroup. The semigroup induced by the solutions of Eq.(6) has the infinitesimal generator $A_\tau(\lambda)$ given by

$$
\begin{aligned}
&A_\tau(\lambda)\phi = \dot{\phi},\\
&\mathscr{D}(A_\tau(\lambda)) = \{\phi \in \mathcal{C}_{\mathbb{C}} \cap \mathcal{C}^1_{\mathbb{C}} :\ \phi(0) \in X_{\mathbb{C}},\ \dot{\phi}(0) = A(\lambda)\phi(0) + \lambda u_\lambda f'(u_\lambda)\phi(-\tau)\},
\end{aligned}
$$

where $\mathcal{C}^1_{\mathbb{C}} = C^1([-\tau,0], Y_{\mathbb{C}})$.

Spectral set

The spectral set $\sigma(A_\tau(\lambda)) = \{\mu \in \mathbb{C} : \ \Delta(\lambda,\mu,\tau)y = 0, \text{ for some } y \in X_{\mathbb{C}} \setminus \{0\}\}$, and

$$\Delta(\lambda,\mu,\tau) = A(\lambda) + \lambda u_\lambda f'(u_\lambda)e^{-\mu\tau} - \mu.$$

The eigenvalues of $A_\tau(\lambda)$ depend continuously on τ. It is clear that $A_\tau(\lambda)$ has an imaginary eigenvalue $\mu = i\nu$ $(\nu \neq 0)$ for some $\tau > 0$ if and only if

$$[A(\lambda) + \lambda u_\lambda f'(u_\lambda)e^{-i\theta} - i\nu]y = 0, \quad y(\neq 0) \in X_{\mathbb{C}} \tag{7}$$

is solvable for some value of $\nu > 0$, $\theta \in [0, 2\pi)$. One can see that if we find a pair of (ν, θ) such that Eq.(7) has a solution y, then

$$\Delta(\lambda, i\nu, \tau_n)y = 0, \qquad \tau_n = \frac{\theta + 2n\pi}{\nu}, \qquad n = 0, 1, 2, \cdots .$$

Decomposition

Suppose that (ν, θ, y) is a solution of Eq.(7) with $y(\neq 0) \in X_{\mathbb{C}}$. Then represented as

$$\begin{aligned} y = \beta \sin(\frac{\pi}{l}(\cdot)) + (\lambda - \lambda_*)z, \quad \langle \sin(\frac{\pi}{l}(\cdot)), z\rangle = 0, \quad \beta \geq 0, \\ \|y\|^2_{Y_{\mathbb{C}}} = \beta^2 \| \sin(\frac{\pi}{l}(\cdot))\|^2_{Y_{\mathbb{C}}} + (\lambda - \lambda_*)^2 \|z\|^2_{Y_{\mathbb{C}}} = \| \sin(\frac{\pi}{l}(\cdot))\|^2_{Y_{\mathbb{C}}}. \end{aligned} \tag{8}$$

Substituting these into Eq.(7), we obtain the equivalent system to Eq.(7):

$$\begin{aligned} g_1(z, \beta, h, \theta, \lambda) &\stackrel{def}{=} (dD^2 + \lambda_*)z + [\beta \sin(\frac{\pi}{l}(\cdot)) + (\lambda - \lambda_*)z] \\ &\cdot \left[1 + \lambda m_1(\xi_\lambda, \alpha_\lambda, \lambda) + \lambda \alpha_\lambda f'(u_\lambda)e^{-i\theta}[\sin(\frac{\pi}{l}(\cdot)) + (\lambda - \lambda_*)\xi_\lambda] - ih\right] = 0, \\ g_2(z) &\stackrel{def}{=} \mathrm{Re}\langle \sin(\frac{\pi}{l}(\cdot)), z\rangle = 0, \quad g_3(z) \stackrel{def}{=} \mathrm{Im}\langle \sin(\frac{\pi}{l}(\cdot)), z\rangle = 0, \\ g_4(z, \beta, \lambda) &\stackrel{def}{=} (\beta^2 - 1)\| \sin(\frac{\pi}{l}(\cdot))\|^2_{Y_{\mathbb{C}}} + (\lambda - \lambda_*)^2 \|z\|^2_{Y_{\mathbb{C}}} = 0. \end{aligned}$$

We define $G : X_{\mathbb{C}} \times \mathbb{R}^3 \times \mathbb{R} \mapsto Y_{\mathbb{C}} \times \mathbb{R}^3$ by $G = (g_1, g_2, g_3, g_4)$ and note

$$z_{\lambda_*} = (1 - i)\xi_{\lambda_*}, \quad \beta_{\lambda_*} = 1, \quad h_{\lambda_*} = 1, \quad \theta_{\lambda_*} = \frac{\pi}{2}, \tag{9}$$

with ξ_{λ_*} defined as in Theorem 1. An easy calculation shows that

$$G(z_{\lambda_*}, \beta_{\lambda_*}, h_{\lambda_*}, \theta_{\lambda_*}, \lambda_*) = 0.$$

Solving eigenvalue problem

Theorem 2. There exists a continuously differentiable mapping $\lambda \mapsto (z_\lambda, \beta_\lambda, h_\lambda, \theta_\lambda)$ from $[\lambda_*, \lambda^*]$ to $X_{\mathbb{C}} \times \mathbb{R}^3$ such that $G(z_\lambda, \beta_\lambda, h_\lambda, \theta_\lambda, \lambda) = 0$. Moreover, if $\lambda \in (\lambda_*, \lambda^*)$, and $(z^\lambda, \beta^\lambda, h^\lambda, \theta^\lambda, \lambda)$ solves the equation $G = 0$ with $h^\lambda > 0$, and $\theta^\lambda \in [0, 2\pi)$, then $(z^\lambda, \beta^\lambda, h^\lambda, \theta^\lambda) = (z_\lambda, \beta_\lambda, h_\lambda, \theta_\lambda)$.

Proof. Using Implicit function theorem.

Corollary. If $0 < \lambda^* - \lambda_* \ll 1$, then for each $\lambda \in (\lambda_*, \lambda^*)$, the eigenvalue problem

$$\Delta(\lambda, i\nu, \tau)y = 0, \qquad \nu \geq 0, \qquad \tau > 0, \qquad y(\neq 0) \in X_{\mathbb{C}}$$

has a solution, or equivalently, $i\nu \in \sigma(A_\tau(\lambda))$ if and only if

$$\nu = \nu_\lambda = (\lambda - \lambda_*)h_\lambda, \qquad \tau = \tau_n = \frac{\theta_\lambda + 2n\pi}{\nu_\lambda}, \qquad n = 0, 1, 2, \cdots \tag{10}$$

and

$$y = r y_\lambda, \qquad y_\lambda = \beta_\lambda \sin(\frac{\pi}{l}(\cdot)) + (\lambda - \lambda_*)z_\lambda,$$

where r is a nonzero constant, and $z_\lambda, \beta_\lambda, h_\lambda, \theta_\lambda$ are defined as in Theorem 2.

Stability of steady state solution

1. If $0 < \lambda^* - \lambda_* \ll 1$ and $\tau \geq 0$, then 0 is not an eigenvalue of $A_\tau(\lambda)$ for $\lambda \in (\lambda_*, \lambda^*]$.
2. If $0 < \lambda^* - \lambda_* \ll 1$ and $\tau = 0$, then all eigenvalues of $A_\tau(\lambda)$ have negative real parts for $\lambda \in (\lambda_*, \lambda^*]$.
3. If $0 < \lambda^* - \lambda_* \ll 1$, then for each fixed $\lambda \in (\lambda_*, \lambda^*]$, $\mu = i\nu_\lambda$ is a simple eigenvalue of A_{τ_n} for $n = 0, 1, 2, \cdots$.
4. Since $\mu = i\nu$ is a simple eigenvalue of A_{τ_n}, by using the implicit function theorem it is not difficult to show that there are a neighborhood $O_n \times D_n \times H_n \subset \mathbb{R} \times \mathbb{C} \times X_{\mathbb{C}}$ of $(\tau_n, i\nu_\lambda, y_\lambda)$ and a continuously differential function $(\mu, y) : O_n \to D_n \times H_n$ such that for each $\tau \in O_n$, the only eigenvalue of $A_\tau(\lambda)$ in D_n is $\mu(\tau)$, and

$$\mu(\tau_n) = i\nu_\lambda, \qquad y(\tau_n) = y_\lambda,$$

$$\Delta(\lambda, \mu(\tau), \tau) = [A(\lambda) + \lambda u_\lambda f'(u_\lambda)e^{-\mu(\tau)\tau} - \mu(\tau)]y(\tau) = 0, \qquad \tau \in O_n. \quad (11)$$

5. If $0 < \lambda^* - \lambda_* \ll 1$, then for each $\lambda \in (\lambda_*, \lambda^*]$,

$$\mathrm{Re}\frac{d\mu(\tau_n)}{d\tau} > 0, \qquad n = 0, 1, 2, \cdots .$$

Hopf bifurcation

1. If $0 < \lambda^* - \lambda_* \ll 1$, then for each fixed $\lambda \in (\lambda_*, \lambda^*]$, the infinitesimal generator $A_\tau(\lambda)$ has exactly $2(n+1)$ eigenvalues with positive real part when $\tau \in (\tau_n, \tau_{\lambda_{n+1}}]$, $n = 0, 1, 2, \cdots$.
2. If $0 < \lambda^* - \lambda_* \ll 1$, then for each fixed $\lambda \in (\lambda_*, \lambda^*]$, the positive steady state solution u_λ of Eq.(1) is asymptotically stable when $\tau \in [0, \tau_0)$ and is unstable when $\tau \in (\tau_0, \infty)$.

Theorem 3. Suppose that $f(u)$ satisfies (A1) and (A2), and define $\lambda_* = d(\pi/l)^2$. Then there is a $\lambda^* > \lambda_*$ with $0 < \lambda^* - \lambda_* \ll 1$, and for each fixed $\lambda \in (\lambda_*, \lambda^*]$, there exist a sequence $\{\tau_n\}_{n=0}^{\infty}$ satisfying $0 < \tau_0 < \tau_1 < \cdots < \tau_n < \cdots$, such that Eq.(1) undergoes a Hopf bifurcation at $(\tau, u) = (\tau_n, u_\lambda)$ for $n = 0, 1, 2, \cdots$. More precisely, there is a family of periodic solutions in form of $(\tau_n(a), u_n(x, t; a))$ with period $T_n(a)$ for $a \in (0, a_1)$ with $a_1 > 0$, such that

$$\begin{aligned} &\tau_n(a) = \frac{\theta_\lambda + 2n\pi}{\nu_\lambda} + a^2 k_n^1(\lambda) + o(a^2), \quad T_n(a) = \frac{2\pi}{\nu_\lambda}\left(1 + a^2 k_n^2(\lambda) + o(a^2)\right), \\ &u_n(x, t; a) = u_\lambda(x) + \frac{a}{2}\left(y_\lambda(x)e^{i\nu_\lambda t} + \overline{y}_\lambda(x)e^{-i\nu_\lambda t}\right) + o(a), \end{aligned} \tag{12}$$

where

$$\begin{aligned} k_n^1(\lambda) &= \frac{d\gamma^*(0)}{da} := k_1(n, \lambda)(\lambda - \lambda_*)^{-3} + o((\lambda - \lambda_*)^{-3}), \\ k_n^2(\lambda) &= \frac{d\delta^*(0)}{da} := k_2(n, \lambda)(\lambda - \lambda_*)^{-2} + o((\lambda - \lambda_*)^{-2}), \end{aligned} \tag{13}$$

Hopf bifurcation (cont.)

$$
\begin{aligned}
k_1(n,\lambda) &= -\frac{\mathrm{Re}\displaystyle\int_0^l f'(u_\lambda)\bar{S}_n m_\lambda^1 \sin(\frac{\pi}{l}x) y_\lambda \bar{y}_\lambda (e^{i\theta_\lambda} + e^{-2i\theta_\lambda})dx}{h_\lambda \left|\displaystyle\int_0^l y_\lambda^2 dx\right| \mathrm{Re}\left\{ i e^{-i(\theta_\lambda+\rho_\lambda)} \displaystyle\int_0^l \frac{u_\lambda f'(u_\lambda) y_\lambda^2}{\lambda-\lambda_*} dx \right\}} \\
&= -\frac{\lambda_*^2 \left[f'(0)\right]^2 [1-3(\frac{\pi}{2}+2n\pi)] \left(\displaystyle\int_0^l \sin^3(\frac{\pi}{l}x)dx\right)^2}{20\left(\displaystyle\int_0^l \sin^2(\frac{\pi}{l}x)dx\right)^2} + o(\lambda-\lambda_*),
\end{aligned}
$$

Hopf bifurcation (cont.)

$$
\begin{aligned}
k_2(n,\lambda) =& \frac{\mathrm{Re}\int_0^l f'(u_\lambda)\bar{S}_n m_\lambda^1 \sin(\frac{\pi}{l}x) y_\lambda \bar{y}_\lambda (e^{i\theta_\lambda} + e^{-2i\theta_\lambda})dx}{h_\lambda^2 \left|S_n\right|^2 \left|\int_0^l y_\lambda^2 dx\right| \mathrm{Re}\left\{ie^{-i(\theta_\lambda+\rho_\lambda)}\int_0^l \frac{u_\lambda f'(u_\lambda)y_\lambda^2}{\lambda-\lambda_*}dx\right\}} \cdot \Bigg(\lambda h_\lambda \left|\int_0^l y_\lambda^2 dx\right| \\
& \cdot \mathrm{Im}\left\{ie^{-i(\theta_\lambda+\rho_\lambda)}\int_0^l \frac{u_\lambda f'(u_\lambda)y_\lambda^2}{\lambda-\lambda_*}dx\right\} + \lambda^2(\theta_\lambda + 2n\pi)\left|\int_0^l \frac{u_\lambda f'(u_\lambda)y_\lambda^2}{\lambda-\lambda_*}dx\right|^2\Bigg) \\
& + \frac{1}{h_\lambda \left|S_n\right|^2} \mathrm{Im}\int_0^l \lambda f'(u_\lambda)\bar{S}_n m_\lambda^1 y_\lambda \bar{y}_\lambda (e^{i\theta_\lambda} + e^{-2i\theta_\lambda})dx \\
=& \frac{\lambda_*^2\,[f'(0)]^2\,[3(\frac{\pi}{2}+2n\pi)^2 - 2(\frac{\pi}{2}+2n\pi) - 3](\int_0^l \sin^3(\frac{\pi}{l}x)dx)^2}{20[1+(\frac{\pi}{2}+2n\pi)]^2(\int_0^l \sin^2(\frac{\pi}{l}x)dx)^2} + o(\lambda-\lambda_*),
\end{aligned}
$$

and $(\theta_\lambda, \nu_\lambda, y_\lambda)$ is the associated eigen-triple. In particular, $k_1(n,\lambda) > 0$ and $k_2(n,\lambda) > 0$ hence the Hopf bifurcation at (τ_n, u_λ) is forward with increasing period.

Nonlocal model

Delayed Fisher equation:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = d\Delta u(x,t) + \lambda u(x,t)\left(1 - u(x,t-\tau)\right), & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{14}$$

It has been pointed out by several authors that, in a reaction-diffusion model with time-delay effect, the effects of diffusion and time delays are not independent of each other, and the individuals which were at location x at previous times may not be at the same point in space presently. Hence the localized density-dependent growth rate per capita $1 - u(x, t-\tau)$ in (14) is not realistic. it is more reasonable to consider the diffusive logistic population model with nonlocal delay effect as follows:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = d\Delta u(x,t) + \lambda u(x,t)\left(1 - \displaystyle\int_\Omega K(x,y)u(y,t-\tau)dy\right), & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{15}$$

where $u(x,t)$ is the population density at time t and location x, $d > 0$ is the diffusion coefficient, $\tau > 0$ is the time delay representing the maturation time, and $\lambda > 0$ is a scaling constant; Ω is a connected bounded open domain in $\mathbb{R}^n$ ($n \geq 1$), with a smooth boundary $\partial\Omega$, and Dirichlet boundary condition is imposed so the exterior environment is hostile; $K(x,y)$ is a kernel function which describes the dispersal behavior of the population. The nonlocal growth rate per capita in (15) incorporates the possible dispersal of the individuals during the maturation period, hence it is a more realistic model than (14).

Hopf bifurcation

Theorem 4. For $\lambda \in (\lambda_*, \lambda^*]$, the positive equilibrium solution u_λ of Eq.(15) is locally asymptotically stable when $\tau \in [0, \tau_0)$ and is unstable when $\tau \in (\tau_0, \infty)$. Moreover at $\tau = \tau_n$, $(n = 0, 1, 2, \cdots)$, a Hopf bifurcation occurs so that a branch of spatially nonhomogeneous periodic orbits of Eq. (15) emerges from (τ_n, u_λ).
More precisely, there exists $\varepsilon_0 > 0$ and continuously differentiable function $[-\varepsilon_0, \varepsilon_0] \mapsto (\tau_n(\varepsilon), T_n(\varepsilon), u_n(\varepsilon, x, t)) \in \mathbb{R} \times \mathbb{R} \times X$ satisfying $\tau_n(0) = \tau_n$, $T_n(0) = 2\pi/\nu_\lambda$, and $u_n(\varepsilon, x, t)$ is a $T_n(\varepsilon)$-periodic solution of Eq.(15) such that $u_n = u_\lambda + \varepsilon v_n(\varepsilon, x, t)$ where v_n satisfies $v_n(0, x, t)$ is a $2\pi/\nu_\lambda$-periodic solution of (5).
Moreover there exists $\delta > 0$ such that if Eq.(15) has a nonconstant periodic solution $u(x, t)$ of period T for some $\tau > 0$ with

$$|\tau - \tau_n| < \delta, \quad \left| T - \frac{2\pi}{\nu_\lambda} \right| < \delta, \quad \max_{t\in\mathbb{R}, x\in\overline{\Omega}} |u(x, t) - u_\lambda(x)| < \delta,$$

then $\tau = \tau_n(\varepsilon)$ and $u(x, t) = u_n(\varepsilon, x, t + \theta)$ for some $|\varepsilon| < \varepsilon_0$ and some $\theta \in \mathbb{R}$.

[Wu, 1995, book]

Homogenous kernel

When $K(x, y) \equiv 1$, $n = 1$ and $\Omega = (0, L)$ where $L > 0$, then the equation becomes

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \dfrac{\partial^2 u(x,t)}{\partial x^2} + \lambda u(x,t) \left(1 - \displaystyle\int_0^\pi u(y, t-\tau) dy \right), & x \in (0,\pi),\ t > 0, \\ u(x,t) = 0, & x = 0,\ \pi,\ t > 0. \end{cases} \tag{16}$$

We can easily verify that Eq. (16) has a unique positive equilibrium solution $u_\lambda(x) = \dfrac{\lambda - 1}{2\lambda} \sin x$ for any $\lambda > 1$ (here $\lambda_* = 1$). Linearizing Eq. (16) at u_λ, we have that

$$\begin{cases} \dfrac{\partial v(x,t)}{\partial t} = \dfrac{\partial^2 v(x,t)}{\partial x^2} + v - \dfrac{\lambda - 1}{2} \sin x \displaystyle\int_0^\pi v(y, t-\tau) dy, & x \in (0,\pi),\ t > 0, \\ v(x,t) = 0, & x = 0,\ \pi,\ t > 0. \end{cases} \tag{17}$$

Note that μ is an eigenvalue of $A_\tau(\lambda)$ if and only if μ is an eigenvalue of the following nonlocal elliptic eigenvalue problem:

$$\begin{cases} \Delta(\lambda, \mu, \tau)\psi := \psi'' + \psi - \dfrac{\lambda - 1}{2} e^{-\mu\tau} \sin x \displaystyle\int_0^\pi \psi(y) dy - \mu\psi = 0, & x \in (0,\pi), \\ \psi(0) = \psi(\pi) = 0. \end{cases} \tag{18}$$

Eigenvalue probelm

Lemma. Suppose that $\lambda > 1$ and $\tau \geq 0$. Then $\mu \in \mathbb{C}$ is an eigenvalue of the problem (18) if and only if one of the following is satisfied:

1. $\mu = -n^2 + 1$ for $n = 2, 3, 4, \cdots$; or
2. μ satisfies

$$(\lambda - 1)e^{-\mu\tau} + \mu = 0. \tag{19}$$

Proof: Substituting the Fourier series $\psi = \sum_{n=1}^{\infty} c_n \sin nx$ into Eq. (18), we have:

$$\sum_{n=2}^{\infty} c_n \left(-n^2 + 1 - \mu\right) \sin nx - \left[(\lambda - 1) \sum_{n=0}^{\infty} \frac{c_{2n+1}}{2n+1} e^{-\mu\tau} + \mu c_1\right] \sin x = 0. \tag{20}$$

<u>Case 1</u>: Suppose that $\mu \in \mathbb{C}$ is an eigenvalue of (18), and $\mu \neq -n^2 + 1$ for each of $n = 2, 3, 4, \cdots$, then (20) implies each $c_n = 0$ for $n \geq 2$, and if $c_1 \neq 0$, then (19) is satisfied, and μ is an eigenvalue with an eigenfunction $\phi_1(x) = \sin x$.
<u>Case 2</u>: If (19) is not satisfied and for some $m = 2, 3, 4, \cdots$, $\mu = -m^2 + 1$, then $c_n = 0$ for $n \geq 2$ and $n \neq m$. If m is even, then $c_1 = 0$ as well, hence $\mu = -m^2 + 1$ is an eigenvalue with an eigenfunction $\phi_m(x) = \sin mx$; if m is odd, then $\mu = -m^2 + 1$ is an eigenvalue with an eigenfunction in form $\phi_m(x) = \sin x + c_m \sin mx$, where c_m satisfies

$$(\lambda - 1)\left(1 + \frac{c_m}{m}\right) e^{(-m^2+1)\tau} - m^2 + 1 = 0.$$

Distribution of eigenvalues

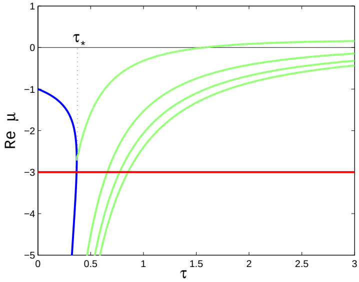


Figure : Relation between $\mathcal{R}e(\mu)$ and τ for Eq. (19). Here $\lambda = 2$. $\mu = -3$ is a fixed real-valued eigenvalue; on the left side of $\tau = \tau_*$ is the curve of real-valued eigenvalues μ satisfying $(\lambda - 1)e^{-\mu\tau} + \mu = 0$; and on the right side of $\tau = \tau_*$ are the curves of real part α_n of complex-valued eigenvalues $\alpha_n \pm i\beta_n$. The curve $\alpha_0(\tau)$ connects with the curve of real eigenvalues at $\tau = \tau_*$, and at $\tau = \pi/2$, $\alpha_0(\tau) = 0$ which gives rise of the first Hopf bifurcation point.

Hopf bifurcation

1. The eigenspace of (18) may not be one-dimensional. When $\mu = -n^2 + 1$ is also a root of (19), the eigenspace is two-dimensional. However as shown in [Davidson-Doods, 2006, AA], usually the eigenspace of such nonlocal problem is at most two-dimensional.
2. The eigenvalue problem (18) with $\tau = 0$ always has a principal eigenvalue μ_0 satisfying (19) with a positive eigenfunction $\sin x$. But μ_0 may not be the largest eigenvalue of (18). For example when $\tau = 0$ and $\lambda < 4$, the maximum eigenvalue of (18) is $1 - \lambda$ which is also the principal eigenvalue; but when $\tau = 0$ and $\lambda \geq 4$, then the maximum eigenvalue is -3 with the corresponding eigenfunction $\sin 2x$, and hence the maximum eigenvalue is not the principal eigenvalue.

Theorem 5. For each $\lambda > 1$, there exist

$$\tau_n(\lambda) = \frac{(4n+1)\pi}{2(\lambda - 1)}, \quad n = 0, 1, 2, \cdots, \tag{21}$$

such that when $\tau = \tau_n(\lambda)$, $n = 0, 1, 2 \cdots$, $A_\tau(\lambda)$ has a pair of simple purely imaginary roots $\pm i\nu_\lambda = \pm i(\lambda - 1)$. Consider the nonlocal problem (16). For each $\lambda > 1$ and $n \in \mathbb{N} \cup \{0\}$, there exists a $\tau_n(\lambda)$ defined as in (21) such that a Hopf bifurcation occurs for Eq. (16) at the unique positive equilibrium solution $u_\lambda = \dfrac{\lambda - 1}{2\lambda} \sin x$ when $\tau = \tau_n(\lambda)$. Moreover, u_λ is locally asymptotically stable when $0 \leq \tau < \tau_0(\lambda)$, and it is unstable when $\tau > \tau_0(\lambda)$.

An observation

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = d\Delta u(x,t) + \lambda u(x,t)\left(1 - \displaystyle\int_\Omega K(x,y)u(y,t-\tau)dy\right), & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{22}$$

suppose that a solution $u(x,t)$ of Eq. (22) is in a separable form

$$u(x,t) = \frac{\lambda - 1}{2\lambda} \sin x \cdot w(t). \tag{23}$$

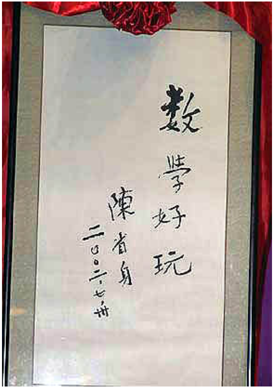
Here we recall that $u_\lambda(x) = \dfrac{\lambda - 1}{2\lambda} \sin x$ is the unique positive equilibrium of Eq. (22) for $\lambda > 1$. Then it is easy to verify that $w(t)$ satisfies the well-known (non-spatial) Hutchinson equation

$$\frac{dw}{dt} = (\lambda - 1)w(t)(1 - w(t - \tau)). \tag{24}$$

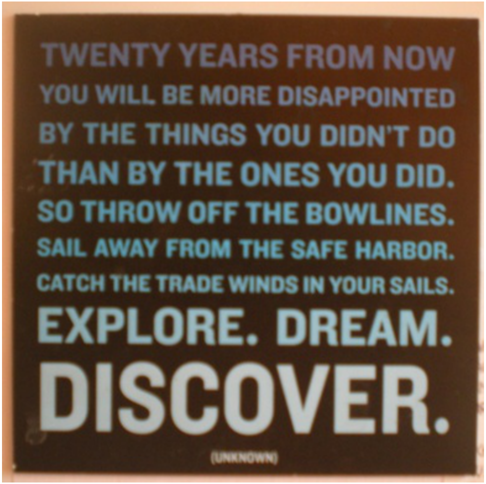
It is also well-known that the Hopf bifurcation points of Eq. (24) are also given by (21), hence all the bifurcating periodic orbits obtained in Theorem 5 are indeed in separable form (23). This shows that the dynamics of Eq. (24) is embedded in the dynamics of Eq. (22) if the initial value is also in separable form (23). This is interesting for a Dirichlet boundary value problem, while it is common for Neumann (no-flux) boundary value problem. It would be interesting to know the stability of periodic solution with such separable form for all $\lambda > 1$, and whether a symmetry-breaking bifurcation can occur so that non-separable periodic orbits can arise.

Conclusion

- (Cliff Taubes) http://w3.math.sinica.edu.tw/media/media.jsp?voln=371
  "What is the most useful mathematics for you?"
  "I think one is fundamental theorem of calculus, and another is maximum principle."
  In my opinion, another two important things are: (i) Taylor expansion of a function; (ii) implicit function theorem.
- (Yuan Wang) The most important thing is: learn by yourself.



Thank you!


TWENTY YEARS FROM NOW
YOU WILL BE MORE DISAPPOINTED
BY THE THINGS YOU DIDN'T DO
THAN BY THE ONES YOU DID.
SO THROW OFF THE BOWLINES.
SAIL AWAY FROM THE SAFE HARBOR.
CATCH THE TRADE WINDS IN YOUR SAILS.
EXPLORE. DREAM.
DISCOVER.
(UNKNOWN)


Thank You!