Reaction-Diffusion Models and Bifurcation Theory
Lecture 1: Basics and Examples from ODEs

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Topic course in Fall 2007
Course Plan

1. Basics and Examples from ODEs
2. Diffusion, advection and cross-diffusion models
3. Delay, chemotaxis and nonlocal models
4. Linear stability (ODE, reaction-diffusion models)
5. Linear stability (delay models)
6. Numerical simulations
7. Analytic bifurcation theory for stationary problems
8. Turing bifurcation and pattern formation
9. Global bifurcation for stationary problems
10. Hopf bifurcation
Early Bifurcation: Buckling

In engineering, buckling is a failure mode characterized by a sudden failure of a structural member subjected to high compressive stresses, where the actual compressive stresses at failure are greater than the ultimate compressive stresses that the material is capable of withstanding. This mode of failure is also described as failure due to elastic instability.
Euler Buckling Equation

\[ \phi'' + \lambda \sin \phi = 0, \; x \in (0, \pi), \; \phi(0) = \phi(\pi) = 0 \]

\( \phi(x) \) is the angle between the tangent to the column’s axis and the \( y \)-axis, and \( \lambda \) is a parameter proportional to the thrust.

The word bifurcation is from the Latin \textit{bifurcus}, which means two forks.

[Da Vinci, \textit{Codex Madrid}, 1478-1519], [Euler, 1757], Galileo, Bernoulli
Da Vinci’s sketching showing buckling

Figure 4 Leonardo da Vinci: drawing to illustrate the bending of a column under a vertical weight [from Codex Madrid I [9], page 117].
Bifurcation in Natural World

Predator-prey interaction: Lotka-Volterra model
Classical examples: Hudson company lynx-hare data in 1800s

\[ \frac{du}{dt} = u(a - bu) - cuv, \quad \frac{dv}{dt} = -dv + fuv. \]

Alfred Lotka  Vito Volterra
Paradox of enrichment: \( \frac{dU}{ds} = \gamma U \left(1 - \frac{U}{K}\right) - \frac{CMUV}{A + U}, \quad \frac{dV}{ds} = -DV + \frac{MUV}{A + U}. \)

Environment is enriched if \( K \) (carrying capacity) is larger, but when \( K \) is small, a coexistence equilibrium is stable; but when \( K \) is larger, the coexistence equilibrium is unstable, and a stable periodic orbit appears as a result of Hopf bifurcation.


2013 is the year of Mathematics for Planet Earth. http://mpe2013.org

A sudden critical transition can bring catastrophic change to the earth ecosystem, thus it is important to predict and prevent it http://www.early-warning-signals.org

Critical Transitions in Nature and Society, by Marten Scheffer
Rich Spatial Patterns in Diffusive Predator-Prey System

Patterns generated by diffusive predator-prey system

\[
\begin{align*}
    u_t - d_1 \Delta u &= u(1 - u) - \frac{muv}{u + a}, & x \in \Omega, \ t > 0, \\
    v_t - d_2 \Delta v &= -\theta v + \frac{muv}{u + a}, & x \in \Omega, \ t > 0, \\
    \partial_{\nu} u &= \partial_{\nu} v = 0, & x \in \partial \Omega, \ t > 0, \\
    u(x, 0) &= u_0(x) \geq 0, \ v(x, 0) = v_0(x) \geq 0, & x \in \Omega.
\end{align*}
\]

Patchiness (spatial heterogeneity) of plankton distributions in phytoplankton-zooplankton interaction

[Medvinsky, et.al., SIAM Review 2002]
Spatiotemporal complexity of plankton and fish dynamics.
Spatial Model: Bifurcation from Grassland to Desert

\[
\frac{\partial w}{\partial t} = a - w - wn^2 + \gamma \frac{\partial w}{\partial x}, \quad \frac{\partial n}{\partial t} = wn^2 - mn + \Delta n, \quad x \in \Omega.
\]

\(w(x, y, t)\): concentration of water; \(n(x, y, t)\): concentration of plant, \(\Omega\): a two-dimensional domain.

\(a > 0\): rainfall; \(-w\): evaporation; \(-wn^2\): water uptake by plants; water flows downhill at speed \(\gamma\); \(wn^2\): plant growth; \(-mn\): plant loss

Regular and Irregular Patterns in Semiarid Vegetation.

Self-Organized Patchiness and Catastrophic Shifts in Ecosystems.
Stability of a Steady State Solution

For a continuous-time evolution equation \( \frac{du}{dt} = F(\lambda, u) \), where \( u \in X \) (state space), \( \lambda \in \mathbb{R} \), a steady state solution \( u^* \) is **locally asymptotically stable** (or just stable) if for any \( \varepsilon > 0 \), then there exists \( \delta > 0 \) such that when \( ||u(0) - u^*||_X < \delta \), then \( ||u(t) - u^*||_X < \varepsilon \) for all \( t > 0 \) and \( \lim_{t \to \infty} ||u(t) - u^*||_X = 0 \). Otherwise \( u^* \) is **unstable**.
Stability of a Steady State Solution

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**Basic Result:** If all the eigenvalues of linearized operator $D_u F(\lambda, u_*)$ have negative real parts, then $u_*$ is locally asymptotically stable.
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**Basic Result**: If all the eigenvalues of linearized operator \( D_u F(\lambda, u_*) \) have negative real parts, then \( u_* \) is locally asymptotically stable.

**Bifurcation** (change of stability): if when the parameter \( \lambda \) changes from \( \lambda_* - \epsilon \) to \( \lambda_* + \epsilon \), the steady state \( u_*(\lambda) \) changes from stable to unstable; and other special solutions (steady states, periodic orbits) may emerge from the known solution \( (\lambda, u_*(\lambda)) \).
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**Steady State Bifurcation** (transcritical/pitchfork): if \( 0 \) is an eigenvalue of \( D_u F(\lambda_*, u_*) \).

**Hopf Bifurcation**: if \( \pm ki \) is a pair of eigenvalues of \( D_u F(\lambda_*, u_*) \).
Structured Mathematical Model

The rate of change of a variable in the model may depend on the quantities of several independent variables.
Structured Mathematical Model

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Possible characters \( x \) (independent variables other than time \( t \)):
1. space location (spatially-structured),
2. age, sex, size of individuals (physiologically-structured, or stage-structured)
3. genetic phenotype (genetically structured)
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variable ($t$ or $x$) is a countable set ($\in \{1, 2, 3, 4, \cdots \}$) or a continuum ($\in (0, T)$)
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2. continuous time and discrete space: lattice differential equations (LDE)
(patch model or ODE system if the space is finite)
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3. discrete time and continuous space: integral projection model (IPM)
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4. discrete time and discrete space: coupled map lattices (CML) (coupled map if the space is finite)
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(variable $u$ is allowed to take continuous real-values)
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Events to be modeled (chemical): birth-death, chemical reaction
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Events to be modeled (physical): dispersal, transport, movement, (mutation)
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**Diffusion**: particles move from regions of higher concentration to regions of lower concentration (random walk, Brownian motion, Fick’s law)
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**Advection**: particles move along the currents in the fluid or along a chemical signal (convection, drift)
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**Long distance dispersal**: movement not necessarily to nearest neighbors
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General Form of the Model: \( u_t = f(t, x, u, u_x, u_{xx}) \)
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**General Form of the Model**: \( u_t = f(t, x, u, u_x, u_{xx}) \)

**Discrete case**: \( u_t = u(n, x) - u(n - 1, x) \),

\( u_x = u(n, x) - u(n, x - 1) \) (advection),

\( u_{xx} = u(n + 1, x) - 2u(n, x) + u(n, x - 1) \) (diffusion)
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**Continuous case**: \( u_t = \frac{du}{dt}, \ u_x = \frac{du}{dx}, \ u_{xx} = \frac{d^2u}{dx^2}. \)
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**continuous case**: \( u_t = \frac{du}{dt}, u_x = \frac{du}{dx}, u_{xx} = \frac{d^2u}{dx^2} \).

**local case**: \( f \) is a function \( f(t, x, u(x), u_x(x), u_{xx}(x)) \)

**nonlocal case**: \( f \) is an operator \( f : u \mapsto f(t, x, u) \)
Structured Mathematical Model: Examples

**Two-Patch model:** \( u_t = f(u) + d(v - u), \ v_t = f(v) + d(u - v) \)
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**Two-Patch model:** \( u_t = f(u) + d(v - u), \quad v_t = f(v) + d(u - v) \)

**Lattice stepping stone model:** \( (u_i)_t = (1 - 2d)u_i + du_{i-1} + du_{i+1} + f(u_i), \quad i = 0, \pm 1, \pm 2, \cdots \)
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**Partial differential equations:**

- **Diffusion:** \( u_t = Du_{xx} \) (\(x\) is 1-D location)
- **Advection:** \( u_t + u_a = f(t) \) (\(a\) is age, age-structured model)
- **Reaction-Diffusion model:** \( u_t = Du_{xx} + u(1 - u) \)
Structured Mathematical Model: Examples

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Integral-differential equation (non-local equation):

\( u_t = \int_{\Omega} k(x, y)u(y, t)dy \)
Structured Mathematical Model: Examples

Two-Patch model: \( u_t = f(u) + d(v - u), \ v_t = f(v) + d(u - v) \)
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Partial differential equations:
Diffusion: \( u_t = Du_{xx} \) (\( x \) is 1-D location)
Advection: \( u_t + u_a = f(t) \) (\( a \) is age, age-structured model)
Reaction-Diffusion model: \( u_t = Du_{xx} + u(1 - u) \)

Integral-differential equation (non-local equation):
\( u_t = \int_\Omega k(x, y)u(y, t)dy \)

Delay equation:
\( u_t = f(u(t), u(t - \tau)) \)

and combination of all above
Consider a scalar ODE
\[
\frac{du}{dt} = f(\lambda, u), \quad \lambda \in \mathbb{R}, u \in \mathbb{R}.
\]

If $f$ is smooth, and $u(t)$ is a bounded solution, then $\lim_{t \to \infty} u(t)$ is an equilibrium. (there is no limit cycle, no chaos)
ODE in $\mathbb{R}^1$

Consider a scalar ODE

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If $f$ is smooth, and $u(t)$ is a bounded solution, then $\lim_{t \to \infty} u(t)$ is an equilibrium. (there is no limit cycle, no chaos)

So the bifurcation of equilibria determines the entire dynamics.

Suppose that $y = y_0$ is an equilibrium.

1. If $f_u(\lambda_0, y_0) < 0$, then $y_0$ is stable (sink);
2. If $f_u(\lambda_0, y_0) > 0$, then $y_0$ is unstable (source).

Solving the bifurcation point:

$$f(\lambda, u) = 0, \quad \frac{\partial f}{\partial u}(\lambda, u) = 0.$$
Basic Bifurcation in $\mathbb{R}^1$

Consider

$$f(\lambda, u) = 0, \quad \lambda \in \mathbb{R}, u \in \mathbb{R}.$$
Basic Bifurcation in \( \mathbb{R}^1 \)

Consider

\[ f(\lambda, u) = 0, \quad \lambda \in \mathbb{R}, u \in \mathbb{R}. \]

Condition for bifurcation: \( f(\lambda_0, u_0) = 0, \quad f_u(\lambda_0, u_0) = 0. \)
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Condition for bifurcation: $f(\lambda_0, u_0) = 0, \quad f_u(\lambda_0, u_0) = 0$.

**Saddle-node bifurcation:**

$$f_{uu}(\lambda_0, u_0) \neq 0, \quad f_\lambda(\lambda_0, u_0) \neq 0.$$
Consider

\[ f(\lambda, u) = 0, \quad \lambda \in \mathbb{R}, u \in \mathbb{R}. \]

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**Saddle-node bifurcation:**

\[ f_{uu}(\lambda_0, u_0) \neq 0, \quad f_\lambda(\lambda_0, u_0) \neq 0. \]

Assume \( f(\lambda, u_0) = 0 \) for \( \lambda \in \mathbb{R}. \)

**Transcritical bifurcation:**

\[ f_{uu}(\lambda_0, u_0) \neq 0, \quad f_\lambda u(\lambda_0, u_0) \neq 0. \]
Basic Bifurcation in $\mathbb{R}^1$

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Transcritical bifurcation:

$$f_{uu}(\lambda_0, u_0) \neq 0, \quad f_{\lambda u}(\lambda_0, u_0) \neq 0.$$ 

Pitchfork bifurcation:

$$f_{uu}(\lambda_0, u_0) = 0, \quad f_{\lambda u}(\lambda_0, u_0) \neq 0, \quad \text{and} \quad f_{uuu}(\lambda_0, u_0) \neq 0.$$
Basic Bifurcation in $\mathbb{R}^1$

Consider

$$f(\lambda, u) = 0, \quad \lambda \in \mathbb{R}, u \in \mathbb{R}.$$ 

Condition for bifurcation: $f(\lambda_0, u_0) = 0$, $f_u(\lambda_0, u_0) = 0$.

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$$f_{uu}(\lambda_0, u_0) \neq 0, \quad f_\lambda(\lambda_0, u_0) \neq 0.$$

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$$f_{uu}(\lambda_0, u_0) \neq 0, \quad f_\lambda(\lambda_0, u_0) \neq 0.$$

**Pitchfork bifurcation:**

$$f_{uu}(\lambda_0, u_0) = 0, \quad f_\lambda(\lambda_0, u_0) \neq 0, \quad \text{and} \quad f_{uuu}(\lambda_0, u_0) \neq 0.$$

**Figure:** Left: saddle-node; Middle: transcritical; Right: pitchfork
Names of bifurcations

**Saddle-node bifurcation** (fold bifurcation, blue sky bifurcation)

A curve \((\lambda(s), u(s))\) near \((\lambda_0, u_0)\):
- \(\lambda(s) > \lambda_0\): supercritical (forward)
- \(\lambda(s) < \lambda_0\): subcritical (backward)

**Transcritical bifurcation/pitchfork bifurcation** near \((\lambda_0, u_0)\):
- Trivial solutions \((\lambda, u_0)\) and nontrivial solutions on a curve \((\lambda(s), u(s))\)
- \(\lambda(s) > \lambda_0\): supercritical (forward) pitchfork
- \(\lambda(s) < \lambda_0\): subcritical (backward) pitchfork

If only "half" of the nontrivial solution curve is of physical importance
- \(\lambda(s) > \lambda_0\): supercritical (forward) transcritical
- \(\lambda(s) < \lambda_0\): subcritical (backward) transcritical

In the transcritical bifurcation/pitchfork bifurcation, sometimes people only use forward/backward as above, but use supercritical (subcritical) if the bifurcating solutions are stable (unstable)
Cusp bifurcation (two parameter bifurcation)

\[ y' = f(\varepsilon, \lambda, y). \]

Left: \( \varepsilon < 0 \) (monotone), middle: \( \varepsilon = 0 \), right: \( \varepsilon > 0 \) (with two saddle-node bifurcation points)

\[ \frac{\partial f}{\partial \lambda}(\varepsilon, \lambda_0, y_0) \neq 0, \quad \frac{\partial^2 f}{\partial y^2}(\varepsilon, \lambda_0, y_0) = 0, \quad \frac{\partial^3 f}{\partial y^3}(\varepsilon, \lambda_0, y_0) \neq 0. \]

http://www.scholarpedia.org/article/Cusp_bifurcation
Imperfect bifurcation (two parameter bifurcation)

\[ y' = f(\varepsilon, \lambda, y). \]

For \( \varepsilon = \varepsilon_0 \), there is a transcritical bifurcation or pitchfork bifurcation. What happens when \( \varepsilon \) is near \( \varepsilon_0 \)?

Typical symmetry breaking of transcritical/pitchfork bifurcation
Saddle-node Bifurcation in $\mathbb{R}^1$

Annual catch of the Peruvian Anchovy Fishery from 1960-1990
Saddle-node Bifurcation in $\mathbb{R}^1$

Annual catch of the Peruvian Anchovy Fishery from 1960-1990

\[
\frac{dP}{dt} = kP \left(1 - \frac{P}{N}\right) - H, \quad \text{steady state: } kP \left(1 - \frac{P}{N}\right) - H = 0
\]

When $H > H_0 \equiv \frac{kN}{4}$, the fishery collapses.

$H_0$ is the maximum sustainable yield (MSY)
A grazing system of herbivore-plant interaction
\[
\frac{dV}{dt} = V(1 - V) - \frac{rV^p}{h^p + V^p}, \quad h, r > 0, \quad p \geq 1.
\]

[May, Nature 1975]
Thresholds and breakpoints in ecosystems with a multiplicity of stable states.

**Catastrophe theory:** Thom, Arnold, Zeeman in 1960-70s
Potential catastrophes: Arctic sea, Greenland ice, Amazon rainforest, etc.
Population models

\[ y' = yf(y), \quad f(y) \text{ is the growth rate per capita} \]
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Generalizing the model:
\[ y' = g(y) - rc(y) \quad (g(y) \text{ growth function; } c(y) \text{ predator functional response; } r \text{ number of predator (bifurcation parameter).}) \]
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\[ y' = g(y) - rc(y) \quad (g(y) \text{ growth function; } c(y) \text{ predator functional response; } r \text{ number of predator (bifurcation parameter)}) \]

Growth function:
Logistic: \( g(y) = yh(y) \): \( h(0) > 0, \; h(N) = 0 \) and \( h(y) \) is decreasing in \((0, N)\).
Allee effect: \( g(y) = yh(y) \), \( \max h(y) > 0, \; h(N) = 0, \; h(y) \) is increasing in \((0, M)\), and \( h(y) \) is decreasing in \((M, N)\).
Functional Responses

[Holling, 1959]

(Type I)

\[ c(y) = \begin{cases} 
    Ey, & 0 < y \leq y^*, \\
    Ey^*, & y > y^* 
\end{cases}, \tag{1} \]

(Type II)

\[ c(y) = \frac{Ay}{B + y}, \quad y > 0, \tag{2} \]

(Type III)

\[ c(y) = \frac{Ay^p}{B^p + y^p}, \quad y > 0, \quad p > 1, \tag{3} \]

Their common characters:

(C1) \( c(0) = 0; \) \( c(y) \) in increasing;

(C2) \( \lim_{y \to \infty} c(y) = c_\infty > 0. \)

Another type II: (Ivlev) \( c(y) = A - Be^{-ry}. \)
Functional Responses

**Generalized Type II:** $c(y)$ satisfies (C1), (C2) and (C3) $c'(0) > 0$, and $c''(y) \leq 0$ for almost every $y \geq 0$;

**Generalized Type III:** $c(y)$ satisfies (C1), (C2) and (C4) $c'(0) = 0$ and there exists $y_* > 0$ such that $c''(y)(y - y_*) \leq 0$ for almost every $y \geq 0$.

**Figure:** (a) constant; (b) linear; (c) Type I; (d) Type II; (e) Type III.
Bifurcation diagram for ecological model

Bifurcation diagram of $y' = g(y) - rc(y)$ can be drawn using the graph of $r = g(y)/c(y)$. But it does not work for systems or PDEs!

$g(y)$ is Logistic, $c(y)$ is Holling type II. Here $r_* = g'(0)/c'(0)$ is a transcritical bifurcation point on $y = 0$ (trivial solution). ($r(y) = g(y)/c(y)$ is concave)

$g(y)$ is Logistic, $c(y)$ is Holling type III. ($r(y) = g(y)/c(y)$ is convex-concave)
Linear systems

For a planar linear system

\[
\begin{align*}
    x' &= ax + by \\
    y' &= cx + dy
\end{align*}
\]
Linear systems

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\[
\begin{align*}
    x' &= ax + by \\
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\end{align*}
\]

An eigenvalue \( \lambda \) satisfies \( \lambda^2 - (a + d)\lambda + ad - bc = 0 \), or let \( T = a + d \) be the trace and let \( D = ad - bc \) be the determinant of the matrix, then

\[
\lambda^2 - T\lambda + D = 0.
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\[
\lambda^2 - T\lambda + D = 0.
\]

Since \( \lambda_1 + \lambda_2 = T \), and \( \lambda_1\lambda_2 = D \), then generic cases are

(A) \( T > 0, \ D > 0, \ T^2 - 4D > 0 \), and \( \lambda_1 > \lambda_2 > 0 \);

(B) \( T < 0, \ D > 0, \ T^2 - 4D > 0 \), and \( \lambda_1 < \lambda_2 < 0 \);

(C) \( D < 0 \), and \( \lambda_1 < 0 < \lambda_2 \);

(D) \( T > 0, \ D > 0, \ T^2 - 4D < 0 \), and \( \lambda_1, \lambda_2 = \alpha \pm i\beta, \ \alpha > 0 \);

(E) \( T < 0, \ D > 0, \ T^2 - 4D < 0 \), and \( \lambda_1, \lambda_2 = \alpha \pm i\beta, \ \alpha < 0 \).

T-D plane (Trace-Determinant plane)
Phase portraits for linear systems

Figure: Phase portraits: (A) $\lambda_1 > \lambda_2 > 0$, (source or unstable node), (B) $\lambda_1 < \lambda_2 < 0$ (sink or stable node), (C) $\lambda_1 < 0 < \lambda_2$ (saddle); (D) $\lambda_1, \lambda_2 = \alpha \pm i\beta$, $\alpha < 0$ (spiral sink or stable spiral), (E) $\lambda_1, \lambda_2 = \alpha \pm i\beta$, $\alpha > 0$ (spiral source or unstable spiral)
Nonlinear planar systems

\[
\begin{align*}
  x' &= f(x, y) \\
  y' &= g(x, y).
\end{align*}
\]

If \((x_0, y_0) \in \mathbb{R}^2\) satisfies \(f(x_0, y_0) = 0\) and \(g(x_0, y_0) = 0\), then \((x_0, y_0)\) is an equilibrium point. The Jacobian matrix \(J\) at \((x_0, y_0)\) is

\[
J = \begin{pmatrix}
  \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\
  \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0)
\end{pmatrix}.
\]

Then the stability of \((x_0, y_0)\) is determined by the linear system \(Y' = J \cdot Y\).
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**Stability Criterion:**

(a) If \(T < 0\) and \(D > 0\), then \((x_0, y_0)\) is stable;
(b) If \(T < 0\), \(D > 0\) or \(D < 0\), then \((x_0, y_0)\) is unstable;
(c) If \(T = 0\) or \(D = 0\), then the stability cannot be determined by \(J\).
**Equilibrium Bifurcation Theorem**

\[ D = 0 \text{ and } T \neq 0. \]

Consider ODE \( x' = f(\lambda, x) \), \( \lambda \in \mathbb{R} \), \( x \in \mathbb{R}^n \), and \( f \) is smooth.

(i) Suppose that for \( \lambda \) near \( \lambda_0 \) the system has a family of equilibria \( x^0(\lambda) \).

(ii) Assume that its Jacobian matrix \( A(\lambda) = f_x(\lambda, x^0(\lambda)) \) has an eigenvalue \( \mu(\lambda) \), \( \mu(\lambda_0) = 0 \), and all other eigenvalues of \( A(\lambda) \) have non-zero real parts for all \( \lambda \) near \( \lambda_0 \).

If \( \mu'(\lambda_0) \neq 0 \), then the system has another family of equilibria \( (\lambda(s), x(s)) \) for \( s \in (-\delta, \delta) \), such that \( \lambda(0) = \lambda_0 \) and \( x(0) = x^0(\lambda_0) \).

This corresponds to the transcritical bifurcation and pitchfork bifurcation. One can also define saddle-node bifurcation (in which there is given family of equilibria).
Hopf Bifurcation Theorem

\[ D > 0 \text{ and } T = 0. \]

Consider ODE \( x' = f(\lambda, x) \), \( \lambda \in \mathbb{R} \), \( x \in \mathbb{R}^n \), and \( f \) is smooth.  
(i) Suppose that for \( \lambda \) near \( \lambda_0 \) the system has a family of equilibria \( x^0(\lambda) \).  
(ii) Assume that its Jacobian matrix \( A(\lambda) = f_x(\lambda, x^0(\lambda)) \) has one pair of complex eigenvalues \( \mu(\lambda) \pm i\omega(\lambda) \), \( \mu(\lambda_0) = 0 \), \( \omega(\lambda_0) > 0 \), and all other eigenvalues of \( A(\lambda) \) have non-zero real parts for all \( \lambda \) near \( \lambda_0 \).  

If \( \mu'(\lambda_0) \neq 0 \), then the system has a family of periodic orbits \( (\lambda(s), x(s)) \) for \( s \in (0, \delta) \) with period \( T(s) \), such that \( \lambda(s) \to \lambda_0 \), \( T(s) \to 2\pi/\omega(\lambda_0) \), and \( ||x(s) - x^0(\lambda_0)|| \to 0 \) as \( s \to 0^+ \).

http://www.scholarpedia.org/article/Andronov-Hopf_bifurcation
Example: Brusselator model

\[ u' = a - (b + 1)u + u^2v, \quad v' = bu - u^2v. \]

Unique equilibrium \((a, b/a)\). Jacobian matrix

\[ L(b) = \begin{pmatrix} b - 1 & a^2 \\ -b & -a^2 \end{pmatrix} \]

So \( T = b - 1 - a^2 = b - (1 + a^2) \), \( D = a^2 \). So \( D > 0 \) always holds and \( T = 0 \) if \( b = 1 + a^2 \). Suppose the eigenvalues are \( \lambda = \alpha(b) \pm \beta(b) \), then \( \lambda_1 + \lambda_2 = 2\alpha(b) = T(b) \). Then \( \alpha'(b) = T'(b)/2 = 1/2 > 0 \). So a Hopf bifurcation occurs at \( b = a^2 + 1 \).
Tools of phase plane analysis

1. Jacobian for local stability of equilibria (last page);
2. Lyapunov functional method (LaSalle Invariance Principle) for global stability of equilibrium;
3. Poincaré-Bendixson Theorem for existence of periodic orbits;
4. Hopf Bifurcation Theorem for existence of periodic orbits;
5. Dulac Criterion for nonexistence of periodic orbits (and global stability of equilibrium;
6. Zhang (Zhifen)’s Uniqueness Theorem for periodic orbit of Lienard system (good for many predator-prey systems).
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**Goal**: to apply these tools to achieve global bifurcation of phase portraits when a parameter varies.

**Important**: The Poincaré-Bendixson theory shows that the $\omega$-limit set for planar system can only be either (i) an equilibrium; or (ii) a periodic orbit; or (iii) a heteroclinic/homoclinic loop.
Rosenzwing-MacArthur Model


\begin{align}
  x' &= x(1 - \frac{x}{K}) - \frac{mxy}{1 + x}, \\
  y' &= -\theta y + \frac{mxy}{1 + x}.
\end{align}

Equilibria: (0, 0), (K, 0), (\lambda, V_\lambda), where

\begin{align*}
  \lambda &= \frac{\theta}{m - \theta}, \\
  V_\lambda &= \frac{(K - \lambda)(1 + \lambda)}{Km},
\end{align*}

We use \(\lambda\) (the horizontal coordinate of coexistence equilibrium) as bifurcation parameter.
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\]

We use \(\lambda\) (the horizontal coordinate of coexistence equilibrium) as bifurcation parameter.

**Case 1: \(\lambda \geq K\)** No coexistence equilibrium, and \((K, 0)\) is globally stable. Use Lyapunov function: 

\[
V(x, y) = \int_{K}^{x} \frac{g(s) - g(K)}{g(s)} ds + y, \text{ where } g(x) = \frac{mx}{1 + x}
\]
Coexistence: global stability

\[ J(\lambda, V_\lambda) = \begin{pmatrix} \frac{\lambda(K - 1 - 2\lambda)}{K(1 + \lambda)} & -\theta \\ \frac{\lambda(K - 1 - 2\lambda)}{K - \lambda} & 0 \end{pmatrix} \]

For \(0 < \lambda < K\), \(\text{Det}(J) = \frac{\theta(K - \lambda)}{K(1 + \lambda)} > 0\), and \(\text{Tr}(J) = \frac{\lambda(K - 1 - 2\lambda)}{K(1 + \lambda)}\). Hence \((\lambda, V_\lambda)\) is stable for \(0 < \lambda < \frac{K - 1}{2}\) and is unstable for \(\frac{K - 1}{2} < \lambda < K\).
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Case 2: \((K - 1)/2 < \lambda < K\) \((\lambda, V_\lambda)\) is locally stable. The global stability can be implied by using Dulac's criterion and the function \(h(x, y) = \left(\frac{mx}{1 + x}\right)^\alpha y^\delta\) for some appropriate \(\alpha, \delta\) [Hsu-Hubble-Waltman, 1978].
Coexistence: global stability

\[ J(\lambda, V_\lambda) = \begin{pmatrix} \frac{\lambda(K - 1 - 2\lambda)}{K(1 + \lambda)} & -\theta \\ \frac{K}{K(1 + \lambda)} & 0 \end{pmatrix} \]

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Lyapunov functions can also be used to prove the global stability. For \(K - 1 \leq \lambda < K\),
\[ V(x, y) = \int_x^\lambda \frac{g(s) - g(\lambda)}{g(s)} ds + \int_y^{V_\lambda} \frac{t - V_\lambda}{t} dt \] where \(g(x) = \frac{mx}{1 + x}\) [Hsu, 1978]; and for \((K - 1)/2 < \lambda < K - 1\),
\[ V(x, y) = y^\alpha \int_x^\lambda \frac{g(s) - g(\lambda)}{g(s)} ds + \int_y^{V_\lambda} t^{\alpha - 1}(t - V_\lambda) dt \] for some \(\alpha > 0\) [Ardito-Ricciardi, 1995].
Unique limit cycle

Case 3: $0 < \lambda < (K - 1)/2$ $(\lambda, V_{\lambda})$ is unstable, hence there exists a periodic orbit from Poincaré-Bendixson Theorem. The system can be converted to a Lienard system so Zhang’s uniqueness theorem implies the uniqueness (and also stability) of periodic orbit. [Cheng, 1981], [Kuang-Freedman, 1988], [Zhang, 1986]
Bifurcation diagram and phase portraits

Figure: (Top) Bifurcation Diagram; (Lower Left) Phase portrait 
$(K - 1)/2 < \lambda < K$; (Lower Right) Phase portrait $0 < \lambda < (K - 1)/2$. 
Summary of Rosenzwing-MacArthur Model

\[ \frac{du}{dt} = u \left( 1 - \frac{u}{K} \right) - \frac{muv}{1 + u}, \quad \frac{dv}{dt} = -\theta v + \frac{muv}{1 + u} \]

Nullcline: \( u = 0, \quad v = \frac{(K - u)(1 + u)}{mK} \); \( v = 0, \quad d = \frac{mu}{1 + u} \).

Solving \( \theta = \frac{mu}{1 + u} \), one have \( u = \lambda \equiv \frac{\theta}{m - \theta} \).

Equilibria: \((0, 0), (K, 0), (\lambda, V_\lambda)\) where \( V_\lambda = \frac{(K - \lambda)(1 + \lambda)}{mK} \)

We take \( \lambda \) as a bifurcation parameter

Case 1: \( \lambda \geq K \): \((K, 0)\) is globally asymptotically stable
Case 2: \( (K - 1)/2 < \lambda < K \): \((\lambda, V_\lambda)\) is globally asymptotically stable
Case 3: \( 0 < \lambda < (K - 1)/2 \): unique limit cycle is globally asymptotically stable
(\( \lambda = (K - 1)/2 \): Hopf bifurcation point)
Generalized Rosenzwing-MacArthur Model

\[
\begin{align*}
    u' &= g(u) (f(u) - v), \\
    v' &= v (g(u) - d).
\end{align*}
\]  

(5)

where \( f, g \) satisfy:

(a1) \( f \in C^1(\mathbb{R}^+) \), \( f(0) > 0 \), there exists \( K > 0 \), such that for any \( u > 0 \), \( u \neq K \), \( f(u)(u - K) < 0 \) and \( f(K) = 0 \); there exists \( \lambda \in (0, K) \) such that \( f'(u) > 0 \) on \([0, \lambda]\), \( f'(u) < 0 \) on \((\lambda, K]\);

(a2) \( g \in C^1(\mathbb{R}^+) \), \( g(0) = 0 \); \( g(u) > 0 \) for \( u > 0 \) and \( g'(u) > 0 \) for \( u \geq 0 \); there exists a unique \( \lambda \in (0, K) \) such that \( g(\lambda) = d \).

Unique positive equilibrium \((\lambda, v_\lambda)\), where \( d = g(\lambda) \), and \( v_\lambda = f(\lambda) \).
Generalized Rosenzwing-MacArthur Model

Complete classification of dynamics:

1. When \( \lambda \geq K \), \((K, 0)\) is globally asymptotically stable;
2. When \( \lambda^0 \leq \lambda < K \), \((\lambda, v_\lambda)\) is globally asymptotically stable, where \( \lambda^0 \in (\bar{\lambda}, K) \) is the unique one satisfied \( f(0) = f(\lambda^0) \);
3. When \( \lambda^0 < \lambda < K \), if one of the followings holds:
   (i) \((uf'(u))'' \leq 0\), \((u/g(u))'' \geq 0\) for \( u \in [0, K]\), and \((uf'(u))' \leq 0\) for \( u \in (\bar{\lambda}, K)\); or
   (ii) \(f'''(u) \leq 0\) and \(g''(u) \leq 0\) for \( u \in [0, K]\), and \(f''(u) \leq 0\) for \( u \in (\bar{\lambda}, K)\), then \((\lambda, v_\lambda)\) is globally asymptotically stable.
4. \(\bar{\lambda}\) is the unique bifurcation point where a backward Hopf bifurcation occurs;
5. When \( 0 < \lambda < \bar{\lambda}\), there is a globally asymptotically stable periodic orbit if \(h_\lambda(u) = f'(u)g(u)/[g(u) - g(\lambda)]\) is nonincreasing in \((0, \lambda) \cup (\lambda, K)\).

[Kuang-Freedman, 1988]
A Rosenzwing-MacArthur model with Allee effect

[Wang-Shi-Wei, 2011, JMB]

\[
\begin{align*}
\frac{du}{dt} &= g(u)(f(u) - v), \\
\frac{dv}{dt} &= v(g(u) - d),
\end{align*}
\]

(6)

where \(f, g\) satisfy: (here \(\mathbb{R} = (0, \infty)\))

(a1) \(f \in C^1(\mathbb{R}^+)\), \(f(b) = f(1) = 0\), where \(0 < b < 1\); \(f(u)\) is positive for \(b < u < 1\), and \(f(u)\) is negative otherwise; there exists \(\lambda \in (b, 1)\) such that \(f'(u) > 0\) on \([b, \lambda)\), \(f'(u) < 0\) on \((\lambda, 1]\);

(a2) \(g \in C^1(\mathbb{R}^+)\), \(g(0) = 0\); \(g(u) > 0\) for \(u > 0\) and \(g'(u) > 0\) for \(u > 0\), and there exists \(\lambda > 0\) such that \(g(\lambda) = d\).

\(g(u)\): predator functional response
\(g(u)f(u)\): the net growth rate of the prey
\(f(u)\): the prey nullcline on the phase portrait
\(\lambda\): a measure of how well the predator is adapted to the prey (main bifurcation parameter)
Theorem: Suppose that \( f(u) \) satisfies

\( (a1') \) \( f \in C^3(\mathbb{R}^+) \), \( f(b) = f(1) = 0 \), where \( 0 < b < 1 \); \( f(u) \) is positive for \( b < u < 1 \), and \( f(u) \) is negative otherwise; there exists \( \bar{\lambda} \in (b, 1) \) such that \( f'(u) > 0 \) on \( [b, \bar{\lambda}) \), \( f'(u) < 0 \) on \( (\bar{\lambda}, 1] \);

\( (a3) \) \( f''(\bar{\lambda}) < 0 \);

\( (a6) \) \( uf'''(u) + 2f''(u) \leq 0 \) for all \( u \in (b, 1) \);

and one of the following:

\( (a8) \) \( (uf'(u))'' \leq 0 \) for \( u \in [b, 1] \), and \( (uf'(u))' \leq 0 \) for \( u \in (\bar{\lambda}, 1) \); or

\( (a9) \) \( f'''(u) \leq 0 \) for \( u \in [b, 1] \), and \( f''(u) \leq 0 \) for \( u \in (\bar{\lambda}, 1) \),

and \( g(u) \) is one of the following:

\[
g(u) = u, \quad \text{or} \quad g(u) = \frac{mu}{a + u}, \quad a, m > 0.
\]

Then \( \cdots \)
Complete classification: Results

**Theorem (cont.):** Then with a bifurcation parameter $\lambda$ defined by

$$
\lambda = d \quad \text{if} \quad g(u) = u, \quad \text{or} \quad \lambda = \frac{ad}{m - d} \quad \text{if} \quad g(u) = \frac{mu}{a + u},
$$

there exist two bifurcation points $\lambda^\#$ and $\bar{\lambda}$ such that the dynamics of the predator-prey system can be classified as follows:

1. If $0 < \lambda < \lambda^\#$, then the equilibrium $(0, 0)$ is globally asymptotically stable;
2. If $\lambda = \lambda^\#$, then there exists a unique heteroclinic loop, and the system is globally bistable with respect to the heteroclinic loop and $(0, 0)$;
3. If $\lambda^\# < \lambda < \bar{\lambda}$, then there exists a unique limit cycle, and the system is globally bistable with respect to the limit cycle and $(0, 0)$;
4. If $\bar{\lambda} < \lambda < 1$, then there is no periodic orbit, and the system is globally bistable with respect to the coexistence equilibrium $(\lambda, v_\lambda)$ and $(0, 0)$;
5. If $\lambda > 1$, then the system is globally bistable with respect to $(1, 0)$ and $(0, 0)$. 
Phase Portraits

Left: $\lambda < b$, Middle: $b < \lambda < \lambda^\#$, $(0,0)$ globally stable; Right: $\lambda^\# < \lambda < \bar{\lambda}$ and close to $\lambda^\#$, large limit cycle

Left: $\lambda < \lambda < \bar{\lambda}$ and close to Hopf point $\bar{\lambda}$, small limit cycle; Middle: $\bar{\lambda} < \lambda < 1$, no limit cycle; Right: $\lambda > 1$
Ordinary Differential Equations

ODE model: \( \frac{dy}{dt} = f(\lambda, y), \quad y \in \mathbb{R}^n, \ f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n \)
Ordinary Differential Equations

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\( n = 4: \lambda^4 + a_1 \lambda^3 + a_2 \lambda^2 + a_3 \lambda + a_4 = 0, a_1 > 0, a_2 > \frac{a_3^2 + a_1^2 a_4}{a_1 a_3}, a_3 > 0, a_4 > 0 \)
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\( n \geq 5: \) check books
Bifurcation Analysis

- 1-D continuous ODE model can be analyzed by calculus. The asymptotical state must be an equilibrium.
- 2-D continuous ODE model can be analyzed by phase plane analysis and techniques mentioned above. The asymptotical state must be an equilibrium or a periodic orbit or a heteroclinic/homoclinic loop.
- $N$-D ($N \geq 3$) continuous ODE models (or $N$-D ($N \geq 1$) discrete models) can sometimes be analyzed, but it is known that chaos can occur for such models. Also more types of bifurcations can occur in these systems. We only consider the bifurcations of equilibria and periodic orbits.
- PDE or other structured models are different as the phase space is infinite dimensional.
Thank You!