The Hopf Bifurcation Theorem in Infinite Dimensions

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Introduction

The Hopf Bifurcation Theorem is the simplest result which guarantees the bifurcation of a family of time periodic solutions of an evolution equation from a family of equilibrium solutions. In this paper we will prove an infinite dimensional version of this theorem. Certain symmetry properties of the bifurcating family of solutions will also be presented together with some new quantitative and qualitative results concerning their linearized stability.

HOPF's pioneering paper [12] giving the basic results on time periodic bifurcation, (i.e. existence and uniqueness, symmetry properties, and stability of the solutions), appeared in 1942. Since then a substantial literature on generalizations and related problems has developed. See e.g. [2, 7, 10, 13–16, 19, 26] and the references cited there. In recent years a considerable amount of work has gone into proving analogues of the Hopf theorem in the context of the Navier-Stokes equations or of abstract evolution equations. It is difficult to compare those papers since they are set in different technical frameworks and have related but differing hypotheses. Some authors [e.g. 10, 13, 20, 23], following HOPF, have approached the bifurcation problem by trying to vary initial conditions and parameters so as to produce a nontrivial time periodic solution. Others [e.g. 7, 15, 19, 24] have introduced the unknown period explicitly as a new parameter in the equations and attempted to find solutions having a known period. We also take the latter approach. Moreover, as in most of these papers, our proof of the existence assertions uses the implicit function theorem.

In earlier work, IUDOVICH [16] treated the Navier-Stokes equations as well as a more general family of evolution equations by working directly in a class of periodic functions. He used the Lyapunov-Schmidt method to reduce the infinite dimensional problem to a one (complex) dimensional problem, which he then solved via the implicit function theorem. He obtained the existence of a bifurcating family of solutions and studied its symmetry properties. Independently of IUDOVICH, SATTINGER [24], and later JOSEPH & SATTINGER [19], studied the Navier-Stokes equations (with a more general class of nonlinearities than the usual one). They also worked directly in a class of periodic functions (but employ Schauder type spaces rather than Sobolev spaces) and use the implicit function theorem to get existence and some symmetry properties for their case. IOOSS [14] obtained an existence result in a Hilbert space setting for a class of equations having real analytic nonlinearities. He combined techniques of HOPF and IUDOVICH in setting up a mapping somewhat like HOPF, but using the Lyapunov-Schmidt procedure as did IUDOVICH. He required our condition (Hβ) (see Section 1) as well as another condition for his work. HENRY [10] treated an abstract evolution equation in a Banach space under hypotheses somewhat like our (HL) (see Section 1) and gets existence of a bifurcating family of solutions using an infinite dimensional version of the center manifold theorem to reduce the problem to a two (real) dimensional one.

Our existence results are clearly related to several previous works [7, 10, 13, 19, 24] but they are more general. We also feel our proofs are more transparent. The existence machinery has the added advantage of being immediately applicable to the linearized stability problem. The stability results given here are new and were motivated by recent work of JOSEPH [18] and our earlier work [4]. They show a precise relationship between the shape of the bifurcating curve and the critical Floquet exponents of the corresponding periodic solutions.

The existence, uniqueness and symmetry results will be presented in Section 1. Linearized stability is studied in Section 2.

This paper is a contracted version of [5], which also presents much simplified proofs of the main results in the finite dimensional case. A summary of our results ([6]) was presented at the International Symposium on Dynamical Systems, Gainsville, Florida, 1976. At that meeting we learned of H. WEINBERGER's approach [27] to the stability material of Section 2, in which the condition (Hβ) of Section 1 is relaxed. Also, subsequent to [5], a volume [21] on the Hopf Bifurcation Theorem has appeared. The invariant manifold approach emphasized in [21] (and also used in [10]) does not appear to be directly applicable in the current generality.

Section 1. The Bifurcation Theorem

We discuss the Hopf bifurcation theorem in infinite dimensions for equations of the form

$$\frac{du}{dt}+Lu+f(\mu,u)=0 \tag{1.1}$$

under assumptions on L and f detailed below. The reader unfamiliar with the finite dimensional case may find that the hypotheses below seem unmotivated. Some motivation can be found in [5], where a simple treatment of the classical Hopf theorem in the spirit of this paper is also given. The framework in which we treat (1.1) involves analytic semigroups and fractional powers of unbounded operators. Familiarity with these notions will be assumed below (see, for example, [8] for more details on the results we will use).

Let X be a real Banach space with the complexification $X_c = X + iX$. If L is a linear operator on X, L will also denote its extension to a linear operator on X_c, $N(L)$ and $R(L)$ will denote its null space and range in X, while $N_c(L)$ and $R_c(L)$ stand for the null space and range in X_c. The spectrum, $\sigma(L)$, is computed with respect to X_c so $\lambda \in \sigma(L)$ if and only if $\bar{\lambda} \in \sigma(L)$.

Let L be a densely defined linear operator on X which satisfies the conditions

(HL) (*i*) $-L$ is the infinitesimal generator of a strongly continuous semigroup $T(t)$ on X,
(*ii*) $T(t)$ is a holomorphic semigroup on X_c,
(*iii*) $(\lambda I - L)^{-1}$ is compact for λ in the resolvent set of L,
(*iv*) i is a simple eigenvalue of L,
(*v*) $n i \notin \sigma(L)$ for $n = 0, 2, 3, \ldots$.

Remarks. By (HL)(*iii*), the condition (HL)(*iv*) is unambiguous and can also be expressed as

$$\begin{aligned} &\dim N_c(L - i I) = 1 = \operatorname{codim} R_c(L - i I) \quad \text{and} \\ &x_0 \in N_c(L - i I) \setminus \{0\} \quad \text{implies } x_0 \notin R_c(L - i I). \end{aligned} \tag{1.2}$$

Another useful equivalent form of (1.2) is

$$\dim N_c((L - i I)^k) = 1 \quad \text{for } k = 1, 2, \ldots. \tag{1.3}$$

It follows from (*i*) and (*ii*) of (HL) that if $r > -\operatorname{Re}\lambda$ for all $\lambda \in \sigma(L)$, then the fractional powers $(L + r I)^\alpha$ are defined for $\alpha \geqq 0$. The Banach spaces $X_\alpha \subset X$ with norms $\|\cdot\|_\alpha$ are defined by

$$\begin{aligned} &X_\alpha = D((L + r I)^\alpha) \\ &\|x\|_\alpha = \|(L + r I)^\alpha x\| \quad \text{for } x \in X_\alpha, \end{aligned} \tag{1.4}$$

where $\|\cdot\|$ is the norm in X. If $\alpha \leqq \gamma$ then $X_\alpha \supseteq X_\gamma$ with continuous injection.

Let $C^k(A, B)$ denote the set of k-times continuously Frechét differentiable maps from A to B, where A and B are subsets of Banach spaces W and Z respectively. The requirements on f in (1.1) are stipulated by condition

(Hf) There is an $\alpha \in [0, 1)$ and a neighborhood Ω of $(0, 0)$ in $\mathbb{R} \times X_\alpha$ such that $f \in C^2(\Omega, X)$. Moreover $f(\mu, 0) = 0$ and $f_x(0, 0) = 0$ if $(\mu, 0) \in \Omega$.

Henceforth α is fixed at the value given by (Hf). Here $f_x(\mu, x)$ denotes the Frechét derivative of the mapping $x \to f(\mu, x)$; it is a bounded linear operator from X_α into X (and a fortiori from X_1 to X). By (HL)(*iv*) we see that i is an I-simple (in the sense of [4]) eigenvalue of L regarded as a mapping of X_{1c} into X_c. Hence if $x_0 \in N_c(L - iI) \setminus \{0\}$, then there are continuously differentiable functions $x(\mu)$, $\beta(\mu)$ defined for small $|\mu|$ such that

$$\begin{aligned} &(L + f_x(\mu, 0))\, x(\mu) = \beta(\mu)\, x(\mu) \\ &x(0) = x_0, \quad \beta(0) = i. \end{aligned} \tag{1.5}$$

Following Hopf, our final assumption is

$$\operatorname{Re}\beta'(0) \neq 0. \tag{Hβ}$$

(Hf) implies that (1.1) has the trivial family of equilibrium solutions $(\mu, 0) \in \Omega$; (HL) and (Hf) imply that (1.1) linearized about $u = 0$ for $\mu = 0$ has nontrivial 2π-

periodic solutions. We now seek nontrivial $2\pi\rho$-periodic solutions of (1.1) with ρ near 1 and (μ, u) near $(0,0)$. With $\tau=\rho^{-1}t$ equation (1.1) can be rewritten

$$u'+\rho\big(Lu+f(\mu,u)\big)=0. \tag{1.6}$$

It remains to give a precise meaning to solutions of (1.6). To that end we have

Lemma 1.7. *Let* $r>0$, *suppose* (HL) *and* (Hf) *hold, and assume* $u\in C([0,r],X_\alpha)$. *Then the following statements are equivalent:*

(1.8) $u'\in C\big((0,r],X\big)$, $u\big((0,r]\big)\subset D(L)$, *and* (1.6) *is satisfied on* $(0,r)$

$$u(\tau)-T(\rho\tau)\,u(0)+\rho\int_0^\tau T\big(\rho(\tau-\xi)\big)f\big(\mu,u(\xi)\big)d\xi=0 \quad \textit{for}\ 0\leqq\tau\leqq r. \tag{1.9}$$

For a proof of Lemma 1.7, see [8], [10]. In view of this lemma we shall say that u is a solution of (1.6) if $u\in C([0,r],X_\alpha)$ and satisfies (1.9).

Let $C_{2\pi}(\mathbb{R},X_\alpha)$ be the Banach space of continuous 2π-periodic functions from $\mathbb{R}$ into X_α and let $C_0([0,2\pi],X_\alpha)$ be the Banach space of continuous h: $[0,2\pi]\to X_\alpha$ such that $h(0)=0$. Define

$$\mathscr{F}(\rho,\mu,u)(\tau)=u(\tau)-T(\rho\tau)\,u(0)+\rho\int_0^\tau T\big(\rho(\tau-\xi)\big)f\big(\mu,u(\xi)\big)d\xi. \tag{1.10}$$

$\mathscr{F}$ is regarded as a mapping of that subset of $\mathbb{R}\times\mathbb{R}\times C_{2\pi}(\mathbb{R},X_\alpha)$ for which (1.10) makes sense, into $C_0([0,2\pi],X_\alpha)$. The properties of $\mathscr{F}$ which we shall require are given in Lemma 1.12 below.

An infinite dimensional version of the Hopf bifurcation theorem can now be stated.

Theorem 1.11. *Let* (HL), (Hf), *and* (Hβ) *be satisfied. Then there are positive numbers* ε, η *and continuously differentiable functions* (ρ,μ,u): $(-\eta,\eta)\to\mathbb{R}\times\mathbb{R}\times C_{2\pi}(\mathbb{R},X_\alpha)$ *with the following properties:*

(*a*) $\mathscr{F}\big(\rho(s),\mu(s),u(s)\big)=0$ *for* $|s|<\eta$

(*b*) $\mu(0)=0$, $u(0)=0$, $\rho(0)=1$, *and* $u(s)\neq 0$ *if* $0<|s|<\eta$

(*c*) *If* $(\mu_1,u_1)\in\mathbb{R}\times C(\mathbb{R},X_\alpha)$ *is a solution of* (1.1) *of period* $2\pi\rho_1$, *where* $|\rho_1-1|<\varepsilon$, $|\mu_1|<\varepsilon$, *and* $\|u_1\|_\alpha<\varepsilon$, *then there exist numbers* $s\in[0,\eta)$ *and* $\theta\in[0,2\pi)$ *such that* $u_1(\rho_1\tau)=u(s)(\tau+\theta)$ *for* $\tau\in\mathbb{R}$.

Moreover, if $f\in C^{k+1}(\Omega,X_\alpha)$ *or if* f *is real analytic, then the functions* $\big(\rho(s),\mu(s),u(s)\big)$ *are respectively of class* C^k *or real analytic.*

The notation in Theorem 1.11 must be read with care, especially in (*c*). Since u: $(-\eta,\eta)\to C_{2\pi}(\mathbb{R},X_\alpha)$, we see that $u(s)\in C_{2\pi}(\mathbb{R},X_\alpha)$ and $u(s)(\tau)\in X_\alpha$ for $s\in(-\eta,\eta)$ and $\tau\in\mathbb{R}$. The proof divides naturally into two parts: First the existence assertions (*a*), (*b*) are established and then the uniqueness property (*c*). The proofs of (*a*), (*b*) are based on three lemmas and a simple application of the implicit function theorem. The first lemma details the regularity properties of $\mathscr{F}$, the second characterizes $N\big(\mathscr{F}_u(1,0,0)\big)$ and $R\big(\mathscr{F}_u(1,0,0)\big)$, and the third supplies an alternate characterization of (Hβ). We first state the lemmas, then prove parts (*a*), (*b*) and the lemmas, and finally prove part (*c*) of the theorem.

Lemma 1.12. *Let* (HL) *and* (Hf) *be satisfied. Then* $\mathscr{F}$ *is a twice continuously differentiable mapping from its domain into* $C_0([0,2\pi], X_\alpha)$, *and* $\mathscr{F}(\rho,\mu,0)=0$ *for* $\rho\in(0,\infty)$ *and* μ *near* 0. *Moreover, for* $v\in C_{2\pi}(\mathbb{R}, X_\alpha)$,

(*i*) $(\mathscr{F}_u(1,0,0)v)(\tau)=v(\tau)-T(\tau)v(0)$

(*ii*) $(\mathscr{F}_{\rho u}(1,0,0)v)(\tau)=\tau LT(\tau)v(0)$

(*iii*) $(\mathscr{F}_{\mu u}(1,0,0)v)(\tau)=\int_0^\tau T(\tau-\xi)f_{\mu x}(0,0)v(\xi)d\xi$.

Next we characterize $N(\mathscr{F}_u(1,0,0))$ and $R(\mathscr{F}_u(1,0,0))$. In the following A^* denotes the adjoint of a linear operator A.

Lemma 1.13. *Let* (HL) *hold. Then*

(*a*) *v belongs to $N(\mathscr{F}_u(1,0,0))$; i.e. $v\in C_{2\pi}(\mathbb{R}, X_\alpha)$ and $v(\tau)-T(\tau)v(0)=0$ for $0\leqq\tau\leqq2\pi$, if and only if $v(\tau)=T(\tau)x$ for some $x\in N(I-T(2\pi))$.*

(*b*) *If $h\in C_0([0,2\pi],X_\alpha)$, then the equation*

$$v(\tau)-T(\tau)v(0)=h(\tau)$$

has a solution $v\in C_{2\pi}(\mathbb{R},X_\alpha)$ if and only if $h(2\pi)\in R(I-T(2\pi))$.

(*c*) *$N(I+L^2)=N(I-T(2\pi))$ and $N(I+L^{*2})=N(I-T(2\pi)^*)$.*

(*d*) *If $x_0\in N(I-T(2\pi))\setminus\{0\}$, then $\{x_0, x_1=Lx_0\}$ is a basis for $N(I-T(2\pi))$. Moreover there exists an element $x_0^*\in N(I-T(2\pi)^*)$ such that*

$$(x_0^*,x_0)=(x_1^*,x_1)=1, \qquad (x_1^*,x_0)=(x_0^*,x_1)=0,$$

where $x_1^*=L^*x_0^*$ and $(\cdot,\cdot)$ denotes the pairing between X^* and X.

(*e*) *If $x_0^*\in N(I+L^{*2})\setminus\{0\}$, then*

$$R(I-T(2\pi))=\{x\in X\,|\,(x_0^*,x)=0=(x_1^*,x)\},$$

where $x_1^*=L^*x_0^*$.

Lemma 1.14. *Let* (HL) *hold and let $L_1\colon X_\alpha\to X$ be bounded and linear. If $x_0\in N(I-T(2\pi))\setminus\{0\}$, $\lambda\in\mathbb{C}$, $(L_1-\lambda I)(x_0-iLx_0)\in R_c(L-iI)$, and x_0^*, x_1^* are as in Lemma 2.13 (d), then*

$$\left(x_0^*, \int_0^{2\pi} T(2\pi-\xi)L_1T(\xi)x_0\,d\xi\right)=2\pi\,\mathrm{Re}\,\lambda,$$

$$\left(x_1^*, \int_0^{2\pi} T(2\pi-\xi)L_1T(\xi)x_0\,d\xi\right)=-2\pi\,\mathrm{Im}\,\lambda.$$

Proof of Theorem 1.11 (a), (b). By Lemmas 1.12 and 1.13 (*a*), (*b*), (*c*), (*d*), *we have*

$$\dim N(\mathscr{F}_u(1,0,0))=\dim N(I-T(2\pi))=\dim N(I+L^2)=2.$$

Also if $x_0 \in N(I+L^2) \setminus \{0\}$, then $\{T(t)x_0, T(t)x_1\}$ is a basis for $N(\mathscr{F}_u(1,0,0))$. Moreover by (*b*), (*d*), (*e*) of Lemma 1.13,

$$R(\mathscr{F}_u(1,0,0)) = \{h \in C_0([0,2\pi], X_\alpha) | (x_0^*, h(2\pi)) = (x_1^*, h(2\pi)) = 0\}. \tag{1.15}$$

Thus $R(\mathscr{F}_u(1,0,0))$ has codimension 2. Let V be a complement of $N(\mathscr{F}_u(1,0,0))$ in $C_{2\pi}(\mathbb{R}, X_\alpha)$, and define

$$\mathscr{G}(s,\rho,\mu,v) = \begin{cases} s^{-1}\mathscr{F}(\rho,\mu,s(T(\cdot)x_0+v)), & s \neq 0 \\ \mathscr{F}_u(\rho,\mu,0)(T(\cdot)x_0+v), & s=0. \end{cases} \tag{1.16}$$

Then by Lemma 1.12, $\mathscr{G}$ is a mapping of class C^1 from a neighborhood of $(0,1,0,0)$ in $\mathbb{R}^3 \times V$ to $C_0([0,2\pi], X_\alpha)$. Obviously $\mathscr{G}(0,1,0,0)=0$, and the Frechét derivative of the map $(\rho,\mu,v) \to \mathscr{G}(s,\rho,\mu,v)$ at $(0,1,0,0)$ is the linear map

$$G(\hat\rho,\hat\mu,\hat v)(\tau) = \hat v(\tau) - T(\tau)\hat v(0) + \hat\rho\tau LT(\tau)x_0 + \hat\mu \int_0^\tau T(\tau-\xi) f_{\mu x}(0,0) T(\xi)x_0 \, d\xi.$$

We claim G is an isomorphism. Once this is shown, the fact that $\mathscr{G}(0,1,0,0)=0$ and the implicit function theorem imply that the solutions (s,ρ,μ,v) of $\mathscr{G}=0$ near $(0,1,0,0)$ are given by continuously differentiable functions $(\rho(s),\mu(s),v(s))$. Then setting

$$u(s)(\tau) = s(T(\tau)x_0 + v(s)(\tau)),$$

we see that $(\rho(s),\mu(s),u(s))$ is the desired curve of solutions of $\mathscr{F}=0$.

Since $\hat v \to \hat v - T(\cdot)\hat v(0)$ maps V isomorphically onto $R(\mathscr{F}_u(1,0,0))$, which has codimension 2 (by the above), it is clear that G is an isomorphism if the relation

$$\hat\rho\tau LT(\tau)x_0 + \hat\mu \int_0^\tau T(\tau-\xi) f_{\mu x}(0,0) T(\xi)x_0 \, d\xi \in R(\mathscr{F}_u(1,0,0)) \tag{1.17}$$

implies $\hat\rho=\hat\mu=0$. By (1.15), the latter implication is equivalent to that the condition

$$\left(x_i^*, 2\pi\hat\rho LT(2\pi)x_0 + \hat\mu \int_0^{2\pi} T(2\pi-\xi) f_{\mu x}(0,0) T(\xi)x_0 \, d\xi\right) = 0, \qquad i=1,2, \tag{1.18}$$

implies $\hat\rho=\hat\mu=0$. Applying Lemma 1.13 (*d*), this reduces to verifying that

$$\left(x_0^*, \int_0^{2\pi} T(2\pi-\xi) f_{\mu x}(0,0) T(\xi)x_0 \, d\xi\right) \neq 0. \tag{1.19}$$

Differentiating (1.5) at $\mu=0$, we see that the hypotheses of Lemma 2.14 are satisfied with $\lambda=\beta'(0)$, $L_1 = f_{\mu x}(0,0)$. Thus the left hand side of (1.19) is $\operatorname{Re}\beta'(0)$. Invoking (H$\beta$) completes the proof.

Proof of Lemma 1.12. We treat the differentiability only, since computation of the derivatives is routine. First, by (Hf) the map $(\mu,u) \to f(\mu,u)$ is of class C^2 from

$$\{(\mu,u) \in \mathbb{R} \times C_{2\pi}(\mathbb{R}; X_\alpha) | (\mu, u(\tau)) \in \Omega \text{ for } \tau \in \mathbb{R}\}$$

into $C_{2\pi}(\mathbb{R}, X)$. We can thus deduce the lemma from the chain rule if enough regularity can be established for the maps

$$(1.20)\qquad \begin{aligned}&(\rho, x)\to T(\rho\tau)x;\ (0,\infty)\times X_\alpha\to C([0,2\pi], X_\alpha)\\ &(\rho, g)\to \int_0^\tau T(\rho(\tau-\xi))\,g(\xi)\,d\xi;\ (0,\infty)\times C([0,2\pi], X)\to C([0,2\pi], X_\alpha).\end{aligned}$$

Henry [10] has noted that in fact the maps (1.20) are analytic. We sketch a proof of this. Due to the linearity in the second argument for each map, it suffices to examine the dependence on the first arguments.

Since $T(t)$ is a holomorphic semigroup, there is a constant $C>0$ such that

$$T(t+\hat{t})=\sum_{k=0}^{\infty}\frac{(-\hat{t}L)^k}{k!}\,T(t)$$

for $t>0$ and $|\hat{t}|\leqq Ct$. Moreover, given $r>0$ there is a constant M_r such that

$$(1.21)\qquad \|L^k T(t)x\|_\alpha=\left\|\left(LT\left(\frac{t}{k}\right)\right)^k x\right\|_\alpha\leqq\frac{M_r^k k^k}{t^k}\,\|x\|_\alpha$$

for $x\in X_\alpha$, $0<t\leqq r$, and $k=1,2,\ldots$, and also such that

$$(1.22)\qquad \|L^k T(t)y\|_\alpha\leqq\frac{M_r^k k^k}{t^{k+\alpha}}\,\|y\|$$

for $y\in X$, $0<t\leqq r$ and $k=1,2,\ldots$. Thus if $\rho>0$ and $0<\tau\leqq 2\pi$ we get

$$(1.23)\qquad T((\rho+\hat{\rho})\tau)x=\sum_{k=0}^{\infty}\frac{(-\hat{\rho}\tau L)^k}{k!}\,T(\rho\tau)x$$

and

$$(1.24)\qquad \int_0^\tau T((\rho+\hat{\rho})(\tau-\xi))\,g(\xi)\,d\xi=\sum_{k=0}^{\infty}\frac{(-\hat{\rho})^k}{k!}\int_0^\tau((\tau-\xi)L)^k\,T(\rho(\tau-\xi))\,g(\xi)\,d\xi$$

for $|\hat{\rho}|$ sufficiently small (depending on ρ) where, by (1.21), (1.22). the series converge in $L^\infty((0,2\pi\rho), X_\alpha)$. By (1.22), each term of (1.24) is in $C([0,2\pi], X_\alpha)$. Moreover, if $x\in X_\alpha$ and $k>0$ the functions $\tau\to\tau^k L^k T(\rho\tau)x$, which are analytic for $\tau>0$, may be extended by continuity to have the value 0 at $\tau=0$. This is obvious for $x\in D(L^{k+1})$. Noting that $D(L^{k+1})$ is dense in X_α and recalling (1.21) we see that the result extends from $D(L^{k+1})$ to all of X_α. Thus in fact each term of the series (1.23), (1.24) is in $C([0,2\pi], X_\alpha)$, and the analyticity of (1.20) therefore follows at once.

Proof of Lemma 1.13. Statement *(a)* is obvious. For *(b)*, if

$$h(\tau)=u(\tau)-T(\tau)\,u(0)$$

and $u\in C_{2\pi}(\mathbb{R}, X_\alpha)$, then

$$h(2\pi)=u(2\pi)-T(2\pi)\,u(0)=(I-T(2\pi))\,u(0).$$

Conversely if $h(2\pi)=(I-T(2\pi))x$, then $u(\tau)=T(\tau)x+h(\tau)$ satisfies $u(\tau)-T(\tau)\,u(0)=h(\tau)$ and $u(2\pi)=u(0)$. Thus (b) holds.

The proofs of (c)–(e) are more involved. For (c), if $x\in N(I+L^2)$, then $u(\tau)=T(\tau)x$ satisfies

$$u''(\tau)=T(\tau)L^2x=-u(\tau).$$

It follows that $u(\tau)=(\cos\tau)x-(\sin\tau)Lx$, so u is 2π-periodic. Thus $x\in N(I-T(2\pi))$ and $N(I+L^2)\subset N(I-T(2\pi))$. For the converse, let $x\in N(I-T(2\pi))$ and $u(\tau)=T(\tau)x$. By (a), u is 2π-periodic and therefore has a Fourier expansion with coefficients

$$a_n=\frac{1}{\pi}\int_0^{2\pi}(\cos n\xi)\,T(\xi)x\,d\xi,\qquad b_n=\frac{1}{\pi}\int_0^{2\pi}(\sin n\xi)\,T(\xi)x\,d\xi$$

for $n=0,1,\ldots$. Two integrations by parts show that

$$(L^2+n^2I)a_n=(L^2+n^2I)b_n=0.$$

By condition (HL)(iv), we have $a_n=b_n=0$ for $n=0,2,3,\ldots$. Therefore

$$T(\tau)x=(\cos\tau)a_1+(\sin\tau)b_1\quad\text{with}\quad a_1,b_1\in N(I+L^2).$$

But $a_1=x$, so that $N(I-T(2\pi))\subset N(I+L^2)$. This proves the first part of (c).

Since the map $x\to x-iLx$ is a linear isomorphism (over $\mathbb{R}$) of $N(I+L^2)$ onto $N_c(L-iI)$ and since $\dim N_c(L-iI)=1$ by (HL)(iv), it follows that $\dim N(I+L^2)=2$.

For the analogous result for L^*, observe first that by the above argument we have $N(I+L^{*2})\subset N(I-T(2\pi)^*)$. Furthermore, by the compactness of $T(2\pi)$,

$$\dim N(I-T(2\pi))=2=\dim N(I-T(2\pi)^*).$$

Similarly

$$\dim N(I+L^2)=2=\dim(I+L^{*2})$$

since L^2 has a compact resolvent. More precisely, the condition $2i\notin\sigma(L)$ implies that the mapping $(2i-L)(-2i+L)=4I+L^2$ is invertible. Now $x\in N(I+L^2)$ is equivalent to $3x=(4I+L^2)x$, which is the case if and only if $x\in N(I-3^{-1}(4I+L^2)^{-1})$. Therefore

$$\begin{aligned}\dim N(I+L^2)=\dim N(I-3^{-1}(4I+L^2)^{-1\,*})&=\dim N(I-3^{-1}(4I+L^{*2})^{-1})\\&=\dim N(I+L^{*2}).\end{aligned}$$

Thus $N(I+L^{*2})=N(I-T(2\pi)^*)$.

Next consider part (d). Since $\dim N(I-T(2\pi))=2$ and $N(I-T(2\pi))=N(I+L^2)$, it is clear that $\{x_0, x_1=-Lx_0\}$ is a basis for $N(I-T(2\pi))$ for any $x_0\in N(I-T(2\pi))\setminus\{0\}$. If $(I+L^2)^2x=0$ and $x\in X$, then

$$(L-iI)^2(L+iI)^2x=0.$$

By (HL)(iv) we have

$$(L-iI)(L+iI)^2x=0=(L+iI)(L^2+I)x.$$

But $L+iI$ has no nontrivial real nullvectors. Hence $(L^2+I)x=0$. Thus $N(I+L^2) = N((I+L^2)^2)$, or equivalently,

(1.25) $$R(I+L^2)\cap N(I+L^2)=\{0\}.$$

By (c), if $(I-T(2\pi))^2x=0$ then

$$0=(I+L^2)(I-T(2\pi))x=(I-T(2\pi))(I+L^2)x=(I+L^2)^2x.$$

Thus we also have $N(I-T(2\pi))=N((I-T(2\pi))^2)$, or equivalently

(1.25′) $$N(I-T(2\pi))\cap R(I-T(2\pi))=\{0\}.$$

Since $T(2\pi)$ is compact by (HL)(*ii*), (*iii*), the set $R(I-T(2\pi))$ is closed. Consequently $R(I-T(2\pi))=N(I-T(2\pi)^*)^\perp$. Thus if $x_0\in N(I-T(2\pi))\setminus\{0\}$, the relation (1.25′) implies that we can find an element $x_0^*\in N(I-T(2\pi)^*)$ such that $(x_0^*,x_0)=1$ and $(x_0^*,x_1)=0$. Note also that

$$(x_1^*,x_1)=(x_0^*,-L^2x_0)=(x_0^*,x_0)=1$$

and similarly $(x_1^*,x_0)=0$. Thus (d) is proved.

Lastly, (e) follows from the relation $R(I-T(2\pi))=N(I-T(2\pi)^*)^\perp$ and the fact that $\{x_0^*, x_1^*=L^*x_0\}$ is a basis for $N(I-T(2\pi)^*)^\perp$, as we see from (c) and what has gone before.

Remark 1.26. In the course of proving (b), (c) above we have also established that

(1.27) $$T(\tau)x=(\cos\tau)x-(\sin\tau)Lx \qquad \text{for } x\in N(I+L^2).$$

Proof of Lemma 1.14. Let $x_1=-Lx_0$ and $y=x_0+ix_1$. Choose $z\in D(L)$ such that

(1.28) $$L_1y=\lambda y+(L-iI)z.$$

By Remark 1.26 we have $T(\xi)x_0=\mathrm{Re}(e^{-i\xi}y)$ (with the obvious meaning). Multiplying (1.28) by $e^{-i\xi}$, we find

(1.29) $$L_1T(\xi)x_0=\mathrm{Re}\big(\lambda e^{-i\xi}y+(L-iI)e^{-i\xi}z\big).$$

From Remark 1.26 again, we see that $T(\tau)y=e^{-i\tau}y$. Hence (1.29) implies

(1.30) $$T(2\pi-\xi)L_1T(\xi)x_0=\mathrm{Re}(\lambda e^{-i\xi}e^{-i(2\pi-\xi)}y)+T(2\pi-\xi)(L-iI)e^{-i\xi}z.$$

But

$$T(2\pi-\xi)(L-iI)e^{-i\xi}z=\frac{d}{d\xi}\big(T(2\pi-\xi)e^{-i\xi}z\big);$$

integrating this over $0\leqq\xi\leqq 2\pi$ then yields

(1.31) $$\int_0^{2\pi} T(2\pi-\xi)L_1T(\xi)x_0\,d\xi=2\pi\,\mathrm{Re}(\lambda y)+\mathrm{Re}\big(z-T(2\pi)z\big)$$
$$=2\pi\big((\mathrm{Re}\,\lambda)x_0-(Im\,\lambda)x_1\big)+\big(I-T(2\pi)\big)(\mathrm{Re}\,z).$$

The lemma now follows on applying x_j^* to (1.31), using Lemma 1.13 (*d*), and noting that $x_j^*(R(I-T(2\pi)))=0$, $j=0,1$.

Proof of Theorem 1.11 (c). The uniqueness will be established in two steps, the main one being based on the following.

Lemma 1.32. *Under the hypotheses of Theorem* 1.11, *there is a neighborhood* $\tilde{\Omega}$ *of* $(1,0,0)$ *in* $\mathbb{R}\times\mathbb{R}\times C_{2\pi}(\mathbb{R},X_\alpha)$ *such that if* $s\in\mathbb{R}$, $\tilde{v}\in V$, $(\tilde{\rho},\tilde{\mu},sT(\cdot)x_0+\tilde{v})\in\tilde{\Omega}$, *and* $\mathscr{F}(\tilde{\rho},\tilde{\mu},sT(\cdot)x_0+\tilde{v})=0$, *then* $\tilde{\rho}=\rho(s)$, $\tilde{\mu}=\mu(s)$, *and* $\tilde{v}=sv(s)$.

Proof. Set $\varphi_0(\tau)=T(\tau)x_0$. It suffices to show that $\tilde{\Omega}$ can be chosen so that $\mathscr{F}(\tilde{\rho},\tilde{\mu},s\varphi_0+\tilde{v})=0$ implies

$$\|\tilde{v}\|_{C_{2\pi}(\mathbb{R},X_\alpha)}+|s|\,|\tilde{\rho}-1|+|s|\,|\tilde{\mu}|\leqq sg(s) \tag{1.33}$$

for some $g\in C(\mathbb{R},\mathbb{R})$ with $g(0)=0$. For then $s=0$ implies $\tilde{v}=0$, while if $s\neq 0$ then $\mathscr{G}(s,\tilde{\rho},\tilde{\mu},s^{-1}\tilde{v})=0$. Choosing $\tilde{\Omega}$ still smaller if necessary, we can then guarantee, via (1.33), that $(s,\tilde{\rho},\tilde{\mu},s^{-1}\tilde{v})$ is in the neighborhood of $(0,1,0,0)$, which by the implicit function theorem contains all the solutions of $\mathscr{G}=0$.

To verify (1.33), we argue as in the proof of Lemma 1.12 of [3]. Since $\mathscr{F}\in C^2$, there exists a function $h\in C(\mathbb{R},\mathbb{R})$ such that $h(0)=0$ and

$$\begin{aligned}
&\|\mathscr{F}(\rho,\mu,s\varphi_0+v)-\mathscr{F}(\rho,\mu,s\varphi_0)-\mathscr{F}_u(\rho,\mu,s\varphi_0)v\|\leqq\|v\|\,h(\|v\|)\\
&\|\mathscr{F}(\rho,\mu,s\varphi_0)-\mathscr{F}(\rho,\mu,0)-\mathscr{F}_u(\rho,\mu,0)s\varphi_0\|\leqq|s|\,h(s)\\
&\|\mathscr{F}_u(\rho,\mu,0)\varphi_0-\mathscr{F}_u(1,0,0)\varphi_0-\mathscr{F}_{\rho u}(1,0,0)(\rho-1)\varphi_0-\mathscr{F}_{\mu u}(1,0,0)\mu\varphi_0\|\\
&\qquad\leqq(|\rho-1|+|\mu|)\,h(|\rho-1|+|\mu|)
\end{aligned} \tag{1.34}$$

for $(\rho,\mu,s\varphi_0+v)\in\Omega$ and near $(1,0,0)$. (At this point we cease to index the various norms involved, the spaces being defined by the context.) Now observe that

$$\begin{aligned}
0&=\mathscr{F}(\tilde{\rho},\tilde{\mu},s\varphi_0+\tilde{v})\\
&=[\mathscr{F}(\tilde{\rho},\tilde{\mu},s\varphi_0+\tilde{v})-\mathscr{F}(\tilde{\rho},\tilde{\mu},s\varphi_0)-\mathscr{F}_u(\tilde{\rho},\tilde{\mu},s\varphi_0)\tilde{v}]\\
&\quad+[(\mathscr{F}_u(\tilde{\rho},\tilde{\mu},s\varphi_0)-\mathscr{F}_u(1,0,0))\tilde{v}]\\
&\quad+[\mathscr{F}(\tilde{\rho},\tilde{\mu},s\varphi_0)-\mathscr{F}(\tilde{\rho},\tilde{\mu},0)-s\mathscr{F}_u(\tilde{\rho},\tilde{\mu},0)\varphi_0]\\
&\quad+s[\mathscr{F}_u(\tilde{\rho},\tilde{\mu},0)\varphi_0-(\tilde{\rho}-1)\mathscr{F}_{\rho u}(1,0,0)\varphi_0-\tilde{\mu}\mathscr{F}_{\mu u}(1,0,0)\varphi_0]\\
&\quad+\mathscr{F}_u(1,0,0)\tilde{v}+s\tilde{\mu}\mathscr{F}_{\mu u}(1,0,0)\varphi_0+s(\tilde{\rho}-1)\mathscr{F}_{\rho u}(1,0,0)\varphi_0.
\end{aligned} \tag{1.35}$$

From the proof of Theorem 1.11, we know that

$$G_0(s\tilde{\rho},s\tilde{\mu},\tilde{v})=\mathscr{F}_u(1,0,0)\tilde{v}+s\mu\mathscr{F}_{\mu u}(1,0,0)\varphi_0+s\tilde{\rho}\mathscr{F}_{\rho u}(1,0,0)\varphi_0$$

is an isomorphism. Hence (1.34)–(1.35) yields the existence of a positive constant c such that

$$\begin{aligned}
&c(\|\tilde{v}\|+|s|\,|\tilde{\rho}-1|+|s\tilde{\mu}|)\\
&\quad\leqq\|\tilde{v}\|\,h(\|\tilde{v}\|)+\|\mathscr{F}_u(\tilde{\rho},\tilde{\mu},s\varphi_0)-\mathscr{F}_u(1,0,0)\|\,\|\tilde{v}\|\\
&\qquad+|s|\,h(s)+|s|(|\tilde{\rho}-1|+|\tilde{\mu}|)\,h(|\tilde{\rho}-1|+|\tilde{\mu}|).
\end{aligned} \tag{1.36}$$

If $\tilde{Q}$ is chosen so that $(\tilde{\rho}, \tilde{\mu}, s\varphi_0+\tilde{v})\in\tilde{Q}$ implies the inequalities: $h(\|\tilde{v}\|)\leqq 4c^{-1}$, $\|\mathscr{F}_u(\tilde{\rho}, \tilde{\mu}, s\varphi_0+\tilde{v})-\mathscr{F}_u(1, 0, 0)\|\leqq 4c^{-1}$, and $h(|\tilde{\rho}-1|+|\tilde{\mu}|)\leqq 4c^{-1}$, then (1.36) gives (1.33) with $g(s)=2c^{-1}h(s)$. The lemma is therefore proved.

Proceeding further, any element $\tilde{u}\in C_{2\pi}(\mathbb{R}, X_\alpha)$ may be uniquely expressed in the form $\varphi+\tilde{v}$ with $v\in V$ and $\varphi\in N(\mathscr{F}_u(1,0,0))$. Utilizing the proof of Lemma 1.13, we see that there exists an element $x\in N(I+L^2)$ such that

$$\varphi(\tau)=(\cos\tau)x-(\sin\tau)Lx=\operatorname{Re}(e^{-i\tau}(x-iLx))=\operatorname{Re}(e^{-i\tau}y),$$

where $y=x-iLx\in N_c(L-iI)$. If $a=x_0-iLx_0$, there are constants $r\geqq 0$ and $\theta\in[0,2\pi)$ such that $e^{-i\theta}y=ra$. Thus

$$M_{-\theta}\varphi=\operatorname{Re}(e^{-i\tau}e^{-i\theta}y)=\operatorname{Re}(e^{-i\tau}ra)=rT(\tau)x_0,$$

where $M_{-\theta}f(\tau)=f(\tau+\theta)$.

Moreover $(\tilde{\rho}, \tilde{\mu}, \tilde{u})$ is a solution of $\mathscr{F}=0$ if and only if $(\tilde{\rho}, \tilde{\mu}, M_{-\theta}\tilde{u})$ is also a solution. This may be deduced either from Lemma 1.7 using the translation invariance of the differential equation and the periodicity of $\tilde{u}$, or directly from the identity

$$\mathscr{F}(\rho,\mu,M_{-\theta}u)(\tau)=M_{-\theta}\mathscr{F}(\rho,\mu,u)(\tau)-T(\tau)\mathscr{F}(\rho,\mu,u)(\theta) \tag{1.37}$$

which is valid for $\theta, \tau\geqq 0$. (Since u is periodic, $M_{-\theta}u$ represents all translates of u as θ varies over $[0,2\pi)$). Choosing

$$V=\left\{v\in C_{2\pi}(\mathbb{R}, X_\alpha)\,\middle|\,\left(x_i^*, \int_0^{2\pi} T(2\pi-\xi)\,v(\xi)\,d\xi\right)=0 \text{ for } i=0,1\right\}, \tag{1.38}$$

we see that V is translation invariant since the functions

$$T(2\pi-\xi)^*x_i^*=(\cos\xi)x_i^*+(\sin\xi)L^*x_i^*$$

span a translation invariant space. Thus

$$M_{-\theta}\tilde{u}=M_{-\theta}\varphi+M_{-\theta}\tilde{v}=rT(\cdot)\varphi_0+M_{-\theta}\tilde{v}$$

is of the form required in Lemma 1.32. Therefore we have shown

Lemma 1.39. *Let V be given by (1.38). Then there is a neighborhood $\tilde{\Omega}$ of $(1,0,0)$ in $\mathbb{R}\times\mathbb{R}\times C_{2\pi}(\mathbb{R}, X_\alpha)$ with the property that if $(\tilde{\rho}, \tilde{\mu}, \tilde{u})\in\tilde{\Omega}$ is a solution of $\mathscr{F}=0$, then there exist constants $s\geqq 0$ and $\theta\in[0,2\pi)$ such that $\tilde{\rho}=\rho(s)$, $\tilde{\mu}=\mu(s)$, and $M_{-\theta}\tilde{u}=sT(\cdot)x_0+sv(s)$.*

This completes the proof of Theorem 1.11(*c*).

Remarks. (*i*) HOPF [12] proved Theorem 1.11 assuming that $X=\mathbb{R}^n$, that $\pm i$ are simple eigenvalues of L and also the only purely imaginary eigenvalues of L, that f is real analytic, and that (Hβ) is satisfied. If $X=\mathbb{R}^n$, then $X_\alpha=\mathbb{R}^n$ for all $\alpha\geqq 0$, so that the hypotheses of Theorem 1.11 are considerably weaker than Hopf's. Of course, other authors have also weakened these hypotheses (see, e.g., [2], [22], [23], [26]).

(*ii*) Theorem 1.11 is a local result, i.e. it assures the existence of periodic solutions of (1.1) near $(\mu, u)=(1,0,0)$. Recently ALEXANDER & YORKE [1] have proved a global version of this result for $X=\mathbb{R}^n$. Their proof has been simplified by IZE [15]. As a nice application of the Hopf theorem, ALEXANDER and YORKE also gave a new proof of the Lyapunov center theorem (see also [26]).

Our next result establishes the precise relationship between the functions $(\rho(s), \mu(s), u(s))$ for $s \geqq 0$ and $s \leqq 0$. First observe that $\mathscr{G}$ in (1.16) can be regarded as depending on $x_0 \in N(I+L^2)$ as well as on (s, ρ, μ, v), that is $\mathscr{G}=\mathscr{G}(s, \rho, \mu, v, x_0)$. Given $x_0 \in N(I+L^2) \setminus \{0\}$ and $\theta \in \mathbb{R}$ we define

$$x_\theta = (\cos\theta)\, x_0 - (\sin\theta)\, L x_0. \tag{1.40}$$

The proof of Theorem 1.11 shows that the equations $\mathscr{G}(s, \rho, \mu, v, x_\theta)=0$ can be solved for $(\rho(s,\theta), \mu(s,\theta), v(s,\theta))$ near $(1,0,0)$ in a strip $|s| \leqq s_0$, $\theta \in \mathbb{R}$. (Since (1.40) is periodic in θ, it is enough to restrict θ to the compact set $[0, 2\pi]$.)

Theorem 1.41. *With the above notation,*

$$\rho(s,\theta)=\rho(-s,\theta+\pi), \qquad \mu(s,\theta)=\mu(-s,\theta+\pi), \qquad v(s,\theta)=-v(-s,\theta+\pi). \tag{1.42}$$

Moreover, if V is given by (1.38), *then $\rho(s,\theta)$, $\mu(s,\theta)$ are even functions of s and $u(-s,\theta)(\tau)=u(s,\theta)(\tau+\pi)$, where $u(s)(\tau)=s\big(T(\tau)\,x_0+v(s)(\tau)\big)$.*

Proof. By the definition of $\mathscr{G}$ and the fact that $x_{\theta+\pi}=-x_\theta$, we have

$$\mathscr{G}(s, \rho, \mu, v, x_\theta) = -\mathscr{G}(-s, \rho, \mu, -v, x_{\theta+\pi}). \tag{1.43}$$

Thus

$$\begin{aligned} 0 &= \mathscr{G}\big(s, \rho(s,\theta), \mu(s,\theta), v(s,\theta), x_\theta\big) \\ &= -\mathscr{G}\big(-s, \rho(s,\theta), \mu(s,\theta), -v(s,\theta), x_{\theta+\pi}\big) \end{aligned} \tag{1.44}$$

and (1.42) follows from the local uniqueness of solutions of $\mathscr{G}=0$. If V is given by (1.38), then it is invariant under the translations $M_\psi f(\tau)=f(\tau-\psi)$. Noting that $M_{-\psi} T(\cdot)\, x_\theta = T(\cdot)\, x_{\psi+\theta}$, we have

$$M_{-\psi}\,\mathscr{G}(s, \rho, \mu, v, x_\theta) = \mathscr{G}(s, \rho, \mu, M_{-\psi}\, v, x_{\psi+\theta}) + T(\cdot)\,\mathscr{G}(s, \rho, \mu, v, x_\theta)(\psi). \tag{1.45}$$

This implies that

$$\rho(s,\theta)=\rho(s,\psi+\theta), \qquad \mu(s,\theta)=\mu(s,\psi+\theta), \qquad M_{-\psi}\, v(s,\theta)=v(s,\psi+\theta); \tag{1.46}$$

in particular;

$$v(s,\theta)(\tau+\psi)=v(s,\psi+\theta)(\tau). \tag{1.47}$$

Choosing $\psi=\pi$ it follows that

$$\rho(s,\theta)=\rho(s,\theta+\pi), \qquad \mu(s,\theta)=\mu(s,\theta+\pi), \qquad v(s,\theta)(\tau+\pi)=v(s,\theta+\pi)(\tau). \tag{1.48}$$

Taken together, (1.43) and (1.48) show that ρ and μ are even functions of s and that $v(-s, \theta)(\tau) = -v(s, \theta)(\tau+\pi)$. Since $u(s, \theta) = s(T(\cdot)x_\theta + v(s, \theta))$, all of the assertions of the theorem are therefore proved.

For the next result we assume that V is given by (1.38).

Corollary 1.47. *If f is real analytic and $\mu(s) \not\equiv 0$ (respectively, $\rho(s) \not\equiv 1$) then $\mu(s) > 0$ or $\mu(s) < 0$ (respectively $\rho(s) > 1$ or $\rho(s) < 1$) for $s \neq 0$ but near* 0.

This is an immediate consequence of the analyticity of μ, ρ, and the fact that they are even functions of s.

Remark 1.48. Corollary 1.47 was proved by HOPF by another argument. Also the one sided nature of the bifurcation with respect to μ fails if analyticity is relaxed. Consider, for example, the system of two equations

$$\frac{d}{d\tau}\begin{pmatrix} x \\ y \end{pmatrix} + \rho\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \mu\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}\right] = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{1.49}$$

If f, g, and their gradients vanish at $(x, y) = (0, 0)$ the assumptions of Theorem 1.11 are satisfied, as is easily verified. Choosing $f(x, y) = x K(r)$ and $g(x, y) = y K(r)$, where $r = (x^2 + y^2)^{\frac{1}{2}}$, we find that (1.49) has the family of solutions $\rho(s) \equiv 1$, $\mu(s) = -K(|s|)$, $x = s\cos\tau$, $y = s\sin\tau$. Now if, for example. $K(r) = \exp(-r^{-2})\sin(r^{-2})$, then $\mu(s)$ takes on positive and negative values in every neighborhood of $s = 0$.

Section 2. Linearized Stability

This section concerns the linearized stability of the periodic solutions found in Theorem 1.11. IUDOVICH [15], JOSEPH & SATTINGER [19], JOSEPH [18], and JOSEPH & NIELD [18′] have also investigated the linearized stability question, while IOOSS [14] and HENRY [10] have considered nonlinear stability as well.

By way of motivation, and to explain what we mean by linearized stability, it may be helpful to give an informal review of the concepts involved. Suppose $A(t)$ is a time-dependent linear operator which is p-periodic in t. The *Floquet multipliers* of the problem

$$\frac{dw}{dt} + A(t)\,w = 0 \tag{2.1}$$

are the eigenvalues of $U(p)$, where $w(t) = U(t)x$ is the solution of (2.1) satisfying $w(0) = x$ (see for example [9], [10]). We say that κ is a *Floquet exponent* of (2.1) if $e^{p\kappa}$ is a Floquet multiplier. Equivalently, $-\kappa$ is a Floquet exponent of (2.1) if and only if the problem

$$\frac{dz}{dt} + A(t)\,z = \kappa z, \qquad z(p) = z(0) \tag{2.2}$$

has a nontrivial solution. Observe that explicit knowledge of $U(p)$ is not required for this alternate characterization of the Floquet exponents. Hence (2.2) may serve as a more convenient vehicle for their calculation than the definition.

If u is a p-periodic solution of the nonlinear problem

$$\frac{du}{dt}+g(u)=0, \tag{2.3}$$

the Floquet exponents and multipliers for u are defined to be the multipliers and exponents of (2.1) with $A(t)=g_u(u(t))$. If $\dot{u}=\frac{du}{dt}\neq 0$, differentiation (2.3) shows that

$$\frac{d\dot{u}}{dt}+g_u(u(t))\,\dot{u}=0,$$

so 0 is a Floquet exponent and 1 is a Floquet multiplier for u. Under appropriate hypotheses it has been shown that the stability properties of a periodic solution of (2.3) are determined by the moduli of its Floquet multipliers (cf. [10, Thm. 8.2.3]), and references here and in the literature to the study of "linearized stability" of periodic solutions mean the study of their Floquet multipliers. In Section 1 we considered 2π-periodic solutions $u(s)$ of (2.3), where $g(u)=\rho(s)(Lu+f(\mu(s),u))$. The Floquet exponents for $u(s)$ are therefore (formally) numbers $-\kappa$ such that the problem

$$\frac{dw}{dt}+\rho(s)\left(Lw+f_u(\mu(s),u(s))\,w\right)=\kappa w, \qquad w(0)=w(2\pi) \tag{2.4}$$

has a nontrivial solution. At $s=0$, (2.4) becomes

$$\frac{dw}{dt}+Lw=\kappa w, \qquad w(0)=w(2\pi); \tag{2.5}$$

the set of values of κ for which (2.5) has a nontrivial solution is $\{\sigma(L)\pm i n \mid n=0,1,2,\ldots\}$, so the corresponding multipliers are $e^{-2\pi\sigma(L)}$. Observe that $1\in e^{-2\pi\sigma(L)}$, while the moduli of the numbers $e^{-2\pi\sigma(L)}\setminus\{1\}$ are bounded away from 1. Moreover, 1 occurs as a multiplier with multiplicity 2 corresponding to the double eigenvalue $\kappa=0$ of (2.5). Therefore we expect the multipliers for the solution $u(s)$ of (1.6) with $\rho=\rho(s)$ and $\mu=\mu(s)$ to consist of a set of numbers uniformly bounded away from 1 in modulus, except for two of them which correspond to small perturbations of the double eigenvalue $\kappa=0$ of (2.5). Since $\dot{u}(s)(\tau)=\frac{d}{d\tau}u(s)(\tau)$ is always an eigenvector for $\kappa=0$ corresponding to the multiplier 1, we seek a second small eigenvalue, which we shall denote by $\kappa(s)$. Of particular interest is the sign of $\operatorname{Re}\kappa(s)$, which determines the modulus of the critical multiplier $e^{-2\pi\kappa(s)}$. We will show below that $\kappa(s)$ is real, and that sign $\kappa(s)$ is determined by sign $\mu'(s)$ for small s. Such a qualitative fact was established in the finite dimensional analytic case by HOPF [12] under the assumption that $\mu''(0)\neq 0$ (see also POORE [22]). A more general result of the same type was presented recently in [18], still however in the analytic case. Here we make precise the result of [18] and extend it to a general class of nonanalytic equations.

To carry out the details of the above program we again work with the integrated form $\mathscr{F}=0$ of (1.6), where $\mathscr{F}$ is given by (1.10). Defining the bounded linear operator $K(r)$ from $C_{2\pi}(\mathbb{R}, X)$ into $C_0([0, 2\pi], X_\alpha)$ by

$$(K(r)\,u)(\tau)=\int_0^\tau T(r(\tau-\xi))\,u(\xi)\,d\xi, \qquad r>0,$$

the corresponding integrated form of (2.4) is

$$\mathscr{F}_u(s)\,w=K(\rho(s))\,\kappa\,w, \qquad w\in C_{2\pi}(\mathbb{R}, X_\alpha), \tag{2.7}$$

where $\mathscr{F}_u(s)=\mathscr{F}_u(\rho(s), \mu(s), u(s))$.

In the following discussion x_0, x_1, x_0^*, x_1^* are as in Lemma 1.13, and

$$\varphi_0(\tau)=T(\tau)\,x_0, \qquad \varphi_1(\tau)=T(\tau)\,x_1=-T(\tau)\,L x_0=\frac{d}{d\tau}\varphi_0. \tag{2.8}$$

The hypotheses of Theorem 1.11 will be assumed to hold, and the notation of its proof will be used. After settling a minor technical point, we characterize $\kappa(s)$ in Lemma 2.10 below.

Lemma 2.9. *Under the above assumptions, the function $s\to s^{-1}\dot{u}(s)=\varphi_1+\dot{v}(s)$ is defined in a neighborhood of $s=0$ and is continuous with values in $C_{2\pi}(\mathbb{R}, X)$.*

The proof of this lemma is sketched at the end of this section.

Lemma 2.10. *Under the above assumptions there exist unique continuous functions $\kappa(s)$, $\eta(s)$, $z(s)$ defined near $s=0$, which have values in $\mathbb{R}$, $\mathbb{R}$ and V respectively, which vanish at $s=0$, and which satisfy*

$$\mathscr{F}_u(s)\,(\varphi_0+z(s))=K(\rho(s))\,[\kappa(s)\,(\varphi_0+z(s))+\eta(s)\,(\varphi_1+\dot{v}(s))], \tag{2.11}$$

where $\mathscr{F}_u(s)=\mathscr{F}_u(\rho(s), \mu(s), u(s))$.

Proof. As noted in the proof of Lemma 1.12, the function $r\to K(r)$ defined in (2.6) is analytic in $r>0$. Now consider the mapping

$$(s, \kappa, \eta, z)\to\mathscr{F}_u(s)\,(\varphi_0+z)-K(\rho(s))\,(\kappa(\varphi_0+z)+\eta(\varphi_1+\dot{v}(s))), \tag{2.12}$$

which takes a neighborhood of $(0,0,0,0)$ in $\mathbb{R}^3\times V$ into $C_0([0, 2\pi], X_\alpha)$ and vanishes at $(0,0,0,0)$. The derivative of (2.12) with respect to (κ, η, z) at $(0,0,0)$ is the map

$$(\hat{\kappa}, \hat{\eta}, \hat{z})\to\mathscr{F}_u(0)\,\hat{z}-K(1)\,(\hat{\kappa}\,\varphi_0+\hat{\eta}\,\varphi_1),$$

which is an isomorphism. As in the proof of Theorem 1.11, one need only show that $K(1)(\hat{\kappa}\,\varphi_0+\hat{\eta}\,\varphi_1)\in R(\mathscr{F}_u(0))$ implies $\hat{\kappa}=\hat{\eta}=0$. This is immediate from the relation $(K(1)\,\varphi_i)(\tau)=\tau\,\varphi_i(\tau)$, (1.15), and Lemma 1.13(d). The continuity of (2.12) and its derivative with respect to (κ, η, z) follows from our general assumptions and Lemma 2.9. One can thus invoke the implicit function theorem (as stated, for example, in [3]) to obtain the desired conclusion.

Remark. That $\kappa(s)$ provides the desired continuation of the second zero eigenvalue of (2.7) can be seen as follows: If $\kappa(s)=0$ for some s, then (2.11) and $\mathscr{F}_u(s)(\varphi_1+\dot v(s))=0$ imply that $\varphi_0+z(s)$ and $\varphi_1+\dot v(s)$ are (obviously independent) null vectors of $\mathscr{F}_u(s)$, which therefore has 0 as a double eigenvalue. If $\kappa(s)\neq 0$, one rewrites (2.11) in the form

$$\begin{aligned}&\mathscr{F}_u(s)\left(\varphi_0+z(s)+\eta\,\kappa(s)^{-1}(\varphi_1+\dot v(s))\right)\\&\qquad=K(\rho(s))\left(\kappa(s)\left(\varphi_0+z(s)+\eta(s)\,\kappa(s)^{-1}(\varphi_1+\dot v(s)\right)\right)\end{aligned}$$

and we have (2.7) with $w=\varphi_0+z(s)+\eta(s)\,\kappa(s)^{-1}(\varphi_1+v(s))$. Hence $-\kappa(s)$ is a Floquet exponent.

In principle the proof of Lemma 2.10 furnishes a constructive means to determine $\kappa(s)$ as well as $\eta(s)$ and $z(s)$. In practice this may be cumbersome, and it is desirable to obtain as much qualitive information as possible about these functions, and in particular about κ, in terms of other data. The following theorem, which is our main linearized stability result, addresses these points.

Theorem 2.13. *Let the hypotheses of Lemma* 2.10 *be satisfied and let* $\kappa(s)$, $\eta(s)$, $z(s)$ *be as in that lemma. Then*

$$|\kappa(s)+(\operatorname{Re}\beta'(0))\,s\,\mu'(s)|\leqq|s\,\mu'(s)|\,o(1)\qquad \text{as } s\to 0. \tag{2.14}$$

In particular, there is a neighborhood of $s=0$ *in which* $\kappa(s)$ *and* $s\,\mu'(s)$ *have the same zeroes, and in which* $\kappa(s)$ *and* $-(\operatorname{Re}\beta'(0))\,s\mu'(s)$ *have the same sign (if they do not vanish). Moreover*

$$\left|\eta(s)-\left(s\frac{\rho'(s)}{\rho(s)}+s\,\mu'(s)\operatorname{Im}\beta'(0)\right)\right|\leqq o(1)\,|s\,\mu'(s)|\qquad \text{as } s\to 0, \tag{2.15}$$

and there is a constant c *such that*

$$\|w'(s)-z(s)\|\leqq c\,|s\,\mu'(s)|, \tag{2.16}$$

for s *near* 0, *where* $w(s)=s\,v(s)=u(s)-s\,\varphi_0$.

Proof. Let differentiation with respect to s be denoted by a prime, and set

$$\mathscr{F}(s)=\mathscr{F}(\rho(s),\mu(s),s(\varphi_0+v(s))),\qquad \mathscr{F}_\rho(s)=\mathscr{F}_\rho(\rho(s),\mu(s),u(s)),\quad \text{etc.}$$

Differentiating the relation $\mathscr{F}(s)=0$ gives

$$\rho'(s)\,\mathscr{F}_\rho(s)+\mu'(s)\,\mathscr{F}_\mu(s)+\mathscr{F}_u(s)\left(\varphi_0+w'(s)\right)=0. \tag{2.17}$$

Subtracting (2.11) from (2.17) yields

$$\begin{aligned}&\rho'(s)\,\mathscr{F}_\rho(s)+\mu'(s)\,\mathscr{F}_\mu(s)\\&\qquad+K(\rho(s))\left[\kappa(s)(\varphi_0+z(s))+\eta(s)(\varphi_1+\dot v(s))\right]+\mathscr{F}_u(s)\left(w'(s)-z(s)\right)=0.\end{aligned} \tag{2.18}$$

A simple computation also shows that

$$\mathscr{F}_\rho(s)=-\rho(s)^{-1}K(\rho(s))\,\dot u(s)=-\rho(s)^{-1}K(\rho(s))\left(\varphi_1+\dot v(s)\right). \tag{2.19}$$

Taken together, (2.18) and (2.19) imply

$$K(\rho(s))\left[\kappa(s)(\varphi_0+z(s))+\left(\eta(s)-\frac{s\rho'(s)}{\rho(s)}\right)(\varphi_1+\dot v(s))\right] + \mathscr{F}_u(s)(w'(s)-z(s))+s\mu'(s)(s^{-1}\mathscr{F}_\mu(s))=0, \tag{2.20}$$

where $s^{-1}\mathscr{F}_\mu(s)$ is assigned its limiting value $K(1)L_1\varphi_0$ $(L_1=f_{\mu x}(0,0))$ at $s=0$. The map of $\mathbb{R}\times\mathbb{R}\times V$ given by

$$(\kappa,\xi,v)\to K(\rho(s))\left[\kappa(\varphi_0+z(s))+\xi(\varphi_1+\dot v(s))\right]+\mathscr{F}_u(s)\,v\in C_0([0,2\pi];X_\alpha) \tag{2.21}$$

is an isomorphism when $s=0$, by Lemmas 1.13 and 1.14. (Since $z(0)=\dot v(0)=0$, this is the map already used in the proof of Lemma 2.10.) Moreover, by earlier results and Lemma 2.9 this linear mapping depends continuously on s. *Thus there is a positive constant* c such that (2.20) implies

$$|\kappa(s)|+\left|\eta(s)-\frac{s\rho'(s)}{\rho(s)}\right|+\|w'(s)-z(s)\|\leqq C|s\mu'(s)| \tag{2.22}$$

for small $|s|$. We have now established (2.16). In view of (2.22), the condition $\dot v(0)=z(0)=0$ together with the continuity of $K(\rho(s))$, $\dot v(s)$, $z(s)$, $s^{-1}\mathscr{F}_\mu(s)$ in s, (2.20) implies that the function

$$g(s)=K(1)\left[\kappa(s)\varphi_0+\left(\eta(s)-\frac{s\rho'(s)}{\rho(s)}\right)\varphi_1+s\mu'(s)L_1\varphi_0\right] \tag{2.23}$$

satisfies

$$\|g(s)-\mathscr{F}_u(0)(w'(s)-z(s))\|\leqq O(1)\,|s\mu'(s)| \qquad \text{as } s\to 0.$$

Hence

$$|(x_i^*,g(s)(2\pi))-(x_i^*,\mathscr{F}_u(0)(w'(s)-z(s))(2\pi))| = |(x_i^*,g(s)(2\pi))|\leqq o(1)\,|s\mu'(s)| \qquad \text{as } s\to 0$$

by (1.15). By (2.23), Lemmas 1.13 and 1.14, and the relation $K(1)\varphi_i(\tau)=\tau\varphi_i(\tau)$, upon choosing $i=0$ we obtain (see the proof of (1.19))

$$|2\pi\kappa(s)+2\pi s\mu'(s)\operatorname{Re}\beta'(0)|\leqq o(1)\,|s\mu'(s)|,$$

which is (2.14). Similarly, choosing $i=1$ we find (2.15).

Remark 2.24. Theorem 2.13 provides useful information about the linearized stability problem. For example, if $\sigma(L)\setminus\{\pm i\}$ is contained in the open right-half plane $\{\zeta\in\mathbb{C}\colon \operatorname{Re}\zeta>0\}$, if $\operatorname{Re}\beta'(0)<0$, and if bifurcation is "supercritical" (i.e. $s\mu'(s)>0$ when $s\neq 0$), then $\kappa(s)>0$. Hence by [10, Theorem 8.2.3] the bifurcating periodic solutions are stable. Similarly, if the bifurcation is "subcritical" (i.e. $s\mu'(s)<0$ for $s\neq 0$), then $\kappa(s)<0$ and the bifurcating solutions are unstable.

Remark 2.25. Equations (2.14)–(2.16) show that the left-hand sides of these expressions divided by $s\mu'(s)$ are bounded for s near 0. In fact these quotients are continuous at $s=0$. This was already proved directly for some analytic equations in [18]. The corresponding theorem in the general case is

Theorem 2.25. *Under the hypotheses of Theorem 2.13, there exist continuous functions $A(s)$, $B(s)$, $C(s)$, with values in $\mathbb{R}$, $\mathbb{R}$, V respectively, which are defined and continuous near $s=0$ and satisfy*

$$\begin{aligned}\kappa(s)&=s\,\mu'(s)\,A(s)\\ \eta(s)&=\frac{s\,\rho'(s)}{\rho(s)}+s\,\mu'(s)\,B(s)\\ z(s)&=w'(s)-s\,\mu'(s)\,C(s)\end{aligned}\tag{2.26}$$

and

$$\begin{aligned}A(0)&=-\operatorname{Re}\beta'(0)\\ B(0)&=-\operatorname{Im}\beta'(0)\\ \mathscr{F}_u(0)\,C(0)&=K(1)\left(\operatorname{Im}\beta'(0)\,\varphi_1-L_1\,\varphi_0+(\operatorname{Re}\beta'(0))\,\varphi_0\right).\end{aligned}\tag{2.27}$$

Proof. Assuming that κ, η, z have the form (2.26), substitution into (2.20) gives

$$\begin{aligned}&\left[A(s)\,K(\rho(s))\left(\varphi_0+w'(s)-s\,\mu'(s)\,C(s)\right)+B(s)\,K(\rho(s))\left(\varphi_1+\dot v(s)\right)\right.\\ &\qquad\qquad\qquad +s^{-1}\mathscr{F}_\mu(s)+\mathscr{F}_\mu(s)\,C(s)\,\rho=0,\end{aligned}\tag{2.28}$$

with the understanding that $s^{-1}\mathscr{F}_\mu(s)$ has its limiting value $\mathscr{F}_{\mu u}(0)\,\varphi_0=K(1)\,L_1\,\varphi_0$ at $s=0$. Define a map of a neighborhood of 0 in $\mathbb{R}\times\mathbb{R}^2\times V$ into $C_0([0,2\pi], X_\alpha)$ by

$$\begin{aligned}(s,\tilde A,\tilde B,\tilde C)\to\tilde A K(\rho(s))\left(\varphi_0+w'(s)-s\,\mu'(s)\,\tilde C\right)+\tilde B K(\rho(s))\left(\varphi_1+\dot v(s)\right)&\\ +s^{-1}\mathscr{F}_\mu(s)+\mathscr{F}_\mu(s)\,\tilde C.&\end{aligned}\tag{2.29}$$

The value of the map at the values $s=0$ and

$$(\tilde A,\tilde B,\tilde C)=\left(-\operatorname{Re}\beta'(0),\,-\operatorname{Im}\beta'(0),\,C(0)\right)$$

(as given (2.27)) is 0, while its derivative with respect to $\tilde A, \tilde B, \tilde C$, at this point is

$$(\tilde A,\tilde B,\tilde C)\to\tilde A K(1)\,\varphi_0+\tilde B K(1)\,\varphi_1+\mathscr{F}_u(0)\,\tilde C,\tag{2.30}$$

which is an isomorphism. Hence by the version of the implicit function theorem used in [3], the zeros of (2.29) near $(0, -\operatorname{Re}\beta'(0), -\operatorname{Im}\beta'(0), C(0))$ are given by continuous functions $A(s)$, $B(s)$, $C(s)$ satisfying (2.27) and (2.28). Given (2.17), the equality (2.20) is equivalent to (2.11). Since $\kappa(s)$, $\mu(s)$, $z(s)$ are the unique solutions of (2.11), it follows from (2.28) and (2.11) that (2.26) holds near $s=0$.

Remark 2.31. As the above argument shows, we could have obtained the more elegant Theorem 2.25 first and then Theorem 2.13 as a corollary. However, Theorem 2.13 already contains the important qualitative information and motivates Theorem 2.25. A version of Theorem 2.25 for analytic nonlinearities in the context of the Navier-Stokes equations was studied in [18]. JOSEPH assumed (2.26) from the beginning and obtained A, B, C by a series expansion method. This motivated in part our Theorems 2.13 and 2.25. We further remark that an analogue of Theorem 2.25 can be given to improve Theorem 1.16 of [4].

Sketch of the proof of Lemma 2.9. Define

$$g(s,w)=\int_0^1 f_x\big(\mu(s), s\,r\,w\big)\,w\,dr,$$

so that $g(s,w)=s^{-1}f\big(\mu(s), sw\big)$ for suitable $w\in X_\alpha$. With

$$w(s)=s^{-1}u(s)=\varphi_0+v(s)$$

the relation $s^{-1}\mathscr{F}\big(\rho(s),\mu(s), s\,w(s)\big)=0$ can be rewritten in the form

$$w(s)(\tau)-T\big(\rho(s)\tau\big)\,x(s)+\rho(s)\int_0^\tau T\big(\rho(s)(\tau-\xi)\big)\,g\big(s, w(s)(\xi)\big)\,d\xi=0,$$

where $x(s)=\varphi_0(0)+v(s)(0)$. By the hypotheses on f and the definition of g, it is clear that $g(s,w)$ is continuously differentiable in the pair (s,w). A result of HENRY [11] shows that the solution of

$$Z(\tau)-T(\rho\tau)x+\rho\int_0^\tau T\big(\rho(\tau-\xi)\big)\,g\big(s,Z(\xi)\big)\,d\xi=0,$$

which we denote by $Z(\rho,s,x,\tau)$, has the property that $\frac{\partial}{\partial\tau}Z(\rho,s,x,\tau)$ exists and $(\rho,s,x,\tau)\to\frac{\partial}{\partial\tau}Z(\rho,s,x,\tau)$ is continuous from $(0,\infty)\times\mathbb{R}\times X_\alpha\times(0,\infty)\big)$ into X_α on the domain of definition of Z. Since

$$w(s)(\tau)=Z\big(\rho(s), s, x(s), \tau\big),$$

the mapping $(s,\tau)\to\partial/\partial\tau\big(w(s)\tau\big)$ exists and is continuous into X_α for $\tau>0$ and $|s|$ small. But $w(s)(\tau)$ is 2π-periodic in τ. Thus the derivative exists and is continuous for $\tau\geqq 0$.

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