Reaction-Diffusion Models and Bifurcation Theory
Lecture 9: Hopf bifurcation

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Stability of a Stationary Solution

For a continuous-time evolution equation $\frac{du}{dt} = F(\lambda, u)$, where $u \in X$ (state space), $\lambda \in \mathbb{R}$, a stationary solution u_* is locally asymptotically stable (or just stable) if for any $\epsilon > 0$, then there exists $\delta > 0$ such that when $||u(0) - u_*||_X < \delta$, then $||u(t) - u_*||_X < \epsilon$ for all $t > 0$ and $\lim_{t\to\infty} ||u(t) - u_*||_X = 0$. Otherwise u_* is unstable.

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Bifurcation: when the parameter λ changes from $\lambda_* - \varepsilon$ to $\lambda_* + \varepsilon$, the stationary solution $u_*(\lambda)$ changes from stable to unstable; and other special solutions (stationary solutions, periodic orbits) may emerge from the known solution $(\lambda, u_*(\lambda))$.

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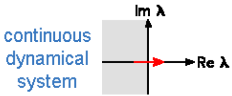
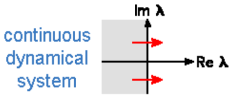
Stationary Bifurcation (transcritical/pitchfolk): if 0 is an eigenvalue of $D_u F(\lambda_*, u_*)$. It generates new stationary (steady state, equilibrium) solutions.
Hopf Bifurcation: if $\pm ki$ ($k > 0$) is a pair of eigenvalues of $D_u F(\lambda_*, u_*)$. It generates new small amplitude periodic orbits.

stationary bifurcation



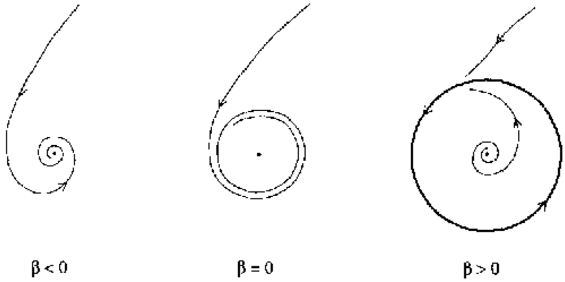
Hopf bifurcation



Poincaré-Andronov-Hopf Bifurcation Theorem

Consider ODE $x' = f(\lambda, x)$, $\lambda \in \mathbb{R}$, $x \in \mathbb{R}^n$, and f is smooth.
(i) Suppose that for λ near λ_0 the system has a family of equilibria $x^0(\lambda)$.
(ii) Assume that its Jacobian matrix $A(\lambda) = f_x(\lambda, x^0(\lambda))$ has one pair of complex eigenvalues $\mu(\lambda) \pm i\omega(\lambda)$, $\mu(\lambda_0) = 0$, $\omega(\lambda_0) > 0$, and all other eigenvalues of $A(\lambda)$ have non-zero real parts for all λ near λ_0.

If $\mu'(\lambda_0) \neq 0$, then the system has a family of periodic solutions $(\lambda(s), x(s))$ for $s \in (0, \delta)$ with period $T(s)$, such that $\lambda(s) \to \lambda_0$, $T(s) \to 2\pi/\omega(\lambda_0)$, and $||x(s) - x^0(\lambda_0)|| \to 0$ as $s \to 0^+$.



Poincaré-Andronov-Hopf bifurcation

Henri Poincaré (1852-1912) Aleksandr Andronov (1901-1952)
Eberhard Hopf (1902-1983)

Andronov, A. A. [1929] "Les cycles limites de Poincaré et la théorie des oscillations auto-entretenues," Comptes Rendus Hebdomadaires de l'Acad'emie des Sciences 189, 559-561. limit cycle in 2-D systems

E. Hopf. [1942] "Abzweigung einer periodischen Lošung von einer stationařen eines Differentialsystems". Ber. Verh. Sačhs. Akad. Wiss. Leipzig. Math.-Nat. Kl. 95, (1943). no. 1, 3-22. limit cycle in *n*-D system

Poincaré, H. [1894] "Les Oscillations 'Electriques" (Charles Maurain, G. Carr'e & C. Naud, Paris).

Proof of Hopf bifurcation theorem: (1) transformation

Consider ODE $x' = f(\lambda, x)$, $\lambda \in \mathbb{R}$, $x \in \mathbb{R}^n$, and f is smooth.
Assumptions:
(i) Suppose that for λ near λ_0 the system has a family of equilibria $x^0(\lambda)$.
(ii) Assume that its Jacobian matrix $A(\lambda) = f_x(\lambda, x^0(\lambda))$ has one pair of complex eigenvalues $\mu(\lambda) \pm i\omega(\lambda)$, $\mu(\lambda_0) = 0$, $\omega(\lambda_0) = \omega_0 > 0$, and all other eigenvalues of $A(\lambda)$ have non-zero real parts for all λ near λ_0.
(iii) $\mu'(\lambda_0) \neq 0$.

Preparation:
1. We can assume $x^0(\lambda) = 0$ (if not we can make a change of variables: $y = x - x^0(\lambda)$), so from now we assume that $f(\lambda, 0) = 0$ for λ near λ_0, and $A(\lambda) = f_x(\lambda, 0)$.
2. A periodic solution $x(t)$ satisfying $x(t + \rho) = x(t)$ for a period ρ. We rescale the time $s = t/\rho$. Then the equation $\dfrac{dx}{dt} = f(\lambda, x)$ becomes $\dfrac{dx}{ds} = \rho f(\lambda, x)$, and now $x(s)$ satisfies $x(s) = x(s + 1)$ for a period 1. From now we consider the equation $x' = \rho f(\lambda, x)$, and we look for periodic solutions with period 1.

Proof of Hopf bifurcation theorem: (2) Setup

Consider ODE $x' = \rho f(\lambda, x)$, $\lambda \in \mathbb{R}$, $x \in \mathbb{R}^n$,
Assumptions:
(i) Suppose that for λ near λ_0, $f(\lambda, 0) = 0$.
(ii) Assume that its Jacobian matrix $A(\lambda) = f_x(\lambda, 0)$ has one pair of complex eigenvalues $\mu(\lambda) \pm i\omega(\lambda)$, $\mu(\lambda_0) = 0$, $\omega(\lambda_0) = \omega_0 > 0$, and all other eigenvalues of $A(\lambda)$ have non-zero real parts for all λ near λ_0.
(iii) $\mu'(\lambda_0) \neq 0$.

Define the spaces

$$X = \{x \in C^1(\mathbb{R} : \mathbb{R}^n) : x(t+1) = x(t)\}, \quad Y = \{y \in C(\mathbb{R}, \mathbb{R}^n) : y(t+1) = y(t)\}.$$

and a mapping $F : U \times V \times X \to Y$, where $\lambda_0 \in U \subset \mathbb{R}$, $\rho_0 = 2\pi/\omega_0 \in V \subset \mathbb{R}$,

$$F(\lambda, \rho, x) = x' - \rho f(\lambda, x).$$

Since the eigenvalues are complex, hence we may consider the linearized equations in

$$X_{\mathbb{C}} = X + iX = \{x_1 + ix_2 : x_1, x_2 \in X\}, \quad Y_{\mathbb{C}} = Y + iY.$$

Proof of Hopf bifurcation theorem: (3) Linearization

Consider $F : U \times V \times X \to Y$, where $\lambda_0 \in U \subset \mathbb{R}$, $\rho_0 = 2\pi/\omega_0 \in V \subset \mathbb{R}$,

$$F(\lambda, \rho, x) = x' - \rho f(\lambda, x).$$

Then

$$F_x(\lambda, \rho, x)[w] = w' - \rho f_x(\lambda, x)w, \quad F_x(\lambda_0, \rho_0, 0)[w] = w' - \frac{2\pi}{\omega_0} f_x(\lambda_0, 0)w.$$

Kernel is two-dimensional:

$$\mathcal{N}(F_x(\lambda_0, \rho_0, 0)) = span\left\{\exp(2\pi it)v_0, \exp(-2\pi it)\overline{v_0}\right\},$$

where $f_x(\lambda_0, 0)v_0 = i\omega_0 v_0$ and $v_0(\neq 0) \in X_{\mathbb{C}}$.

Range is codimensional two:

$$\mathcal{R}(F_x(\lambda_0, \rho_0, 0)) = \left\{h \in Y_{\mathbb{C}} : h\exp(2\pi it)v_0 = 0, \ h\exp(-2\pi it)\overline{v_0} = 0\right\},$$

or more precisely $h = \sum_{k \in \mathbb{Z}} h_k \exp(2k\pi it)$ (Fourier series), $h_{-k} = \overline{h_k}$, $h_1 \cdot v_0 = 0$.

Proof of Hopf bifurcation theorem: (4) New spaces

For

$$X = \{x \in C^1(\mathbb{R} : \mathbb{R}^n) : x(t+1) = x(t)\}, \quad Y = \{y \in C(\mathbb{R}, \mathbb{R}^n) : y(t+1) = y(t)\},$$

there are the space decompositions:

$$X = \mathcal{N}(F_x(\lambda_0, \rho_0, 0)) + Z, \quad Y = \mathcal{R}(F_x(\lambda_0, \rho_0, 0)) + W,$$

where Z and W are complements of $\mathcal{N}(F_x(\lambda_0, \rho_0, 0))$ and $\mathcal{R}(F_x(\lambda_0, \rho_0, 0))$ respectively.

Let $w_0 = \dfrac{\exp(2\pi it)v_0 + \exp(-2\pi it)\overline{v_0}}{2} = \cos(2\pi t)u_0$ $(u_0 \in \mathbb{R}^n)$, and let $X_1 = span\{w_0\} + Z$.

We restrict $F(\lambda, \rho, x) = x' - \rho f(\lambda, x)$ for $x \in X_1$. Then $\mathcal{N}(F_x(\lambda_0, \rho_0, 0)) = span\{w_0\}$.

Define $Y_1 = \{y \in Y : \sum_{k\neq 1} y_k \exp(2k\pi it) + y_1 \cos(2\pi t)\}$. Then $F : U \times V \times X_1 \to Y_1$ satisfies $codim(\mathcal{R}(F_x(\lambda_0, \rho_0, 0))) = 1$. Indeed $\mathcal{R}(F_x(\lambda_0, \rho_0, 0)) = \{y \in Y_1 : y_1 \cdot x_0 = 0\}$.

Bifurcation from simple eigenvalue with two parameters

Theorem 7.6. [Crandall-Rabinowitz, 1971, JFA]
Let U be a neighborhood of (λ_0, u_0) in $\mathbb{R} \times X$, and let $F : U \to Y$ be a continuously differentiable mapping such that $F_{\lambda u}$ exists and continuous in U. Assume that $F(\lambda, u_0) = 0$ for $(\lambda, u_0) \in U$. At (λ_0, u_0), F satisfies

(F1) $dim\mathcal{N}(F_u(\lambda_0, u_0)) = codim\mathcal{R}(F_u(\lambda_0, u_0)) = 1$, and

(F3) $F_{\lambda u}(\lambda_0, u_0)[w_0] \notin \mathcal{R}(F_u(\lambda_0, u_0))$, where $w_0 \in \mathcal{N}(F_u(\lambda_0, u_0))$,

Let Z be any complement of $\mathcal{N}(F_u(\lambda_0, u_0)) = span\{w_0\}$ in X. Then the solution set of $F(\lambda, u) = 0$ near (λ_0, u_0) consists precisely of the curves $u = u_0$ and $\{(\lambda(s), u(s)) : s \in I = (-\epsilon, \epsilon)\}$, where $\lambda : I \to \mathbb{R}$, $z : I \to Z$ are continuous functions such that $u(s) = u_0 + sw_0 + sz(s)$, $\lambda(0) = \lambda_0$, $z(0) = 0$.

two-parameter case. [Shearer, 1978, MPCPS] Let U be a neighborhood of (λ_0, ρ_0, u_0) in $\mathbb{R} \times \mathbb{R} \times X$, and let $F : U \to Y$ be a continuously differentiable mapping such that $F_{\lambda u}$ and $F_{\rho u}$ exist and continuous in U. Assume that $F(\lambda, \rho, u_0) = 0$ for $(\lambda, \rho, u_0) \in U$. At (λ_0, ρ_0, u_0), F satisfies

(F1) $dim\mathcal{N}(F_u(\lambda_0, \rho_0, u_0)) = codim\mathcal{R}(F_u(\lambda_0, \rho_0, u_0)) = 1$, and

(F3) there exists $(a_1, a_2) \in \mathbb{R}^2$ such that

$a_1 F_{\lambda u}(\lambda_0, u_0)[w_0] + a_2 F_{\rho u}(\lambda_0, \rho_0, u_0)[w_0] \notin \mathcal{R}(F_u(\lambda_0, u_0))$, where $w_0 \in \mathcal{N}(F_u(\lambda_0, u_0))$,
Let Z be any complement of $\mathcal{N}(F_u(\lambda_0, u_0)) = span\{w_0\}$ in X. Then the solution set of $F(\lambda, \rho, u) = 0$ near (λ_0, ρ_0, u_0) consists precisely of the set $u = u_0$ and a curve $\{(\lambda(s), \rho(s), u(s)) : s \in I = (-\epsilon, \epsilon)\}$, where $\lambda, \rho : I \to \mathbb{R}$, $z : I \to Z$ are continuous functions such that $u(s) = u_0 + sw_0 + sz(s)$, $\lambda(0) = \lambda_0$, $\rho(0) = \rho_0$, $z(0) = 0$.

Proof of Hopf bifurcation theorem: (5)

For the mapping $F : U \times V \times X_1 \to Y_1$, $F(\lambda, \rho, x) = x' - \rho f(\lambda, x)$, **(F1)** is satisfied.

$F_{\rho u}(\lambda_0, \rho_0, 0)[w_0] = -f_x(\lambda_0, 0)w_0 = 0$,
$F_{\lambda u}(\lambda_0, \rho_0, 0)[w_0] = -\rho_0 f_{\lambda x}(\lambda_0, 0)w_0$

Let $f_x(\lambda, 0)[w(\lambda)] = (\alpha(\lambda) + i\beta(\lambda))w(\lambda)$. By differentiating with respect to λ, we get
$f_{\lambda x}(\lambda_0, 0)[\exp(2\pi it)v_0] =$
$(\alpha'(\lambda_0) + i\beta'(\lambda_0))\exp(2\pi it)v_0 - [f_x(\lambda_0, 0)w'(\lambda_0) - (\alpha(\lambda_0) + i\beta(\lambda_0))w'(\lambda_0)]$. Then
$f_{\lambda x}(\lambda_0, 0)w_0 = \alpha'(\lambda_0)w_0 + z$ for some $z \in \mathcal{R}(F_u(\lambda_0, \rho_0, 0))$, hence
$f_{\lambda x}(\lambda_0, 0)w_0 \notin \mathcal{R}(F_u(\lambda_0, \rho_0, 0))$ since $\alpha'(\lambda_0) \neq 0$.

From the bifurcation from simple eigenvalue with two-parameter theorem, all nontrivial solutions of $F(\lambda, \rho, x) = 0$ are on a curve $\{(\lambda(s), \rho(s), x(s)) : |s| < \delta\}$.

In this way, we prove the periodic solutions in X_1 are all on the curve $\{(\lambda(s), \rho(s), x(s)) : |s| < \delta\}$. Note that with different choice of X_1 and Y_1, different periodic solutions can be obtained, but they are only the same as the ones in X_1 after a time phase shift.

Dynamical system approach

Consider ODE $x' = \rho f(\lambda, x)$, $\lambda \in \mathbb{R}$, $x \in \mathbb{R}^n$,
Assumptions:
(i) Suppose that for λ near λ_0, $f(\lambda, 0) = 0$.
(ii) Assume that its Jacobian matrix $A(\lambda) = f_x(\lambda, 0)$ has one pair of complex eigenvalues $\mu(\lambda) \pm i\omega(\lambda)$, $\mu(\lambda_0) = 0$, $\omega(\lambda_0) = \omega_0 > 0$, and all other eigenvalues of $A(\lambda)$ have non-zero real parts for all λ near λ_0.
(iii) $\mu'(\lambda_0) \neq 0$.

More non-degeneracy condition: $l_1(0) \neq 0$ (where $l_1(\alpha)$ is the first Lyapunov coefficient), then according to the Center Manifold Theorem, there is a family of smooth two-dimensional invariant manifolds W_c^α near the origin. The n-dimensional system restricted on W_c^α is two-dimensional.

Moreover, under the non-degeneracy conditions, the n-dimensional system is locally topologically equivalent near the origin to the suspension of the normal form by the standard saddle, i.e.
$\dot{y}_1 = \beta y_1 - y_2 + \sigma y_1(y_1^2 + y_2^2)$, $\dot{y}_2 = y_1 + \beta y_2 + \sigma y_2(y_1^2 + y_2^2)$, (center manifold)
$\dot{y}^s = -y^s$, (stable manifold), $\dot{y}^u = +y^u$ (unstable manifold)

Whether Andronov-Hopf bifurcation is subcritical or supercritical is determined by σ, which is the sign of the "first Lyapunov coefficient" $l_1(0)$ of the dynamical system near the equilibrium.

First Lyapunov coefficient

Write the Taylor expansion of $f(x,0)$ at $x=0$ as

$$f(x,0) = A_0 x + \frac{1}{2}B(x,x) + \frac{1}{6}C(x,x,x) + O(\|x\|^4),$$

where $B(x,y)$ and $C(x,y,z)$ are the multilinear functions with components

$$B_j(x,y) = \sum_{k,l=1}^{n} \left.\frac{\partial^2 f_j(\xi,0)}{\partial\xi_k \partial\xi_l}\right|_{\xi=0} x_k y_l \ ,$$

$$C_j(x,y,z) = \sum_{k,l,m=1}^{n} \left.\frac{\partial^3 f_j(\xi,0)}{\partial\xi_k \partial\xi_l \partial\xi_m}\right|_{\xi=0} x_k y_l z_m \ ,$$

where $j = 1,2,\ldots,n$. Let $q \in \mathbb{C}^n$ be a complex eigenvector of A_0 corresponding to the eigenvalue $i\omega_0$: $A_0 q = i\omega_0 q$. Introduce also the adjoint eigenvector $p \in \mathbb{C}^n$: $A_0^T p = -i\omega_0 p$, $\langle p,q\rangle = 1$. Here $\langle p,q\rangle = \bar{p}^T q$ is the inner product in $\mathbb{C}^n$. Then (see, for example, [Kuznetsov, 2004, book])

$$l_1(0) = \frac{1}{2\omega_0}\mathrm{Re}\left[\langle p, C(q,q,\bar{q})\rangle - 2\langle p, B(q, A_0^{-1}B(q,\bar{q}))\rangle + \langle p, B(\bar{q},(2i\omega_0 I_n - A_0)^{-1}B(q,q))\rangle\right]$$

where I_n is the unit $n\times n$ matrix. Note that the value (but not the sign) of $l_1(0)$ depends on the scaling of the eigenvector q. The normalization $\langle q,q\rangle = 1$ is one of the options to remove this ambiguity.

Rosenzweig-MacArthur model

$$\frac{du}{dt} = u\left(1-\frac{u}{k}\right) - \frac{muv}{1+u}, \quad \frac{dv}{dt} = -\theta v + \frac{muv}{1+u}$$

Nullcline: $u = 0$, $v = \dfrac{(k-u)(1+u)}{m}$; $v = 0$, $\theta = \dfrac{mu}{1+u}$.

Solving $\theta = \dfrac{mu}{1+u}$, one have $u = \lambda \equiv \dfrac{\theta}{m-\theta}$.

Equilibria: $(0,0)$, $(k,0)$, (λ, v_λ) where $v_\lambda = \dfrac{(k-\lambda)(1+\lambda)}{m}$

We take λ as a bifurcation parameter

Case 1: $\lambda \geq k$: $(k,0)$ is globally asymptotically stable
Case 2: $(k-1)/2 < \lambda < k$: $(k,0)$ is a saddle, and (λ, v_λ) is a globally stable equilibrium
Case 3: $0 < \lambda < (k-1)/2$: $(k,0)$ is a saddle, and (λ, v_λ) is an unstable equilibrium
($\lambda = \lambda_0 = (k-1)/2$ is a Hopf bifurcation point)

$$A_0 = L_0(\lambda_0) := \begin{pmatrix} \dfrac{\lambda_0(k-1-2\lambda_0)}{k(1+\lambda)} & -\theta \\ \dfrac{k-\lambda_0}{k(1+\lambda_0)} & 0 \end{pmatrix}.$$

Normal form (1)

[Yi-Wei-Shi, 2009, JDE]
Eigenvector: $A_0 q = i\omega_0 q$, $A_0^* q^* = -i\omega q^*$, $\langle q, q^* \rangle = 1$.

$$q := \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = \begin{pmatrix} 1 \\ -i\omega_0/\theta \end{pmatrix}, \quad \text{and} \quad q^* := \begin{pmatrix} a_0^* \\ b_0^* \end{pmatrix} = \begin{pmatrix} 1/2 \\ -\theta i/(2\omega_0) \end{pmatrix},$$

where $\omega_0 = \sqrt{\theta/k}$.

$$\begin{aligned} f(\lambda, u, v) &= (u+\lambda)\left(1 - \frac{u+\lambda}{k}\right) - \frac{m(u+\lambda)(v+v_\lambda)}{1+u+\lambda}, \\ g(\lambda, u, v) &= -\theta(v+v_\lambda) + \frac{m(u+\lambda)(v+v_\lambda)}{1+u+\lambda}, \end{aligned} \tag{1}$$

then we have,

$$\begin{aligned} c_0 &= \frac{-2(k-1)^2 + 8i\omega_0 k}{k(k-1)(k+1)}, \quad d_0 = -\frac{4(k-1) + 8i\omega_0 k}{k(k-1)(k+1)}, \\ e_0 &= \frac{2(1-k)}{k(k+1)}, \quad f_0 = -\frac{4}{k(k+1)}, \quad g_0 = -h_0 = -\frac{24(k-1) + 16i\omega_0 k}{k(k-1)(k+1)^2}. \end{aligned} \tag{2}$$

Normal form (2)

and,

$$\begin{aligned}
\langle q^*, Q_{qq}\rangle &= \frac{4\theta\omega_0 k - (k-1)^2\omega_0 + 2\theta(3-k)i}{k(k-1)(k+1)\omega_0}, \\
\langle q^*, Q_{q\overline{q}}\rangle &= \frac{(1-k)\omega_0 - 2\theta i}{k(k+1)\omega_0}, \\
\langle \overline{q}^*, Q_{qq}\rangle &= -\frac{(k-1)^2\omega_0 + 2\theta k\omega_0 - 4\theta k i}{k(k-1)(k+1)\omega_0}, \\
\langle \overline{q}^*, C_{qq\overline{q}}\rangle &= \frac{-12(k-1)\omega_0 - 8\theta k\omega_0 + 4\theta(3k-5)i}{k(k-1)(k+1)^2\omega_0}.
\end{aligned} \tag{3}$$

$$\begin{aligned}
H_{20} &= \begin{pmatrix} c_0 \\ d_0 \end{pmatrix} - \langle q^*, Q_{qq}\rangle \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} - \langle \overline{q}^*, Q_{qq}\rangle \begin{pmatrix} \overline{a_0} \\ \overline{b_0} \end{pmatrix} = 0, \\
H_{11} &= \begin{pmatrix} e_0 \\ f_0 \end{pmatrix} - \langle q^*, Q_{q\overline{q}}\rangle \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} - \langle \overline{q}^*, Q_{q\overline{q}}\rangle \begin{pmatrix} \overline{a_0} \\ \overline{b_0} \end{pmatrix} = 0,
\end{aligned} \tag{4}$$

which implies that $w_{20} = w_{11} = 0$. So

$$\langle q^*, Q_{w_{11},q}\rangle = \langle q^*, Q_{w_{20},\overline{q}}\rangle = 0. \tag{5}$$

Normal form (3)

Therefore

$$
\begin{aligned}
\mathrm{Re}(c_1(\lambda_0)) =& \mathrm{Re}\left\{ \frac{i}{2\omega_0}\langle q^*, Q_{qq}\rangle \cdot \langle q^*, Q_{q\bar{q}}\rangle + \frac{1}{2}\langle q^*, C_{q,q,\bar{q}}\rangle \right\} \\
=& \frac{\theta(4\theta k - (k-1)^2 - (3-k)(1-k))}{k^2(k-1)(k+1)^2\omega_0^2} + \frac{6\omega_0(1-k) - 4\theta\omega_0 k}{k(k-1)(k+1)^2\omega_0} \\
=& \frac{\theta(4\theta k - (k-1)^2 - (3-k)(1-k))}{k^2(k-1)(k+1)^2\omega_0^2} - \frac{6(k-1) + 4\theta k}{k(k-1)(k+1)^2} \qquad (6) \\
=& \frac{4\theta k - (k-1)^2 - (3-k)(1-k) - 6(k-1) - 4\theta k}{k(k-1)(k+1)^2} \\
=& -\frac{2(k-1)(k+1)}{k(k-1)(k+1)^2} = -\frac{2}{k(k+1)} < 0
\end{aligned}
$$

The bifurcation is supercritical(resp. subcritical) if

$$\frac{1}{\alpha'(\lambda_0)}\mathrm{Re}(c_1(\lambda_0)) < 0(\textit{resp.} > 0); \qquad (7)$$

see also [Kuznetsov, 2004, book]

Higher dimension

ODE model: $\dfrac{dy}{dt} = f(\lambda, y), \quad y \in \mathbb{R}^n,\ f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$

Equilibrium: $y = y_0$ so that $f(\lambda_0, y_0) = 0$

Jacobian Matrix: $J = f_y(\lambda_0, y_0)$ is an $n \times n$ matrix

Characteristic equation:
$P(\lambda) = Det(\lambda I - J) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n$

Routh-Hurwitz criterion: complicated for general n

$n = 1$: $\lambda + a_1 = 0$, $\underline{a_1 > 0}$
$n = 2$: $\lambda^2 + a_1\lambda + a_2 = 0$, $\underline{a_1 > 0,\ a_2 > 0}$ Trace-determinant plane
$n = 3$: $\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$, $\underline{a_1 > 0,\ a_2 > \dfrac{a_3}{a_1},\ a_3 > 0}$
$n = 4$: $\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0$, $\underline{a_1 > 0,\ a_2 > \dfrac{a_3^2 + a_1^2 a_4}{a_1 a_3},\ a_3 > 0,\ a_4 > 0}$
$n \geq 5$: check books

3D system

$n = 3$: $\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$, $\underline{a_1 > 0,\ a_2 > \frac{a_3}{a_1},\ a_3 > 0}$

Hopf bifurcation point: $a_1 > 0$, $a_3 > 0$, $a_1 a_2 - a_3 = 0$.
Eigenvalues: $\lambda_1 = \beta i$, $\lambda_2 = -\beta i$, and $\lambda_3 = -\alpha$ (for $\alpha, \beta > 0$) Then

$$a_1 = -(\lambda_1+\lambda_2+\lambda_3) = \alpha > 0,\ a_2 = \lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3 = \beta^2 > 0,\ a_3 = -\lambda_1\lambda_2\lambda_3 = \alpha\beta^2 > 0.$$

And $a_1 a_2 - a_3 = 0$.

Example: (Lorenz system) $x' = \sigma(y - x)$, $y' = rx - y - xz$, $z' = xy - bz$.

Basic dynamics:
equilibria: $C_0 = (0, 0, 0)$, $C_\pm = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r - 1)$.
global stability: when $0 < r < 1$, C_0 is globally stable
Jacobian: $\begin{pmatrix} -\sigma & \sigma & 0 \\ r - z & -1 & -x \\ y & x & -b \end{pmatrix}$, characteristic equation at $C_\pm$:
$\lambda^3 + (\sigma + b + 1)\lambda^2 + (r + \sigma)b\lambda + 2b\sigma(r - 1) = 0$

Hopf bifurcation: $a_1 = \sigma + b + 1 > 0$, $a_3 = 2b\sigma(r - 1) > 0$,
$a_1 a_2 - a_3 = (\sigma + b + 1)(r + \sigma)b - 2b\sigma(r - 1) = 0$
Hopf bifurcation point: $r = \dfrac{\sigma(\sigma + b + 3)}{\sigma - b - 1}$. It is a subcritical bifurcation.

Global bifurcation of periodic orbits

Consider ODE $x' = f(\lambda, x)$, $\lambda \in \mathbb{R}$, $x \in \mathbb{R}^n$, and f is smooth.
Assumptions:
(i) Suppose that for λ near λ_0 the system has a family of equilibria $x^0(\lambda)$.
(ii) Assume that its Jacobian matrix $A(\lambda) = f_x(\lambda, x^0(\lambda))$ has one pair of complex eigenvalues $\mu(\lambda) \pm i\omega(\lambda)$, $\mu(\lambda_0) = 0$, $\omega(\lambda_0) = \omega_0 > 0$, and all other eigenvalues of $A(\lambda)$ have non-zero real parts for all λ near λ_0.
(iii) $\mu'(\lambda_0) \neq 0$.

Let $x(\lambda, t; x_0)$ be the solution of the equation with initial condition $x(\lambda, 0; x_0) = x_0$.
We say (λ, x_0) is stationary if $x(\lambda, t; x_0) = x_0$ for all $t \geq 0$.
We say (λ, x_0) is periodic if it is not stationary, and there exists $T > 0$ such that $x(\lambda, T; x_0) = x_0$.
If (λ, x_0) is periodic, then all positive $T > 0$ such that $x(\lambda, T; x_0) = x_0$ are the periods. The smallest positive period is the least period.

Define

$$\Sigma = \{(\lambda, T, x_0) \in \mathbb{R} \times (0, \infty) \times \mathbb{R}^n : x(\lambda, T; x_0) = x_0 \ \text{ and } \ (\lambda, x_0) \ \text{ is periodic}\}.$$

Global bifurcation of periodic orbits

[Alexander-Yorke, 1978, AJM]
Consider ODE $x' = f(\lambda, x)$, $\lambda \in \mathbb{R}$, $x \in \mathbb{R}^n$, and f is smooth.
Assumptions:
(i) Suppose that for λ near λ_0 the system has a family of equilibria $x^0(\lambda)$.
(ii) Assume that its Jacobian matrix $A(\lambda) = f_x(\lambda, x^0(\lambda))$ has one pair of complex eigenvalues $\mu(\lambda) \pm i\omega(\lambda)$, $\mu(\lambda_0) = 0$, $\omega(\lambda_0) = \omega_0 > 0$, and all other eigenvalues of $A(\lambda)$ have non-zero real parts for all λ near λ_0.
(iii) $\mu'(\lambda_0) \neq 0$.

Define

$$S = \{(\lambda, T, x_0) \in \mathbb{R} \times (0, \infty) \times \mathbb{R}^n : x(\lambda, T; x_0) = x_0 \ \text{ and } \ (\lambda, x_0) \ \text{ is periodic}\}.$$

1. There exists connected component S_0 of $S \cup \{y_0 \equiv (\lambda_0, 2\pi/\omega_0, x^0(\lambda_0))\}$ containing y_0 and at least one periodic solution. Near y_0, every $y = (\lambda, T, x_0)(\neq y_0) \in S_0$ is periodic with the least period T.
2. One or both of the following are satisfied: (i) S_0 is not contained in any compact subset of $\mathbb{R} \times (0, \infty) \times \mathbb{R}^n$; (ii) there exists a point $(\lambda_*, T_*, x_{0*}) \in \overline{S_0} \backslash S_0$.
3. For any $(\lambda_*, T_*, x_{0*}) \in \overline{S} \backslash S$, (λ_*, x_{0*}) is stationary. For any $\varepsilon > 0$, there is a neighborhood U_ε of (λ_*, T_*, x_{0*}) such that for any $(\lambda, T, x_0) \in U_\varepsilon \cap S$, all points of the orbit $x(\lambda, t; x_0)$ are of distance less than ε from x_{0*}.

Remarks

1. The assumptions (ii) and (iii) can be generalized to: there are k pairs of purely imaginary eigenvalues of $A(\lambda_0)$ in form $\{i\beta_j\omega_0 : 1 \leq j \leq k\}$ with $1 \leq \beta_1 \leq \beta_2 \leq \cdots \leq \beta_k$, and the change of the number of such eigenvalues with positive real part from $\lambda = \lambda_0 - \varepsilon$ to $\lambda = \lambda_0 + \varepsilon$ is odd.
2. The proofs of the result use homotopy theory or Fuller index or other topological invariants.
3. The theorem states that either the connected component S_0 contains another stationary solution, or it is unbounded in the sense that

$$\sup_{(\lambda,T,x_0)\in S_0, t\in\mathbb{R}} \left(|\lambda| + |T| + |T^{-1}| + |x(\lambda,t;x_0)|\right) = \infty.$$

4. If $(\lambda, T, x_0) \in S_0$, then T is not necessarily the least period of the periodic solution $x(\lambda, T; x_0)$. If $i\omega_0$ is an simple simple eigenvalue of $A(\lambda_0)$, then near y_0, T is the least period. Note that if T is a period, so is kT for $k \in \mathbb{N}$, so the periods are always unbounded. The main point of the theorem is the periods can be unbounded continuously.
5. The notation $x(\lambda, t; x_0)$ is a periodic solution, and $\{x(\lambda, t; x_0) : t \in \mathbb{R}\}$ ia a periodic orbit. It is clear that for any $x_t = x(\lambda, t; x_0)$, $x(\lambda, t; x_t)$ is also a periodic solution, but it has the same orbit as $x(\lambda, t; x_0)$. For a fixed λ, the periodic solution is never unique, but the periodic orbit may be unique. Hence it is wrong to say "there exists at least two periodic solutions" in a theorem, and you should say "there exists at least two periodic orbits".

Example

Rosenzweig-MacArthur model

$$\frac{du}{dt} = u\left(1 - \frac{u}{k}\right) - \frac{muv}{1+u}, \quad \frac{dv}{dt} = -\theta v + \frac{muv}{1+u}.$$

Parameter: $\lambda \equiv \dfrac{\theta}{m-\theta}$.

Equilibria: $(0,0)$, $(k,0)$, (λ, v_λ) where $v_\lambda = \dfrac{(k-\lambda)(1+\lambda)}{m}$

Case 1: $\lambda \geq k$: $(k,0)$ is globally asymptotically stable
Case 2: $(k-1)/2 < \lambda < k$: (λ, v_λ) is a globally stable equilibrium
Case 3: $0 < \lambda < (k-1)/2$: $(k,0)$ and (λ, v_λ) are both unstable
($\lambda = \lambda_0 = (k-1)/2$ is a Hopf bifurcation point)

There exists a branch of periodic orbits $S_0 = \{(\lambda, T, x_0) : 0 < \lambda < (k-1)/2\}$. One can show that $|x_0|$ is bounded for S_0, so T is unbounded when $\lambda \to 0$. In this case, the limit of the orbits $\{x(\lambda, t; x_0) : t \in \mathbb{R}\}$ when $\lambda \to 0$ is not an orbit.
[Hsu-Shi, 2009, DCDS-B]

Sometimes if $T \to \infty$ as $\lambda \to \lambda_*$, the limit of the orbits $\{x(\lambda, t; x_0) : t \in \mathbb{R}\}$ when $\lambda \to \lambda_*$ is a homoclinic orbit or a heteroclinic loop of the system.
[Wang-Shi-Wei, 2011, JMB]

Abstract version: Hopf bifurcation theorem

[Crandall-Rabinowitz, ARMA, 1977]
Consider an evolution equation in Banach space X:

$$\frac{du}{dt} + Lu + f(\mu, u) = 0. \tag{8}$$

Here X is a Banach space, and $X_{\mathbb{C}} = X + iX$ is the complexification of X; $L : X \to X$ is a linear operator and it can be extended to $X_{\mathbb{C}}$ naturally. The spectral set $\sigma(L) \subseteq \mathbb{C}$, and $\lambda \in \sigma(L)$ if and only if $\bar{\lambda} \in \sigma(L)$.

Conditions on L (HL):

1. $-L$ is the infinitesimal generator of a strongly continuous semigroup $T(t)$ on X;
2. $T(t)$ is a holomorphic (analytic) semigroup on $X_{\mathbb{C}}$;
3. $(\lambda I - L)^{-1}$ is compact for $\lambda \notin \sigma(L)$;
4. i is a simple eigenvalue of L (with eigenvector $w_0 \neq 0$);
5. $ni \notin \sigma(L)$ for $n = 0$ and $n = 2, 3, \cdots$.

Abstract version: Hopf bifurcation theorem

Conditions on f: (Hf)

1. There exists $\alpha \in (0,1)$ and a neighborhood U of $(\mu, u) = (0,0)$ in $\mathbb{R} \times X^\alpha$ such that $f \in C^2(U, X)$;
2. $f(\mu, 0) = 0$ for $(\mu, 0) \in U$ and $f_u(0,0) = 0$.

(HL) and (Hf) imply that there exists C^1 functions $(\beta(\mu), v(\mu))$ for $\mu \in (-\delta, \delta)$ such that

$$[L + f_u(\mu, 0)]v(\mu) = \beta(\mu)v(\mu), \;\; \beta(0) = i, \;\; v(0) = w_0.$$

Condition on β: (Hβ)

1. Re $\beta'(0) \neq 0$.

Rescaling $\tau = \rho^{-1}t$: change the period of periodic orbit to a parameter

$$\frac{du}{d\tau} + \rho L u + \rho f(\mu, u) = 0. \tag{9}$$

Looking for a period-1 periodic orbit for the rescaled equation.

Convert it to integral equation $u(\tau)$ is a solution to (9) for $\tau \in [0, r]$ if and only if, for $\tau \in [0, r]$,

$$F(\rho, \mu, u) \equiv u(\tau) - T(\rho\tau)u(0) + \rho \int_0^\tau T(\rho(\tau - \xi)) f(\mu, u(\xi)) d\xi = 0.$$

Abstract version: Hopf bifurcation theorem

Let $C_{2\pi}(\mathbb{R}, X_\alpha)$ be the set of 2π-periodic continuous functions, and let $C_0([0,2\pi], X_\alpha) = \{h : [0,2\pi] \to X_\alpha, h(0) = 0, h \text{ is continuous }\}$. Then

$$F(\rho,\mu,u) \equiv u(\tau) - T(\rho\tau)u(0) + \rho\int_0^\tau T(\rho(\tau-\xi))f(\mu,u(\xi))d\xi$$

is well-defined so that $F : \mathbb{R}\times\mathbb{R}\times C_{2\pi}(\mathbb{R}, X_\alpha) \to C_0([0,2\pi], X_\alpha)$.

Theorem. Let (HL), (Hf) and (Hβ) be satisfied. Then there exist $\varepsilon, \eta > 0$ and C^1 functions $(\rho,\mu,u) : (-\eta,\eta) \to \mathbb{R}\times\mathbb{R}\times C_{2\pi}(\mathbb{R}, X_\alpha)$ such that

1. $F(\rho(s),\mu(s),u(s)) = 0$ for $|s| < \eta$.
2. $\mu(0) = 0,\ u(0) = 0,\ \rho(0) = 1$ and $u(s) \neq 0$ for $0 < |s| < \eta$.
3. If $(\mu_1, u_1) \in \mathbb{R}\times C(\mathbb{R}, X_\alpha)$ is a solution of (8) with period $2\pi\rho_1$, where $|\rho_1 - 1| < \varepsilon$, $|\mu_1| < \varepsilon$, and $||u_1||_\alpha < \varepsilon$, then there exist $s \in [0,\eta)$ and $\theta \in [0,2\pi)$ such that $u(\rho_1\tau) = u(s)(\tau+\theta)$ for $\tau \in \mathbb{R}$.

Note: There is a relation between the solutions with $s \in (0,\eta)$ and $s \in (-\eta,0)$, and they are the same orbit with different phases.

Reaction-Diffusion systems

[Yi-Wei-Shi, 2009, JDE]
A general reaction-diffusion system subject to Neumann boundary condition on spatial domain $\Omega = (0, \ell\pi)$.

$$\begin{cases} u_t - d_1 u_{xx} = f(\lambda, u, v), & x \in (0, \ell\pi),\ t > 0, \\ v_t - d_2 v_{xx} = g(\lambda, u, v), & x \in (0, \ell\pi),\ t > 0, \\ u_x(0, t) = v_x(0, t) = 0,\ u_x(\ell\pi, t) = v_x(\ell\pi, t) = 0, & t > 0, \\ u(x, 0) = u_0(x),\ v(x, 0) = v_0(x), & x \in (0, \ell\pi), \end{cases} \tag{10}$$

where $d_1, d_2, \lambda \in \mathbb{R}^+$, $f, g : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ are $C^k(k \geq 3)$ with $f(\lambda, 0, 0) = g(\lambda, 0, 0) = 0$. Define the real-valued Sobolev space

$$X := \{(u, v) \in H^2(0, \ell\pi) \times H^2(0, \ell\pi) | (u_x, v_x)|_{x=0,\ \ell\pi} = 0\}. \tag{11}$$

The linearized operator of the steady state system of (10) evaluated at $(\lambda, 0, 0)$ is,

$$L(\lambda) := \begin{pmatrix} d_1 \dfrac{\partial^2}{\partial x^2} + A(\lambda) & B(\lambda) \\ C(\lambda) & d_2 \dfrac{\partial^2}{\partial x^2} + D(\lambda) \end{pmatrix}, \tag{12}$$

with the domain $D_{L(\lambda)} = X_{\mathbb{C}}$, where $A(\lambda) = f_u(\lambda, 0, 0)$,$B(\lambda) = f_v(\lambda, 0, 0)$, $C(\lambda) = g_u(\lambda, 0, 0)$, and $D(\lambda) = g_v(\lambda, 0, 0)$.

Hopf bifurcations

We assume that for some $\lambda_0 \in \mathbb{R}$, the following condition holds:
($\mathbf{H}_1$): There exists a neighborhood O of λ_0 such that for $\lambda \in O$, $L(\lambda)$ has a pair of complex, simple, conjugate eigenvalues $\alpha(\lambda) \pm i\omega(\lambda)$, continuously differentiable in λ, with $\alpha(\lambda_0) = 0, \omega(\lambda_0) = \omega_0 > 0$, and $\alpha'(\lambda_0) \neq 0$; all other eigenvalues of $L(\lambda)$ have non-zero real parts for $\lambda \in O$.

Theorem. Suppose that the assumption ($\mathbf{H}_1$) holds. Then there is a family of periodic orbits $S = \{(\lambda(s), T(s), u(s, x, t), v(s, x, t)) : 0 < s < \delta\}$ with $\lambda(s), T(s), u(s, \cdot, \cdot), v(s, \cdot, \cdot)$ differentiable in s, $(u(s, x, t + T(s)), v(s, x, t + T(s))) = (u(s, x, t), v(s, x, t))$, and

$$\lim_{s\to 0} \lambda(s) = \lambda_0, \ \lim_{s\to\infty} T(s) = \frac{2\pi}{\omega_0}, \ \lim_{s\to 0} |u(s, x, t)| + |v(s, x, t)| = 0,$$

uniformly for $x \in [0, \ell\pi]$ and $t \in \mathbb{R}$. All periodic orbits of the system are time phase shifts of the ones on S.

Normal form calculations: [Yi-Wei-Shi, 2009, JDE]

Remarks

1. The result for (semilinear) reaction-diffusion systems can be extended to quasilinear systems with cross-diffusion, self-diffusion, chemotaxis.
   [Liu-Shi-Wang, 2013, preprint]
   [Amann, 1991, book chapter] [Da Prado-Lunardi, 1985, AIHP] [Simonett, 1995, DIE]
2. The Hopf bifurcation from non-constant equilibria are much difficult to obtain since the linearized operator cannot be decomposed with Fourier series.
3. The stability of the bifurcating periodic orbits are difficult to analyze except near the Hopf bifurcation points.
4. The Hopf bifurcation theorem is also extended to delay differential equations (see next lecture), and delayed reaction-diffusion equations (see last lecture).
5. The uniqueness of limit cycle is difficult in general.

References

1. (general Hopf bifurcation) [Marsden-McCracken, book, 1976] [Hassard-Kazarinoff-Wan, book, 1981] [Kielhofer, book, 2004] [Kuznetsov, book, 2004]
2. (Hopf bifurcation for reaction-diffusion systems) [Crandall-Rabinowitz, 1977, ARMA] [Henry, book, 1981]
3. (Navier-Stokes equations) [Iudovich, 1971, JAMM] [Sattinger, 1971, ARMA], [Iooss, 1972, ARMA] [Joseph-Sattinger, 1972, ARMA]
4. (Delay differential equations) [Hale, 1977, book] [Kuang, 1993, book] [Wu, 1995, book]
5. (Global Hopf bifurcation) [Alexander-Yorke, 1978, AJM] [Chow-Mallet-Paret, 1978, JDE] [Nussbaum, 1978, TAMS] [Ize, 1979, CPDE] [Wu, 1998, TAMS]
6. (Hopf bifurcation with symmetry) [Vanderbauwhede, 1982, book] [Golubitsky-Stewart, 1985, ARMA]
7. (non-densely defined Cauchy problems) [Magal-Ruan, 2009, AMS-Memoir] [Liu-Magal-Ruan, 2011, ZAMP] [Liu-Magal-Ruan, 2012, CAMQ]