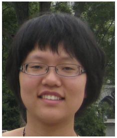
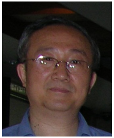
Reaction-Diffusion Models and Bifurcation Theory Lecture 11: Bifurcation in delay differential equations

Junping Shi

College of William and Mary, USA

Collaborators/Support

Shanshan Chen (PhD, 2012; Harbin Institute of Technology, Harbin/Weihai, China)
Junjie Wei (Harbin Institute of Technology, Harbin/Weihai, China)

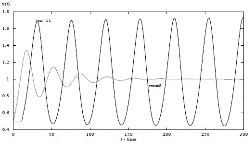
Chen, Shanshan; Shi, Junping; Wei, Junjie, Time delay induced instabilities and Hopf bifurcations in general reaction-diffusion systems. *Journal of Nonlinear Science*. 23 (2013), no. 1, 1–38.

Grant Support: NSF (US): DMS-1022648; NSFC (China): 11071051; China Scholarship Council.

Hutchinson Equation

In the Logistic equation, the growth rate per capita is a decreasing function of the current population size. But in the reality, the female individual may need some maturing time to be able to reproduce. Hence in some cases, the growth rate per capita should instead depend on the population size of a past time. That is the delay effect in the density-dependent population growth. In 1948, British-American biologist George Evelyn Hutchinson (1903-1991) proposed the Logistic equation with delay (now called Hutchinson equation) (τ is the time delay).

$$\frac{dP(t)}{dt} = aP(t)\left(1 - \frac{P(t-\tau)}{K}\right)$$



Left: George Evelyn Hutchinson (1903-1991) Right: Simulation of Hutchinson equation

Mackey-Glass Equation and Nicholson's Blowfly equation

In 1977, Mackey and Glass constructed an equation of physiological control (for respiratory studies, or for white blood cells): $\frac{dx}{dt} = \lambda - \frac{\alpha V_m x(t-\tau)}{\theta^n + x^n(t-\tau)}$. It was shown that when λ increases, a sequence of period-doubling Hopf bifurcations occurs and chaotic behavior exists for some parameter values. A similar equation is Nicholson's equation for blowfly population $\frac{dx}{dt} = \beta x^n(t-\tau)\exp(-x(t-\tau)) - \alpha x(t)$

M.C. Mackey, L. Glass, Oscillation and chaos in physiological control systems. Science, 1977.
A. Nicholson, An outline of the dynamics of animal populations, Aust. J. Zool., 1954.
W. Gurney, S. Blythe, R. Nisbet, Nicholson's blowflies revisited, Nature, 1980.

Left: Michael Mackey; Right: Leon Glass

Delay-induced instability

$$\frac{dP(t)}{dt} = aP(t)\,(1 - P(t-\tau))$$

Linearization at $P = 1$ without time delay:
$v'(t) = -av(t)$ (so $P = 1$ is stable when there is no time delay)

Delay-induced instability

$$\frac{dP(t)}{dt} = aP(t)\,(1 - P(t-\tau))$$

Linearization at $P = 1$ without time delay:
$v'(t) = -av(t)$ (so $P = 1$ is stable when there is no time delay)

Linearization at $P = 1$ with time delay:
$v'(t) = -av(t-\tau)$
Characteristic equation: $\lambda + ae^{-\lambda\tau} = 0$.

Delay-induced instability

$$\frac{dP(t)}{dt} = aP(t)\,(1 - P(t-\tau))$$

Linearization at $P = 1$ without time delay:
$v'(t) = -av(t)$ (so $P = 1$ is stable when there is no time delay)

Linearization at $P = 1$ with time delay:
$v'(t) = -av(t-\tau)$
Characteristic equation: $\lambda + ae^{-\lambda\tau} = 0$.

neutral stability: $\lambda = \beta i \quad \cos(\beta\tau) = 0,\ a\sin(\beta\tau) = \beta$

Delay-induced instability

$$\frac{dP(t)}{dt} = aP(t)\left(1 - P(t-\tau)\right)$$

Linearization at $P = 1$ without time delay:
$v'(t) = -av(t)$ (so $P = 1$ is stable when there is no time delay)

Linearization at $P = 1$ with time delay:
$v'(t) = -av(t-\tau)$
Characteristic equation: $\lambda + ae^{-\lambda\tau} = 0$.

neutral stability: $\lambda = \beta i$ $\cos(\beta\tau) = 0$, $a\sin(\beta\tau) = \beta$

So $\tau_0 = \dfrac{\pi}{2a}$ is the value where the stability is lost when $\tau > \tau_0$. And $\tau_n = \dfrac{(2n+1)\pi}{2a}$ is a Hopf bifurcation point.
(Thus $P = 1$ is stable when the time delay $\tau < \dfrac{\pi}{2a}$, but it is unstable if $\tau > \dfrac{\pi}{2a}$.)

Delay-induced instability

$$\frac{dP(t)}{dt} = aP(t)\left(1 - P(t-\tau)\right)$$

Linearization at $P = 1$ without time delay:
$v'(t) = -av(t)$ (so $P = 1$ is stable when there is no time delay)

Linearization at $P = 1$ with time delay:
$v'(t) = -av(t-\tau)$
Characteristic equation: $\lambda + ae^{-\lambda\tau} = 0$.

neutral stability: $\lambda = \beta i \quad \cos(\beta\tau) = 0,\ a\sin(\beta\tau) = \beta$

So $\tau_0 = \dfrac{\pi}{2a}$ is the value where the stability is lost when $\tau > \tau_0$. And $\tau_n = \dfrac{(2n+1)\pi}{2a}$ is a Hopf bifurcation point.
(Thus $P = 1$ is stable when the time delay $\tau < \dfrac{\pi}{2a}$, but it is unstable if $\tau > \dfrac{\pi}{2a}$.)
Basic lesson: a large delay destabilizes an equilibrium

Another example

$$\frac{du}{dt} = ru(t)[1 - au(t) - bu(t-\tau)].$$

Here a and b represent the portions of instantaneous and delayed dependence of the growth rate respectively, and we assume that $a, b \in (0,1)$ and $a + b = 1$. Then $u_* = 1$ is an equilibrium.

Another example

$$\frac{du}{dt} = ru(t)[1 - au(t) - bu(t-\tau)].$$

Here a and b represent the portions of instantaneous and delayed dependence of the growth rate respectively, and we assume that $a, b \in (0,1)$ and $a + b = 1$. Then $u_* = 1$ is an equilibrium.

Linearization at $u = 1$ with time delay:
$v'(t) = -arv(t) - brv(t-\tau)$
Characteristic equation: $\lambda + ar + bre^{-\lambda\tau} = 0$.

Another example

$$\frac{du}{dt} = ru(t)[1 - au(t) - bu(t-\tau)].$$

Here a and b represent the portions of instantaneous and delayed dependence of the growth rate respectively, and we assume that $a, b \in (0,1)$ and $a + b = 1$. Then $u_* = 1$ is an equilibrium.

Linearization at $u = 1$ with time delay:
$v'(t) = -arv(t) - brv(t-\tau)$
Characteristic equation: $\lambda + ar + bre^{-\lambda\tau} = 0$.

neutral stability: $\lambda = \beta i$
$\cos(\beta\tau) = -\dfrac{a}{b}$, $\sin(\beta\tau) = \dfrac{\beta}{br}$

Another example

$$\frac{du}{dt} = ru(t)[1 - au(t) - bu(t-\tau)].$$

Here a and b represent the portions of instantaneous and delayed dependence of the growth rate respectively, and we assume that $a, b \in (0, 1)$ and $a + b = 1$. Then $u_* = 1$ is an equilibrium.

Linearization at $u = 1$ with time delay:
$v'(t) = -arv(t) - brv(t - \tau)$
Characteristic equation: $\lambda + ar + bre^{-\lambda\tau} = 0$.

neutral stability: $\lambda = \beta i$
$\cos(\beta\tau) = -\dfrac{a}{b}$, $\sin(\beta\tau) = \dfrac{\beta}{br}$
$\beta = r\sqrt{b^2 - a^2}$.

If $a \geq b$, then the neutral stability condition cannot be achieved. Indeed one can prove u_* is globally stable for any $\tau > 0$.
If $a < b$, then $\tau_0 = \dfrac{1}{r\sqrt{b^2 - a^2}} \arccos\left(-\dfrac{a}{b}\right)$ is the value where the stability is lost when $\tau > \tau_0$. And $\tau_n = \dfrac{1}{r\sqrt{b^2 - a^2}} \left(\arccos\left(-\dfrac{a}{b}\right) + 2n\pi\right)$ is a Hopf bifurcation point.

General case

$$\frac{du}{dt} = f(u(t), u(t-\tau)).$$

Here $f = f(u, w)$ is a smooth function, and we assume that $u = u_*$ is an equilibrium.

General case

$$\frac{du}{dt} = f(u(t), u(t-\tau)).$$

Here $f = f(u, w)$ is a smooth function, and we assume that $u = u_*$ is an equilibrium.

Linearization at $u = u_*$ with time delay:

$v'(t) = f_u(u_*, u_*)v(t) + f_w(u_*, u_*)v(t-\tau)$

Characteristic equation: $\lambda - f_u - f_w e^{-\lambda\tau} = 0$.

General case

$\frac{du}{dt} = f(u(t), u(t-\tau))$.
Here $f = f(u, w)$ is a smooth function, and we assume that $u = u_*$ is an equilibrium.

Linearization at $u = u_*$ with time delay:
$v'(t) = f_u(u_*, u_*)v(t) + f_w(u_*, u_*)v(t-\tau)$
Characteristic equation: $\lambda - f_u - f_w e^{-\lambda\tau} = 0$.

neutral stability: $\lambda = \beta i$
$\cos(\beta\tau) = -\frac{f_u}{f_w}$, $\sin(\beta\tau) = -\frac{\beta}{f_w}$

General case

$$\frac{du}{dt} = f(u(t), u(t-\tau)).$$

Here $f = f(u, w)$ is a smooth function, and we assume that $u = u_*$ is an equilibrium.

Linearization at $u = u_*$ with time delay:
$v'(t) = f_u(u_*, u_*)v(t) + f_w(u_*, u_*)v(t-\tau)$
Characteristic equation: $\lambda - f_u - f_w e^{-\lambda\tau} = 0$.

neutral stability: $\lambda = \beta i$
$\cos(\beta\tau) = -\frac{f_u}{f_w}$, $\sin(\beta\tau) = -\frac{\beta}{f_w}$
$\beta = \sqrt{f_w^2 - f_u^2}$.

General case

$$\frac{du}{dt} = f(u(t), u(t-\tau)).$$

Here $f = f(u, w)$ is a smooth function, and we assume that $u = u_*$ is an equilibrium.

Linearization at $u = u_*$ with time delay:
$v'(t) = f_u(u_*, u_*)v(t) + f_w(u_*, u_*)v(t-\tau)$
Characteristic equation: $\lambda - f_u - f_w e^{-\lambda\tau} = 0$.

neutral stability: $\lambda = \beta i$
$\cos(\beta\tau) = -\dfrac{f_u}{f_w}$, $\sin(\beta\tau) = -\dfrac{\beta}{f_w}$
$\beta = \sqrt{f_w^2 - f_u^2}$.

If $|f_u| \geq |f_w|$, then the neutral stability condition cannot be achieved. Hence u_* is locally stable for any $\tau > 0$.

If $|f_u| < |f_w|$, then $\tau_0 = \dfrac{1}{\sqrt{f_w^2 - f_u^2}} \arccos\left(-\dfrac{f_u}{f_w}\right)$ is the value where the stability is lost when $\tau > \tau_0$.

General case

$$\frac{du}{dt} = f(u(t), u(t-\tau)).$$

Here $f = f(u, w)$ is a smooth function, and we assume that $u = u_*$ is an equilibrium.

Linearization at $u = u_*$ with time delay:
$v'(t) = f_u(u_*, u_*)v(t) + f_w(u_*, u_*)v(t-\tau)$
Characteristic equation: $\lambda - f_u - f_w e^{-\lambda\tau} = 0$.

neutral stability: $\lambda = \beta i$
$\cos(\beta\tau) = -\dfrac{f_u}{f_w}$, $\sin(\beta\tau) = -\dfrac{\beta}{f_w}$
$\beta = \sqrt{f_w^2 - f_u^2}$.

If $|f_u| \geq |f_w|$, then the neutral stability condition cannot be achieved. Hence u_* is locally stable for any $\tau > 0$.

If $|f_u| < |f_w|$, then $\tau_0 = \dfrac{1}{\sqrt{f_w^2 - f_u^2}} \arccos\left(-\dfrac{f_u}{f_w}\right)$ is the value where the stability is lost when $\tau > \tau_0$.

Lesson: If the strength of instantaneous dependence is stronger than the delayed dependence, then the equilibrium is always stable; and if it is weaker, then the equilibrium loses the stability with a larger delay.

Stability

An equation with k different delays, and variable $x \in \mathbb{R}^n$:

$$\dot{x}(t) = f(x(t), x(t-\tau_1), \cdots, x(t-\tau_k)). \quad (1)$$

Stability

An equation with k different delays, and variable $x \in \mathbb{R}^n$:

$$\dot{x}(t) = f(x(t), x(t-\tau_1), \cdots, x(t-\tau_k)). \quad (1)$$

The characteristic equation takes the form

$$det\left(\lambda I - A_0 - \sum_{j=1}^{m} A_j e^{-\lambda \eta_j}\right) = 0,$$

where A_j $(0 \le j \le m)$ is an $n \times n$ constant matrix, $\eta_j > 0$.

[Brauer, 1987, JDE], [Ruan, 2001, Quer.Appl.Math]
An steady state $x = x_*$ of system (1) is said to be absolutely stable (i.e., asymptotically stable independent of the delays) if it is asymptotically stable for all delays $\tau_j \ge 0$ $(1 \le j \le k)$; and $x = x_*$ is said to be conditionally stable (i.e., asymptotically stable depending on the delays) if it is asymptotically stable for τ_j $(1 \le j \le k)$ in some intervals, but not necessarily for all delays.

Transcendental characteristic equation

The characteristic equation takes the form

$$det\left(\lambda I - A_0 - \sum_{j=1}^{m} A_j e^{-\lambda \eta_j}\right) = 0,$$

where A_j $(0 \leq j \leq m)$ is an $n \times n$ constant matrix, $\eta_j > 0$.

Transcendental characteristic equation

The characteristic equation takes the form

$$det\left(\lambda I - A_0 - \sum_{j=1}^{m} A_j e^{-\lambda\eta_j}\right) = 0,$$

where A_j $(0 \leq j \leq m)$ is an $n \times n$ constant matrix, $\eta_j > 0$.

Most previous work considers $n \leq 3$ and $m \leq 2$.
Books: Hale-Verduyn Lunel [1993], Kuang [1993], Wu [1996], Smith [2011]
Hale-Huang [1993], Belair-Campbell [1994], Li-Ruan-Wei [1999]
Ruan [2001], Ruan-Wei [2001, 2003], Li-Wei [2005] many many others

Transcendental characteristic equation

The characteristic equation takes the form

$$det\left(\lambda I - A_0 - \sum_{j=1}^{m} A_j e^{-\lambda \eta_j}\right) = 0,$$

where A_j $(0 \leq j \leq m)$ is an $n \times n$ constant matrix, $\eta_j > 0$.

Most previous work considers $n \leq 3$ and $m \leq 2$.
Books: Hale-Verduyn Lunel [1993], Kuang [1993], Wu [1996], Smith [2011]
Hale-Huang [1993], Belair-Campbell [1994], Li-Ruan-Wei [1999]
Ruan [2001], Ruan-Wei [2001, 2003], Li-Wei [2005] many many others

Most work has a characteristic equation with only one transcendental term:

$$P(\lambda) + e^{-\lambda \tau} Q(\lambda) = 0,$$

where P and Q are polynomials of λ, and the degree of P is greater than that of Q.

Transcendental characteristic equation

The characteristic equation takes the form

$$det\left(\lambda I - A_0 - \sum_{j=1}^{m} A_j e^{-\lambda \eta_j}\right) = 0,$$

where A_j $(0 \leq j \leq m)$ is an $n \times n$ constant matrix, $\eta_j > 0$.

Most previous work considers $n \leq 3$ and $m \leq 2$.
Books: Hale-Verduyn Lunel [1993], Kuang [1993], Wu [1996], Smith [2011]
Hale-Huang [1993], Belair-Campbell [1994], Li-Ruan-Wei [1999]
Ruan [2001], Ruan-Wei [2001, 2003], Li-Wei [2005] many many others

Most work has a characteristic equation with only one transcendental term:

$$P(\lambda) + e^{-\lambda\tau} Q(\lambda) = 0,$$

where P and Q are polynomials of λ, and the degree of P is greater than that of Q.

1. scalar equations with a single delay or planar systems with only one delay term
2. planar system: $\dot{x}(t) = f(x(t), y(t-\tau_1))$, $\dot{y}(t) = g(x(t-\tau_2), y(t))$
3. planar system: $\dot{x}(t) = f(x(t), y(t)) \pm k_1 g(x(t-\tau), y(t-\tau))$,
$\dot{y}(t) = h(x(t), y(t)) \pm k_2 g(x(t-\tau), y(t-\tau))$.

General form

[Cooke-Grossman, 1982, JMAA], [Ruan, 2001, Quar.Appl.Math]
Characteristic equation

$$\lambda^2 + a\lambda + b + (c\lambda + d)e^{-\lambda\tau} = 0. \tag{2}$$

Neutral stability: $\pm i\omega$, $(\omega > 0)$, is a pair of roots.
$-\omega^2 + a\omega i + b + (c\omega i + d)e^{-i\omega\tau} = 0.$

General form

[Cooke-Grossman, 1982, JMAA], [Ruan, 2001, Quar.Appl.Math]
Characteristic equation

$$\lambda^2 + a\lambda + b + (c\lambda + d)e^{-\lambda\tau} = 0. \tag{2}$$

Neutral stability: $\pm i\omega$, $(\omega > 0)$, is a pair of roots.
$-\omega^2 + a\omega i + b + (c\omega i + d)e^{-i\omega\tau} = 0.$

$-d\cos(\omega\tau) + c\omega\sin(\omega\tau) = b - \omega^2$, $-c\omega\cos(\omega\tau) - d\sin(\omega\tau) = a\omega$

General form

[Cooke-Grossman, 1982, JMAA], [Ruan, 2001, Quar.Appl.Math]
Characteristic equation

$$\lambda^2 + a\lambda + b + (c\lambda + d)e^{-\lambda\tau} = 0. \tag{2}$$

Neutral stability: $\pm i\omega$, $(\omega > 0)$, is a pair of roots.
$-\omega^2 + a\omega i + b + (c\omega i + d)e^{-i\omega\tau} = 0.$

$-d\cos(\omega\tau) + c\omega\sin(\omega\tau) = b - \omega^2$, $-c\omega\cos(\omega\tau) - d\sin(\omega\tau) = a\omega$

$\omega^4 - (c^2 - a^2 + 2b)\omega^2 + (b^2 - d^2) = 0.$

General form

[Cooke-Grossman, 1982, JMAA], [Ruan, 2001, Quar.Appl.Math]
Characteristic equation

$$\lambda^2 + a\lambda + b + (c\lambda + d)e^{-\lambda\tau} = 0. \tag{2}$$

Neutral stability: $\pm i\omega$, $(\omega > 0)$, is a pair of roots.
$-\omega^2 + a\omega i + b + (c\omega i + d)e^{-i\omega\tau} = 0.$

$-d\cos(\omega\tau) + c\omega\sin(\omega\tau) = b - \omega^2$, $-c\omega\cos(\omega\tau) - d\sin(\omega\tau) = a\omega$

$\omega^4 - (c^2 - a^2 + 2b)\omega^2 + (b^2 - d^2) = 0.$

Let $T = c^2 - a^2 + 2b$, and $D = b^2 - d^2$. Then there is no positive root ω^2 if (i) $T < 0$ and $D > 0$; or (ii) $T^2 - 4D < 0$.

Theorem. If $a + c > 0$, $b + d > 0$, and either (i) $T < 0$ and $D > 0$; or (ii) $T^2 - 4D < 0$ is satisfied, then all roots of (2) have negative real part for any $\tau \geq 0$.

General form

[Cooke-Grossman, 1982, JMAA], [Ruan, 2001, Quar.Appl.Math]
Characteristic equation

$$\lambda^2 + a\lambda + b + (c\lambda + d)e^{-\lambda\tau} = 0. \tag{2}$$

Neutral stability: $\pm i\omega$, $(\omega > 0)$, is a pair of roots.
$-\omega^2 + a\omega i + b + (c\omega i + d)e^{-i\omega\tau} = 0.$

$-d\cos(\omega\tau) + c\omega\sin(\omega\tau) = b - \omega^2$, $-c\omega\cos(\omega\tau) - d\sin(\omega\tau) = a\omega$

$\omega^4 - (c^2 - a^2 + 2b)\omega^2 + (b^2 - d^2) = 0.$

Let $T = c^2 - a^2 + 2b$, and $D = b^2 - d^2$. Then there is no positive root ω^2 if (i) $T < 0$ and $D > 0$; or (ii) $T^2 - 4D < 0$.

Theorem. If $a + c > 0$, $b + d > 0$, and either (i) $T < 0$ and $D > 0$; or (ii) $T^2 - 4D < 0$ is satisfied, then all roots of (2) have negative real part for any $\tau \geq 0$.

On the other hand, if (i) $D < 0$ or (ii) $T > 0$, $D > 0$ and $T^2 - 4D \geq 0$, then $\omega^4 - (c^2 - a^2 + 2b)\omega^2 + (b^2 - d^2) = 0$ has one or two positive roots. And the critical delay value can be solved:

$$\tau_n = \frac{1}{\omega}\left(\arccos\left(\frac{(d - ac)\omega^2 - bd}{d^2 + c^2\omega^2}\right) + 2n\pi\right).$$

Example 1: Rosenzwing-MacArthur Model

[Chen-Shi-Wei, 2012, CPAA]

$$\begin{cases} \dfrac{\partial u}{\partial t} - d_1 u_{xx} = u\left(1 - \dfrac{u}{K}\right) - \dfrac{muv}{u+1}, & x \in (0, l\pi),\ t > 0, \\ \dfrac{\partial v}{\partial t} - d_2 v_{xx} = -dv + \dfrac{mu(t-\tau)v}{u(t-\tau)+1}, & x \in (0, l\pi),\ t > 0, \\ \dfrac{\partial u(x,t)}{\partial x} = \dfrac{\partial v(x,t)}{\partial x} = 0, & x = 0, l\pi,\ t > 0, \\ u(x,t) = u_0(x,t) \geq 0, v(x,t) = v_0(x,t) \geq 0, & x \in (0, l\pi),\ t \in [-\tau, 0], \end{cases}$$

Constant steady state: (λ, v_λ) where $\lambda = \dfrac{d}{m-d}$ and $v_\lambda = \dfrac{(K-\lambda)(1+\lambda)}{Km}$.

Example 1: Rosenzwing-MacArthur Model

[Chen-Shi-Wei, 2012, CPAA]

$$\begin{cases} \dfrac{\partial u}{\partial t} - d_1 u_{xx} = u\left(1 - \dfrac{u}{K}\right) - \dfrac{muv}{u+1}, & x \in (0, l\pi),\ t > 0, \\ \dfrac{\partial v}{\partial t} - d_2 v_{xx} = -dv + \dfrac{mu(t-\tau)v}{u(t-\tau)+1}, & x \in (0, l\pi),\ t > 0, \\ \dfrac{\partial u(x,t)}{\partial x} = \dfrac{\partial v(x,t)}{\partial x} = 0, & x = 0, l\pi,\ t > 0, \\ u(x,t) = u_0(x,t) \geq 0, v(x,t) = v_0(x,t) \geq 0, & x \in (0, l\pi),\ t \in [-\tau, 0], \end{cases}$$

Constant steady state: (λ, v_λ) where $\lambda = \dfrac{d}{m-d}$ and $v_\lambda = \dfrac{(K-\lambda)(1+\lambda)}{Km}$.

Main result: For any $\lambda \in ((K-1)/2, K)$, there exists $\tau_0(\lambda) > 0$ such that (λ, v_λ) is stable when $\tau < \tau_0(\lambda)$, and (λ, v_λ) is unstable when $\tau > \tau_0(\lambda)$. Moreover $\lim\limits_{\lambda \to (K-1)/2} \tau_0(\lambda) = 0$, and $\lim\limits_{\lambda \to K} \tau_0(\lambda) = \infty$; At $\tau = \tau_0(\lambda)$, a branch of homogenous periodic orbits bifurcate from (λ, v_λ).

There is no parameter region in which the stability persists for all delay $\tau > 0$ (not absolutely stable).

Calculation

The characteristic equation

$$\Delta_n(\lambda,\tau) = \lambda^2 + A_n\lambda + B_n + Ce^{-\lambda\tau} = 0, \quad n = 0, 1, 2, \cdots,$$

where

$$A_n = \frac{(d_1+d_2)n^2}{l^2} - \frac{\beta(k-1-2\beta)}{k(1+\beta)},$$

$$B_n = \frac{d_1 d_2 n^4}{l^4} - \frac{d_2 n^2}{l^2}\frac{\beta(k-1-2\beta)}{k(1+\beta)}, \quad C = \frac{r(k-\beta)}{k(\beta+1)}.$$

If $\pm i\sigma(\sigma > 0)$ is a pair of roots of characteristic equation, then we have

$$\begin{cases} \sigma^2 - B_n = C\cos\sigma\tau, \\ \sigma A_n = C\sin\sigma\tau, \end{cases} \quad n = 0, 1, 2, \cdots,$$

which leads to

$$\sigma^4 + (A_n^2 - 2B_n)\sigma^2 + B_n^2 - C^2 = 0, \quad n = 0, 1, 2, \cdots,$$

where

$$A_n^2 - 2B_n = \frac{d_2^2 n^4}{l^4} + \left(\frac{d_1 n^2}{l^2} - \frac{\beta(k-1-2\beta)}{k(1+\beta)}\right)^2,$$

$$B_n^2 - C^2 = \frac{d_2^2 n^4}{l^4}\left(\frac{d_1 n^2}{l^2} - \frac{\beta(k-1-2\beta)}{k(1+\beta)}\right)^2 - \frac{r^2(k-\beta)^2}{k^2(\beta+1)^2}.$$

Delayed Diffusive Leslie-Gower Predator-Prey Model

[Chen-Shi-Wei, 2012, IJBC]

$$
\begin{cases}
\dfrac{\partial u(t,x)}{\partial t} - d_1 \Delta u(t,x) = u(t,x)(p - \alpha u(t,x) - \beta v(t-\tau_1, x)), & x \in \Omega,\ t > 0, \\
\dfrac{\partial v(t,x)}{\partial t} - d_2 \Delta v(t,x) = \mu v(t,x) \left(1 - \dfrac{v(t,x)}{u(t-\tau_2, x)}\right), & x \in \Omega,\ t > 0, \\
\dfrac{\partial u(t,x)}{\partial \nu} = \dfrac{\partial v(t,x)}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\
u(x,t) = u_0(x,t) \geq 0, & x \in \Omega,\ t \in [-\tau_2, 0]. \\
v(x,t) = v_0(x,t) \geq 0, & x \in \Omega,\ t \in [-\tau_1, 0].
\end{cases}
$$

Constant steady state: $(u_*, v_*) = \left(\dfrac{p}{\alpha+\beta}, \dfrac{p}{\alpha+\beta}\right)$.

Delayed Diffusive Leslie-Gower Predator-Prey Model

[Chen-Shi-Wei, 2012, IJBC]

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} - d_1\Delta u(t,x) = u(t,x)(p - \alpha u(t,x) - \beta v(t-\tau_1,x)), & x \in \Omega,\ t>0, \\ \dfrac{\partial v(t,x)}{\partial t} - d_2\Delta v(t,x) = \mu v(t,x)\left(1 - \dfrac{v(t,x)}{u(t-\tau_2,x)}\right), & x \in \Omega,\ t>0, \\ \dfrac{\partial u(t,x)}{\partial \nu} = \dfrac{\partial v(t,x)}{\partial \nu} = 0, & x \in \partial\Omega,\ t>0, \\ u(x,t) = u_0(x,t) \geq 0, & x \in \Omega,\ t \in [-\tau_2, 0]. \\ v(x,t) = v_0(x,t) \geq 0, & x \in \Omega,\ t \in [-\tau_1, 0]. \end{cases}$$

Constant steady state: $(u_*, v_*) = \left(\dfrac{p}{\alpha+\beta}, \dfrac{p}{\alpha+\beta}\right)$.

Main result: (a) If $\alpha > \beta$, then (u_*, v_*) is globally asymptotically stable for any $\tau_1 \geq 0,\ \tau_2 \geq 0$. (proved with upper-lower solution method)
(b) If $\alpha < \beta$, then there exists $\tau_* > 0$ such that (u_*, v_*) is stable for $\tau_1 + \tau_2 < \tau_*$, and it is unstable for $\tau_1 + \tau_2 > \tau_*$.

Delayed Diffusive Leslie-Gower Predator-Prey Model

[Chen-Shi-Wei, 2012, IJBC]

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} - d_1\Delta u(t,x) = u(t,x)(p - \alpha u(t,x) - \beta v(t-\tau_1,x)), & x \in \Omega,\ t > 0, \\ \dfrac{\partial v(t,x)}{\partial t} - d_2\Delta v(t,x) = \mu v(t,x)\left(1 - \dfrac{v(t,x)}{u(t-\tau_2,x)}\right), & x \in \Omega,\ t > 0, \\ \dfrac{\partial u(t,x)}{\partial \nu} = \dfrac{\partial v(t,x)}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,t) = u_0(x,t) \geq 0, & x \in \Omega,\ t \in [-\tau_2, 0]. \\ v(x,t) = v_0(x,t) \geq 0, & x \in \Omega,\ t \in [-\tau_1, 0]. \end{cases}$$

Constant steady state: $(u_*, v_*) = \left(\dfrac{p}{\alpha+\beta}, \dfrac{p}{\alpha+\beta}\right)$.

Main result: (a) If $\alpha > \beta$, then (u_*, v_*) is globally asymptotically stable for any $\tau_1 \geq 0,\ \tau_2 \geq 0$. (proved with upper-lower solution method)
(b) If $\alpha < \beta$, then there exists $\tau_* > 0$ such that (u_*, v_*) is stable for $\tau_1 + \tau_2 < \tau_*$, and it is unstable for $\tau_1 + \tau_2 > \tau_*$.

[Du-Hsu, 2004, JDE] When $\tau_1 = \tau_2 = 0$, if $\alpha > s_0\beta$, for some $s_0 \in (1/5, 1/4)$, then (u_*, v_*) is globally asymptotically stable. (proved with Lyapunov function, and it is conjectured that the global stability holds for all $\alpha, \beta > 0$.)

Bifurcation diagram of Leslie-Gower system

There is a parameter region in which the global stability persists for all delay $\tau > 0$ (absolutely stable). The other region conditionally stability holds.

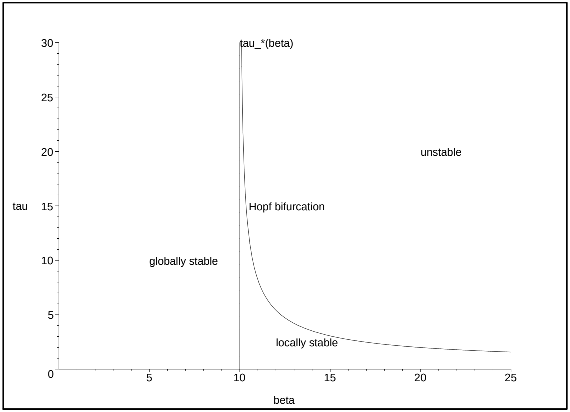


Figure : Bifurcation Diagram with parameters β and $\tau = \tau_1 + \tau_2$. Here $d_1 = 0.1,\ d_2 = 0.2,\ \alpha = 10,\ \mu = 1,\ p = 2.$

Calculation

The characteristic equation

$$\Delta_n(\lambda, \tau) = \lambda^2 + A_n\lambda + B_n + Ce^{-\lambda\tau} = 0, \quad n = 0, 1, 2, \cdots,$$

where

$$A_n = \frac{\alpha}{\alpha+\beta}p + \mu + (d_1 + d_2)\lambda_n, \quad B_n = \left(\lambda_n d_1 + \frac{\alpha}{\alpha+\beta}p\right)(\lambda_n d_2 + \mu),$$

$$C = \mu\frac{\beta}{\alpha+\beta}p, \quad \text{and} \quad \tau = \tau_1 + \tau_2.$$

If $\pm i\sigma(\sigma > 0)$ is a pair of roots of the characteristic equation, then we have

$$\begin{cases} \sigma^2 - B_n = C\cos\sigma\tau, \\ \sigma A_n = C\sin\sigma\tau, \end{cases} \quad n = 0, 1, 2, \cdots,$$

which lead to

$$\sigma^4 + (A_n^2 - 2B_n)\sigma^2 + B_n^2 - C^2 = 0, \quad n = 0, 1, 2, \cdots,$$

where

$$A_n^2 - 2B_n = \left(d_1\lambda_n + \frac{\alpha}{\alpha+\beta}p\right)^2 + (d_2\lambda_n + \mu)^2,$$

$$B_n^2 - C^2 = \left(\lambda_n d_1 + \frac{\alpha}{\alpha+\beta}p\right)^2(\lambda_n d_2 + \mu)^2 - \left(\mu\frac{\beta}{\alpha+\beta}p\right)^2.$$

Motivation of this work

Gierer-Meinhardt system with the gene expression time delays
Gierer-Meinhardt [1972], Seirin Lee et.al. [2010]

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \epsilon^2 D \dfrac{\partial^2 u(x,t)}{\partial x^2} + \gamma \left(p - qu(x,t) + \dfrac{u^2(x,t-\tau)}{v(x,t-\tau)} \right), & x \in (0,\pi),\ t > 0, \\ \dfrac{\partial v(x,t)}{\partial t} = D \dfrac{\partial^2 v(x,t)}{\partial x^2} + \gamma(u^2(x,t-\tau) - v(x,t)), & x \in (0,\pi),\ t > 0, \\ \dfrac{\partial u(0,t)}{\partial x} = \dfrac{\partial u(\pi,t)}{\partial x} = \dfrac{\partial v(0,t)}{\partial x} = \dfrac{\partial v(\pi,t)}{\partial x} = 0, & t > 0, \\ u(x,t) = \phi_1(x,t) \geq 0, v(x,t) = \phi_2(x,t) \geq 0, & x \in (0,\pi),\ t \in [-\tau, 0], \end{cases}$$

where D, ϵ, p, q, γ and τ are positive parameters.

Motivation of this work

Gierer-Meinhardt system with the gene expression time delays
Gierer-Meinhardt [1972], Seirin Lee et.al. [2010]

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \epsilon^2 D \dfrac{\partial^2 u(x,t)}{\partial x^2} + \gamma \left(p - qu(x,t) + \dfrac{u^2(x,t-\tau)}{v(x,t-\tau)} \right), & x \in (0,\pi),\ t > 0, \\ \dfrac{\partial v(x,t)}{\partial t} = D \dfrac{\partial^2 v(x,t)}{\partial x^2} + \gamma (u^2(x,t-\tau) - v(x,t)), & x \in (0,\pi),\ t > 0, \\ \dfrac{\partial u(0,t)}{\partial x} = \dfrac{\partial u(\pi,t)}{\partial x} = \dfrac{\partial v(0,t)}{\partial x} = \dfrac{\partial v(\pi,t)}{\partial x} = 0, & t > 0, \\ u(x,t) = \phi_1(x,t) \geq 0, v(x,t) = \phi_2(x,t) \geq 0, & x \in (0,\pi),\ t \in [-\tau, 0], \end{cases}$$

where D, ϵ, p, q, γ and τ are positive parameters. Or more general (for non-PDE):

$$\begin{cases} \dot{x}(t) = f(x(t), y(t), x(t-\tau), y(t-\tau)), \\ \dot{y}(t) = g(x(t), y(t), x(t-\tau), y(t-\tau)). \end{cases}$$

The corresponding characteristic equation:

$$\lambda^2 + a\lambda + b + (c\lambda + d)e^{-\lambda\tau} + he^{-2\lambda\tau} = 0. \tag{3}$$

Here $a, b, c, d, h \in \mathbb{R}$, and $\tau > 0$.

$c = d = 0$ or $h = 0$: considered previously
$a = b = 0$: Hu-Li-Yan [2009]
We consider the full case for any $a, b, c, d, h \in \mathbb{R}$, with at least one of c and d is not zero, and h is not zero.

Solving characteristic equation

Characteristic equation: $\lambda^2 + a\lambda + b + (c\lambda + d)e^{-\lambda\tau} + he^{-2\lambda\tau} = 0.$
Neutral stability: $\pm i\omega$, $(\omega > 0)$, is a pair of roots.

$$-\omega^2 + a\omega i + b + (c\omega i + d)e^{-i\omega\tau} + he^{-2i\omega\tau} = 0.$$

Solving characteristic equation

Characteristic equation: $\lambda^2 + a\lambda + b + (c\lambda + d)e^{-\lambda\tau} + he^{-2\lambda\tau} = 0$.
Neutral stability: $\pm i\omega$, $(\omega > 0)$, is a pair of roots.

$$-\omega^2 + a\omega i + b + (c\omega i + d)e^{-i\omega\tau} + he^{-2i\omega\tau} = 0.$$

If $\frac{\omega\tau}{2} \neq \frac{\pi}{2} + j\pi$, $j \in \mathbb{Z}$, then let $\theta = \tan\frac{\omega\tau}{2}$, and we have $e^{-i\omega\tau} = \frac{1-i\theta}{1+i\theta}$.
Separating the real and imaginary parts, we obtain that θ satisfies

$$\begin{cases} (\omega^2 - b + d - h)\theta^2 - 2a\omega\theta = \omega^2 - b - d - h, \\ (c\omega - a\omega)\theta^2 + (-2\omega^2 + 2b - 2h)\theta = -(a\omega + c\omega). \end{cases}$$

Define

$$D(\omega) = det\begin{pmatrix} \omega^2 - b + d - h & -2a\omega \\ (c-a)\omega & -2\omega^2 + 2b - 2h \end{pmatrix},$$

$$E(\omega) = det\begin{pmatrix} \omega^2 - b - d - h & -2a\omega \\ -(c+a)\omega & -2\omega^2 + 2b - 2h \end{pmatrix},$$

$$F(\omega) = det\begin{pmatrix} \omega^2 - b + d - h & \omega^2 - b - d - h \\ (c-a)\omega & -(c+a)\omega \end{pmatrix}.$$

Quartic equation

ω satisfies $D(\omega)E(\omega) = F(\omega)^2$, and ω^2 is a positive root of

$$z^4 + s_1 z^3 + s_2 z^2 + s_3 z + s_4 = 0,$$

where

$$\begin{aligned}
s_1 =& 2a^2 - 4b - c^2, \\
s_2 =& 6b^2 - 2h^2 - 4ba^2 - d^2 + a^4 - a^2c^2 + 2c^2b + 2hc^2, \\
s_3 =& 2d^2b - a^2d^2 - 4b^3 + 2b^2a^2 - c^2b^2 - 2bc^2h \\
& + 4acdh - 2d^2h + 4bh^2 - 2h^2a^2 - c^2h^2, \\
s_4 =& b^4 - d^2b^2 - 2b^2h^2 + 2bd^2h - d^2h^2 + h^4 = (b-h)^2[-d^2 + (b+h)^2],
\end{aligned}$$

Lemma. If $\pm i\omega$, $(\omega > 0)$, is a pair of purely imaginary roots of the characteristic equation, then ω^2 is a positive root of the above quartic polynomial equation.

Lemma. If the quartic equation has a positive root ω_N^2, $(\omega_N > 0)$, and $D(\omega_N) \neq 0$, then the equation of θ has a unique real root $\theta_N = \dfrac{F(\omega_N)}{D(\omega_N)}$ when $\omega = \omega_N$. Hence the characteristic equation has a pair of purely imaginary roots $\pm i\omega_N$ when

$$\tau = \tau_N^j = \frac{2\arctan\theta_N + 2j\pi}{\omega_N}, \quad j \in \mathbb{Z}. \tag{4}$$

Non-degeneracy, transversality, quartic eq

Non-degenerate root: $D(\omega) \neq 0$

Lemma. Suppose the quartic equation has a positive root ω^2 for some $\omega > 0$. Then $D(\omega) \neq 0$ if

(1) $c \neq 0$, $b + h \leq \dfrac{ad}{c}$;

(2) $c \neq 0$, $\dfrac{d}{c}\left(2h - \dfrac{ad}{c}\right) - a\left(b + h - \dfrac{ad}{c}\right) \neq 0$ and $a \neq c$;

(3) $c = 0$ and $a \neq 0$;

(4) $c = 0$, $a = 0$, $b + h - d \leq 0$, and $b - h \leq 0$.

Non-degeneracy, transversality, quartic eq

Non-degenerate root: $D(\omega) \neq 0$

Lemma. Suppose the quartic equation has a positive root ω^2 for some $\omega > 0$. Then $D(\omega) \neq 0$ if

(1) $c \neq 0,\ b + h \leq \dfrac{ad}{c}$;

(2) $c \neq 0,\ \dfrac{d}{c}\left(2h - \dfrac{ad}{c}\right) - a\left(b + h - \dfrac{ad}{c}\right) \neq 0$ and $a \neq c$;

(3) $c = 0$ and $a \neq 0$;

(4) $c = 0,\ a = 0,\ b + h - d \leq 0$, and $b - h \leq 0$.(degenerate case $D(\omega) = 0$ can also be analyzed similarly.)

Transversality: the pair of complex eigenvalue moving across the imaginary axis.

Lemma. Suppose that (ω^2, θ) are solved from the procedure. Define

$$\begin{aligned}\mathcal{G}(\omega,\theta) =& [d(1+\theta^2) + 2h(1-\theta^2)] \cdot [2\omega(1-\theta^2) + 2a\theta] \\ & - [c\omega(1+\theta^2) - 4h\theta] \cdot [a(1-\theta^2) - 4\omega\theta + c(1+\theta^2)].\end{aligned}$$

If $\mathcal{G}(\omega,\theta) \neq 0$, then $i\omega$ is a simple root of the characteristic equation for $\tau = \tau^j$ and there exists $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ which is the unique root of the characteristic equation for $\tau \in (\tau^j - \epsilon, \tau^j + \epsilon)$ for some small $\epsilon > 0$ satisfying $\alpha(\tau^j) = 0$ and $\omega(\tau^j) = \omega$.

Solving quartic equation: standard procedure and conditions guaranteeing existence of a positive root.

Summary of one of routes

Theorem. Suppose that $a, b, c, d, h \in \mathbb{R}$ satisfy

(i) $c \neq 0$ and $h \neq 0$;

(ii) $b \neq h$ and $d^2 > (b+h)^2$;

(iii) $b + h \leq \dfrac{ad}{c}$ or $\left(\dfrac{d}{c}\left(2h - \dfrac{ad}{c}\right) - a\left(b + h - \dfrac{ad}{c}\right)\right) \cdot (a - c) \neq 0$.

Then

1. The quartic equation has a root ω_N^2 for $\omega_N > 0$ with $D(\omega_N) \neq 0$.
2. Let $\theta_N = \dfrac{F(\omega_N)}{D(\omega_N)}$, and $\tau = \tau_N^j = \dfrac{2\arctan\theta_N + 2j\pi}{\omega_N}$, where $j \in \mathbb{Z}$. Then the characteristic equation has a pair of roots $\pm i\omega_N$ when $\tau = \tau_N^j$.
3. If $\mathcal{G}(\omega_N, \theta_N) \neq 0$, then $i\omega_N$ is a simple root of the characteristic equation for $\tau = \tau_N^j$ and there exists $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ which is the unique root for τ near τ_N^j satisfying $\alpha(\tau_N^j) = 0$, $\omega(\tau_N^j) = \omega_N$ and $\alpha'(\tau_N^j) \neq 0$.

Moreover if $a, b, c, d, h \in \mathbb{R}$ also satisfy

(iv) $a + c > 0$ and $b + d + h > 0$,

then there exists $\tau_* > 0$ such that when $\tau \in [0, \tau_*)$, all roots have negative real parts; if $\mathcal{G}(\theta_*, \omega_*) \neq 0$, then when $\tau = \tau_*$, it has one pair of simple purely imaginary roots, and for $\tau \in (\tau_*, \tau_* + \epsilon)$, it has one pair of complex roots with positive real parts.

Corollary: Define a subset in the parameter space

$$P = \{(a, b, c, d, h) \in \mathbb{R}^5 : a + c > 0,\ b + d + h > 0,\ b - d + h < 0\}.$$

Then the above results occur for almost any $(a, b, c, d, h) \in P$.

Setup

$$\begin{cases} \dfrac{\partial u}{\partial t} = D_1 \Delta u + f(u, v, u_\tau, v_\tau), & x \in \Omega,\ t > 0, \\ \dfrac{\partial v}{\partial t} = D_2 \Delta v + g(u, v, u_\tau, v_\tau), & x \in \Omega,\ t > 0, \\ \dfrac{\partial u(x,t)}{\partial \nu} = \dfrac{\partial v(x,t)}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,t) = \phi_1(x,t) \geq 0, v(x,t) = \phi_2(x,t) \geq 0, & x \in \Omega,\ t \in [-\tau, 0], \end{cases} \tag{5}$$

- $u = u(x,t)$,$v = v(x,t)$, $u_\tau = u(x, t-\tau)$, and $v_\tau = v(x, t-\tau)$;
- Ω is a bounded connected domain in $\mathbb{R}^n$ ($n \geq 1$) with smooth boundary $\partial\Omega$;
- Δ is the Laplace operator in $\mathbb{R}^n$, and $\partial w/\partial \nu$ is the outer normal derivative of $w = u, v$;
- The functions $f(u,v,w,z)$ and $g(u,v,w,z)$ are continuously differentiable in $\mathbb{R}^4$.
- There exist $u^* > 0$ and $v^* > 0$ such that

$$f(u^*, v^*, u^*, v^*) = 0, \quad g(u^*, v^*, u^*, v^*) = 0.$$

Then (u^*, v^*) is a constant positive equilibrium.

Linearization

$$\frac{d}{dt}\begin{pmatrix} \phi(t) \\ \psi(t) \end{pmatrix} = D\begin{pmatrix} \Delta\phi(t) \\ \Delta\psi(t) \end{pmatrix} + L_1\begin{pmatrix} \phi(t) \\ \psi(t) \end{pmatrix} + L_2\begin{pmatrix} \phi(t-\tau) \\ \psi(t-\tau) \end{pmatrix},$$

where

$$D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}, \quad L_1 = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}, \quad L_2 = \begin{pmatrix} f_w & f_z \\ g_w & g_z \end{pmatrix},$$

Let the eigenvalues of $-\Delta$ with Neumann boundary condition be μ_n, and the corresponding eigenfunction are $\gamma_n(x)$, $n \in \mathbb{N}_0 = \mathbb{N}\bigcup\{0\}$. Then for a fixed $n \in \mathbb{N}_0$, the characteristic equation at (u_*, v_*) is

$$det\begin{pmatrix} \lambda + D_1\mu_n - f_u - f_w e^{-\lambda\tau} & -f_v - f_z e^{-\lambda\tau} \\ -g_u - g_w e^{-\lambda\tau} & \lambda + D_2\mu_n - g_v - g_z e^{-\lambda\tau} \end{pmatrix} = 0,$$

or

$$K(\lambda,\tau,n) \equiv \lambda^2 + a_n\lambda + b_n + (c_n\lambda + d_n)e^{-\lambda\tau} + h_n e^{-2\lambda\tau} = 0, \quad n \in \mathbb{N}_0,$$

where

$$\begin{aligned} a_n =& (D_1 + D_2)\mu_n - (f_u + g_v), \\ b_n =& D_1 D_2 \mu_n^2 - (D_1 g_v + D_2 f_u)\mu_n + f_u g_v - f_v g_u, \\ c_n =& -(f_w + g_z), \\ d_n =& -(D_1 g_z + D_2 f_w)\mu_n + (f_u g_z - f_z g_u) + (f_w g_v - f_v g_w), \\ h_n =& f_w g_z - f_z g_w. \end{aligned}$$

Hierarchy of stability

$$K(\lambda,\tau,n) \equiv \lambda^2 + a_n\lambda + b_n + (c_n\lambda + d_n)e^{-\lambda\tau} + h_n e^{-2\lambda\tau} = 0.$$

Define the spectrum set for a fixed $\tau \in \overline{\mathbb{R}^+}$ and $n \in \mathbb{N}_0$ by

$$S_{\tau,n} = \{\lambda \in \mathbb{C} : K(\lambda,\tau,n) = 0\},$$

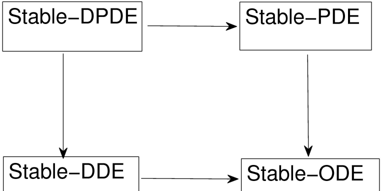
and for a fixed $\tau \in \overline{\mathbb{R}^+}$,

$$S_\tau = \bigcup_{n\in\mathbb{N}_0} S_{\tau,n}.$$

Define $\mathbb{C}^- \equiv \{x + iy : x, y \in \mathbb{R}, x < 0\}$.

- stable w.r.t. ODE (Ordinary Differential Equation) if $S_{0,0} \subseteq \mathbb{C}^-$;
- stable w.r.t. DDE (Delay Differential Equation) for $\tau > 0$ if $S_{\tau,0} \subseteq \mathbb{C}^-$;
- stable w.r.t. PDE (Partial Differential Equation) if $S_0 \subseteq \mathbb{C}^-$;
- stable w.r.t. DPDE (Delay Partial Differential Equation) for $\tau > 0$ if $S_\tau \subseteq \mathbb{C}^-$.

Turing instability: stable w.r.t. ODE but unstable for PDE



Instability w.r.t. DDE

$$\begin{cases} u_t = f(u, v, u_\tau, v_\tau), & t > 0, \\ v_t = g(u, v, u_\tau, v_\tau), & t > 0, \\ u(t) = \phi_1(t) \geq 0, v(t) = \phi_2(t) \geq 0, & t \in [-\tau, 0], \end{cases} \tag{6}$$

where $u = u(t)$,$v = v(t)$, $u_\tau = u(t - \tau)$, and $v_\tau = v(t - \tau)$.

$$K(\lambda, \tau, 0) \equiv \lambda^2 + a_0\lambda + b_0 + (c_0\lambda + d_0)e^{-\lambda\tau} + h_0 e^{-2\lambda\tau} = 0, \quad n \in \mathbb{N}_0,$$

where

$$\begin{aligned} a_0 &= -Tr(L_1), \quad b_0 = Det(L_1), \quad c_0 = -Tr(L_2), \\ d_0 &= \frac{1}{2}\left[Det(L_1 + L_2) - Det(L_1 - L_2)\right], \quad h_0 = Det(L_2). \end{aligned}$$

Recall that

$$L_1 = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}, \quad L_2 = \begin{pmatrix} f_w & f_z \\ g_w & g_z \end{pmatrix},$$

Instability w.r.t. DDE

Theorem. Assume that

$$Tr(L_2) \neq 0, \quad Tr(L_2) \neq Tr(L_1), \quad Det(L_2) \neq 0, \quad Det(L_2) \neq Det(L_1),$$

and

$$b_0 + h_0 \leq \frac{a_0 d_0}{c_0} \quad \text{or} \quad \frac{d_0}{c_0}\left(2h_0 - \frac{a_0 d_0}{c_0}\right) - a_0\left(b_0 + h_0 - \frac{a_0 d_0}{c_0}\right) \neq 0.$$

If L_1 and L_2 satisfy

$$Tr(L_1 + L_2) < 0, \quad Det(L_1 + L_2) > 0, \quad \text{and} \quad Det(L_1 - L_2) < 0, \tag{7}$$

then there exists $\tau_0 > 0$, the equilibrium (u^*, v^*) is stable for the DDE when $0 \leq \tau < \tau_0$, but it is unstable when $\tau \in (\tau_0, \tau_0 + \epsilon)$. Moreover, a Hopf bifurcation for (6) occurs at $\tau = \tau_0$.

The conditions in (7) lead to delay-induced instability. The matrix L_1 is the "non-delay" part, and L_2 is the "delay" part of the linearization.

Diffusive case

Theorem. Assume that D_1, D_2 are the diffusion coefficients, and μ_n is a simple eigenvalue of $-\Delta$ with Neumann boundary condition for $n \in \mathbb{N}_0$. Assume that

$$Tr(L_2) \neq 0, \quad Tr(L_2) \neq Tr(L_1) - (D_1 + D_2)\mu_n,$$
$$Det(L_2) \neq 0, \quad Det(L_2) \neq Det(L_1) - (D_1 g_v + D_2 f_u)\mu_n + D_1 D_2 \mu_n^2,$$

and

$$b_n + h_n \leq \frac{a_n d_n}{c_n} \quad \text{or} \quad \frac{d_n}{c_n}\left(2h_n - \frac{a_n d_n}{c_n}\right) - a_n\left(b_n + h_n - \frac{a_n d_n}{c_n}\right) \neq 0.$$

If (D_1, D_2), L_1 and L_2 satisfy

$$Tr(L_1 + L_2) < (D_1 + D_2)\mu_n,$$
$$Det(L_1 + L_2) > [D_1(g_v + g_z) + D_2(f_u + f_w)]\mu_n - D_1 D_2 \mu_n^2,$$
$$\text{and} \quad Det(L_1 - L_2) < [D_1(g_v - g_z) + D_2(f_u - f_w)]\mu_n - D_1 D_2 \mu_n^2,$$

then there exists $\tau_n > 0$, the equilibrium (u^*, v^*) is stable in mode-n when $0 \leq \tau < \tau_n$, but it is unstable in mode-n when $\tau \in (\tau_n, \tau_n + \epsilon)$. Moreover, a Hopf bifurcation occurs at $\tau = \tau_n$, and the bifurcating periodic orbits have the spatial profile of $\gamma_n(x)$.

Delayed Gierer-Meinhardt system

Gierer-Meinhardt system with gene expression time delays

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \epsilon^2 D \dfrac{\partial^2 u(x,t)}{\partial x^2} + \gamma \left(p - qu(x,t) + \dfrac{u^2(x,t-\tau)}{v(x,t-\tau)} \right), & x \in (0,\pi),\ t > 0, \\ \dfrac{\partial v(x,t)}{\partial t} = D \dfrac{\partial^2 v(x,t)}{\partial x^2} + \gamma (u^2(x,t-\tau) - v(x,t)), & x \in (0,\pi),\ t > 0, \\ \dfrac{\partial u(0,t)}{\partial x} = \dfrac{\partial u(\pi,t)}{\partial x} = \dfrac{\partial v(0,t)}{\partial x} = \dfrac{\partial v(\pi,t)}{\partial x} = 0, & t > 0, \\ u(x,t) = \phi_1(x,t) \geq 0, v(x,t) = \phi_2(x,t) \geq 0, & x \in (0,\pi),\ t \in [-\tau, 0], \end{cases}$$

Unique positive equilibrium $(u^*, v^*) = \left(\dfrac{p+1}{q}, \left(\dfrac{p+1}{q} \right)^2 \right)$.

Delayed Gierer-Meinhardt system

Gierer-Meinhardt system with gene expression time delays

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \epsilon^2 D \dfrac{\partial^2 u(x,t)}{\partial x^2} + \gamma\left(p - qu(x,t) + \dfrac{u^2(x,t-\tau)}{v(x,t-\tau)}\right), & x \in (0,\pi),\ t > 0, \\ \dfrac{\partial v(x,t)}{\partial t} = D \dfrac{\partial^2 v(x,t)}{\partial x^2} + \gamma(u^2(x,t-\tau) - v(x,t)), & x \in (0,\pi),\ t > 0, \\ \dfrac{\partial u(0,t)}{\partial x} = \dfrac{\partial u(\pi,t)}{\partial x} = \dfrac{\partial v(0,t)}{\partial x} = \dfrac{\partial v(\pi,t)}{\partial x} = 0, & t > 0, \\ u(x,t) = \phi_1(x,t) \geq 0, v(x,t) = \phi_2(x,t) \geq 0, & x \in (0,\pi),\ t \in [-\tau, 0], \end{cases}$$

Unique positive equilibrium $(u^*, v^*) = \left(\dfrac{p+1}{q}, \left(\dfrac{p+1}{q}\right)^2\right)$.

For DDE: $\dfrac{du}{dt} = \gamma\left(p - qu(t) + \dfrac{u^2(t-\tau)}{v(t-\tau)}\right)$ $\dfrac{dv}{dt} = \gamma(u^2(t-\tau) - v(t))$:
Characteristic equation: $\lambda^2 + \gamma(q+1)\lambda + \gamma^2 q + \frac{2q\gamma}{p+1}[-(\lambda+\gamma)e^{-\lambda\tau} + \gamma e^{-2\lambda\tau}] = 0$.

- If $p > \dfrac{q-1}{q+1}$, then (u^*, v^*) is local asymptotically stable for ODE.
- There exist $p_0(q) > 0$ such that for any $p > p_0(q)$, (u^*, v^*) is locally asymptotically stable for DDE with any $\tau \geq 0$.

Hence delay-induced instability can occur for $\dfrac{q-1}{q+1} < p < p_0(q)$.

Numerical example

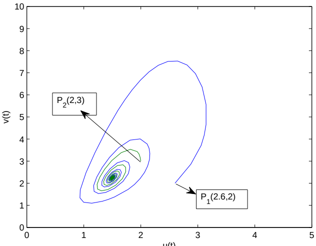
Assume that $p = 0.2$, $q = 0.8$, and $\gamma = 1$. Then $(u^*, v^*) = (1.5, 2.25)$.

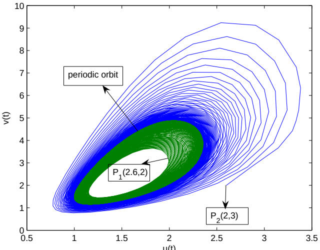
$$\begin{aligned} &\lambda^2 + 1.8\lambda + 0.8 + (-1.3333\lambda - 1.3333)e^{-\lambda\tau} + 1.3333e^{-2\lambda\tau} = 0, \\ &z^4 + 1.5022z^3 + 0.4615z^2 - 2.4124z + 0.7886 = 0. \end{aligned} \tag{8}$$

Positive roots of the quartic equation: $\omega_1^2 \approx 0.4194$ and $\omega_2^2 \approx 0.6441$
$\omega_1 \approx 0.6476$ and $\omega_2 \approx 0.8026$.

Two sequences of Hopf bifurcation points: $\tau = \tau_1^j \approx 0.4243 + \dfrac{2j\pi}{0.6476}$,
$\tau = \tau_2^j \approx 5.7817 + \dfrac{2j\pi}{0.8026}$, $j = 0, 1, 2, \cdots$.

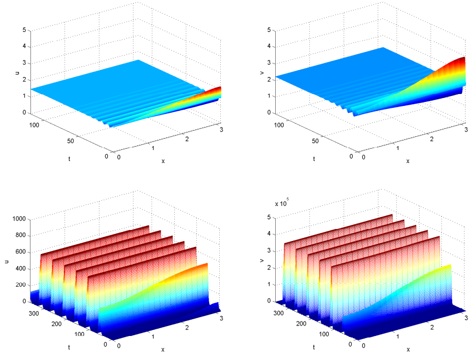
Phase portraits: Left: $\tau = 0.2$; Right: $\tau = 0.43$





Simulation with diffusion

$p = 0.2,\ q = 0.8$, and $\gamma = 1,\ \epsilon^2 = 0.1,\ D = 0.3$
Upper: $\tau = 0.2$, Lower: $\tau = 6$ (left $u(x,t)$, right $v(x,t)$)



Conclusion

- The stability of an equilibrium in a delayed system is usually difficult to determine if there is more than one transcendental terms in the characteristic equation. A systematic approach to solve the purely imaginary roots of a second order transcendental polynomial is provided here to consider the stability of a (constant) equilibrium in a (reaction-diffusion) planar system with a simultaneous delay.

Conclusion

- The stability of an equilibrium in a delayed system is usually difficult to determine if there is more than one transcendental terms in the characteristic equation. A systematic approach to solve the purely imaginary roots of a second order transcendental polynomial is provided here to consider the stability of a (constant) equilibrium in a (reaction-diffusion) planar system with a simultaneous delay.
- Our approach is easy to be applied to a specific model from application as the coefficients in the transcendental polynomial depend only on the linearization of the system, and a complete set of conditions on the coefficients leading to instability are proved. Such conditions are easy to verify and numerical algorithms of finding bifurcation values are given so the sequence of Hopf bifurcation points can be explicitly calculated.

Conclusion

- The stability of an equilibrium in a delayed system is usually difficult to determine if there is more than one transcendental terms in the characteristic equation. A systematic approach to solve the purely imaginary roots of a second order transcendental polynomial is provided here to consider the stability of a (constant) equilibrium in a (reaction-diffusion) planar system with a simultaneous delay.
- Our approach is easy to be applied to a specific model from application as the coefficients in the transcendental polynomial depend only on the linearization of the system, and a complete set of conditions on the coefficients leading to instability are proved. Such conditions are easy to verify and numerical algorithms of finding bifurcation values are given so the sequence of Hopf bifurcation points can be explicitly calculated.

Future work

- Our work here is still a special case of the characteristic equation with two delays (in our case, the two delays are τ and 2τ), and a complete analysis for the case of two arbitrary delays is still out of reach.
- Our general analysis for delayed reaction-diffusion systems shows that the equilibrium loses its stability at a lowest delay value $\tau_* > 0$. In all our examples, τ_* is identical to τ_0, where spatially homogeneous periodic orbits bifurcate from the equilibrium. The possibility of equilibrium first loses stability to spatially nonhomogeneous periodic orbits remains an open problem.
- Global bifurcation such as stability switches and higher co-dimensional bifurcations such as double Hopf bifurcation or Turing-Hopf bifurcation.

References

- Shanshan Chen, Junping Shi, *Stability and Hopf Bifurcation in a Diffusive Logistic Population Model with Nonlocal Delay Effect*. **Journal of Differential Equations**, 253 (2012), 3440–3470.
- Shanshan Chen, Junping Shi, Junjie Wei, *Time delay induced instabilities and Hopf bifurcations in general reaction-diffusion systems*. **Journal of Nonlinear Science**. 23 (2013), 1–38.
- Shanshan Chen, Junping Shi, Junjie Wei, *Global stability and Hopf bifurcation in a delayed diffusive Leslie-Gower predator-prey system*. **International Journal of Bifurcation and Chaos**, 22, (2012), 1250061.
- Shanshan Chen, Junping Shi, Junjie Wei, *The effect of delay on a diffusive predator-prey system with Holling type-II predator functional response*. **Communications on Pure and Applied Analysis**, 12, (2013), 481–501.
- Junping Shi, *Absolute Stability and Conditional Stability in General Delayed Differential Equations*. (15 pages) To appear in **Mathematical & Statistical Research With Applications to Physical & Life Sciences, Engineering & Technology**, Springer Proceedings of Mathematics and Statistics.