Reaction-Diffusion Models and Bifurcation Theory
Lecture 1: Basics and Examples from ODEs

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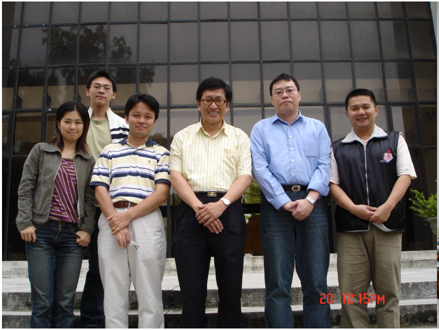
Topic course in Fall 2007

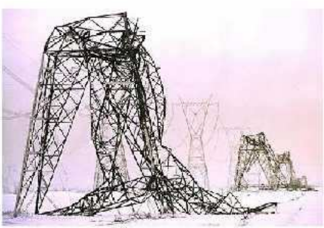
My visits to Taiwan: 2005, 2007, 2009, 2010, 2013



Course Plan

1. Basics and Examples from ODEs
2. Diffusion, advection and cross-diffusion models
3. Delay, chemotaxis and nonlocal models
4. Linear stability (ODE, reaction-diffusion models)
5. Linear stability (delay models)
6. Numerical simulations
7. Analytic bifurcation theory for stationary problems
8. Turing bifurcation and pattern formation
9. Global bifurcation for stationary problems
10. Hopf bifurcation

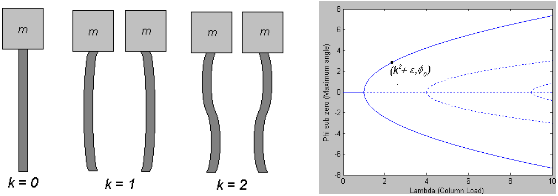
Earliest Bifurcation: Buckling



(Left: Buckling due to glaze icing; Right: Buckling of rails)

In engineering, buckling is a failure mode characterized by a sudden failure of a structural member subjected to high compressive stresses, where the actual compressive stresses at failure are greater than the ultimate compressive stresses that the material is capable of withstanding. This mode of failure is also described as failure due to elastic instability.

Euler Buckling Equation



(Left: buckling under different thrusts; Right: bifurcation diagram (λ, ϕ))

$\phi'' + \lambda \sin\phi = 0$, $x \in (0, \pi)$, $\phi(0) = \phi(\pi) = 0$

$\phi(x)$ is the angle between the tangent to the column's axis and the y-axis, and λ is a parameter proportional to the thrust.

The word bifurcation is from the Latin *bifurcus*, which means two forks.

[Da Vinci, *Codex Madrid*, 1478-1519], [Euler, 1757], Galileo, Bernoulli

Da Vinci's sketching showing buckling

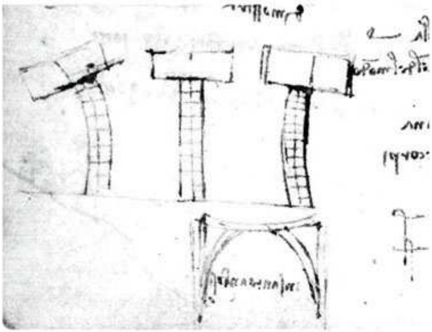


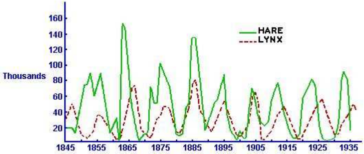
Figure 4 Leonardo da Vinci: drawing to illustrate the bending of a column under a vertical weight [from *Codex Madrid I* [9], page 117].

Bifurcation in Natural World

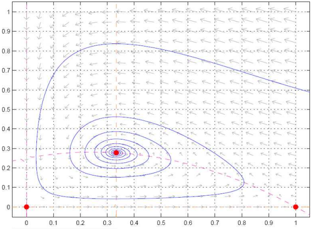
Predator-prey interaction: Lotka-Volterra model
Classical examples: Hudson company lynx-hare data in 1800s

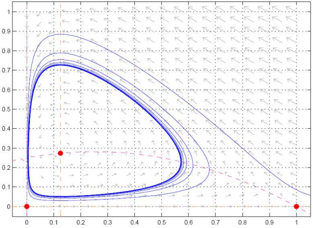
Alfred Lotka Vito Volterra

$$\frac{du}{dt} = u(a - bu) - cuv, \quad \frac{dv}{dt} = -dv + fuv.$$



Bifurcation to Oscillations





Paradox of enrichment: $\frac{dU}{ds} = \gamma U \left(1 - \frac{U}{K}\right) - \frac{CMUV}{A+U}, \quad \frac{dV}{ds} = -DV + \frac{MUV}{A+U}.$
Environment is enriched if K (carrying capacity) is larger, but when K is small, a coexistence equilibrium is stable; but when K is larger, the coexistence equilibrium is unstable, and a stable periodic orbit appears as a result of Hopf bifurcation.

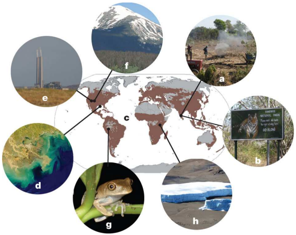
[Rosenzweig, *Science* 1971] Paradox of enrichment: destabilization of exploitation ecosystems in ecological time.
[May, *Science* 1972] Limit cycles in predator-prey communities.

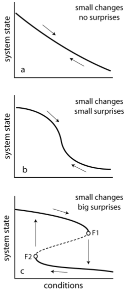
Planet earth, global warming, critical transition

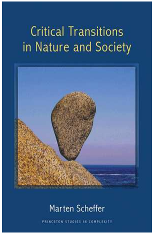
2013 is the year of Mathematics for Planet Earth. http://mpe2013.org

A sudden critical transition can bring catastrophic change to the earth ecosystem, thus it is important to predict and prevent it http://www.early-warning-signals.org

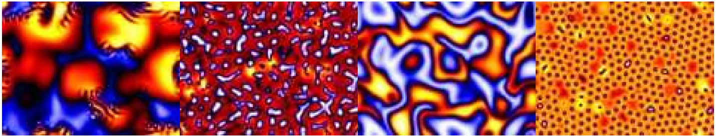
Critical Transitions in Nature and Society, by Marten Scheffer
http://www.sparcs-center.org/about-us/staff-contacts/marten-scheffer.html







Rich Spatial Patterns in Diffusive Predator-Prey System

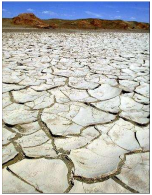


Patterns generated by diffusive predator-prey system

$$\begin{cases} u_t - d_1 \Delta u = u(1-u) - \dfrac{muv}{u+a}, & x \in \Omega,\ t > 0, \\ v_t - d_2 \Delta v = -\theta v + \dfrac{muv}{u+a}, & x \in \Omega,\ t > 0, \\ \partial_\nu u = \partial_\nu v = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) \geq 0,\ v(x,0) = v_0(x) \geq 0, & x \in \Omega. \end{cases}$$

Patchiness (spatial heterogeneity) of plankton distributions in phytoplankton-zooplankton interaction
[Medvinsky, et.al., *SIAM Review* 2002]
Spatiotemporal complexity of plankton and fish dynamics.

Spatial Model: Bifurcation from Grassland to Desert



$$\frac{\partial w}{\partial t} = a - w - wn^2 + \gamma \frac{\partial w}{\partial x}, \quad \frac{\partial n}{\partial t} = wn^2 - mn + \Delta n, \quad x \in \Omega.$$

$w(x, y, t)$:concentration of water; $n(x, y, t)$: concentration of plant,
Ω: a two-dimensional domain.
$a > 0$: rainfall; $-w$: evaporation; $-wn^2$: water uptake by plants; water flows downhill at speed γ; wn^2: plant growth; $-mn$: plant loss
[Klausmeier, *Science*, 1999]
Regular and Irregular Patterns in Semiarid Vegetation.
[Rietkerk, et.al. *Science*, 2004]
Self-Organized Patchiness and Catastrophic Shifts in Ecosystems.

Stability of a Steady State Solution

For a continuous-time evolution equation $\dfrac{du}{dt} = F(\lambda, u)$, where $u \in X$ (state space), $\lambda \in \mathbb{R}$, a steady state solution u_* is locally asymptotically stable (or just stable) if for any $\epsilon > 0$, then there exists $\delta > 0$ such that when $||u(0) - u_*||_X < \delta$, then $||u(t) - u_*||_X < \epsilon$ for all $t > 0$ and $\lim_{t\to\infty} ||u(t) - u_*||_X = 0$. Otherwise u_* is unstable.

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Bifurcation (change of stability): if when the parameter λ changes from $\lambda_* - \varepsilon$ to $\lambda_* + \varepsilon$, the steady state $u_*(\lambda)$ changes from stable to unstable; and other special solutions (steady states, periodic orbits) may emerge from the known solution $(\lambda, u_*(\lambda))$.

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Steady State Bifurcation (transcritical/pitchfolk): if 0 is an eigenvalue of $D_u F(\lambda_*, u_*)$.
Hopf Bifurcation: if $\pm ki$ is a pair of eigenvalues of $D_u F(\lambda_*, u_*)$.

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(patch model or ODE system if the space is finite)

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$u_x = u(n, x) - u(n, x-1)$ (advection),
$u_{xx} = u(n+1, x) - 2u(n, x) + u(n, x-1)$ (diffusion)

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continuous case: $u_t = \dfrac{du}{dt}$, $u_x = \dfrac{du}{dx}$, $u_{xx} = \dfrac{d^2u}{dx^2}$.

local case: f is a function $f(t, x, u(x), u_x(x), u_{xx}(x))$
nonlocal case: f is an operator $f : u \mapsto f(t, x, u)$

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$u_t = \int_\Omega k(x, y)u(y, t)dy$

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Delay equation:
$u_t = f(u(t), u(t - \tau))$

and combination of all above

ODE in $\mathbb{R}^1$

Consider a scalar ODE

$$\frac{du}{dt} = f(\lambda, u), \quad \lambda \in \mathbb{R}, u \in \mathbb{R}.$$

If f is smooth, and $u(t)$ is a bounded solution, then $\lim_{t\to\infty} u(t)$ is an equilibrium. (there is no limit cycle, no chaos)

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So the bifurcation of equilibria determines the entire dynamics.

Suppose that $y = y_0$ is an equilibrium.
1. If $f_u(\lambda_0, y_0) < 0$, then y_0 is stable (sink);
2. If $f_u(\lambda_0, y_0) > 0$, then y_0 is unstable (source).

Solving the bifurcation point:

$$f(\lambda, u) = 0, \quad \frac{\partial f}{\partial u}(\lambda, u) = 0.$$

Basic Bifurcation in $\mathbb{R}^1$

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Pitchfork bifurcation:

$$f_{uu}(\lambda_0, u_0) = 0, \; f_{\lambda u}(\lambda_0, u_0) \neq 0, \quad \text{and } f_{uuu}(\lambda_0, u_0) \neq 0.$$

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$$f_{uu}(\lambda_0, u_0) \neq 0, \; f_{\lambda u}(\lambda_0, u_0) \neq 0.$$

Pitchfork bifurcation:

$$f_{uu}(\lambda_0, u_0) = 0, \; f_{\lambda u}(\lambda_0, u_0) \neq 0, \quad \text{and } f_{uuu}(\lambda_0, u_0) \neq 0.$$

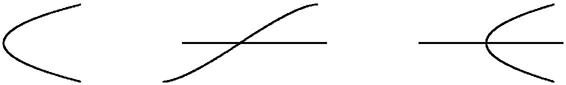


Figure: Left: saddle-node; Middle: transcritical; Right: pitchfork

Names of bifurcations

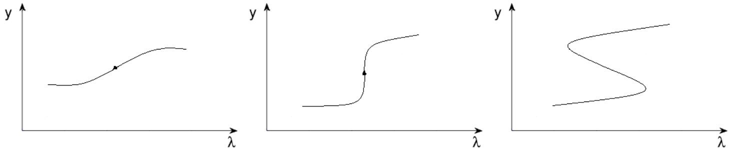
Saddle-node bifurcation (fold bifurcation, blue sky bifurcation)
a curve $(\lambda(s), u(s))$ near (λ_0, u_0):
$\lambda(s) > \lambda_0$: supercritical (forward)
$\lambda(s) < \lambda_0$: subcritical (backward)

transcritical bifurcation/pitchfork bifurcation near (λ_0, u_0):
trivial solutions (λ, u_0) and nontrivial solutions on a curve $(\lambda(s), u(s))$
$\lambda(s) > \lambda_0$: supercritical (forward) pitchfork
$\lambda(s) < \lambda_0$: subcritical (backward) pitchfork

If only "half" of the nontrivial solution curve is of physical importance
$\lambda(s) > \lambda_0$: supercritical (forward) transcritical
$\lambda(s) < \lambda_0$: subcritical (backward) transcritical

In the transcritical bifurcation/pitchfork bifurcation, sometimes people only use forward/backward as above, but use supercritical (subcritical) if the bifurcating solutions are stable (unstable)

Cusp bifurcation (two parameter bifurcation)



$$y' = f(\varepsilon, \lambda, y).$$

Left: $\varepsilon < 0$ (monotone), middle: $\varepsilon = 0$, right: $\varepsilon > 0$ (with two saddle-node bifurcation points)

$$\frac{\partial f}{\partial \lambda}(\varepsilon, \lambda_0, y_0) \neq 0, \; \frac{\partial^2 f}{\partial y^2}(\varepsilon, \lambda_0, y_0) = 0, \; \frac{\partial^3 f}{\partial y^3}(\varepsilon, \lambda_0, y_0) \neq 0.$$

`http://www.scholarpedia.org/article/Cusp_bifurcation`

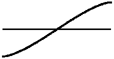
Imperfect bifurcation (two parameter bifurcation)

$$y' = f(\varepsilon, \lambda, y).$$

For $\varepsilon = \varepsilon_0$, there is a transcritical bifurcation or pitchfork bifurcation.
What happens when ε is near ε_0?

Typical symmetry breaking of transcritical/pitchfork bifurcation
[Shi, JFA, 1999], [Liu-Shi-Wang, JFA, 2007; JFA, 2013 to appear]

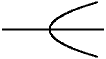
$\varepsilon < \varepsilon_0$



$\varepsilon = \varepsilon_0$

$\varepsilon > \varepsilon_0$

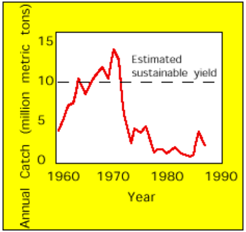
$\varepsilon < \varepsilon_0$



$\varepsilon = \varepsilon_0$

$\varepsilon > \varepsilon_0$

Saddle-node Bifurcation in $\mathbb{R}^1$



Annual catch of the Peruvian Anchovy Fishery from 1960-1990

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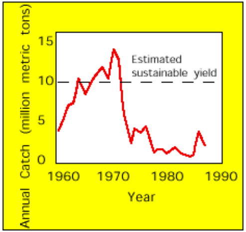


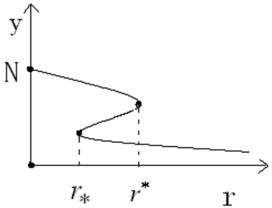
Annual catch of the Peruvian Anchovy Fishery from 1960-1990

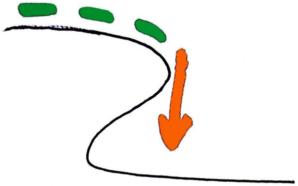
$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right) - H, \quad \text{steady state: } kP\left(1 - \frac{P}{N}\right) - H = 0$$

When $H > H_0 \equiv \dfrac{kN}{4}$, the fishery collapses.

H_0 is the maximum sustainable yield (MSY)

Hysteresis Bifurcation in $\mathbb{R}^1$





A grazing system of herbivore-plant interaction

$$\frac{dV}{dt} = V(1-V) - \frac{rV^p}{h^p + V^p}, \quad h, r > 0, \quad p \geq 1.$$

[Noy-Meir, *J. Ecology* 1975]
Stability of Grazing Systems: An Application of Predator-Prey Graphs.
[May, *Nature* 1975]
Thresholds and breakpoints in ecosystems with a multiplicity of stable states.

Catastrophe theory: Thom, Arnold, Zeeman in 1960-70s

Potential catastrophes: Arctic sea, Greenland ice, Amazon rainforest, etc.

Population models

$y' = yf(y)$, $\quad f(y)$ is the growth rate per capita

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Growth function:
Logistic: $g(y) = yh(y)$: $h(0) > 0$, $h(N) = 0$ and $h(y)$ is decreasing in $(0, N)$.
Allee effect: $g(y) = yh(y)$, $\max h(y) > 0$, $h(N) = 0$, $h(y)$ is increasing in $(0, M)$, and $h(y)$ is decreasing in (M, N).

Functional Responses

[Holling, 1959]

(Type I)

$$c(y) = \begin{cases} Ey, & 0 < y \le y_*, \\ Ey_*, & y \ge y_*, \end{cases} \tag{1}$$

(Type II)

$$c(y) = \frac{Ay}{B + y}, \qquad y > 0, \tag{2}$$

(Type III)

$$c(y) = \frac{Ay^p}{B^p + y^p}, \quad y > 0, p > 1, \tag{3}$$

Their common characters:

(C1) $c(0) = 0$; $c(y)$ in increasing;

(C2) $\lim\limits_{y \to \infty} c(y) = c_\infty > 0$.

Another type II: (Ivlev) $c(y) = A - Be^{-ry}$.

Functional Responses

Generalized Type II: $c(y)$ satisfies (C1), (C2) and

(C3) $c'(0) > 0$, and $c''(y) \leq 0$ for almost every $y \geq 0$;

Generalized Type III: $c(y)$ satisfies (C1), (C2) and

(C4) $c'(0) = 0$ and there exists $y_* > 0$ such that $c''(y)(y - y_*) \leq 0$ for almost every $y \geq 0$.

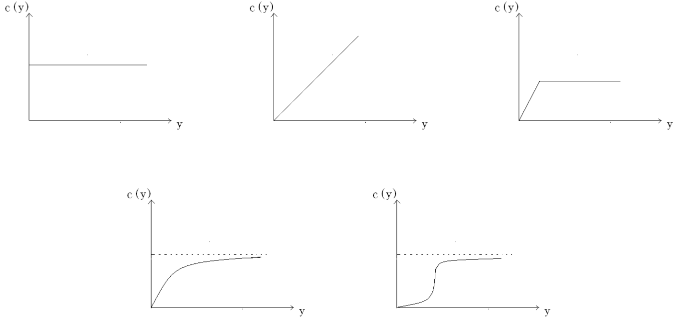
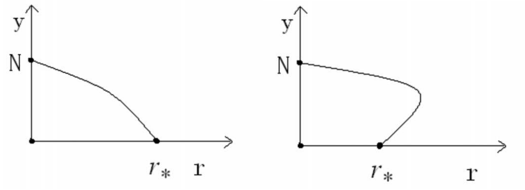


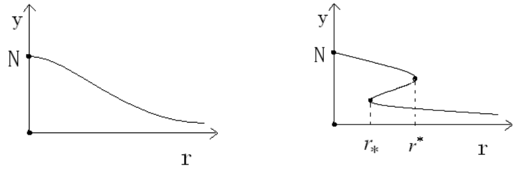
Figure: (a) constant; (b) linear; (c) Type I; (d) Type II; (e) Type III.

Bifurcation diagram for ecological model

Bifurcation diagram of $y' = g(y) - rc(y)$ can be drawn using the graph of $r = g(y)/c(y)$. But it does not work for systems or PDEs!



$g(y)$ is Logistic, $c(y)$ is Holling type II. Here $r_* = g'(0)/c'(0)$ is a transcritical bifurcation point on $y = 0$ (trivial solution). ($r(y) = g(y)/c(y)$ is concave)



$g(y)$ is Logistic, $c(y)$ is Holling type III. ($r(y) = g(y)/c(y)$ is convex-concave)

Linear systems

For a planar linear system

$$\begin{cases} x' = ax + by \\ y' = cx + dy \end{cases}$$

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An eigenvalue λ satisfies $\lambda^2 - (a+d)\lambda + ad - bc = 0$, or let $T = a + d$ be the trace and let $D = ad - bc$ be the determinant of the matrix, then

$$\lambda^2 - T\lambda + D = 0.$$

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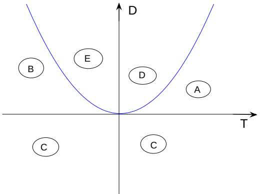
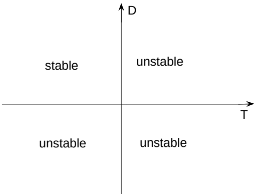
$$\lambda^2 - T\lambda + D = 0.$$

Since $\lambda_1 + \lambda_2 = T$, and $\lambda_1\lambda_2 = D$, then generic cases are

(A) $T > 0,\ D > 0,\ T^2 - 4D > 0$, and $\lambda_1 > \lambda_2 > 0$;
(B) $T < 0,\ D > 0,\ T^2 - 4D > 0$, and $\lambda_1 < \lambda_2 < 0$;
(C) $D < 0$, and $\lambda_1 < 0 < \lambda_2$;
(D) $T > 0,\ D > 0,\ T^2 - 4D < 0$, and $\lambda_1, \lambda_2 = \alpha \pm i\beta,\ \alpha > 0$;
(E) $T < 0,\ D > 0,\ T^2 - 4D < 0$, and $\lambda_1, \lambda_2 = \alpha \pm i\beta,\ \alpha < 0$.

T-D plane (Trace-Determinant plane)





Phase portraits for linear systems

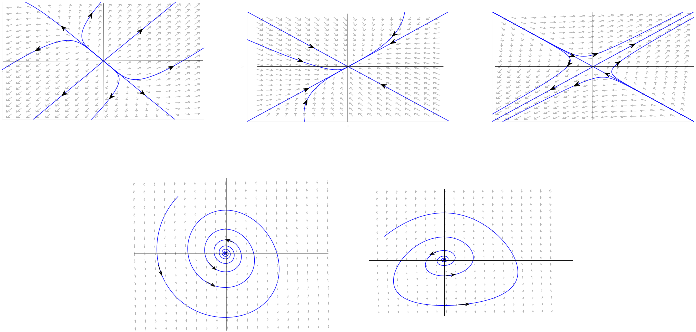


Figure: Phase portraits: (A) $\lambda_1 > \lambda_2 > 0$, (source or unstable node), (B) $\lambda_1 < \lambda_2 < 0$ (sink or stable node), (C) $\lambda_1 < 0 < \lambda_2$ (saddle); (D) $\lambda_1, \lambda_2 = \alpha \pm i\beta,\ \alpha < 0$ (spiral sink or stable spiral), (E)$\lambda_1, \lambda_2 = \alpha \pm i\beta,\ \alpha > 0$ (spiral source or unstable spiral)

Nonlinear planar systems

$$\begin{cases} x' = f(x, y) \\ y' = g(x, y). \end{cases}$$

If $(x_0, y_0) \in \mathbb{R}^2$ satisfies $f(x_0, y_0) = 0$ and $g(x_0, y_0) = 0$, then (x_0, y_0) is an equilibrium point. The Jacobian matrix J at (x_0, y_0) is

$$J = \begin{pmatrix} \dfrac{\partial f}{\partial x}(x_0, y_0) & \dfrac{\partial f}{\partial y}(x_0, y_0) \\ \dfrac{\partial g}{\partial x}(x_0, y_0) & \dfrac{\partial g}{\partial y}(x_0, y_0) \end{pmatrix}.$$

Then the stability of (x_0, y_0) is determined by the linear system $Y' = J \cdot Y$.

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Stability Criterion:

(a) If $T < 0$ and $D > 0$, then (x_0, y_0) is stable;

(b) If $T < 0$, $D > 0$ or $D < 0$, then (x_0, y_0) is unstable;

(c) If $T = 0$ or $D = 0$, then the stability cannot be determined by J.

Equilibrium Bifurcation Theorem

$D = 0$ and $T \neq 0$.

Consider ODE $x' = f(\lambda, x)$, $\lambda \in \mathbb{R}$, $x \in \mathbb{R}^n$, and f is smooth.
(i) Suppose that for λ near λ_0 the system has a family of equilibria $x^0(\lambda)$.
(ii) Assume that its Jacobian matrix $A(\lambda) = f_x(\lambda, x^0(\lambda))$ has an eigenvalue $\mu(\lambda)$, $\mu(\lambda_0) = 0$, and all other eigenvalues of $A(\lambda)$ have non-zero real parts for all λ near λ_0.

If $\mu'(\lambda_0) \neq 0$, then the system has another family of equilibria $(\lambda(s), x(s))$ for $s \in (-\delta, \delta)$, such that $\lambda(0) = \lambda_0$ and $x(0) = x^0(\lambda_0)$.

This corresponds to the transcritical bifurcation and pitchfork bifurcation. One can also define saddle-node bifurcation (in which there is given family of equilibria).

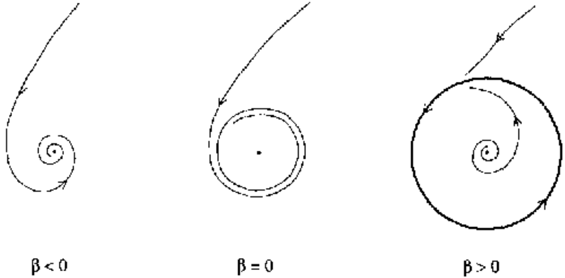
Hopf Bifurcation Theorem

$D > 0$ and $T = 0$.

Consider ODE $x' = f(\lambda, x)$, $\lambda \in \mathbb{R}$, $x \in \mathbb{R}^n$, and f is smooth.
(i) Suppose that for λ near λ_0 the system has a family of equilibria $x^0(\lambda)$.
(ii) Assume that its Jacobian matrix $A(\lambda) = f_x(\lambda, x^0(\lambda))$ has one pair of complex eigenvalues $\mu(\lambda) \pm i\omega(\lambda)$, $\mu(\lambda_0) = 0$, $\omega(\lambda_0) > 0$, and all other eigenvalues of $A(\lambda)$ have non-zero real parts for all λ near λ_0.

If $\mu'(\lambda_0) \neq 0$, then the system has a family of periodic orbits $(\lambda(s), x(s))$ for $s \in (0, \delta)$ with period $T(s)$, such that $\lambda(s) \to \lambda_0$, $T(s) \to 2\pi/\omega(\lambda_0)$, and $||x(s) - x^0(\lambda_0)|| \to 0$ as $s \to 0^+$.

http://www.scholarpedia.org/article/Andronov-Hopf_bifurcation



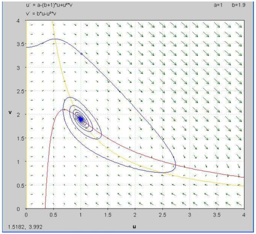
Example: Brusselator model

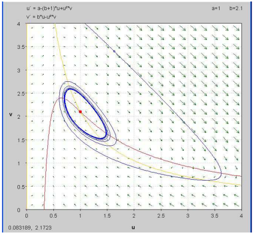
$$u' = a - (b+1)u + u^2 v, \quad v' = bu - u^2 v.$$

Unique equilibrium $(a, b/a)$. Jacobian matrix

$$L(b) = \begin{pmatrix} b-1 & a^2 \\ -b & -a^2 \end{pmatrix}$$

So $T = b - 1 - a^2 = b - (1 + a^2)$, $D = a^2$. So $D > 0$ always holds and $T = 0$ if $b = 1 + a^2$. Suppose the eigenvalues are $\lambda = \alpha(b) \pm \beta(b)$, then $\lambda_1 + \lambda_2 = 2\alpha(b) = T(b)$. Then $\alpha'(b) = T'(b)/2 = 1/2 > 0$. So a Hopf bifurcation occurs at $b = a^2 + 1$.





Tools of phase plane analysis

1. Jacobian for local stability of equilibria (last page);
2. Lyapunov functional method (LaSalle Invariance Principle) for global stability of equilibrium;
3. Poincaré-Bendixson Theorem for existence of periodic orbits;
4. Hopf Bifurcation Theorem for existence of periodic orbits;
5. Dulac Criterion for nonexistence of periodic orbits (and global stability of equilibrium;
6. Zhang (Zhifen)'s Uniqueness Theorem for periodic orbit of Lienard system (good for many predator-prey systems).

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Goal: to apply these tools to achieve global bifurcation of phase portraits when a parameter varies.

Important: The Poincaré-Bendixson theory shows that the ω-limit set for planar system can only be either (i) an equilibrium; or (ii) a periodic orbit; or (iii) a heteroclinic/homoclinic loop.

Rosenzwing-MacArthur Model

[Rosenzwing-MacArthur, Amer. Naturalist, 1963], [Rosenzweig, Science, 1971]

$$\begin{cases} x' &= x(1-\frac{x}{K}) - \frac{mxy}{1+x}, \\ y' &= -\theta y + \frac{mxy}{1+x}. \end{cases} \tag{4}$$

Equilibria: $(0,0)$, $(K,0)$, (λ, V_λ), where

$$\lambda = \frac{\theta}{m-\theta}, \; V_\lambda = \frac{(K-\lambda)(1+\lambda)}{Km},$$

We use λ (the horizontal coordinate of coexistence equilibrium) as bifurcation parameter.

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Case 1: $\lambda \geq K$ No coexistence equilibrium, and $(K,0)$ is globally stable. Use Lyapunov function: $V(x,y) = \int_K^x \frac{g(s)-g(K)}{g(s)} ds + y$, where $g(x) = \frac{mx}{1+x}$

Coexistence: global stability

$$J(\lambda, V_\lambda) = \begin{pmatrix} \dfrac{\lambda(K-1-2\lambda)}{K(1+\lambda)} & -\theta \\ \dfrac{K-\lambda}{K(1+\lambda)} & 0 \end{pmatrix}$$

For $0 < \lambda < K$, $\mathrm{Det}(J) = \dfrac{\theta(K-\lambda)}{K(1+\lambda)} > 0$, and $\mathrm{Tr}(J) = \dfrac{\lambda(K-1-2\lambda)}{K(1+\lambda)}$. Hence (λ, V_λ) is stable for $0 < \lambda < \dfrac{K-1}{2}$ and is unstable for $\dfrac{K-1}{2} < \lambda < K$.

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Case 2: $(K-1)/2 < \lambda < K$ (λ, V_λ) is locally stable. The global stability can be implied by using Dulac's criterion and the function $h(x,y) = \left(\dfrac{mx}{1+x}\right)^\alpha y^\delta$ for some appropriate α, δ [Hsu-Hubble-Waltman, 1978].

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Lyapunov functions can also be used to prove the global stability. For $K-1 \le \lambda < K$,
$V(x,y) = \displaystyle\int_x^\lambda \frac{g(s)-g(\lambda)}{g(s)}ds + \int_y^{V_\lambda} \frac{t-V_\lambda}{t}dt$ where $g(x) = \dfrac{mx}{1+x}$. [Hsu, 1978];
and for $(K-1)/2 < \lambda < K-1$,
$V(x,y) = y^\alpha \displaystyle\int_x^\lambda \frac{g(s)-g(\lambda)}{g(s)}ds + \int_y^{V_\lambda} t^{\alpha-1}(t-V_\lambda)dt$ for some $\alpha > 0$
[Ardito-Ricciardi, 1995].

Unique limit cycle

Case 3: $0 < \lambda < (K-1)/2$ (λ, V_λ) is unstable, hence there exists a periodic orbit from Poincaré-Bendixson Theorem. The system can be converted to a Lienard system so Zhang's uniqueness theorem implies the uniqueness (and also stability) of periodic orbit. [Cheng, 1981], [Kuang-Freedman, 1988], [Zhang, 1986]

Bifurcation diagram and phase portraits

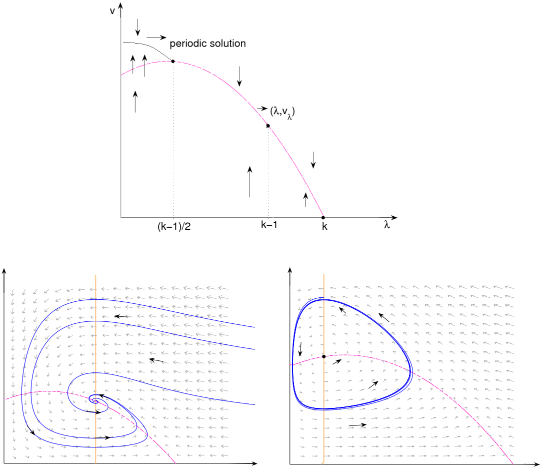


Figure: (Top) Bifurcation Diagram; (Lower Left)Phase portrait $(K-1)/2 < \lambda < K$; (Lower Right) Phase portrait $0 < \lambda < (K-1)/2$.

Summary of Rosenzwing-MacArthur Model

$$\frac{du}{dt} = u\left(1 - \frac{u}{K}\right) - \frac{muv}{1+u}, \quad \frac{dv}{dt} = -\theta v + \frac{muv}{1+u}$$

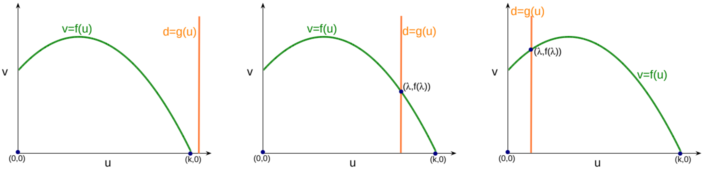
Nullcline: $u = 0$, $v = \dfrac{(K-u)(1+u)}{mK}$; $v = 0$, $d = \dfrac{mu}{1+u}$.

Solving $\theta = \dfrac{mu}{1+u}$, one have $u = \lambda \equiv \dfrac{\theta}{m-\theta}$.

Equilibria: $(0,0)$, $(K,0)$, (λ, V_λ) where $v_\lambda = \dfrac{(K-\lambda)(1+\lambda)}{mK}$

We take λ as a bifurcation parameter

Case 1: $\lambda \geq K$: $(K,0)$ is globally asymptotically stable
Case 2: $(K-1)/2 < \lambda < K$: (λ, V_λ) is globally asymptotically stable
Case 3: $0 < \lambda < (K-1)/2$: unique limit cycle is globally asymptotically stable
($\lambda = (K-1)/2$: Hopf bifurcation point)



Generalized Rosenzwing-MacArthur Model

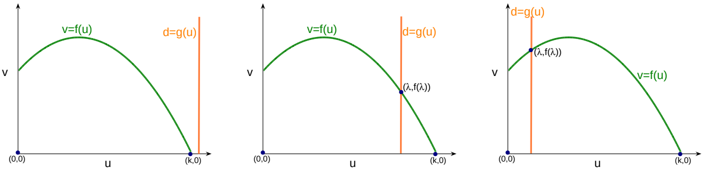
$$\begin{cases} u' = g(u)\,(f(u) - v)\,, \\ v' = v\,(g(u) - d)\,. \end{cases} \tag{5}$$

where f, g satisfy:

(a1) $f \in C^1(\overline{\mathbf{R}}^+)$, $f(0) > 0$, there exists $K > 0$, such that for any $u > 0$, $u \neq K$, $f(u)(u - K) < 0$ and $f(K) = 0$; there exists $\bar{\lambda} \in (0, K)$ such that $f'(u) > 0$ on $[0, \bar{\lambda})$, $f'(u) < 0$ on $(\bar{\lambda}, K]$;

(a2) $g \in C^1(\overline{\mathbf{R}}^+)$, $g(0) = 0$; $g(u) > 0$ for $u > 0$ and $g'(u) > 0$ for $u \geq 0$; there exists a unique $\lambda \in (0, K)$ such that $g(\lambda) = d$.

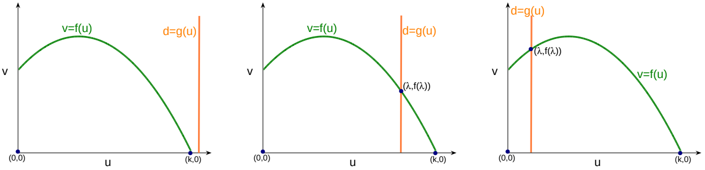
Unique positive equilibrium (λ, v_λ), where $d = g(\lambda)$, and $v_\lambda = f(\lambda)$.



Generalized Rosenzwing-MacArthur Model

Complete classification of dynamics:

1. when $\lambda \geq K$, $(K, 0)$ is globally asymptotically stable;
2. when $\lambda^0 \leq \lambda < K$, (λ, v_λ) is globally asymptotically stable, where $\lambda^0 \in (\bar{\lambda}, K)$ is the unique one satisfied $f(0) = f(\lambda^0)$;
3. when $\lambda^0 < \lambda < K$, if one of the followings holds:
   (i) $(uf'(u))'' \leq 0$, $(u/g(u))'' \geq 0$ for $u \in [0, K]$, and $(uf'(u))' \leq 0$ for $u \in (\bar{\lambda}, K)$; or
   (ii)$f'''(u) \leq 0$ and $g''(u) \leq 0$ for $u \in [0, K]$, and $f''(u) \leq 0$ for $u \in (\bar{\lambda}, K)$,
   then (λ, v_λ) is globally asymptotically stable.
4. $\bar{\lambda}$ is the unique bifurcation point where a backward Hopf bifurcation occurs;
5. when $0 < \lambda < \bar{\lambda}$, there is a globally asymptotically stable periodic orbit if $h_\lambda(u) = f'(u)g(u)/[g(u) - g(\lambda)]$ is nonincreasing in $(0, \lambda) \cup (\lambda, K)$.
   [Kuang-Freedman, 1988]



A Rosenzwing-MacArthur model with Allee effect

[Wang-Shi-Wei, 2011, JMB]

$$\begin{cases} \dfrac{du}{dt} = g(u)(f(u) - v), \\ \dfrac{dv}{dt} = v(g(u) - d), \end{cases} \tag{6}$$

where f, g satisfy: (here $\mathbb{R} = (0, \infty)$)

(a1) $f \in C^1(\overline{\mathbb{R}^+})$, $f(b) = f(1) = 0$, where $0 < b < 1$; $f(u)$ is positive for $b < u < 1$, and $f(u)$ is negative otherwise; there exists $\bar{\lambda} \in (b, 1)$ such that $f'(u) > 0$ on $[b, \bar{\lambda})$, $f'(u) < 0$ on $(\bar{\lambda}, 1]$;

(a2) $g \in C^1(\overline{\mathbb{R}^+})$, $g(0) = 0$; $g(u) > 0$ for $u > 0$ and $g'(u) > 0$ for $u > 0$, and there exists $\lambda > 0$ such that $g(\lambda) = d$.

$g(u)$: predator functional response
$g(u)f(u)$: the net growth rate of the prey
$f(u)$: the prey nullcline on the phase portrait
λ: a measure of how well the predator is adapted to the prey (main bifurcation parameter)

Complete classification: conditions

Theorem: Suppose that $f(u)$ satisfies

(a1') $f \in C^3(\overline{\mathbb{R}^+})$, $f(b) = f(1) = 0$, where $0 < b < 1$; $f(u)$ is positive for $b < u < 1$, and $f(u)$ is negative otherwise; there exists $\bar{\lambda} \in (b, 1)$ such that $f'(u) > 0$ on $[b, \bar{\lambda})$, $f'(u) < 0$ on $(\bar{\lambda}, 1]$;

(a3) $f''(\bar{\lambda}) < 0$;

(a6) $uf'''(u) + 2f''(u) \leq 0$ for all $u \in (b, 1)$;

and one of the following:

(a8) $(uf'(u))'' \leq 0$ for $u \in [b, 1]$, and $(uf'(u))' \leq 0$ for $u \in (\bar{\lambda}, 1)$; or

(a9) $f'''(u) \leq 0$ for $u \in [b, 1]$, and $f''(u) \leq 0$ for $u \in (\bar{\lambda}, 1)$,

and $g(u)$ is one of the following:

$$g(u) = u, \quad \text{or} \quad g(u) = \frac{mu}{a + u}, \quad a, m > 0.$$

Then $\cdots$

Complete classification: Results

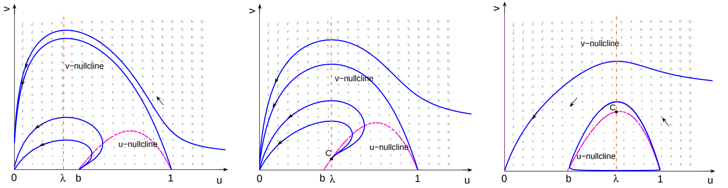
Theorem (cont.): Then with a bifurcation parameter λ defined by

$$\lambda = d \text{ if } g(u) = u, \quad \text{or } \lambda = \frac{ad}{m-d} \text{ if } g(u) = \frac{mu}{a+u},$$

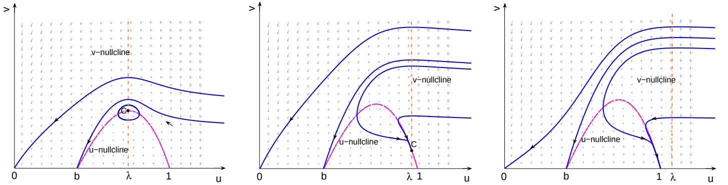
there exist two bifurcation points $\lambda^\sharp$ and $\bar{\lambda}$ such that the dynamics of the predator-prey system can be classified as follows:

1. If $0 < \lambda < \lambda^\sharp$, then the equilibrium $(0,0)$ is globally asymptotically stable;
2. If $\lambda = \lambda^\sharp$, then there exists a unique heteroclinic loop, and the system is globally bistable with respect to the heteroclinic loop and $(0,0)$;
3. If $\lambda^\sharp < \lambda < \bar{\lambda}$, then there exists a unique limit cycle, and the system is globally bistable with respect to the limit cycle and $(0,0)$;
4. If $\bar{\lambda} < \lambda < 1$, then there is no periodic orbit, and the system is globally bistable with respect to the coexistence equilibrium (λ, v_λ) and $(0,0)$;
5. If $\lambda > 1$, then the system is globally bistable with respect to $(1,0)$ and $(0,0)$.

Phase Portraits



Left: $\lambda < b$, Middle: $b < \lambda < \lambda^\sharp$, $(0,0)$ globally stable;
Right: $\lambda^\sharp < \lambda < \bar{\lambda}$ and close to $\lambda^\sharp$, large limit cycle



Left: $\lambda^\sharp < \lambda < \bar{\lambda}$ and close to Hopf point $\bar{\lambda}$, small limit cycle;
Middle: $\bar{\lambda} < \lambda < 1$, no limit cycle; Right: $\lambda > 1$

Ordinary Differential Equations

ODE model: $\frac{dy}{dt} = f(\lambda, y), \quad y \in \mathbb{R}^n,\ f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$

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$P(\lambda) = Det(\lambda I - J) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n$

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$n = 1$: $\lambda + a_1 = 0$, $\underline{a_1 > 0}$

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$n = 1$: $\lambda + a_1 = 0$, $\underline{a_1 > 0}$
$n = 2$: $\lambda^2 + a_1\lambda + a_2 = 0$, $\underline{a_1 > 0,\ a_2 > 0}$

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$n = 3$: $\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$, $\underline{a_1 > 0,\ a_2 > \dfrac{a_3}{a_1},\ a_3 > 0}$

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$n = 4$: $\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0$, $\underline{a_1 > 0,\ a_2 > \dfrac{a_3^2 + a_1^2 a_4}{a_1 a_3},\ a_3 > 0,\ a_4 > 0}$

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$n \geq 5$: check books

Bifurcation Analysis

- 1-D continuous ODE model can be analyzed by calculus. The asymptotical state must be an equilibrium.
- 2-D continuous ODE model can be analyzed by phase plane analysis and techniques mentioned above. The asymptotical state must be an equilibrium or a periodic orbit or a heteroclinic/homoclinic loop.
- N-D ($N \geq 3$) continuous ODE models (or N-D ($N \geq 1$) discrete models) can sometimes be analyzed, but it is known that chaos can occur for such models. Also more types of bifurcations can occur in these systems. We only consider the bifurcations of equilibria and periodic orbits.
- PDE or other structured models are different as the phase space is infinite dimensional.


Thank You!