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Bifurcation from Simple Eigenvalues

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Let G be a mapping of a subset of a Banach space W into a Banach space Y. Let C be a curve in W such that $G(C) = \{0\}$. A general version of the main problem of bifurcation theory may be stated: Given $p \in C$, determine the structure of $G^{-1}\{0\}$ in some neighborhood of p. In this work simple conditions are given under which there is a neighborhood N_p of p such that $G^{-1}\{0\} \cap N_p$ is topologically (or diffeomorphically) equivalent to the subset $(-1, 1) \times \{0\} \cup \{0\} \times (-1, 1)$ of the plane, and the first order behavior of G on $G^{-1}\{0\} \cap N_p$ as well as the set itself is studied.

The results obtained help unify that part of bifurcation theory commonly called "bifurcation from a simple eigenvalue" as well as they extend its applicability. A broad spectrum of examples is offered, including some generalizations of known results concerning nonlinear eigenvalue problems for ordinary and partial differential equations.

INTRODUCTION

Bifurcation phenomena arise in many parts of mathematical physics and an understanding of their nature is of practical as well as theoretical importance. This paper develops the theory of "bifurcation at a simple eigenvalue," which has received much attention in the literature, in a

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quite general setting. Many papers (e.g., [2, 3, 5, 6, 8, 12]) have recently appeared which contain and use special cases of our results. It is our hope that by collecting the most important facts concerning bifurcation at a simple eigenvalue in one place, establishing them in a generality sufficient to lay bare the essential features of the problem, and offering a sufficient spectrum of examples, we will help to eliminate this duplication of effort and set straight a few misconceptions.

We turn now to a more precise description of our work. Let W and Y be real Banach spaces, Ω an open subset of W and $G : \Omega \to Y$ be a continuous map. Suppose there is a simple arc C in Ω given by $C = \{w(t) : t \in I\}$, where I is an interval, such that $G(w) = 0$ for $w \in C$. If there is a number $\tau \in I$ such that every neighborhood of $w(\tau)$ contains zeros of G not lying on C, then $w(\tau)$ is called a *bifurcation point* for the equation $G(w) = 0$ with respect to the curve C. In many situations W is of the form $\mathbf{R} \times X$, where X is a real Banach space and $C = \{(\lambda, 0) \mid \lambda \in \mathbf{R}, 0 \in X\}$. The basic problem of bifurcation theory is that of finding the bifurcation points for $G = 0$ with respect to C and studying the structure of $G^{-1}\{0\}$ near such points. In the special case $W = \mathbf{R} \times X$ above, it is easily shown that if $G_x(\lambda, 0)$, the Fréchet derivative of the map $x \to G(\lambda, x)$ at $(\lambda, 0)$, is an isomorphism of X onto Y, then $(\lambda, 0)$ is not a bifurcation point. Under further restrictions, e.g., $X = Y$,

$$G(\lambda, x) = Bx - \lambda x + H(x) + R(\lambda, x),$$

B is linear, H is homogeneous of some order, R is a small remainder,

$$G_x(\lambda_0, 0) = B - \lambda_0 I$$

has zero as a simple eigenvalue, the range of $B - \lambda_0 I$ has codimension 1, and a certain nondegeneracy condition is satisfied, it is known (see, e.g., [7, Theorem 6.12]) that $(\lambda_0, 0)$ is a bifurcation point and in addition to the line C, the zeros of G near $(\lambda_0, 0)$ consist of a continuous curve passing through $(\lambda_0, 0)$. One can then ask whether this new curve of solutions can be continued, and whether there are bifurcations (so-called "secondary bifurcations") with respect to it. (Under certain compactness assumptions on G, this bifurcation phenomena is global in a sense. See [9].)

It is our opinion that most of the existing literature places too much emphasis on parameterizing solutions of $G = 0$ by λ. This is far too restrictive, and is mathematically inadequate to handle even the simple situation $H = R = 0$ above. In line with this is the overemphasis placed on studying $G_x(\lambda, x)$ in treating the continuation and secondary bifurcation questions, when in fact G', the entire Fréchet derivative of G, should be of concern.

Our approach treats bifurcation and secondary bifurcation simultaneously. In addition, some standard assumptions are dispensed with and the conclusions usually drawn from these assumptions are shown to be consequences of continuity and the theory of Fredholm operators. The main result is:

THEOREM 1. *Let W, Y be Banach spaces, Ω an open subset of W and $G : \Omega \to Y$ be twice continuously differentiable. Let $w : [-1, 1] \to \Omega$ be a simple continuously differentiable arc in Ω such that $G(w(t)) = 0$ for $|t| \leqslant 1$. Suppose*

(a) $w'(0) \neq 0$,

(b) $\dim N(G'(w(0))) = 2$, $\operatorname{codim}(R(G'(w(0)))) = 1$,

(c) $N(G'(w(0)))$ *is spanned by* $w'(0)$ *and* v, *and*

(d) $G''(w(0))(w'(0), v) \notin R(G'(w(0)))$.

Then $w(0)$ is a bifurcation point of $G(w) = 0$ with respect to $C = \{w(t) : t \in [-1, 1]\}$ and in some neighborhood of $w(0)$ the totality of solutions of $G(w) = 0$ form two continuous curves intersecting only at $w(0)$.

Here dim and codim are abbreviations for dimension and codimension, respectively, (the codimension of a subspace Z of Y is the dimension of Y/Z), and $N(T)$, $R(T)$ denote the null space and range of a linear operator T. G' and G'' are the first and second Fréchet derivatives of G. Their values are linear operators from W into Y and bilinear operators from $W \times W$ into Y, respectively. For example, $G''(w(0))(w'(0), v)$ is the value of the bilinear operator $G''(w(0))$ at $(w'(0), v) \in W \times W$.

In Section 1 a simple change of variables is used to demonstrate that Theorem 1 is a special case of a more general and explicit result, Theorem 1.7. This latter result is proved, in turn, by an elementary application of the implicit function theorem in conjunction with several estimates. Supplementary results concerning the nature of the bifurcating curve and the behavior of G' along this curve are then obtained. Section 2 is devoted to examples and applications, which range from studying mappings in finite dimensions to boundary-value problems for nonlinear ordinary and partial differential equations. The study of problems involving differential operators is put in our framework by employing the graph topologies, and a useful theorem concerning this setting is established as a simple corollary of the results in Section 1. (During the course of this work we learned of the paper [11] by Recken, in which the graph topology is used in a bifurcation problem.)

1. The Main Results

We use the notation of Theorem 1 and assume its hypotheses. Let X be a complement of span$\{w'(0)\}$ in W and consider the map

$$(t, x) \to w(t) + x \tag{1.1}$$

of $[-1, 1] \times X$ into W. The Fréchet derivative of the transformation (1.1) at $(0, 0) \in \mathbf{R} \times X$ is the linear map

$$(t^*, x^*) \to t^* w'(0) + x^* \tag{1.2}$$

of $\mathbf{R} \times X$ into W. Clearly, (1.2) is an isomorphism of $\mathbf{R} \times X$ onto W, so (1.1) defines a C^1 diffeomorphism (see [4, Theorem 10.2.5]) of a neighborhood of $(0, 0) \in \mathbf{R} \times X$ onto a neighborhood of $w(0) \in W$. Thus studying the equation

$$F(t, x) = G(w(t) + x) = 0, \qquad (t, x) \in [-1, 1] \times X \tag{1.3}$$

in a neighborhood of $(0, 0)$ is C^1 equivalent to the study of $G(w) = 0$ near $w(0)$. Observe that since G is twice continuously differentiable and w is once continuously differentiable, F is once continuously differentiable. Moreover, the second-order "mixed partial" derivative F_{tx} exists and is continuous, since it involves G'' and w', but not w''.

At this point we have made two choices in the development, that of v and X, whose roles warrant clarification. Since $N(G'(w(0))) =$ span$\{w'(0), v\}$ is two-dimensional, we could have chosen X to contain v. In any case, X will contain a vector of the form $v + \beta w'(0)$ in $N(G'(w(0)))$. We assumed

$$G''(w(0))(w'(0), v) \notin R(G'(w(0))) \tag{1.4}$$

and will show

$$G''(w(0))(w'(0), w'(0)) \in R(G'(w(0))). \tag{1.5}$$

Using (1.4), (1.5) and

$$G''(w(0))(w'(0), v + \beta w'(0)) = \beta G''(w(0))(w'(0), w'(0)) + G''(w(0))(w'(0), v),$$

it follows that (1.4) holds if v is replaced by $v + \beta w'(0)$. To show (1.5), observe $G(w(t)) = 0$ implies $G'(w(\tau))\, w'(\tau) = 0$ and hence

$$\frac{[G'(w(\tau)) - G'(w(0))]}{\tau}\, w'(\tau) = G'(w(0))\, \frac{(w'(0) - w'(\tau))}{\tau}. \tag{1.6}$$

The right side of (1.6) is in $R(G'(w(0)))$, which is closed (since it is of finite codimension) and the limit as $\tau \to 0$ of the left side is $G''(w(0))(w'(0), w'(0))$, establishing (1.5). Thus, whatever the initial choice of X and v, we may assume, without loss of generality, that $v \in X$. Observe next that $F_x(0, 0)X = G'(w(0))X = R(G'(w(0)))$ and

$$F_{tx}(0, 0)v = G''(w(0))(w'(0), v).$$

Theorem 1 is therefore an immediate corollary of Theorem 1.7 below, which gives more precise information.

THEOREM 1.7. *Let X, Y be Banach spaces, V a neighborhood of* 0 *in X and*

$$F : (-1, 1) \times V \to Y$$

have the properties

(a) $F(t, 0) = 0$ *for* $|t| < 1$,

(b) *The partial derivatives* F_t, F_x *and* F_{tx} *exist and are continuous,*

(c) $N(F_x(0, 0))$ *and* $Y/R(F_x(0, 0))$ *are one-dimensional.*

(d) $F_{tx}(0, 0)\, x_0 \notin R(F_x(0, 0))$, *where*

$$N(F_x(0, 0)) = \operatorname{span}\{x_0\}.$$

If Z is any complement of $N(F_x(0, 0))$ in X, then there is a neighborhood U of $(0, 0)$ *in* $\mathbf{R} \times X$, *an interval* $(-a, a)$, *and continuous functions* $\varphi : (-a, a) \to \mathbf{R}$, $\psi : (-a, a) \to Z$ *such that* $\varphi(0) = 0$, $\psi(0) = 0$ *and*

$$F^{-1}(0) \cap U = \{(\varphi(\alpha), \alpha x_0 + \alpha\psi(\alpha)) : |\alpha| < a\} \cup \{(t, 0) : (t, 0) \in U\}. \tag{1.8}$$

If F_{xx} is also continuous, the functions φ and ψ are once continuously differentiable.

Proof. Define a function f by

$$f(\alpha, t, z) = \begin{cases} \alpha^{-1}F(t, \alpha x_0 + \alpha z) & \text{if } \alpha \neq 0 \\ F_x(t, 0)(x_0 + z) & \text{if } \alpha = 0 \end{cases} \tag{1.9}$$

for those $(\alpha, t, z) \in \mathbf{R} \times \mathbf{R} \times Z$ such that $\alpha(x_0 + z) \in V$, and $|t| < 1$.

Observe that the partial derivatives f_t and f_z are continuous in (α, t, z). In addition,

$$f(0,0,0) = F_x(0,0)\,x_0 = 0, \tag{1.10}$$

and the Fréchet derivative of the map $(t, z) \to f(0, t, z)$ at $(t, z) = (0, 0)$ is the linear map

$$(t^*, z^*) \to t^* F_{tx}(0,0)\,x_0 + F_x(0,0)\,z^* \tag{1.11}$$

of $\mathbf{R} \times Z$ into Y. Assumptions (c) and (d) of Theorem 1.7 imply (1.11) is an isomorphism onto Y, and the implicit function theorem (see the Appendix to this paper) implies the existence of the functions φ, ψ possessing the properties asserted in the theorem, save for (1.8). (Observe the continuity of F_{xx} implies the continuity of f_α.) We only know $F^{-1}(0)$ contains the right side of (1.8) at this point, and need to look more closely to obtain the equality (1.8) for some U. The implicit function theorem as used here implies that the zeros of f near $(0, 0, 0)$ form a continuous curve which may be parameterized by α, or, in view of (1.9), that the zeros of $F(t, \alpha x_0 + z)$, $z \in Z$, near $t = 0$, $\alpha x_0 + z = 0$, *restricted by any estimate of the form* $\| z \| \leqslant g(\alpha) \mid \alpha \mid$ where $\lim_{\alpha \to 0} g(\alpha) = 0$ are either of the form $(t, 0)$ or the form $(\varphi(\alpha), \alpha x_0 + \alpha\psi(\alpha))$. The equality (1.8) for some U, therefore, follows from

LEMMA 1.12. *Let F satisfy the hypotheses of Theorem 1.7. Then there is a neighborhood U_1 of $(0, 0) \in \mathbf{R} \times V$ and a continuous function g on $\mathbf{R}$, $g(0) = 0$, such that $F(t, \alpha x_0 + z) = 0$, $z \in Z$ and $(t, \alpha x_0 + z) \in U_1$ implies*

$$\| z \| + |\alpha|\,|t| \leqslant |\alpha|\, g(\alpha).$$

Proof. In this proof and later we will use the existence of constants $k, K > 0$ satisfying

$$\begin{aligned} k(|\alpha| + \|z\|) &\leqslant \|\alpha x_0 + z\| \leqslant K(|\alpha| + \|z\|) \\ k(|\mu| + \|z\|) &\leqslant \|F_x(0,0)z + \mu F_{tx}(0,0)\,x_0\| \leqslant K(|\mu| + \|z\|). \end{aligned} \tag{1.13}$$

We write $\| \;\; \|$ for both the norm in X and the norm in Y as well as for the norms of linear operators. Our assumptions on F imply there is a neighborhood U_2 of $(0, 0) \in \mathbf{R} \times V$ and a continuous function h on $\mathbf{R}$, $h(0) = 0$, such that if $(t, \alpha x_0 + z) \in U_2$, then

$$\begin{aligned} &\text{(i)} \quad \|(F(t, \alpha x_0 + z) - F(t, \alpha x_0)) - F_x(t, \alpha x_0)z\| \leqslant \|z\|\, h(\|z\|) \\ &\text{(ii)} \quad \|(F(t, \alpha x_0) - F(t, 0)) - \alpha F_x(t, 0)\, x_0\| \leqslant |\alpha|\, h(|\alpha|) \\ &\text{(iii)} \quad \|F_x(t, 0)\, x_0 - tF_{tx}(0, 0)\, x_0\| \leqslant |t|\, h(|t|). \end{aligned} \tag{1.14}$$

Thus

$$
\begin{aligned}
0 &= F(t, \alpha x_0 + z) = F(t, \alpha x_0 + z) - F(t, \alpha x_0) + F(t, \alpha x_0) - F(t, 0) \\
&= \{(F(t, \alpha x_0 + z) - F(t, \alpha x_0)) - F_x(t, \alpha x_0)z\} \\
&\quad + \{[F_x(t, \alpha x_0) - F_x(0, 0)]z\} + \{(F(t, \alpha x_0) - F(t, 0)) - \alpha F_x(t, 0)\, x_0\} \\
&\quad + \{\alpha(F_x(t, 0)\, x_0 - tF_{tx}(0, 0)\, x_0)\} + F_x(0, 0)z + t\alpha F_{tx}(0, 0)\, x_0
\end{aligned}
$$

together with (1.13) and (1.14) implies

$$
\begin{aligned}
k\{\| z \| + | t\alpha |\} \leqslant \| z \| \, h(\| z \|) + \| F_x(t, \alpha x_0) - F_x(0, 0) \| \, \| z \| \\
+ | \alpha | \, h(| \alpha |) + | \alpha t | \, h(| t |).
\end{aligned}
\tag{1.15}
$$

Restricting $(t, \alpha x_0 + z)$ to a sufficiently small neighborhood U_1 of $(0, 0)$, we will have $h(\| z \|) \leqslant k/4$, $\| F_x(t, \alpha x_0) - F_x(0, 0) \| \leqslant k/4$ and $h(| t |) \leqslant k/2$, so that (1.15) implies

$$
\| z \| + | t\alpha | \leqslant (2/k)\, h(| \alpha |)\, | \alpha |,
$$

and the proof is complete.

Remark. Suppose $X = Y$ and $H(\lambda, x) = x - \lambda(Lx + M(x))$ where L is a compact linear map, M is continuously Fréchet differentiable near $0 \in X$ and $M(0) = 0$, $M_x(0) = 0$. Then (a) and (b) of Theorem 1.7 are satisfied with $F(t, x) = H(\lambda_0 + tx)$ and $\lambda_0 \in \mathbf{R}$ arbitrary. If, in addition, λ_0 is a simple characteristic value for L, then (c) and (d) of Theorem 1.7 are also satisfied. This case is frequently encountered in applications. A much more general version of this result will be given in Section 2.

With the basic existence and uniqueness Theorems 1 and 1.7 now in hand, we will establish a variety of useful supplementary facts. These results are listed below in the setting of Theorem 1.7. Each statement that follows has a straightforward analog in the setting of Theorem 1. The proofs conclude this section.

THEOREM 1.16. *Under the hypotheses of Theorem* 1.7 *there is a constant* $\delta > 0$ *such that* $F_x(t, 0)$ *is an isomorphism of* X *onto* Y *for* $0 < | t | < \delta$.

In the setting of Theorem 1, Theorem 1.16 says that $G'(w(t))$ is onto Y and $N(G'(w(t))) = \operatorname{span}\{w'(t)\}$ for $0 < | t | < \delta$. This will imply

THEOREM 1.17. *In addition to the assumptions of Theorem* 1.7, *let*

F be twice continuously differentiable. If φ, ψ are the functions of Theorem 1.7, *then there is a $\delta > 0$ such that $\varphi'(\alpha) \neq 0$ and $0 < |\alpha| < \delta$ implies $F_x(\varphi(\alpha), \alpha x_0 + \alpha\psi(\alpha))$ is an isomorphism of X onto Y.*

Results similar to Theorem 1.17 have been used under the assumption $\varphi'(0) \neq 0$, a simple subcase. See, e.g., Refs. [10, 11]. Finally, we look more closely at the functions φ, ψ of Theorem 1.7. Since solutions of $F(t, x) = 0$ have the form $(\varphi(\alpha), \alpha x_0 + \alpha\psi(\alpha))$ near $(0, 0)$ and $\psi(0) = 0$, we know the principal part, αx_0, of the second component for small α. Similar information (and more) is obtained for φ in the next theorem.

THEOREM 1.18. *In addition to the assumptions of Theorem* 1.7, *suppose F has n continuous derivatives with respect to (t, x) and $n + 1$ continuous derivatives with respect to x. Then the functions (φ, ψ) have n continuous derivatives with respect to α. If*

$$\begin{aligned} &F_x^{(j)}(0,0)(x_0)^j = 0 \quad \textit{for} \quad 1 \leqslant j \leqslant n \quad \textit{then} \quad \varphi^{(j)}(0) = 0 \\ &\textit{and} \quad \psi^{(j)}(0) = 0 \quad \textit{for} \quad 1 \leqslant j \leqslant n-1 \quad \textit{and} \end{aligned} \tag{1.19}$$

$$(1/(n+1))F_x^{(n+1)}(0,0)(x_0)^{n+1} + F_x(0,0)\,\psi^{(n)}(0) + \varphi^{(n)}(0)F_{tx}(0,0)\,x_0 = 0. \tag{1.20}$$

The notation $F_x^{(j)}(0, 0)(x_0)^j$ means the value of the j-th Fréchet derivative of the map $x \to F(0, x)$ at $(0, 0)$ evaluated at the j-tuple each of whose entries is x_0.

We begin the proofs.

Proof of Theorem 1.16. The Fredholm index of $F_x(0, 0)$ is zero by assumption, so the Fredholm index of $F_x(t, 0)$ will also be zero if $|t|$ is small. It therefore suffices to prove that $F_x(t, 0)$ is one-to-one. We have

$$\|F_x(t,0) - (F_x(0,0) + tF_{tx}(0,0))\| = o(t) \qquad \text{as} \qquad t \to 0.$$

So for $z \in Z$,

$$\begin{aligned} &\|F_x(t,0)(\alpha x_0 + z)\| \\ &\quad \geqslant \|F_x(0,0)(\alpha x_0 + z) + t\alpha F_{tx}(0,0)\,x_0 + tF_{tx}(0,0)z\| \\ &\qquad - o(t)(|\alpha| + \|z\|) \\ &\quad \geqslant \|F_x(0,0)z + t\alpha F_{tx}(0,0)\,x_0\| - |t|\,\|F_{tx}(0,0)\|\,\|z\| - o(t)(|\alpha| + \|z\|) \\ &\quad \geqslant k(\|z\| + |t\alpha|) - (\|F_{tx}(0,0)\|\,|t|\,\|z\| + o(t)(|\alpha| + \|z\|)). \end{aligned}$$

Thus estimating $\| F_{tx}(0, 0)\| \, | t | + o(t) < k/2$ and $o(t) < k \, | t |/2$ for $| t | < \delta$, when δ is sufficiently small, yields

$$\| F_x(t, 0)(\alpha x_0 + z)\| \geq (k/2)(\| z \| + | t | \, | \alpha |) \qquad \text{if} \quad | t | < \delta.$$

Then F_x is one-to-one if $0 < | t | < \delta$, and the proof is complete.

Proof of Theorem 1.17. Let w be the curve in $(-a, a) \times V$ given by $w(t) = (\varphi(t), tx_0 + t\psi(t))$. The map $F : (-a, a) \times V \to Y$ and curve w satisfy the assumptions on G and w in Theorem 1. By the remark following Theorem 1.16, $F'(w(t))$ annihilates only $w'(t) = (\varphi'(t), x_0 + t\psi'(t) + \psi(t))$ and is onto Y if $0 < | t | < \delta$. Since

$$F'(w(t))(t^*, x^*) = F_t(w(t))\, t^* + F_x(w(t))\, x^*,$$

and $(0, x^*)$ is not a multiple of $w'(t)$ unless $x^* = 0$ or $\varphi'(t) = 0$, it follows that $F_x(w(t))$ is an isomorphism of X onto Y if $\varphi'(t) \neq 0$ and $0 < | t | < \delta$.

Proof of Theorem 1.18. Under the regularity assumptions on F in this theorem, the function f of (1.9) has n continuous derivatives with respect to (α, t, z); so the regularity assertions concerning (φ, ψ) are immediate consequences of the implicit function theorem. Moreover, if f is given by (1.9), then

$$f(\alpha, 0, 0) = \alpha^{-1} F(0, \alpha x_0)$$

and the assumption (1.19) implies that $f(\alpha, 0, 0)$ has a zero of order $(n - 1)$ at $\alpha = 0$. The first nonvanishing derivative at $\alpha = 0$ of the implicit function $(\varphi(\alpha), \psi(\alpha))$ determined by $f(\alpha, \varphi(\alpha), \psi(\alpha)) = 0$ therefore has order at least n and is found by solving

$$\frac{\partial^n f}{\partial \alpha^n}(0, 0, 0) + f_t(0, 0, 0)\, \varphi^{(n)}(0) + f_z(0, 0, 0)\, \psi^{(n)}(0) = 0,$$

which is precisely (1.20).

2. Applications and Examples

Several applications to and examples from the theory of nonlinear eigenvalue problems will be given to illustrate the results developed in the previous section.

Let X_1 and Y be Banach spaces and L_1, L_2 two closed linear maps defined on $D(L_1) \subset D(L_2) \subset X_1$ taking values in Y. Let N_1, $N_2 : D(L_1) \to Y$ and consider the equation

$$H(\lambda, u) = L_1 u + N_1(u) - \lambda(L_2 u + N_2(u)) = 0 \tag{2.1}$$

for $(\lambda, u) \in \mathbf{R} \times D(L_1)$. It is often assumed that $X_1 = Y$ and that for some μ $L_1 - \mu L_2$ has a bounded (frequently even compact) inverse so that (2.1) can be converted to

$$T^{-1}F(\lambda, u) = u + T^{-1}N_1(u) - (\lambda - \mu)\, T^{-1}L_2 u + \lambda T^{-1}N_2(u) = 0, \tag{2.2}$$

where $T = L_1 - \mu L_2$. In addition, it may be possible to extend $T^{-1}N_1$, $T^{-1}N_2$ and $T^{-1}L_2$ to all of X by continuity in such a way that (2.1) and (2.2) are equivalent after this extension.

Rather than pursue this approach, we take a different point of view. Let $X = D(L_1)$ under the graph topology, i.e.,

$$\| u \|_X = \| u \|_{X_1} + \| L_1 u \|_Y, \qquad u \in D(L_1). \tag{2.3}$$

We then have the following result:

THEOREM 2.4. *Let $N_i : X \to Y$ be once continuously differentiable and $N_i(0) = 0$, $N_i'(0) = 0$, $i = 1, 2$. Regarding $L_1 - \lambda_0 L_2$ as a map from X to Y, suppose $N(L_1 - \lambda_0 L_2)$ and $Y/R(L_1 - \lambda_0 L_2)$ are one dimensional,*

$$N(L_1 - \lambda_0 L_2) = \operatorname{span}\{u_0\} \quad \text{and} \quad (L_1 - \lambda_0 L_2)X = \{y \in Y : y^*(y) = 0\}$$

for some fixed $y^* \in Y^*$. If $y^*(L_2 u_0) \neq 0$, then $(\lambda_0, 0)$ is a bifurcation point for (2.1) with respect to the curve $\{(\lambda, 0) : \lambda \in \mathbf{R}, 0 \in X\}$ and there is a unique curve of solutions $(\lambda(\alpha), u(\alpha))$ passing through $(\lambda_0, 0)$ at $\alpha = 0$, as in Theorem 1.7.

Proof. Set

$$F(t, x) = L_1 x + N_1(x) - (\lambda_0 + t)(L_2 x + N_2(x)). \tag{2.5}$$

Since a closed linear operator is continuous in the graph topology and $D(L_1) \subset D(L_2)$, L_1 and L_2 are continuous from X to Y. The definition (2.5) and the assumptions of Theorem (2.4) yield

$$\begin{aligned} F_t(t, x) &= -(L_2 x + N_2(x)) \\ F_x(t, x) &= L_1 + N_1'(x) - (\lambda_0 + t)(L_2 + N_2'(x)) \\ F_{tx}(t, x) &= -(L_2 + N_2'(x)), \end{aligned}$$

verifying the differentiability conditions of Theorem 1.7. Moreover, $F_x(0, 0) = L_1 - \lambda_0 L_2$ satisfies, by assumption, the criteria of Theorem 1.7. It remains to check that

$$F_{tx}(0, 0)\, u_0 = -L_2 u_0 \notin R(F_x(0, 0)) = R(L_1 - \lambda_0 L_2),$$

which has been assumed in the form $y^*(L_2 u_0) \neq 0$, and the proof is complete.

Remark 2.6. If $X \subset Y$ and $L_2 = I$, the identity on Y, $u_0 \notin R(L_1 - \lambda_0 L_2)$ means simply that $N((L_1 - \lambda_0 I)^2) = N(L_1 - \lambda_0 I)$, i.e., that λ_0 is a simple eigenvalue of L_1. Note also that the functions N_i of Theorem 2.4 can depend on λ if $N_i(\lambda, 0) = 0$, $N_{iu}(\lambda, 0) = 0$, $i = 1, 2$, and the proof is the same.

Theorem 2.4 was a trivial corollary to Theorem 1.7, and was stated because equations of this simple form frequently occur. We give some examples. Let $X_1 = Y = C([0, \pi])$ under the maximum norm and $L_1 u = -(pu')' + qu$ where p is continuously differentiable, positive and q is continuous on $[0, \pi]$. As $D(L_1)$ we take $\{u \in C^2([0, \pi]) : u(0) = u(\pi) = 0\}$. (These boundary conditions can be replaced by any separated ones.) Let $L_2 = I$ and consider

$$H(\lambda, u) = L_1 u + g(x, u, u', u'') - \lambda(u + f(x, u, u', u'')). \tag{2.7}$$

As is well-known, the map $u \to (L_1 - \lambda_0 I)u$ is an isomorphism of $D(L_1)$ onto $C([0, \pi])$ or λ_0 is a simple eigenvalue of L_1 (in the sense of Remark 2.6) and $L_1 - \lambda_0 I$ satisfies the conditions of Theorem 2.4. Suppose the second alternative prevails and that $f(x, \xi, \eta, \rho)$, $g(x, \xi, \eta, \rho)$ are continuously differentiable in (ξ, η, ρ) and f, g and their first derivatives with respect to (ξ, η, ρ) vanish at $(x, 0, 0, 0)$. Then $N_1(u) = g(x, u, u', u'')$ and $N_2(u) = f(x, u, u', u'')$ satisfy the conditions of Theorem 2.4, and that theorem is applicable here.

An interesting special case of (2.7) is $L_1 u = -u''$, $g(x, u, u', u'') = h(u^2 + u'^2)u$, $f(x, u, u', u'') = k(u^2 + u'^2)u$, $h(0) = k(0) = 0$, and $\lambda_0 = 1$, so that $N(L_1 - \lambda_0 I) = \operatorname{span}\{\sin x\}$. To find the unique curve of solutions of

$$\begin{gathered} -u'' + h(u^2 + u'^2)u - \lambda(u + k(u^2 + u'^2)u) = 0, \\ u(0) = u(\pi) = 0 \end{gathered} \tag{2.8}$$

bifurcating from $(\lambda_0, 0)$ we try $u = c \sin x$, c a constant, in (2.8). This gives

$$1 + h(c^2) - \lambda(1 + k(c^2)) = 0,$$

or

$$\lambda(c) = \frac{1 + h(c^2)}{1 + k(c^2)}. \tag{2.9}$$

Due to the freedom in choosing h, k a wide variety of behaviour is possible for λ. Taking $k = 0$ and

$$h(c^2) = \begin{cases} \exp(-1/c^2)\sin(1/c^2) & c \neq 0 \\ 0 & c = 0 \end{cases}$$

yields

$$\lambda(c) = 1 + \exp(-1/c^2)\sin(1/c^2).$$

Theorem 1.17 is quite interesting as applied to this example. Moreover, the inadequacy of the "classification diagrams" given in Refs. [2, p. 153; 7, p. 211], where the concern is with parameterizing in terms of λ, is also illustrated.

As our next example, we consider a system of partial differential equations that arises in describing convective phenomena in fluid dynamics. The equations are

$$\begin{aligned} &\text{(i)} \quad \Delta\vec{U} - \nabla p + R\theta\vec{e} = (\vec{U}\cdot\nabla)\vec{U}, \quad -\infty < x_1, \quad x_2 < \infty, \quad |x_3| < 1, \\ &\text{(ii)} \quad \Delta\theta + u_3 = P(\vec{U}\cdot\nabla)\theta \qquad (2.10) \\ &\text{(iii)} \quad \nabla\cdot\vec{U} = 0, \end{aligned}$$

where $\Delta = \sum_{i=1}^{3} \partial^2/\partial x_i^2$, $\nabla = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$, $\vec{U} = (u_1, u_2, u_3)$, $\vec{e} = (0, 0, 1)$, p and θ are real-valued and P and R are real parameters, called the Prandtl and Rayleigh numbers, respectively. Note that (2.10)(i) is a vector equation with three components. The unknowns are $\vec{U}$, p and θ. As boundary conditions, we take

$$\vec{U} = 0, \quad \theta = 0 \qquad \text{for} \quad x_3 = \pm 1, \quad -\infty < x_1, \quad x_2 < \infty. \tag{2.11}$$

In addition, we impose periodicity conditions

$$\varphi\left(x_1 + \frac{2\pi}{a}, x_2, x_3\right) = \varphi\left(x_1, x_2 + \frac{2\pi}{b}, x_3\right) = \varphi(x_1, x_2, x_3) \tag{2.12}$$

for $\varphi = \vec{U}$, φ or p, together with the symmetry conditions

$$\begin{aligned} &(1) \quad u_1(x_1, x_2, x_3) = -u_1(-x_1, x_2, x_3) = u_1(x_1, -x_2, x_3) \qquad (2.13) \\ &(2) \quad u_2(x_1, x_2, x_3) = u_2(-x_1, x_2, x_3) = -u_2(x_1, -x_2, x_3) \\ &(3) \quad \varphi(x_1, x_2, x_3) = \varphi(-x_1, x_2, x_3) = \varphi(x_1, -x_2, x_3), \\ &\qquad \text{for } \varphi = u_3, \theta \text{ or } p. \end{aligned}$$

The parameters a, b in (2.12) are both nonnegative and at least one is positive. (For example, "$b = 0$" means φ is independent of x_2.) For a description of the physical situation described by these equations, see [8].

Putting this problem in the framework of Theorem 2.4 requires some care in the choice of spaces. Let D denote the set

$$D = \{(x_1, x_2, x_3) : |x_3| \leqslant 1, -\infty < x_2, x_1 < \infty\},$$

C the cell

$$C = \left\{(x_1, x_2, x_3) : |x_1| \leqslant \frac{2\pi}{a}, |x_2| \leqslant \frac{2\pi}{b}, |x_3| \leqslant 1\right\}$$

in D, and set

$$\|\varphi\|_k^2 = \sum_{|\sigma| \leqslant k} \int_C |D^\sigma \varphi|^2 \, dx + \int_C |\varphi|^2 \, dx \tag{2.14}$$

for any smooth function φ on C with values in $\mathbf{R}^n$. Here the usual multi-index notation is being employed in writing D^σ. Let $H_k{}^l$ denote the completion of smooth mappings of D into $\mathbf{R}^l$ which satisfy (2.12) under the norm $\| \quad \|_k$ of (2.14). $H_k{}^1$ is abbreviated to H_k. We will be taking $H_0{}^4$ as our range space Y, writing elements of $H_0{}^4$ as

$$H_0{}^4 = \{(f, \varphi) \mid f = (f_1, f_2, f_3) \in H_0{}^3, \varphi \in H_0\}.$$

The setting up of the domain space X requires more care. The conditions (2.11), (2.12) and (2.13) will be incorporated in the definition of X, as well as the condition $\nabla \cdot \vec{U} = 0$. The space $H_{k,i}$ is the closure in H_k of the set of smooth functions on D satisfying (2.12) and condition (2.13)(i) for $i \in \{1, 2, 3\}$. The space $\mathring{H}_1$ is the closure of smooth functions on D satisfying (2.12) and which also vanish near $|x_3| = 1$ under the norm $\| \quad \|_1$. We set

$$\begin{aligned} X = \{(\vec{U}, \theta, p) \mid \vec{U} &= (u_1, u_2, u_3), u_i \in H_{2,i} \cap \mathring{H}_1, \\ i &= 1, 2, 3, \nabla \cdot \vec{U} = 0, \theta \in H_{2,3} \cap \mathring{H}_1, p \in H_1 \cap H_{1,3}\} \end{aligned} \tag{2.15}$$

and use the norm

$$\|(\vec{U}, \theta, p)\|_X^2 = \|\vec{U}\|_2^2 + \|\theta\|_2^2 + \|p\|_1^2.$$

Observe that every component of an element of X has at least its first derivatives in $\mathscr{L}^2(C)$, so $\nabla \cdot \vec{U} = 0$ has a meaning. The operator L_1 is defined by setting

$$L_1 w = (\Delta \vec{U} - \nabla p, \Delta\theta + u_3)$$

where $w = (\vec{U}, \theta, p)$, $\vec{U} = (u_1, u_2, u_3)$, and L_2 is given by

$$L_2 w = (0, 0, -\theta, 0).$$

If X is given by (2.15) and $Y = H_0{}^4$, then L_1 and L_2 are continuous linear maps of X into Y. The problem (2.10), (2.11), (2.12), (2.13) can now be stated as

$$L_1 w - RL_2 w - N_1(w) = 0, \qquad w \in X,$$

where $N_1(\vec{U}, \theta, p) = ((\vec{U} \cdot \nabla)\vec{U}, P(\vec{U} \cdot \nabla)\theta)$. Now $N_1(w)$ is of the form $N_1 w = B(w, w)$ where B is a symmetric bilinear mapping on $X \times X$. Therefore, to verify the continuous differentiability of N_1 and $N_1'(0) = 0$ we need only observe that B is continuous, and for this it is enough to check that N_1 takes values in Y and is bounded on bounded sets. The last assertions follow at once from the Sobolev inequalities. Let us assume that at some value λ_0 of λ, $N(L_1 - \lambda_0 L_2)$ is nontrivial. It is shown in [8] that then there are integers j, m such that replacing (a, b) in (2.12) by (ja, mb) (and hence restricting attention to certain subspaces of X, Y which we assume, for convenience, are X, Y themselves) one has that

$$N(L_1 - \lambda_0 L_2) = \operatorname{span}\{(\vec{U}_0, \theta_0, p_0)\}$$

is one-dimensional and

$$R(L_1 - \lambda_0 L_2) = \left\{(f, \varphi) \in Y : \int_C (\vec{U}_0 \cdot f + \lambda_0 \theta_0 \varphi)\, dx = y^*(f, \varphi) = 0\right\}.$$

Using $\Delta\theta_0 + u_{03} = 0$, we find

$$y^*(L_2(\vec{U}_0, \theta_0, p_0)) = \int_C u_{03}\theta_0\, dx = \int_C -\theta_0\, \Delta\theta_0\, dx = \int_C |\nabla\theta_0|^2\, dx \neq 0,$$

and the conditions of Theorem 2.4 have all been verified.

Our next application will be to a problem that does not quite fall into the framework of Theorem 2.4, but to which Theorem 1.7 is applicable. This problem was the subject of a recent paper by H. Keller [6] and deals with a situation in which the eigenvalue parameter appears in a nonlinear fashion.

Let $\Omega \subseteq \mathbf{R}^n$ be a bounded domain with smooth boundary and

$$Lu = \sum_{|\sigma|, |\rho| \leqslant m} (-1)^{|\sigma|} D^\sigma (a_{\sigma\rho}(x)\, D^\rho u)$$

for $u \in C^{2m}(\Omega)$, where the usual multi-index notation is being employed. We assume that the coefficients of L are smooth and

$$\sum_{|\sigma|,|\rho|=m} a_{\sigma\rho}(x)\, \xi^{\sigma+\rho} \geqslant \mu \mid \xi \mid^{2m}$$

for all $x \in \bar{\Omega}$ and $\xi \in \mathbf{R}^n$, where $\mu > 0$ is a constant, i.e., L is uniformly elliptic in Ω. We further assume L is formally self-adjoint and is self-adjoint with respect to the elliptic boundary conditions $B_j u = 0$, $1 \leqslant j \leqslant m - 1$, (see [1]).

Consider the nonlinear eigenvalue problem

$$\begin{aligned} F(\lambda, u) &= Lu + \lambda g(\lambda, x, u) + h(\lambda, x, u, Du, ...) = 0 \\ B_j u &= 0 \quad \text{on} \quad \partial\Omega, \qquad 1 \leqslant j \leqslant m - 1, \end{aligned} \tag{2.16}$$

where h depends on u and its derivatives up to order $2m$. The problem (2.16) is treated in [6] for the case $h \equiv 0$ and under several technical assumptions, some of which we will not need here. Sufficient conditions on g, h for our purposes are described below. For $g(\lambda, x, u)$, we require

$$\begin{aligned} &\text{(i)} \quad g \text{ is three times continuously differentiable} \\ &\text{(ii)} \quad g(\lambda, x, 0) = 0 \\ &\text{(iii)} \quad g_u(\lambda, x, 0) > 0. \end{aligned} \tag{2.17}$$

Writing $h(\lambda, x, u, Du, ...)$ as $h(\lambda, x, \eta)$ evaluated at $\eta = (u, Du, ...)$, where $\eta \in R^s$ and s is the number of distinct multi-indices β such that $0 \leqslant |\beta| \leqslant 2m$, we require that

$$\begin{aligned} &\text{(i)} \quad h \text{ and } h_{D^\beta u} \text{ are three times continuously differentiable if } |\beta| = 2m, \\ &\text{(ii)} \quad h(\lambda, x, 0) = 0, \\ &\text{(iii)} \quad h_\eta(\lambda, x, 0) = 0. \end{aligned} \tag{2.18}$$

The conditions (2.17) on g are more than sufficient for our purposes and are stronger than those required in [6]. However, it should be noted that the arguments of [6] are not complete. In the paragraph following equation (3.76) of [6], a compactness theorem of [1] is invoked and not all the hypotheses of this theorem are verified in [6]. The small gap may easily be closed upon assuming more regularity for g.

For $\alpha \in (0, 1)$ let $C^{k+\alpha}(\bar{\Omega})$ denote the set of k-times continuously differentiable functions on $\bar{\Omega}$ whose k-th order derivatives are Hölder continuous in $\bar{\Omega}$ with exponent α. $C^{k+\alpha}(\bar{\Omega})$ is a Banach space under the usual norm

$$\| u \|_{k+} = \sup_{x\in\Omega} \sum_{|\sigma|\leqslant k} | D^\sigma u(x)| + \sup \left\{ \frac{| D^\sigma u(x) - D^\sigma u(y)|}{| x - y |^\alpha} : x, y \in \Omega, x \neq y, | \sigma | \leqslant k \right\}.$$

Let $X = \{u \in C^{2m+\alpha}(\bar{\Omega}) : B_j u = 0, 1 \leqslant j \leqslant m-1\}$ and $Y = C^\alpha(\bar{\Omega})$. The assumptions (2.17) and (2.18) are more than enough to guarantee that the map $(\lambda, u) \to F(\lambda, u)$ defined by (2.16) satisfies the smoothness requirements of Theorem 1.7 for X, Y as above. The verification of this is left to the interested reader. Moreover,

$$F_u(\lambda, 0)\, u^* = (L + \lambda g_u(\lambda, x, 0))\, u^* \tag{2.19}$$

for $u^* \in X$. Keller [6] essentially assumes

$$N(F_u(\lambda_0, 0)) = \operatorname{span}\{\varphi_0\}$$

is one-dimensional and

$$R(F_u(\lambda_0, 0)) = \left\{ v \in Y : \int_\Omega v(x)\, \varphi_0(x)\, dx = 0 \right\}$$

for some λ_0. We have

$$F_{\lambda u}(\lambda_0, 0)\, \varphi_0 = g_u(\lambda_0, x, 0)\, \varphi_0 + \lambda_0 g_{\lambda u}(\lambda_0, x, 0)\, \varphi_0,$$

which is not in the range of $F_u(\lambda_0, 0)$ if

$$\int_\Omega (g_u(\lambda_0, x, 0)\, \varphi_0^2(x) + \lambda_0 g_{\lambda u}(\lambda_0, x, 0)\, \varphi_0^2(x))\, dx \neq 0, \tag{2.20}$$

and this condition is implied by the more stringent requirement (2.12) of [6]. A simple sufficient condition for (2.20) to hold is that

$$\lambda_0 g_{\lambda u}(\lambda_0, x, 0) + g_u(\lambda_0, x, 0) > 0 \qquad \text{for} \quad x \in \Omega.$$

Under these assumptions, Theorem 1.7 applies directly to yield a unique curve of solutions crossing the curve $(\lambda, 0)$ at $(\lambda_0, 0)$, generalizing the main result of [6]. If we take $\{u \in X : \int_D u\varphi_0\, dx = 0\}$ as the Z of Theorem 1.7, our solutions are parameterized in the same fashion as in [6], and formula (1.10b) of [6] is just Theorem 1.18 in this special case.

Our last examples will use Theorem 1 directly, rather than Theorem 1.7. Consider a closed linear operator L with domain $D(L)$ in a Banach space Y and values in Y. We take $X = D(L)$ with the graph topology, and assume $N(L) = \text{span}\{u_0\}$ and $Y/R(L)$ are one-dimensional. Let

$$G(\lambda, u) = Lu - \lambda(u + f(u)), \tag{2.21}$$

where $f : D(L) \to Y$ and $G : \mathbf{R} \times D(L) \to Y$. It is assumed that f is continuously differentiable, $f(0) = 0$ and $f'(0) = 0$. The curve $u = 0$ is a curve of zeros of G, and we ask if $(0, 0)$ is a bifurcation point with respect to this curve. Theorem 1.7 is trivially applicable here if $u_0 \notin \mathbf{R}(L)$, but the existence assertion of Theorem 1.7 is also trivial (even if $u_0 \in R(L)$) in this case, for $\{0, \alpha u_0) : \alpha \in \mathbf{R}\}$ is a curve of zeros of G. (This case is usually excluded in bifurcation studies, since the bifurcating curve obviously cannot be parameterized by λ.) We ask about "secondary bifurcation" with respect to the curve $\alpha \to (0, \alpha u_0)$. Note that

$$G'(0, \alpha u_0)(\lambda^*, u^*) = Lu^* - \lambda^*(\alpha u_0 + f(\alpha u_0)); \tag{2.22}$$

so $(0, u_0)$ is in the null space of $G'(0, \alpha u_0)$. In order for a secondary bifurcation to occur, we need a second null vector (λ_1, u_1) for $G'(0, \alpha u_0)$. A necessary and sufficient condition for (λ_1, u_1) to exist is

$$\alpha_0 u_0 + f(\alpha_0 u_0) \in R(L) \tag{2.23}$$

for some $\alpha_0 \in \mathbf{R}$. If (2.23) holds and $Lv = \alpha_0 u_0 + f(\alpha_0 u_0)$, then $G'(0, \alpha_0 u_0)(1, v) = 0$.

The condition

$$G''(0, \alpha_0 u_0)((0, u_0), (1, v)) \notin R(G'(0, \alpha_0 u_0))$$

is

$$u_0 + f'(\alpha_0 u_0)\, u_0 \notin R(G'(0, \alpha_0 u_0)),$$

or

$$u_0 + f'(\alpha_0 u_0)\, u_0 \notin R(L). \tag{2.24}$$

As a simple example, consider $Lu = -u'' - u$, $Y = C([0, \pi])$, and

$$D(L) = C^2([0, \pi]) \cap \{u : u(0) = u(\pi) = 0\}.$$

Let $f(u) = -u^3$. Then $u_0 = \sin x$ and (2.23) becomes

$$\alpha_0 \sin x - \alpha_0{}^3(\sin x)^3 \in R(L),$$

or

$$\alpha_0 \int_0^{\pi} (\sin x)^2 \, dx - \alpha_0^3 \int_0^{\pi} (\sin x)^4 \, dx = 0;$$

so

$$\alpha_0 = + \left(\frac{\int_0^{\pi} (\sin x)^2 \, dx}{\int_0^{\pi} (\sin x)^4 \, dx} \right)^{1/2}.$$

Moreover, (2.24) simply requires that

$$\int_0^{\pi} (\sin x)^2 \, dx - 3\alpha_0^2 \int_0^{\pi} (\sin x)^4 \, dx \neq 0,$$

which is certainly the case here.

We conclude with an elementary but computationally instructive example of the use of Theorem 1. Here we take $W = \mathbf{R}^3$, $Y = \mathbf{R}^2$ and set

$$G(\lambda, x, y) = \begin{pmatrix} x + \lambda(x^3 - x + f(x, y) y) \\ 10y - \lambda(y + g(x, y) y) \end{pmatrix} = \begin{pmatrix} G_1(\lambda, x, y) \\ G_2(\lambda, x, y) \end{pmatrix} \tag{2.25}$$

for $(\lambda, x, y) \in \mathbf{R}^3$. Here f, g are arbitrary C^2 functions such that

$$g(x, 0) = 2x^2, \qquad f(0, 0) = 0. \tag{2.26}$$

Clearly, $G(\lambda, 0, 0) = 0$ and for bifurcation with respect to the curve $x = y = 0$ to occur at $(\lambda_0, 0, 0)$ we need $G'(\lambda_0, 0, 0)$ to have two null vectors. Since

$$G'(\lambda_0, 0, 0)(\lambda^*, x^*, y^*) = \begin{pmatrix} (1 - \lambda_0)\, x^* \\ (10 - \lambda_0)\, y^* \end{pmatrix}$$

$N(G'(\lambda_0, 0, 0))$ is two dimensional if $\lambda_0 = 1$ or 10. Choosing $\lambda_0 = 1$, the curve of bifurcating solutions through $(1, 0, 0)$ can be described in the form

$$x_{\pm}(\lambda) = \pm((\lambda - 1)/\lambda)^{1/2}, \qquad y(\lambda) = 0, \qquad \lambda \geqslant 1. \tag{2.27}$$

(Described in terms of the projection on $(0, 1, 0)$, the parameterization will be smooth.) We look for secondary bifurcations along the arc

$$x = x_+(\lambda) = ((\lambda - 1)/\lambda)^{1/2}, \qquad y(\lambda) = 0, \qquad \lambda > 1.$$

Now

$$G'(\lambda, x_+(\lambda), 0)(\lambda^*, x^*, y^*)$$
$$= \begin{pmatrix} -(\lambda - 1)^{1/2} \lambda^{-3/2} \lambda^* + 2(\lambda - 1)\, x^* + \lambda f(x_+(\lambda), 0)\, y^* \\ (10 - \lambda(1 + g(x_+(\lambda), 0)))\, y^* \end{pmatrix}. \tag{2.28}$$

The vector $(2\lambda^2((\lambda - 1)/\lambda)^{1/2}, 1, 0)$ is annihilated by $G'(\lambda, x_+(\lambda), 0)$ for each $\lambda > 1$. In view of (2.28), (2.27) and (2.26), β (below) is also a null vector of $G'(\lambda, x_+(\lambda), 0)$ for the value $\lambda = 4$. Letting

$$\begin{aligned} \alpha &= (16\sqrt{3}, 1, 0), \\ \beta &= (48 + 32f(\sqrt{3}/2, 0), \sqrt{3}, \sqrt{3}), \\ \gamma &= (4, \sqrt{3}/2, 0) \end{aligned} \tag{2.29}$$

the conditions of Theorem 1 are satisfied if

$$G''(\gamma)(\alpha, \beta) \notin R(G'(\gamma)).$$

However, $R(G'(\gamma))$ consists of the vectors in $\mathbf{R}^2$ whose second components vanish, and $G''(\gamma)(\alpha, \beta)$ has -168 as its second component. It is instructive to verify the above calculations.

Appendix A

The Implicit Function Theorem

We state below the version of the implicit function theorem used in this work.

THEOREM A. *Let E, F, G be three Banach spaces, f a continuous mapping of an open subset A of $E \times F$ into G. Let the map $y \to f(x, y)$ of*

$$A_x = \{y \in F : (x, y) \in A\}$$

into G be differentiable in A_x for each $x \in E$ such that $A_x \neq \varnothing$, and assume the derivative of this map (denoted by f_y) is continuous on A. Let $(x_0, y_0) \in A$ be such that $f(x_0, y_0) = 0$, and $f_y(x_0, y_0)$ is a linear homeomorphism of F onto G. Then there are neighborhoods U of x_0 in E and V of y_0 in F such that

(i) *$U \times V \subset A$.*

(ii) *There is exactly one function $u : U \to V$ satisfying $f(x, u(x)) = 0$ for $x \in U$.*

(iii) *The mapping u of* (ii) *is continuous.*

If, moreover, the mapping f is k-times continuously differentiable on A, then (iii) *above may be replaced by*

(iv) *u is k-times continuously differentiable.*

This theorem is certainly well-known, but we did not find a convenient reference for the precise statement we have used. However, the proof of Theorem (10.2.1) of [4] may be used without change. Also note the theorem (save the last paragraph) remains true if E is any topological space.

References

1. S. Agmon, A. Douglis, and L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I, *Comm. Pure Appl. Math.* **12** (1959), 623–727.
2. M. S. Berger, A bifurcation theory for nonlinear elliptic partial differential equations and related systems, *in* "Bifurcation Theory and Nonlinear Eigenvalue Problems," (J. B. Keller and S. Antman, Eds.), pp. 113–191, Benjamin, New York, 1969.
3. M. G. Crandall and P. H. Rabinowitz, Nonlinear Sturm–Liouville eigenvalue problems and topological degree, *J. Math. Mech.* **19** (1970), 1083–1103.
4. J. Dieudonné, "Foundations of Modern Analysis," Academic Press, New York, 1960.
5. P. C. Fife, The Bénard problem for general fluid dynamical equations and remarks on the Boussinesq approximation, *Indiana U. Math. J.* **20** (1970), 303–326.
6. H. B. Keller, Nonlinear Bifurcation, *J. Differential Equations* **7** (1970), 417–434.
7. M. A. Krasnoselskii, "Positive Solutions of Operator Equations," Noordhoff, Groningen, 1964.
8. P. H. Rabinowitz, Existence and nonuniqueness of rectangular solutions of the Bénard problem, *Arch. Rational Mech. Anal.* **29** (1968), 32–57.
9. P. H. Rabinowitz, Some global results for nonlinear eigenvalue problems, *J. Func. Anal.* **7** (1971), 487–513.
10. P. H. Rabinowitz, Nonlinear Sturm–Liouville problems for second-order ordinary differential equations, *Comm. Pure Appl. Math.*, to appear.
11. M. Recken, A general theorem on bifurcation and its application to the Hartree equation of the Helium atom, Math. Report No. 36, Battelle Institute, Advance Studies Center, Geneva, Feb. 1970.
12. D. H. Sattinger, Stability of bifurcating solutions by Leray–Schauder degree, *Archive Rat. Mech. Anal.*, to appear.

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