# Contents

*Preface* vii

1. Introduction 1

1.1 Existence and multiplicity of solutions . . . . . . . . . . . 1
1.2 Bifurcation . . . . . . . . . . . . . . . . . . . . . . . . . 3
1.3 Reaction-diffusion equations . . . . . . . . . . . . . . . . 6
1.4 Examples of reaction-diffusion equations . . . . . . . . . . 9

2. Preliminaries 15

2.1 Linearized operator . . . . . . . . . . . . . . . . . . . . 15
2.2 Variational methods . . . . . . . . . . . . . . . . . . . . 21
2.3 Maximum principles . . . . . . . . . . . . . . . . . . . . 23
2.4 Moving plane method . . . . . . . . . . . . . . . . . . . 28
2.5 Monotone methods . . . . . . . . . . . . . . . . . . . . . 33
2.6 Bounds of solution set . . . . . . . . . . . . . . . . . . . 36

3. Abstract Bifurcation Theory 39

3.1 Banach spaces and implicit function theorem . . . . . . . 39
3.2 Bifurcations on $\mathbf{R}^1$ . . . . . . . . . . . . . . . . . . . . . 43
3.3 Saddle-node bifurcation . . . . . . . . . . . . . . . . . . 46
3.4 Transcritical and pitchfork bifurcations . . . . . . . . . . . 49
3.5 Global bifurcation . . . . . . . . . . . . . . . . . . . . . 59
3.6 Bifurcation from infinity . . . . . . . . . . . . . . . . . . 65
3.7 Stability . . . . . . . . . . . . . . . . . . . . . . . . . . 71

4. Bifurcation Diagrams for General Domains 79

4.1 Nonlinearities . . . 80
4.2 Rellich-Pohozaev type identities . . . 83
4.3 Sublinear nonlinearities (logistic) . . . 88
4.4 Sup-sub nonlinearities (Allee effect) . . . 98
4.5 Superlinear nonlinearities . . . 110
4.6 Convex nonlinearities with $f(0) > 0$ . . . 115
4.7 Sub-sup nonlinearities . . . 119

5. Radially Symmetric Solutions 121

5.1 Parameterized solution set . . . 121
5.2 Shooting method . . . 123
5.3 Sturm comparison lemmas . . . 125
5.4 Linearized equation . . . 126
5.5 Basic bifurcations of radial solutions . . . 129
5.6 Exact multiplicity: an example and sketch of the proof . . 131
5.7 Determining the turning direction . . . 133
5.8 Horizontal asymptotes . . . 135

6. Disconjugacy 141

6.1 Review of the recipe . . . 141
6.2 One dimensional case . . . 142
6.3 Test functions for conjugacy . . . 142
6.4 Different approaches . . . 148
6.4.1 Result in Tang . . . 148
6.4.2 Result of Tang . . . 150
6.4.3 Result of Shi . . . 153
6.5 Open problems . . . 154

7. Exact multiplicity for ball domains 159

8. Radially symmetric entire solutions 161

8.1 Solutions in $\mathbf{R}^1$ . . . 161
8.2 Characterization of radially symmetric solutions in $\mathbf{R}^n$ . . 162
8.3 The decay rates . . . 165
8.4 Existence of radially symmetric solution in $\mathbf{R}^n$ . . . 166
8.5 Explicit solutions . . . 168
8.6 Uniqueness of radially symmetric solution in $\mathbf{R}^n$ . . . 170
8.7 Stability of the entire solutions . . . 172

8.8 Weak stability of radial entire solutions . . . . . . . . . . . 174
8.9 Liouville type theorems . . . . . . . . . . . . . . . . . . . . . 178
8.10 Monotonicity . . . . . . . . . . . . . . . . . . . . . . . . . . 180
8.11 Stability . . . . . . . . . . . . . . . . . . . . . . . . . . . . 180

9. Dynamical system approaches 183

9.1 Time mapping . . . . . . . . . . . . . . . . . . . . . . . . . 184

10. Stable solutions of singularly perturbed problem 187

*Bibliography* 189

# Chapter 1

# Introduction

## 1.1 Existence and multiplicity of solutions

This book intends to introduce the readers to the existence, multiplicity, and in particular, exact multiplicity of solutions to semilinear elliptic equation. Very often but not always, we will only consider positive solutions. For simplicity of the presentation, we will limit us to the equation:

$$\Delta u(x) + \lambda f(u(x)) = 0, \qquad \text{in} \quad \Omega, \tag{1.1}$$

when $\Delta$ is the Laplacian operator defined by $\Delta u = \sum_{i=1}^{n} \partial^2 u/\partial x_i^2$, $\Omega$ is an open subset in $\mathbf{R}^n$, $n \geq 1$, $\lambda > 0$ is a parameter, and $f$ is a smooth function if not otherwise defined. If $\Omega$ has a boundary $\partial\Omega$, then we assume that $\partial\Omega$ is smooth, and satisfies the conditions usually required for linear elliptic equation theory. On the boundary, we usually assume the Dirichlet boundary condition:

$$u(x) = 0, \quad x \in \partial\Omega, \tag{1.2}$$

but we will also consider the equation on the whole space $\mathbf{R}^n$.

An equation like (1.1) can be solved analytically only in a very few accidental examples if the domain is also a special one. Numerical solutions to (1.1) would be very useful, but a good understanding of the existence and multiplicity of solutions is needed to guide the numerical calculation. There is no complete answer to the question of existence for general nonlinear function $f(u)$, but for most nonlinearities arising from applications, there has been an answer for existence/nonexistence. The uniqueness of solution can be shown for some particular cases, but for most nonlinear problems, there are more than one solutions, and some results of multiplicity have also been found in the last a few decades. The ultimate goal is to determine the exact multiplicity of the solutions, which is rarely achieved. It is the goal of this book to give a systematic approach to these problems. In summary, this line of questions is to attempt to obtain a complete description of the solution set of the equation like (1.1), and when that is too hard to achieve, a partial description with as much as possible information on the solution set.

As a simplest possible example, let's consider the linear equation:

$$\begin{cases} \Delta\phi + \lambda\phi = 0 & \text{in} \quad \Omega, \\ \phi = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.3}$$

It is well-known that the operator $-\Delta$ has a sequence of eigenvalues:

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n \leq \cdots. \tag{1.4}$$

In particular, from Krein-Rutman Theorem, the principal eigenvalue $\lambda_1$ is simple, in the sense that the eigenspace is one dimensional, and the eigenfunction $\phi_1(x)$ is of one sign for $x \in \Omega$, thus it can be chosen as positive. Thus (1.3) is solvable if and only if $\lambda = \lambda_k$, and it has a positive solution only when $\lambda = \lambda_1$. On the other hand, we could consider the same equation on an unbounded domain, say $\mathbf{R}^n$:

$$\Delta\phi + \lambda\phi = 0, \text{ in } \quad \mathbf{R}^n. \tag{1.5}$$

When there is no any restriction on the solutions, one can easily find many solutions of (1.5), for example:

$$\phi(x) = \begin{cases} c_1\cos(\sqrt{\lambda}x_i) + c_2\sin(\sqrt{\lambda}x_i), & \lambda > 0; \\ c_1 + c_2 x_i, & \lambda = 0; \\ c_1\exp(\sqrt{-\lambda}x_i) + c_2\exp(-\sqrt{-\lambda}x_i) & \lambda < 0, \end{cases} \tag{1.6}$$

where $x = (x_1, \cdots, x_n)$, and $c_1, c_2$ are constants. However, more often we are only interested in the bounded solutions of equation. In that case, we cannot find bounded solutions when $\lambda < 0$ (use Liouville's Theorem, see [GT]), and for $\lambda \geq 0$, we still have solutions:

$$\phi(x) = \begin{cases} c_1\cos(\sqrt{\lambda}x_i) + c_2\sin(\sqrt{\lambda}x_i), & \lambda > 0; \\ c_1, & \lambda = 0. \end{cases} \tag{1.7}$$

In fact, the spectrum of $-\Delta$ on the whole space is $[0, \infty)$, which is an example of continuous spectrum. Finally we can only find positive solutions when $\lambda = 0$, and the solutions are $u(x) = c_1 > 0$. From this example, one can see that solution set depends on the parameter $\lambda$, and we will study the phenomenon of *bifurcation*, which describes the changes of the solution set when the parameter changes. Also when comparing the results for a bounded domain and an unbound one, we can find there are drastic differences between them, and this will be demonstrated in this notes. In Section 1.2, we use examples of ordinary differential equations to illustrate the basic types of bifurcation. Solutions of equation (1.1) are steady state solutions to a class of important evolution equations–reaction-diffusion equations. We review the derivation of reaction-diffusion equations in Section 1.3, and list several examples of reaction-diffusion equations in Section 1.4.

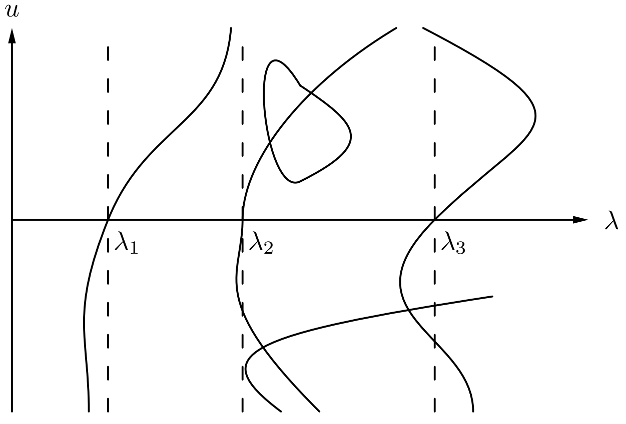


Fig. 1.1 Bifurcation diagrams of linear and nonlinear problems: dashed lines represent the solution set of linear problem, and the solid curves represent that of nonlinear one

## 1.2 Bifurcation

In a system of equations, if one or more parameters vary, then the qualitative behavior of the system may change. Such a change is called a *bifurcation*. In this book, we concentrate on the bifurcations of solutions to an equation of form

$$f(\lambda, u) = 0, \tag{1.8}$$

where $\lambda$ is a parameter. The number of solutions for the equation changes when the parameter changes. For example, consider an ordinary differential equation

$$\frac{dP}{dt} = aP\left(1 - \frac{P}{N}\right) - K. \tag{1.9}$$

From simple algebra, we conclude that when $K < aN/4$, (1.9) has two equilibrium solutions $P_K^{\pm}$, and when $K > aN/4$, (1.9) has no equilibrium solution. This phenomenon can be depicted in Fig. 1.2.

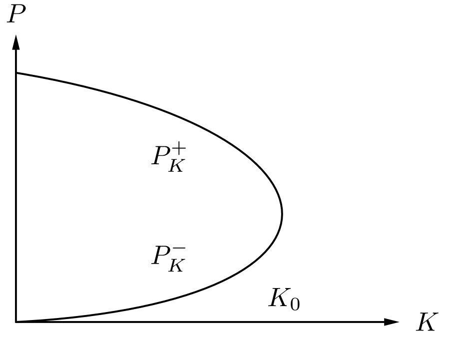


Fig. 1.2 Bifurcation diagram of logistic equation with harvesting

This elementary example shows the basic ingredients of dynamical systems and bifurcation theory. The parabola curve on the diagram is called the *bifurcation curve*, the parameter $K$ is a *bifurcation parameter*, and the diagram is called a *bifurcation diagram*. The point $K_0 = aN/4$ is a *bifurcation point* since the number of equilibrium solutions (or the solutions to an equation $aP(1 - P/N) - K = 0$) changes as $K$ across $K_0$. Moreover, the equilibrium $P_K^+$ is *stable* since all solutions with initial values near $P_K^+$ tend to $P_K^+$ as $t \to \infty$, and $P_K^-$ is *unstable*.

The main task of this book is to determine the bifurcation diagrams of some partial differential equations. We need to study the existence, multiplicity and stability of equilibrium solutions of some dynamical equations. The bifurcation theory for ordinary differential equations or finite dimensional dynamical systems has been well established. For detailed discussions, see Chow and Hale [CH], Strogatz [Str], Wiggins [Wi] or other books in dynamical systems. From the implicit function theorem, the necessary condition for bifurcation is

$$f(\lambda_0, u_0) = 0, \ f_u(\lambda_0, u_0) = 0, \tag{1.10}$$

and in the following examples in this section, we always assume (1.10) is satisfied. Here we list a few well-known generic bifurcations for future reference.

**Example 1.1.** (Saddle-node bifurcation) The bifurcation in the example above is called a *saddle-node bifurcation*. On the two sides of the bifurcation point, there are zero or two equilibrium solutions respectively. If there are two equilibrium solutions on the left neighborhood of the bifurcation point, the bifurcation is *subcritical*, otherwise it is *supercritical* (see Fig. 1.3.) The conditions of a saddle-node bifurcation for a scalar equation $u' = f(\lambda, u)$ $(\lambda, u \in \mathbf{R})$ are

$$f_\lambda(\lambda_0, u_0) \neq 0, \quad \text{and } f_{uu}(\lambda_0, u_0) \neq 0. \tag{1.11}$$

Fig. 1.3 (a) Subcritical saddle-node; (b) Supercritical saddle-node.

**Example 1.2.** (Transcritical bifurcation) A *transcritical* bifurcation occurs if there are two equilibrium solutions on both sides of bifurcation point, but there is only one at the bifurcation point. It occurs under the following conditions:

$$f_\lambda(\lambda_0, u_0) = 0, \ f_{\lambda u}(\lambda_0, u_0) \neq 0, \quad \text{and } f_{uu}(\lambda_0, u_0) \neq 0. \tag{1.12}$$

It is not totally trivial to show there are exactly two curves of equilibrium solutions crossing at the bifurcation point just under these conditions. It would be easier if we assume that $f(\lambda, u_0) = 0$ for all $\lambda$ near $\lambda_0$. We shall discuss this problem in Section 3.3.

Fig. 1.4 Transcritical bifurcations

**Example 1.3.** (Pitchfork bifurcation) Pitchfork bifurcation is similar to transcritical bifurcation, and it occurs if

$$\begin{aligned} &f_\lambda(\lambda_0, u_0) = f_{uu}(\lambda_0, u_0) = 0, \\ &f_{\lambda u}(\lambda_0, u_0) \neq 0, \quad \text{and } f_{uuu}(\lambda_0, u_0) \neq 0. \end{aligned} \tag{1.13}$$

In Chapter 3, we will formulate the infinite dimensional version of these basic bifurcation theorems, and applications to semilinear equations will be found in later chapters.

Fig. 1.5 (a) Supercritical pitchfork; (b) Subcritical pitchfork.

## 1.3 Reaction-diffusion equations

Diffusion mechanism models the movement of individuals of a certain species in an environment or media. The individuals can be very small such as bacteria, molecules, cells, seeds, or very large objects such as individual or groups of animals, or certain kind of events like epidemics and rumors. The particles reside in a region, which we call $\Omega$, and we assume that $\Omega$ is an open subset of $\mathbf{R}^n$ with $n \geq 1$. We use the density function of the particles $P(t,x)$, where $t$ is the time, and $x \in \Omega$.

How do the particles move? A Chinese proverb is "People goes to high place, water flows to low place". It is a natural phenomenon that a substance goes from high density region to low density region. The movement of $P(t,x)$ is called the flux of the population density, which is a vector. The "high to low" principle now means that, the flux always points to the most rapid decreasing direction of $P(t,x)$, which is the negative gradient of $P(t,x)$. This principle is called *Fick's law*, and it can be represented as

$$\mathbf{J}(t,x) = -d(x)\nabla_x P(t,x), \tag{1.14}$$

where $\mathbf{J}$ is the flux of $P$, $d(x)$ is the diffusion coefficient at $x$ and $\nabla_x$ is the gradient operator.

On the other hand, the density of particles at any point may change because of other reasons like birth, death, hunting, or chemical reactions. We assume that the rate of change of the density function due to these reasons is $f(t,x,P)$, which we usually call the reaction rate. Now we derive a differential equation using the balanced law. We choose any subregion $O$ of $\Omega$, then the total population in $O$ is $\int_O P(t,x)dx$, and the rate of change of the total population is

$$\frac{d}{dt}\int_O P(t,x)dx. \tag{1.15}$$

The net growth of the population inside the region $O$ is

$$\int_O f(t,x,P(t,x))dx, \tag{1.16}$$

and the total out flux is

$$\int_{\partial O} \mathbf{J}(t,x) \cdot \mathbf{n}(x) dS, \tag{1.17}$$

where $\partial O$ is the boundary of $O$, and $\mathbf{n}(x)$ is the outer normal direction at $x$. Then the balance law implies

$$\frac{d}{dt}\int_O P(t,x)dx = -\int_{\partial O} \mathbf{J}(t,x) \cdot \mathbf{n}(x) dS + \int_O f(t,x,P(t,x))dx \tag{1.18}$$

From the divergence theorem, we have

$$\int_{\partial O} \mathbf{J}(t,x) \cdot \mathbf{n}(x) dS = \int_O div(\mathbf{J}(t,x))dx. \tag{1.19}$$

Combining (1.14), (1.18) and (1.19), and interchanging the order of differentiation and integration, we obtain

$$\int_O \frac{\partial P(t,x)}{\partial t} dx = \int_O \left[div(d(x)\nabla_x P(t,x)) + f(t,x,P(t,x))\right] dx. \tag{1.20}$$

Since the choice of the subregion $O$ is arbitrary, then the differential equation

$$\frac{\partial P(t,x)}{\partial t} = div(d(x)\nabla_x P(t,x)) + f(t,x,P(t,x)) \tag{1.21}$$

holds for any $(t,x)$. The equation (1.21) is a *reaction diffusion equation.* For other ways of deriving (1.21) and related equations in mathematical biology, chemical reaction theory, see Murray [Mu], Okubo and Levin [OL], and Fife [F3].

The diffusion coefficient $d(x)$ is not a constant in general since the environment is usually heterogeneous. (Indeed the diffusion coefficient can even depend on the density itself, see [Mu; OL].) But when the region of the diffusion is approximately homogeneous, we can assume that $d(x) \equiv d$, then (1.21) can be simplified to

$$\frac{\partial P}{\partial t} = d\Delta P + f(t,x,P), \tag{1.22}$$

where $\Delta P = div(\nabla P) = \sum_{i=1}^{n} \frac{\partial^2 P}{\partial x_i^2}$ is the Laplacian operator. In classical mathematical physics, the equation $u_t = \Delta u$ is called heat equation. So sometimes (1.22) is also called a nonlinear heat equation. Conduction of heat can be considered as a form of diffusion of heat.

While the spatial domain $\Omega$ can be the whole space $\mathbf{R}^n$, more realistic domains are usually bounded. In most parts of this book, we assume that $\Omega$ is a bounded smooth domain in $\mathbf{R}^n$, *i.e.* $\Omega$ is a bounded open subset of

$\mathbf{R}^n$, and the boundary of $\Omega$ is a differentiable manifold of dimension $n-1$ with sufficient smoothness. For a differential equation like (1.22) to be well-posed, certain boundary conditions need to be added. Typical boundary conditions are

$$\text{(Dirichlet) } P(t,x) = \phi(x), \quad x \in \partial\Omega; \tag{1.23}$$

$$\text{(Neumann) } \nabla_x P(t,x) \cdot \mathbf{n}(x) = \phi(x), \quad x \in \partial\Omega; \tag{1.24}$$

$$\text{or (Robin) } \nabla_x P(t,x) \cdot \mathbf{n}(x) + aP(t,x) = \phi(x), \quad x \in \partial\Omega. \tag{1.25}$$

In this book, we will only consider the homogeneous Dirichlet case in (1.2).

A solution $u(x)$ of (1.1) is an equilibrium solution (time-independent solution) of a reaction-diffusion equation:

$$\frac{\partial u(x,t)}{\partial t} = \Delta_x u(x,t) + \lambda f(u(x,t)), \tag{1.26}$$

or a nonlinear wave equation:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \Delta_x u(x,t) + \lambda f(u(x,t)), \tag{1.27}$$

which have been interesting subjects for many disciplines of sciences and engineering. We will not discuss much about the dynamics of (1.26) or (1.27) in this notes, but the readers can find such discussions in Hale [Ha1], Henry [He] and Smoller [Sm]. For the transparency of the presentation, we limit our attention to only (1.1), but in applications, the equations may be more complicated. For examples, (1.1) is the simplification of the following equilibrium equations:

$$\Delta u + \overrightarrow{V}(x) \cdot \nabla u + \lambda f(u) = 0, \qquad \text{(convection)}; \tag{1.28}$$

$$\Delta u + \lambda f(x,u) = 0, \qquad \text{(non-homogeneous reaction)}; \tag{1.29}$$

$$\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \lambda f(u) = 0, \qquad \text{(non-homogeneous diffusion)}; \tag{1.30}$$

$$\Delta(D(u)) + \lambda f(u) = 0, \qquad \text{(density-dependent diffusion)}; \tag{1.31}$$

$$div(|\nabla u|^{p-2}\nabla u) + \lambda f(u) = 0, \quad p > 1, \qquad \text{(p-Laplacian)}, \tag{1.32}$$

$$\int_{\mathbf{R}^n} J(x-y)u(y)dy + \lambda f(u) = 0, \qquad \text{(non-local interaction)}, \tag{1.33}$$

$$\sum_{|j| \le k} a_j D^j u + \lambda f(u) = 0, \qquad \text{(higher order equation)}, \tag{1.34}$$

and the combinations of all above. Another generalization is on the number of variables (density functions), to the reaction-diffusion systems. Also boundary conditions may not be standard. In this book, we will only consider (1.1). Many results can be generalized to the equations list above, sometimes even trivial, but some most elegant ones may not be easily generalized, or even impossible. On the other hand, in these more complicated models, we usually try to discover the effect of these new structure, like drifting, spatial heterogeneity, nonlinear diffusion, and nonlocal interactions. We believe that the methods for (1.1) serve as the guide and the model for these more complicated yet realistic equations.

## 1.4 Examples of reaction-diffusion equations

To motivate the studies of in the following chapters, we briefly introduce a few well-known examples of nonlinear reaction-diffusion equations from ecology, chemical reactions, and neuron conduction, etc. In the following, we always assume that $d > 0$ is the diffusion coefficient.

(1) (Linear model) If the growth rate per capita is a constant, the diffusive population grows obeying the Malthusian equation (Malthus 1798 [Ma]):

$$P_t = d\Delta P + aP, \quad a > 0; \tag{1.35}$$

and Fourier's law of heat transfer and Newton's law of cooling give rise an equation of the heat loss:

$$h_t = d\Delta h - k(h - h_0), \quad k > 0, \tag{1.36}$$

where $h_0$ is the constant temperature of the environment, and the constant $k$ measures the rate of converging to $h_0$. These linear equations are solvable with appropriate boundary conditions by the classic theory, especially when the domain is special.

(2) (Logistic model) In 1838, Verhulst [V] first introduced logistic population growth model $P' = aP(1 - P/N)$, where $a > 0$ is growth rate per capita coefficient, and $N > 0$ is the carrying capacity. It was rediscovered by Pearl, Lotka in early 20th century, and experiments of Gause in 1930s confirmed logistic type growth of bacteria population. In 1937, Fisher [Fis] and Kolmogoroff, Petrovsky, and Piscounoff [KPP] derived a diffusive logistic equation for the concentration $P(x,t)$ of a dominant gene over a spatial territory:

$$P_t = d\Delta P + aP\,(1 - P)\,, \quad a > 0. \tag{1.37}$$

In 1953, Skellam [Sk] used (1.35) and (1.37) for the model of population dispersal, which is the beginning of the study of biological invasion of a foreign species into a new territory. Logistic equation imposes a limit for the growth, thus it is more realistic than the unbounded Malthusian growth.

(3) (Allee effect) In the logistic equation the growth rate per capita $g(P) = k(1 - P/N)$ is a strictly decreasing function with respect to the population density $P$, which considers the crowding effect. For some species, a small or sparse population may not be favorable, since for example, mating may be difficult. This is called Allee effect after American ecologist Allee [All]. Mathematically $g(P)$ will not have maximum value at $P = 0$. If the growth rate per capita $g(P)$ is negative when $P$ is small, we call such a growth pattern has a strong Allee effect. Typical example growth rate $f(P) = Pg(P)$ is

$$f(P) = kP\left(1 - \frac{P}{N}\right)\left(\frac{P}{M} - 1\right), \tag{1.38}$$

where $0 < M < N$, $M$ is the sparsity constant and $N$ is the carrying capacity. If the growth rate per capita $g(P)$ is smaller than the maximum but still positive for small $P$, we call such a growth pattern has a weak Allee effect. An example is the function in (1.38) with $M < 0 < N$ (see [SS3]). The corresponding reaction diffusion equation is

$$P_t = d\Delta P + aP\left(1 - \frac{P}{N}\right)\left(\frac{P}{M} - 1\right), \tag{1.39}$$

where $a > 0$, $N > -M > 0$. Ecological model with Allee effect is one example of equation with bistability. Allen-Cahn equation [AC] (or real Ginzburger-Landau equation) describing the dynamics of binary alloy shares similar bistable feature:

$$u_t = d\Delta u + au(1 - u)(1 + u), \tag{1.40}$$

and equation with cubic nonlinearity like (1.38) also appears in the FitzHugh-Nagumo model of neuron conduction (FitzHugh [Fiz], Nagumo et.al. [Nag]).

(4) (Harvesting) In the population model considered above, effect of harvesting can be added if the population is the victim to predation/harvesting. Ignoring the dependence of harvesting on the temporal and spatial changes, we consider only the dependence of harvesting on its own density:

$$P_t = d\Delta P + aP\left(1 - \frac{P}{N}\right) - h(P), \tag{1.41}$$

$h(P)$ can be a constant to have a constant yield harvesting when a harvesting quota is fixed, or it can be a constant effort harvesting, in which the rate of harvesting is proportional to the population density: $h(P) = kP$. Holling [Ho] suggested a few forms of $h(P)$, which is called the functional response to the predator. The typical harvesting rate $h(P)$ is, type (I): $h(P) = hP$, if $0 \leq P \leq c$, and $h(P) = hc$, when $P \geq c$; type (II): $h(P) = bP/(1+cP)$ or $h(P) = b - be^{-cP}$; or type (III): $h(P) = bP^2/(1+cP^2)$, here $b, c > 0$. The common feature is the saturating harvesting rate by the predator when prey is abundant. In (1.41), the choice of harvesting function $h(P)$ can be many of predator response functionals defined by ecologists, see for example Turchin [Tur] for as many as 20 choices.

(5) (Autocatalytic chemical reaction) Autocatalysis is the process that a chemical is involved in its own production, and it is a simple and important mechanism to provide feedback control in many biological systems (see [Mu]). An isothermal autocatalytic chemical reaction can be written as

$$A + pB \to (p+1)B. \tag{1.42}$$

Let $b(x,t)$ be the percentage of the autocatalyst $B$ in the reactor. Then the percentage of reactant $A$ is $a(x,t) = 1 - b(x,t)$, and the reaction rate is $kab^p = k(1-b)b^p$, where $p \geq 1$ is the order of the reaction with respect to the autocatalytic species. Then the reaction can be described by

$$b_t = d\Delta b + k(1-b)b^p, \quad k > 0. \tag{1.43}$$

Notice that for quadratic reaction ($p = 1$), we rediscover the logistic equation. The origin of autocatalysis can be traced back to Lotka [Lo]. More on autocatalytic chemical reaction can be found in Gray and Scott [GS].

(6) (Combustion model) Combustion is an extremely rapid exothermic chemical reaction. A full fluid mechanical and chemical kinetic system can be derived. If we assume that there is only one chemical species, and it is non-compressible with constant density, then a renormalized temperature function $\theta(x,t)$ satisfies

$$\theta_t = d\Delta\theta + \delta\exp(\theta), \tag{1.44}$$

where $\delta > 0$ is the Frank-Kamenetski parameter. (1.44) is the solid fuel ignition model. Its steady state solutions satisfy Gelfand equation:

$$d\Delta u + \lambda\exp(u) = 0, \quad x \in \Omega, \tag{1.45}$$

and a perturbed Gelfand equation is

$$d\Delta u + \lambda \exp(u/(1+\varepsilon u)) = 0, \quad x \in \Omega. \tag{1.46}$$

More can be found in Bebernes and Eberly [BE], and in 1963 Gelfand [Ge] first studied the solutions to (1.45).

(7) (Activator-inhibitor in pattern formation) In 1952, Turing [Tu] suggested that different diffusion rates of two chemical can destabilize a uniform equilibrium which is stable with respect to spatial uniform perturbation, and it implies the formation of spatial heterogeneous patterns. Such diffusion induced instability requires the two chemicals to be a pair of an activator which promotes or activates its own formation and an inhibitor which inhibits its own formation. One concrete example of Turing type equation is Gierer-Meinhardt system [GM]:

$$u_t = d_1\Delta u - u + u^p/v^q, \quad v_t = d_2\Delta v - v + u^r/v^s, \tag{1.47}$$

where all parameters are positive except $s$ being non-negative. For the parameters in Turing instability ranges, the system is approximated by its shadow system as $d_2 \to \infty$, and the steady state solutions of the shadow system can be further approximated by a scalar semilinear equation:

$$d\Delta u - u + u^p = 0, \tag{1.48}$$

for $p > 1$. Survey papers of Ni [Ni3; Ni4] are the best source and reference for this model. Equation (1.48) also arises from several problems in mathematical physics. One is a model of charged Bose gas by Dyson [Dy] (see also Lieb and Solovej [LiS]), and the other comes from studying the solitary waves of nonlinear Klein-Gordon equation or Schrödinger equation, see Strauss [Stra].

(8) (Emden-Fowler equation) Stars form from collapsing clouds of gas and dust. The equilibrium state of the stellar structure is governed by several physical laws. Let $r$ be the radius of a 3-dimensional star from the center, let $\rho$ and $P$ be the density and pressure at $r$, and let $M(r)$ be the total mass of radius $r$. Then the conservation law of mass and hydrostatic law gives

$$\frac{dM}{dr} = 4\pi r^2\rho, \quad 4\pi r^2\frac{dP}{dM} = -\frac{GM}{r^2}. \tag{1.49}$$

We also assume that for some positive constants $K, \gamma$,

$$P = K\rho^{(\gamma+1)/\gamma}. \tag{1.50}$$

Then with a change of variable $\rho = u^\gamma$ and some rescaling, we find that the equilibrium configuration of polytropic and isothermal gas spheres is given by the equation

$$\Delta u + \lambda u^\gamma = 0, \tag{1.51}$$

for some $\gamma, \lambda > 0$. (1.53) is known as the Emden-Fowler equation in astrophysics (see Emden [Em] and Fowler [Fo]). Chandrasekhar [Cha] contains a more comprehensive description of the model. Chandrasekhar theory of stellar collapse makes a variant of (1.53)

$$\Delta u + \lambda(2u + u^2)^{3/2} = 0, \tag{1.52}$$

see Lieb and Yau [LiY]. Another extension of (1.53) is the Hénon equation [Hen]:

$$\Delta u + \lambda |x|^l u^\gamma = 0. \tag{1.53}$$

It is also worth mentioning that the solutions of (1.53) are the steady state solutions of Fujita equation of parabolic type [Fu]:

$$u_t = d\Delta u + u^p, \quad p > 1, \tag{1.54}$$

which plays an important role in studying the blow-up phenomenon of nonlinear parabolic equations.

More general equation of temperature (or density) evolution of the plasma takes the form (see Wilhelmsson and Lazzaro [WL]):

$$\frac{\partial T}{\partial t} = d\nabla(T^\delta \nabla T) + kT^p, \tag{1.55}$$

for positive constants $\delta, k, p$. Some self-similar solutions of (1.55) satisfy

$$\nabla(T^\delta \nabla T) - k_1 T + k_2 T^p = 0, \tag{1.56}$$

and a change variable yields the semilinear equation:

$$\Delta u - k_3 u^p + k_4 u^q = 0, \tag{1.57}$$

where $k_i > 0$ and $q > p > 1$. Notice the similarity of (1.57) to (1.48).

(9) (Buckling problem) An elastic rod is straight in its natural state. It remains straight if a small compressive thrust is applied to its ends. But if the thrust is slowly increased beyond a certain critical value, then the rod will bend and it assumes a buckled state. This process is called buckling, and Euler was the first one to calculate the critical buckling load. A simplified model of buckling state can be formulated as follows: suppose that the length of the elastic column is normalized

so that $0 \leq x \leq 1$, $\theta(x)$ is the angle between the tangent to the column's axis and the $x$-axis, and $\lambda$ is a parameter proportional to the thrust. Then $\theta(x)$ satisfies

$$\theta'' + \lambda \sin\theta = 0, \quad 0 < x < 1, \quad \theta(0) = \theta(1) = 0. \tag{1.58}$$

Here the modulus of elasticity and the area moment of inertia are assumed to be constants; and both ends of the column are fixed (other boundary conditions are also possible). More models of elasticity can be found in Antman [Ant].

The sample equations given here certainly do not exhaust all possible examples. But from the diversity of the nonlinear equations, one can conclude that it is worthwhile to classify these nonlinearities into categories by their mathematical properties, which will streamline the studies to the nonlinear equations. In Chapter 5, we will make some definitions of function classes.

# Chapter 2

# Preliminaries

In this chapter, basic analytical methods for the equation

$$\begin{cases} \Delta u + \lambda f(u) = 0, & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{2.1}$$

are introduced. Many results are from standard textbooks of partial differential equations and functional analysis, thus they are given without proofs but references are given for the source. The properties of linearized operator associated with (2.1) are discussed in Section 2.1; in particular, we show that the linearized operator is a Fredholm operator with index zero when defined in appropriate Sobolev or Hölder spaces; variational methods for the existence of solutions to (2.1) are reviewed in Section 2.2; and in Section 2.3, the maximum principles for the linearized elliptic operator are recalled and several useful variants are also given for applications later; in Section 2.4 we apply the maximum principle to prove the symmetric properties of positive solutions of (2.1) by using the celebrated moving plane method; the monotone methods for the existence of solutions to (2.1) based on the maximum principle are given in Section 2.5; and we conclude in Section 2.6 with some necessary conditions on $f(u)$ for the existence of positive solutions of (2.1).

## 2.1 Linearized operator

Let $\Omega$ be a bounded domain in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$ so that the elliptic regularity theory is valid. We use the notation of Sobolev spaces $W^{k,p}(\Omega)$, $W_0^{k,p}(\Omega)$ and $L^p(\Omega)$ for $k \in \mathbf{N}$ and $1 \le p \le \infty$; and we use $||\cdot||_{k,p}$ and $||\cdot||_p$ as the norms in $W^{k,p}(\Omega)$ and $L^p(\Omega)$. We also use $\langle \cdot, \cdot \rangle_k$ and $\langle \cdot, \cdot \rangle$ as the inner products of $W^{k,2}(\Omega)$ and $L^2(\Omega)$ respectively. Pioncaré's inequality states that for any $u \in W_0^{1,p}(\Omega)$,

$$\int_\Omega |u(x)|^p dx \leq C \int_\Omega |\nabla u(x)|^p dx, \tag{2.2}$$

where $C$ is a positive constant only depending on $p$ and $\Omega$. We also use the notation of Hölder spaces $C^{k,\alpha}(\overline{\Omega})$, $C_0^{k,\alpha}(\overline{\Omega})$, $C^k(\overline{\Omega})$ and $C_0^k(\overline{\Omega})$ for $k \in \mathbf{N} \cup \{0\}$ and $\alpha \in (0,1]$; and we use $||\cdot||_{C^{k,\alpha}}$ and $||\cdot||_{C^k}$ as the norms in $C^{k,\alpha}(\overline{\Omega})$ and $C^k(\overline{\Omega})$.

We consider some basic properties of Laplacian operator $\Delta$. For some later propose we consider a more general operator $L_c(u) = -\Delta u + c(x)u$ where $c(x) \in L^\infty(\Omega)$, and here we assume $\partial\Omega$ is of $C^2$ class. For all $u \in C_0^\infty(\overline{\Omega})$, $L_c(u)$ is well-defined. Considered as an operator on the Hilbert space $L^2(\Omega)$, $L_c$ is densely defined and symmetric since $\langle L_c u, v\rangle = \langle u, L_c v\rangle$ for all $u, v \in C_0^\infty(\overline{\Omega})$ from Green's theorem and integration by parts. The operator $L_c$ is associated with a quadratic form:

$$b_c(u,v) = \int_\Omega [\nabla u(x) \cdot \nabla v(x) + c(x)u(x)v(x)]dx, \tag{2.3}$$

in the sense that $\langle L_c u, v\rangle = b_c(u,v)$. Then $b_c$ is closable on $C_0^\infty(\overline{\Omega})$, and the domain of its closure in $L^2(\Omega)$ is the Hilbert space $W_0^{1,2}(\Omega)$. Let $c_0 = ||c(x)||_\infty$. Then

$$\widetilde{b_c}(u,v) = b_c(u,v) + c_0 \int_\Omega u(x)v(x)dx \tag{2.4}$$

is positive in the sense that there exists $\delta > 0$ such that $\widetilde{b_c}(u,u) \geq \delta ||u||_{1,2}^2$ for any $u \in W_0^{1,2}(\Omega)$. From Riesz representation theorem, for any $f \in L^2(\Omega)$, there exists a unique $u_f \in W_0^{1,2}(\Omega)$ such that $\widetilde{b_c}(u_f, v) = \langle f, v\rangle$. Moreover $u_f \in W^{2,2}(\Omega)$ from the elliptic regularity theory. Then from Lax-Milgram theorem, there exists a linear self-adjoint operator $A : W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega) \to L^2(\Omega)$ such that $\widetilde{b_c}(u,v) = \langle Au, v\rangle$, and $A$ is bijective with a bounded inverse. $A$ is the Friedrichs extension of $L_c + c_0 I$, so in the following we do not distinguish between $L_c + c_0 I$ and $A$. In particular, when $c(x) \equiv 0$, we have $L_0 = -\Delta : W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega) \to L^2(\Omega)$ is self-adjoint and invertible.

The inverse $B = (L_c + c_0 I)^{-1} : L^2(\Omega) \to W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ is a compact symmetric operator from $L^2(\Omega)$ to $L^2(\Omega)$, since the embedding from $W_0^{1,2}(\Omega)$ to $L^2(\Omega)$ is compact from Rellich theorem. From Hilbert-Schmidt theorem, the spectrum of $B$ is a collection of countable many points $\{\eta_i\}$ such that $\eta_1 > \eta_2 > \cdots > \eta_i > \eta_{i+1} > \cdots$ and $\lim_{i\to\infty} \eta_i = 0$, and each null space $N(B - \eta_i I)$ is finite dimensional from Riesz-Schauder theory for compact operator. For the operator $B-\eta I$, we have the Fredholm alternatives: the equation $(B-\eta I)\phi = f$ is uniquely solvable if $\eta \neq \eta_i$, and when $\eta = \eta_i$, then $(B-\eta I)\phi = f$ is solvable if and only if $\langle f, \psi\rangle = 0$ for all $\psi \in N(B-\eta_i I)$.

The spectral properties of $L_c$ can be easily obtained from that of $B = (L_c + c_0 I)^{-1}$, since $L_c\phi = \lambda\phi$ if and only if $B\phi = (\lambda + c_0)^{-1}\phi$. Let $\lambda_i = \eta_i^{-1} - c_0$. Then the spectrum of $L_c$ consists of a sequence of real numbers $\{\lambda_i\}$ such that $\lambda_1 < \lambda_2 < \cdots < \lambda_i < \lambda_{i+1} < \cdots$ and $\lim_{i\to\infty} \lambda_i = \infty$. For $\lambda \neq \lambda_i$, the equation $(L_c - \lambda I)u = f$ is uniquely solvable for any $f \in L^2(\Omega)$ and the solution $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. For each eigenvalue $\lambda = \lambda_i$, the eigenspace $N(L_c - \lambda_i I)$ is finite dimensional. The equation $(L_c - \lambda_i I)u = f$ is equivalent to $(B-\eta_i I)u = -Bf$, and the eigenspace $N(B-\eta_i I)$ is same as $N(L_c - \lambda_i I)$, thus the solvability condition of the equation $(L_c - \lambda_i I)u = f$ is equivalent to

$$0 = \langle -Bf, \phi\rangle = -\langle f, B\phi\rangle = -\langle f, \eta_i\phi\rangle = -\eta_i\langle f, \phi\rangle, \qquad (2.5)$$

for any $\phi \in N(B-\eta_i I) = N(L_c - \lambda_i I)$. In particular, we have showed that $dim N(L_c - \lambda_i I) = codim R(L_c - \lambda_i I)$, and

$$R(L_c - \lambda_i I) = \left\{ f \in L^2(\Omega) : \int_\Omega f\phi dx = 0,\ \forall \phi \in N(L_c - \lambda_i I) \right\}. \qquad (2.6)$$

Therefore we can conclude that for any $\lambda \in \mathbf{R}$, $L_c - \lambda I = -\Delta + [c(x) - \lambda]I : W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega) \to L^2(\Omega)$ is a Fredholm operator of index zero.

The spectral properties and Fredholmness of $L_c$ also hold for other spaces. Suppose that there exist Banach spaces $X$ and $Y$ such that $X \subset Y \subset L^2(\Omega)$, such that

- $Y$ is continuously embedded into $L^2(\Omega)$, $X$ is compactly embedded into $Y$, and $L_c : X \to Y$ is continuous.
- (Elliptic regularity) If $f \in Y \subset L^2(\Omega)$, then the solution $u$ of $(L_c + c_0 I)u = f$ satisfies $u \in X$, and there exists a positive constant $c(p)$ such that

$$||u||_X \le c(p)||f||_Y. \qquad (2.7)$$

The invertibility of $L_c + c_0 I$ in $L^2(\Omega)$ and the elliptic regularity imply that $L_c + c_0 I : X \to Y$ is bijective and it has a continuous inverse. Since the embedding of $X$ into $Y$ is compact, then $(L_c + c_0 I)^{-1} : Y \to Y$ is compact. Here we use $B_p = (L_c + c_0 I)^{-1}|_Y$ and $B_2 = (L_c + c_0 I)^{-1}|_{L^2(\Omega)}$. Because $B_p$ is compact, the operator $B_p - \eta I$ is a Fredholm operator with index 0 for any $\eta \in \mathbf{R}$. On the other hand, for any $\eta \in \mathbf{R}$, $N(B_2 - \eta I) \supset N(B_p - \eta I)$ and $R(B_2 - \eta I) \supset R(B_p - \eta I)$. Hence we have

$$\begin{aligned} &codimR(B_2 - \eta I) \leq codimR(B_p - \eta I) \\ =&dimN(B_p - \eta I) \leq dimN(B_2 - \eta I). \end{aligned}$$

But $codimR(B_2 - \eta I) = dimN(B_2 - \eta I)$ since $B_2 - \eta I$ is a Fredholm operator with index 0. Therefore we must have $N(B_2 - \eta I) = N(B_p - \eta I)$ and $codimR(B_2 - \eta I) = codimR(B_p - \eta I)$. This implies that for any $\lambda \neq \lambda_i$, $L_c u = f$ is uniquely solvable for any $f \in Y$ and the solution $u \in X$; for $\lambda = \lambda_i$, the eigenspace $N(L_c - \lambda_i I)$ in $Y$ is identical to the one in $L^2(\Omega)$, and it is easy to verify that

$$R((L_c - \lambda_i I)|_X) = \left\{ f \in Y : \int_\Omega f\phi dx = 0, \ \forall \phi \in N(L_c - \lambda_i I) \right\}. \qquad (2.8)$$

The two important examples of $X$ and $Y$ are (a) $X = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and $Y = L^p(\Omega)$ for $p > 2$; and (b) $X = C_0^{2,\alpha}(\overline{\Omega})$ and $Y = C^\alpha(\overline{\Omega})$ for $\alpha \in (0,1)$. The conditions above can be verified through the Sobolev embedding theorems, elliptic $L^p$ estimates and Hölder estimates. Note that we do need the boundary $\partial\Omega$ to have sufficient smoothness for the elliptic regularity. We will always assume $\partial\Omega$ is of class $C^{2,\alpha}$ unless specified otherwise. For future reference, we summarize the results above:

**Proposition 2.1.** *Assume that $\Omega$ is a bounded domain in $\mathbf{R}^n$ $(n \geq 2)$ with $C^{2,\alpha}$ boundary. Let $c \in L^\infty(\Omega)$, and let $X = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and $Y = L^p(\Omega)$ for $p \geq 2$ or $X = C_0^{2,\alpha}(\overline{\Omega})$ and $Y = C^\alpha(\overline{\Omega})$ for $\alpha \in (0,1)$. Define $L_c : X \to Y$ by $L_c u = -\Delta u + c(x)u$. Then*

- *$L_c$ is a linear continuous Fredholm operator of index zero;*
- *The spectrum of $L_c$ consists of a sequence of real numbers $\lambda_1(c) < \cdots < \cdots < \lambda_i(c) < \cdots$ and $\lim_{i\to\infty} \lambda_i(c) = \infty$;*
- *For each $i \in \mathbf{N}$, the null space $N(L_c - \lambda_i(c)I)$ is finite dimensional, and the range space can be characterized by*

$$R(L_c - \lambda_i(c)I) = \left\{ f \in Y : \int_\Omega f\phi dx = 0, \ \forall \phi \in N(L_c - \lambda_i(c)I) \right\}. \qquad (2.9)$$

From Hilbert-Schmidt theory of the compact symmetric operators on Hilbert space, the collection of all eigenfunctions of $L_c$ is an orthonormal basis of $L^2(\Omega)$, and the eigenvalues $\lambda_i(c)$ can be characterized by the

Courant-Fischer principle: for $u \in W_0^{1,2}(\Omega)\backslash\{0\}$, define the Rayleigh quotient

$$R_c(u) = \frac{\int_\Omega (|\nabla u|^2 + c(x)u^2)dx}{\int_\Omega u^2 dx}. \tag{2.10}$$

Then

$$\lambda_i(c) = \max_{S_i} \min_{u \in S_i} R_c(u) = \min_{S_{n-1}} \max_{u \perp S_{n-1}} R_c(u), \tag{2.11}$$

where $S_i$ is any linear subspace of $W_0^{1,2}(\Omega)$ of dimension $i$. In particular,

$$\lambda_1(c) = \min_{u \in W_0^{1,2}(\Omega)\backslash\{0\}} R_c(u). \tag{2.12}$$

Notice that a special case of (2.12) is the Poincaré inequality for $p = 2$:

$$\lambda_1(0) \int_\Omega [u(x)]^2 dx \le \int_\Omega |\nabla u(x)|^2 dx, \quad \forall u \in W_0^{1,2}(\Omega). \tag{2.13}$$

In the remaining part of the book, we will always use $\lambda_i(c)$ to denote the eigenvalues of $L_c$, and use $\lambda_i$ to denote $\lambda_i(0)$, unless specified otherwise.

A function $u(x)$ is a *classical solution* of (2.1), if $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ and $u$ satisfies the equation and boundary conditions. Very often abstract formulations are used to consider the solutions of (2.1), and weak solutions in Sobolev spaces are obtained. However elliptic estimates can be used to proved the regularity of weak solutions. On the other hand, the Hölder estimates of elliptic equations make it necessary to consider solutions in $C^{2,\alpha}(\overline{\Omega})$ instead of $C^2(\overline{\Omega})$ when the spatial dimension $n \ge 2$. In the following we often consider a nonlinear operator $F$ defined on $\mathbf{R} \times C_0^{2,\alpha}(\overline{\Omega})$ for $\alpha \in (0,1)$:

$$F(\lambda, u) = \Delta u + \lambda f(u). \tag{2.14}$$

One could also consider the same operator in the Sobolev space $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. But since we are only interested in the solutions of $F(\lambda, u) = 0$, we usually can use regularity theory to prove $W^{2,p}$ solutions are indeed classical solution in the Hölder spaces under reasonable conditions on $f$. Hence we will only consider the operator $F(\lambda, u)$ defined in a Hölder space. Sobolev setting is still needed when other abstract methods such as variational methods are used.

**Proposition 2.2.** *Let $F$ be defined as in* (2.14).

*(1) Suppose that $f \in C^k(\overline{\Omega})$ for $k \in \mathbf{N}$, then $F : \mathbf{R} \times C_0^{2,\alpha}(\overline{\Omega}) \to C^\alpha(\overline{\Omega})$ is of class $C^k$ for any $\alpha \in (0,1)$. Moreover for any $(\lambda, u) \in \mathbf{R} \times C_0^{2,\alpha}(\overline{\Omega})$, the partial derivative $F_u(\lambda, u) = \Delta + \lambda f'(u) : C_0^{2,\alpha}(\overline{\Omega}) \to C^\alpha(\overline{\Omega})$ is a Fredholm operator with index zero.*

(2) *Suppose that* $f \in C^{k,\beta}(\overline{\Omega})$ *for* $k \in \mathbf{N} \cup \{0\}$ *and* $\beta \in (0,1)$, *then* $F : \mathbf{R} \times C_0^{2,\alpha}(\overline{\Omega}) \to C^{\gamma}(\overline{\Omega})$ *is of class* $C^k$ *with* $\gamma = \alpha\beta$ *for any* $\alpha \in (0,1)$.

The proof of the smoothness is a rather standard argument and thus omitted, and the Fredholm property has been proved earlier. When $n = 1$, we can replace $C^{2,\alpha}$ and $C^{\alpha}$ spaces by $C^2$ and $C^0$ spaces respectively.

The stability of solutions to (2.1) is an important issue, especially when we consider the corresponding parabolic equation:

$$\begin{cases} u_t = \Delta u + \lambda f(u), & t > 0, \ x \in \Omega, \\ u(x,0) = u_0(x), & x \in \Omega, \\ u(x,t) = 0, & t > 0, \ x \in \partial\Omega. \end{cases} \tag{2.15}$$

Suppose that $(\lambda_*, u_*)$ is an equilibrium solution of (2.15). Then it can be showed that the local stability of $(\lambda_*, u_*)$ can be determined through the eigenvalue problem:

$$\begin{cases} -\Delta\phi - \lambda_* f'(u_*)\phi = \mu\phi, & \text{in} \quad \Omega, \\ \phi = 0, & \text{on} \quad \partial\Omega. \end{cases} \tag{2.16}$$

From the discussions above, the eigenvalue problem (2.16) has a sequence of real eigenvalues $\mu_1 < \mu_2 \le \cdots \le \mu_n \le \cdots \to \infty$, and from Proposition 2.4 which will be proved in Section 2.3, $\mu_1$ is a simple eigenvalue with a positive eigenfunction $\phi_1 > 0$.

**Definition 2.1.** Let $(\lambda_*, u_*)$ be a solution to (2.1).

(1) If $\mu_1(u_*) > 0$, then we say that $(\lambda_*, u_*)$ is *stable*; if $\mu_1(u_*) < 0$, it is *unstable*; and when $\mu_1(u_*) = 0$, it is *neutrally stable*. (Sometimes we still regard neutrally stable as unstable.)
(2) If $(\lambda_*, u_*)$ is unstable, then the number of negative eigenvalues of (2.16) (counting the multiplicity) is the *Morse index* $M(u_*)$ of $(\lambda_*, u_*)$.
(3) If $\mu_i(u_*) = 0$ for some $i$, then $(\lambda_*, u_*)$ is *degenerate*, otherwise it is *non-degenerate*.

As an example, we consider the stability of the constant solution $u = 0$ of (2.1) assuming that $f(0) = 0$. From (2.16), it is easy to see that $\mu_i = \lambda_i - \lambda f'(0)$, and in particular $\mu_1 = \lambda_1 - \lambda f'(0)$. If $f'(0) \le 0$, then $u = 0$ is stable for all $\lambda > 0$; if $f'(0) > 0$, then $u = 0$ is stable if $\lambda < \lambda_1/f'(0)$, and $u = 0$ is unstable if $\lambda > \lambda_1/f'(0)$. At $\lambda = \lambda_1/f'(0)$, $u = 0$ is neutrally stable and degenerate. When $\lambda \ne \lambda_i/f'(0)$, $u = 0$ is non-degenerate, otherwise it is degenerate. When $\lambda \in (\lambda_i/f'(0), \lambda_{i+1}/f'(0)]$, the Morse index of $u = 0$ is exactly $i$.

## 2.2 Variational methods

Besides directly considering functional equation $F(\lambda, u) = 0$ as defined in (2.14), one can use several other approaches for the solutions of (2.1). Here we introduce the variational methods. We define

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega F(u)dx, \tag{2.17}$$

where $u \in W_0^{1,2}(\Omega)$ and $F(u) = \int_0^u f(t)dt$. One can verify that for any $w \in W_0^{1,2}(\Omega)$,

$$I'(u)[w] = \int_\Omega [\nabla u \cdot \nabla w - \lambda f(u)w]dx. \tag{2.18}$$

Then $u \in W_0^{1,2}(\Omega)$ is a weak solution of (2.1) if and only if $u$ is a critical point of $I(u)$, which means $I'(u)[w] = 0$ for any $w \in W_0^{1,2}(\Omega)$. If the nonlinear function $f \in C^1(\mathbf{R})$ and satisfies some growth condition (see (2.22)), then one can show that a weak solution $u$ is a classical one given that the domain is also sufficiently smooth.

To obtain critical points of (2.17), we introduce two most fundamental theorems of variational methods. In a more general setting, we consider a $C^1$ functional $I : X \to \mathbf{R}$, where $X$ is a Banach space. The key to the theory of critical points is the Palais-Smale (PS) condition:

Suppose that $\{u_m\}$ is a sequence in $X$ such that

$$|I(u_m)| \le C \text{ for all } m \in \mathbf{N},\ ||I'(u_m)|| \to 0 \text{ as } m \to \infty, \tag{2.19}$$

then $\{u_m\}$ contains a convergent subsequence.

The following basic results would be sufficient for our need in this book, and the readers should consult standard references in variational methods for more advanced results and proofs of these results, like Rabinowitz [R3], Struwe [St] and Chang [Ch1; Ch2].

**Theorem 2.1. (Minimization)** *Suppose that $X$ is a real Banach space and $I \in C^1(X, \mathbf{R})$ satisfies (PS) condition. If $I(u) \ge C$ for some $C \in \mathbf{R}$ and all $u \in X$, then*

$$c = \inf_{u \in X} I(u) \tag{2.20}$$

*is a critical value, and c can be achieved by $u \in X$.*

**Theorem 2.2. (Mountain Pass Lemma)** *Suppose that $X$ is a real Banach space and $I \in C^1(X, \mathbf{R})$ satisfies (PS) condition. Suppose*

($I_1$) *there exists* $u_0 \in X$ *such that there are constants* $\rho, \alpha > 0$ *such that if* $||u - u_0|| = \rho$, *then* $I(u) \geq I(u_0) + \alpha$, *and*
($I_2$) *there exists* $u_1 \in X$ *such that* $||u_1 - u_0|| > \rho$, *and* $I(u_1) \leq I(u_0)$.

*Then* $I$ *possesses a critical value* $c \geq I(u_0) + \alpha$, *and* $c$ *can be characterized as*

$$c = \inf_{g \in \Gamma} \max_{u \in g([0,1])} I(u), \tag{2.21}$$

*where*

$$\Gamma = \{g \in C^0([0,1], X) : g(0) = u_0, g(1) = u_1\}.$$

Theorem 2.1 is one of the results of direct methods of calculus of variations, and a proof of this result and other minimization principles can be found in Rabinowitz [R3] and Struwe [St]. The Mountain Pass Lemma in this form was first proved by Ambrosetti and Rabinowitz [AR].

To apply these variational principles to the functional $I(u)$ defined in (2.17), we need to verify the functional is differentiable and it satisfies the (PS) condition. The following result is from [R3] Propositions B.10, B.34 and B.35:

**Proposition 2.3.** *Suppose that* $\Omega$ *is a bounded smooth domain in* $\mathbf{R}^n$. *If* $f \in C^0(\mathbf{R})$, *and there exists constants* $a_1, a_2 > 0$ *such that*

$$|f(u)| \leq a_1 + a_2|u|^s, \tag{2.22}$$

*where* $0 \leq s \leq (n+2)/(n-2)$ *when* $n \geq 3$, *or* $0 \leq s < \infty$ *when* $n = 1, 2$. *Then* $I$ *defined in* (2.17) *is a* $C^1$ *functional from* $X \equiv W_0^{1,2}(\Omega)$ *to* $\mathbf{R}$, $I'(u)[w]$ *is defined as in* (2.18) *for any* $w \in X$, *and*

$$J(u) = \int_\Omega F(u(x))dx \tag{2.23}$$

*is weakly continuous and* $J'(u)$ *is compact. Moreover if* $\{u_m\}$ *is a bounded sequence in* $X$ *such that* $I'(u_m) \to 0$ *as* $m \to \infty$, *then* $\{u_m\}$ *has a convergent subsequence. If in addition,* $f \in C^1(\mathbf{R})$ *and there exists constants* $a_1, a_2 > 0$ *such that*

$$|f'(u)| \leq a_1 + a_2|u|^{s-1}, \tag{2.24}$$

*where* $0 \leq s \leq (n+2)/(n-2)$ *when* $n \geq 3$, *or* $0 \leq s < \infty$ *when* $n = 1, 2$, *then* $I \in C^2(X, R)$ *and*

$$I''(u)[v, w] = \int_\Omega \nabla v(x) \nabla w(x) dx - \lambda \int_\Omega f'(u(x)) v(x) w(x) dx, \tag{2.25}$$

*for any* $v, w \in X$.

When considering the positive solutions of (2.1), one can consider a modified functional defined as

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega \tilde{F}(u)dx, \tag{2.26}$$

where $\tilde{F}(u) = \int_0^u \tilde{f}(t)dt$, $\tilde{f}(u) = f(u)$ if $u \geq 0$ and $\tilde{f}(u) = 0$ if $u < 0$. If $f$ is continuous and $f(0) = 0$, then $\tilde{f}$ is continuous thus $I$ in (2.26) is still $C^1$ under conditions in Proposition 2.3; and if $f$ is differentiable and $f(0) = 0$, then $\tilde{f}$ is piecewisely differentiable, and one can still show $I \in C^2$ as in Proposition 2.3.

It is useful to gain stability information about the solution obtained through minimization or minimax procedures. The Morse index can be defined as the dimension of the negative space ($v$ belongs to the negative space if $I''(u)[v,v] < 0$) of the bilinear form $I''(u)$ at a critical point $u$, and one can verify that this definition is coincident to the one in Definition 2.1 since $I''(u)[v,v]$ is identical to the Rayleigh expression (2.10). For example one can show that the minimizer of the energy functional is always stable or neutrally stable:

**Theorem 2.3.** *Let* $(\lambda_*, u_*)$ *be a solution of* (2.1) *which we obtain from Theorem 2.1, i.e.* $I(u_*) = \inf I(u)$. *Then* $\mu_1(u_*) \geq 0$*;*

***Proof.*** Let $X = W_0^{1,2}(\Omega)$. If $u_*$ is a minimizer, then for any $v \in X$, the function $g(t) = I(u_* + tv) - I(u_*)$ satisfies $g(0) = g'(0) = 0$ and $g''(0) \geq 0$ from the minimizing property of $u_*$. It is easy to verify that $g''(0) = I''(u_*)[v,v]$. Hence $I''(u_*)[v,v] \geq 0$ for any $v \in X$, and $\mu_1(u_*) = \min_{v\in X\backslash\{0\}} I''(u_*)[v,v]/\int_\Omega v^2 \geq 0$. □

For the Mountain Pass critical value, usually there exists a mountain-pass type critical point which is not stable, but in general minimizer is still possible for Mountain Pass critical value. Related results can be found in Struwe [St] (Chapter 2 Theorems 10.2 and 10.3), and Chang [Ch1] (Chapter 2). In applications, Mountain Pass Lemma often provides the existence of a critical point other than the known minimizers.

## 2.3 Maximum principles

It is easy to observe that a concave function $f : [a,b] \to \mathbf{R}$ achieves its minimum value at one of boundary points. In particular, if $f$ is concave, $f(a) \geq 0$ and $f(b) \geq 0$, then $f > 0$ for $x \in (a,b)$. The generalization of this simple fact to multi-variable functions and its variants are called maximum principles, which play an essential role in theory of linear and nonlinear elliptic PDEs. Here for the purpose of this book, we restrict our attention to the linear differential operator defined by $L_c = -\Delta + c(x)$. The basic maximum principle is as follows:

**Theorem 2.4.** *Suppose that $\Omega$ is a bounded connected domain in $\mathbf{R}^n$ with $n \geq 2$, $c \in L^\infty(\Omega)$, and $c(x) \geq 0$ for $x \in \overline{\Omega}$. If $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, $L_c u \geq 0$ in $\Omega$, $u(x) \geq 0$ on $\partial\Omega$, then*

*(1) (weak maximum principle) $u(x) \geq 0$ for $x \in \Omega$;*
*(2) (strong maximum principle) If there exists $x_* \in \Omega$ such that $u(x_*) = 0$, then $u(x) \equiv 0$ for $x \in \overline{\Omega}$;*
*(3) (Hopf boundary lemma) If there is a ball $B_r \subset \Omega$ such that $x_* \in \partial B_r \cap \partial\Omega$, and $u(x) > u(x_*)$ for all $x \in B_r$, then for any outward direction $\nu$ at $x_*$ with respect to $\partial B_r$ (i.e., $\nu \cdot n(x_*) > 0$ for the outer normal vector $n(x_*)$ of $\partial B_r$),*

$$\limsup_{t \to 0} \frac{u(x_*) - u(x_* - t\nu)}{t} < 0; \tag{2.27}$$

*if $u \in C^1(\overline{\Omega})$, then*

$$\frac{\partial u}{\partial \nu}(x_*) < 0. \tag{2.28}$$

The proof of these standard maximum principles can be found in any textbook of PDEs, for example, Evans [E], Gilbarg and Trudinger [GT], and Han and Lin[HL]. We remark that all results in Theorem 2.4 remain true if we replace the operator $L_c = -\Delta + c(x)$ by a more general second order elliptic operator $L = -\sum_{i,j=1}^n a_{ij}(x)\partial_{ij} + \sum_{i=1}^n b_i(x)\partial_i + c(x)$, where $a_{ij}, b_i \in L^\infty(\Omega)$, $\partial_{ij} = \partial^2/\partial x_i \partial x_j$ and $\partial_i = \partial/\partial x_i$.

The positivity of $c(x)$ is restrictive in applications, and in the following we shall show how to avoid this restriction. The connection between the weak and strong maximum principle can be established without the restriction on the sign of $c$:

**Theorem 2.5.** *If $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, $L_c u \geq 0$ in $\Omega$, $u(x) \geq 0$ on $\partial\Omega$. If $u(x) \geq 0$ for all $x \in \Omega$, then either $u(x) > 0$ for all $x \in \Omega$, or $u(x) \equiv 0$. If $u(x) > 0$ in $\Omega$, $\partial\Omega$ is of class $C^{2,\alpha}$, $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$, and there exists $x_* \in \partial\Omega$ such that $u(x_*) = 0$, then*

$$\frac{\partial u}{\partial \nu}(x_*) < 0. \tag{2.29}$$

***Proof.*** We write $c(x)$ into the positive and negative parts: $c(x) = c^+(x) - c^-(x)$. Then $-\Delta u + c^+(x)u \geq c^-(x)u \geq 0$ for any $x \in \Omega$. Then we can apply strong maximum principle and Hopf lemma in Theorem 2.4 to $-\Delta + c^+(x)$. Note that interior ball condition for Hopf lemma is satisfied if $\partial\Omega$ is $C^{2,\alpha}$ smooth. □

The maximum principle can be cleverly applied when studying nonlinear problems. We will derive several variants of the maximum principle. For the convenience, we define

**Definition 2.2.** We say that the *maximum principle* holds for $L_c$ in $\Omega$ if $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, $L_c u \geq 0$ in $\Omega$, $u \geq 0$ on $\partial\Omega$, then $u \geq 0$ in $\Omega$.

Notice that from Theorem 2.5, we do not need to distinguish weak and strong maximum principle. First we show that the maximum principle implies the eigenfunction corresponding to the principal eigenvalue $\lambda_1(c)$ does not change sign:

**Proposition 2.4.** *Suppose that $\Omega$ is a bounded connected domain in $\mathbf{R}^n$ ($n \geq 2$) with $C^{1,1}$ boundary, $c \in L^\infty(\Omega)$. Then for $L_c = -\Delta + c(x)$ : $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega) \to L^2(\Omega)$, the eigenfunction $\phi_1$ corresponding to $\lambda_1(c)$ belongs to $W^{2,p}(\Omega) \cap C_0^{1,\alpha}(\overline{\Omega})$ for $p > 2$ and $\alpha \in (0,1)$; $\phi_1$ can be chosen as positive ($\phi_1(x) > 0$ for $x \in \Omega$), and for any $x \in \partial\Omega$, $\partial\phi_1(x)/\partial\nu < 0$, where $\nu$ is the outer normal direction. Moreover $dim(N(L_c - \lambda_1(c))) = 1$, and for any other eigenvalue $\lambda_i(c) > \lambda_1(c)$ with $i > 1$, the eigenfunction $\phi_i$ always changes sign in $\Omega$. If in addition, $c \in C^\alpha(\overline{\Omega})$ and $\partial\Omega$ is of $C^{2,\alpha}$, then $\phi_1 \in C_0^{2,\alpha}(\Omega)$.*

***Proof.*** Suppose that $\phi_1$ is a non-trivial eigenfunction corresponding to $\lambda_1(c)$. Then for $u = \phi_1$, the Rayleigh quotient $R_c(u)$ achieves the minimum as in (2.12). It is clear that $|\phi_1|$ also achieves the same minimum, and $|\phi_1|$ also belongs to $W_0^{1,2}(\Omega)$. Moreover from the minimizing property, $|\phi_1|$ is also a weak solution of $L_c\phi = \lambda_1(c)\phi$ in $\Omega$ and $\phi = 0$ on $\partial\Omega$. From elliptic regularity theory ([GT] Theorem 8.34 and Theorem 9.13), $|\phi_1| \in W^{2,p}(\Omega) \cap C_0^{1,\alpha}(\Omega)$ for $p > 2$ and $\alpha \in (0,1)$. Thus $(L_c - \lambda_1(c))|\phi_1| = 0$ in $\Omega$ and $|\phi_1| = 0$ on $\partial\Omega$. Since $|\phi_1| \geq 0$ in $\Omega$, from Theorem 2.5, either $|\phi_1| \equiv 0$ or $|\phi_1| > 0$ in $\Omega$, and the latter must be the case for a non-trivial eigenfunction. Therefore $\phi_1(x) \neq 0$ for $x \in \Omega$. The claim of $\partial\phi_1(x)/\partial\nu < 0$ for any $x \in \partial\Omega$ also follows from Theorem 2.5. Since any eigenfunction $\phi_1$ associated with $\lambda_1(c)$ is of one sign, then $dim(L_c - \lambda_1(c)) = 1$. Otherwise there are two orthogonal eigenfunction $\phi_{1,1}$ and $\phi_{1,2}$ but we can assume both are positive, thus $\int_\Omega \phi_{1,1}\phi_{1,2}dx > 0$ that is a contradiction. For the same reason eigenfunction $\phi_i$ $(i \geq 2)$ cannot be of one sign either. If $c \in C^\alpha(\overline{\Omega})$ and $\partial\Omega$ is $C^{2,\alpha}$, then $\phi_1 \in C_0^{2,\alpha}(\Omega)$ from Hölder estimates ([GT] Theorem 6.14). □

The positiveness of $\phi_1$ can be used to improve the maximum principle:

**Theorem 2.6.** *Suppose that $\Omega$ is a bounded connected domain in $\mathbf{R}^n$ $(n \geq 2)$ with $C^{2,\alpha}$ boundary. Then the maximum principle holds in $\Omega$ if the principal eigenvalue $\lambda_1(c)$ is positive. If $\lambda_1(c) = 0$, and these exists $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, $L_c u \geq 0$ in $\Omega$, $u \geq 0$ on $\partial\Omega$, then $u = c\phi_1$, where $\phi_1$ is an eigenfunction corresponding to $\lambda_1(c)$.*

***Proof.*** First we assume $\lambda_1(c) > 0$. Suppose that the maximum principle does not hold in $\Omega$. Then there exists $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, $L_c u \geq 0$ in $\Omega$, $u \geq 0$ on $\partial\Omega$, but $u < 0$ for some $x \in \Omega$. Let $\Omega_0$ be a connected component of $\{x \in \Omega : u(x) < 0\}$. Let $\phi_1$ be the positive eigenfunction of $L_c$ on $\Omega$. Then by integrating $\phi_1 L_c u - u L_c \phi_1$ on $\Omega_1$, and using the Green's theorem, one obtain

$$-\int_{\partial\Omega_1} \phi_1 \frac{\partial u}{\partial \nu} ds = \int_{\Omega_1} (\phi_1 L_c u - \lambda_1(c) u \phi_1) dx. \tag{2.30}$$

The left hand of (2.30) is non-positive since $\partial u/\partial\nu \geq 0$ and $\phi_1 > 0$ on $\partial\Omega_1$, and the right hand side is positive since $\phi_1 > 0$, $L_c u \geq 0$, $\lambda_1(c) > 0$ and $u < 0$ in $\Omega$ (the integral $\int_{\Omega_1} u\phi_1 dx > 0$). That is a contradiction. Therefore the maximum principle holds.

If $\lambda_1(c) = 0$, we still use the same proof to reach (2.30). But the equation only holds if $u = \partial u/\partial\nu \equiv 0$ on $\partial\Omega$ and $L_c u \equiv 0$ in $\Omega$. This implies $\lambda_1(c,\Omega) \leq \lambda_1(c,\Omega_1) \leq 0$. Combining with the assumption $\lambda_1(c,\Omega) = 0$, we obtain $\lambda_1(c,\Omega) = \lambda_1(c,\Omega_1) = 0$. From the uniqueness of eigenfunction in Proposition 2.4, we must have $\Omega = \Omega_1$ and $u = c\phi_1$. □

We point out that the application of Green's theorem in the proof above is not in a classical sense, as the boundary of $\Omega_1$ (which is the level set of a smooth function $u$) may not be as smooth as required in standard Green's theorem in calculus. However this can be validated by an approximation process, see Dancer [D6] page 21-22. It is easy to see that Theorem 2.6 is an improvement than the weak maximum principle in Theorem 2.4, since for $c \geq 0$, $\lambda_1(c) > \lambda_1(0) > 0$. Similar proof can be used to prove:

**Proposition 2.5.** *Suppose that there exists $g \in C^2(\Omega) \cap C^0(\overline{\Omega})$, $g > 0$ and $L_c g \geq 0$ in $\Omega$. Then one of the following two statements is true:*

*(1) The maximum principle holds and* $\lambda_1(c) > 0$*; or*

*(2)* $g \equiv 0$ *on* $\partial\Omega$ *and* $L_c g \equiv 0$ *in* $\Omega$*, i.e.* $\lambda_1(c) = 0$ *and* $g$ *is a principal eigenfunction of* $L_c$.

*This implies that if there exists* $g \in C^2(\Omega) \cap C(\overline{\Omega})$, $g > 0$ *and* $Lg \geq 0$ *in* $\Omega$, *and* $g \not\equiv 0$ *on* $\partial\Omega$. *Then the maximum principle holds and* $\lambda_1(L, \Omega) > 0$.

Another way to state the idea above is

**Corollary 2.1.**

*(1)* $\lambda_1(c) \geq 0$ *if and only if there exists* $w(x) > 0$ *in* $\Omega$ *such that* $L_c w = 0$ *in* $\Omega$*;*

*(2)* $\lambda_1(c) < 0$ *if and only if there exists a sign-changing* $w(x)$ *such that* $L_c w = 0$ *in* $\Omega$.

Another useful maximum principle holds for domain which is narrow in one direction (say $x_n$):

**Proposition 2.6.** *There exists* $\delta > 0$ *which only depends on* $||c||\infty$ *such that if* $\Omega \subset \{x \in \mathbf{R}^n : |x_n| < \delta\}$, *then the maximum principle holds for* $L_c$ *in* $\Omega$.

***Proof.*** We choose $k > 0$ so that $k^2 > ||c||_\infty$. Then for $w(x) = \cos(kx_n)$, $-\Delta w = [k^2 - c(x)]\cos(kx_n)$. If we choose $\delta = \pi/4k$, then $w(x) > 0$ and $-\Delta w(x) \geq 0$ in $\overline{\Omega}$, then from Theorem 2.4, the maximum principle holds for $L_c$ in $\Omega$. □

Another maximum principle requires only the smallness of the volume of $\Omega$, and it is a consequence of Alexandroff weak maximum principle, and we refer it to [BNV; HL]. In this section we only review the maximum principles for the symmetric operator $L_c$ which is needed for later purpose in this book. Most results here can be extended to more general non-symmetric second order differential operators and also weak solutions. We refer to Berestycki, Nirenberg and Varadhan [BNV] for more general results.

To conclude this section, we apply the maximum principle to equation (2.1):

**Theorem 2.7.** *Suppose that* $f \in C^1(\mathbf{R}^+)$ *and* $\Omega$ *is a bounded connected domain in* $\mathbf{R}^n$ $(n \geq 2)$ *with* $C^{2,\alpha}$ *boundary. Suppose that* $\lambda > 0$, $u \in C_0^{2,\alpha}(\overline{\Omega})$ *is a solution of* (2.1) *satisfying* $u(x) \geq 0$ *for* $x \in \Omega$.

*(1) If* $f(0) \geq 0$, *then either* $u(x) > 0$ *for all* $x \in \Omega$ *and* $\partial u(x)/\partial\nu < 0$ *for all* $x \in \partial\Omega$, *or* $u(x) \equiv 0$ *for* $x \in \Omega$.

*(2) If $u(x) \not\equiv 0$, and $u_M = u(x_0) = \max_{x\in\overline{\Omega}} u(x)$, then $f(u_M) > 0$.*

***Proof.*** We first prove the first part. Suppose that there exists $x_* \in \Omega$ such that $u(x_*) = 0$, and $u \not\equiv 0$ in $\Omega$. Let $\Omega_1$ be a connected component of $\{x \in \Omega : u(x) > 0\}$. Then there exists $y_* \in \Omega_1$ such that $dist(y_*, \partial\Omega_1) = dist(y_*, z_*)$ where $z_* \in \partial\Omega_1 \bigcap \Omega$. Apparently $u(z_*) = 0$, and there exists a ball $B_r(y_*) \subset \Omega_1$ such that $z_* \in \partial B_r(y_*)$. If $f(0) > 0$, we can find a smaller ball $B_{r1}(y_{**}) \subset B_r(y_*)$ and $z_* \in \partial B_{r1}(y_{**})$ such that $-\Delta u(x) = f(u(x)) > 0$ for $x \in B_{r1}(y_{**})$. From Hopf lemma in Theorem 2.5, $\partial u(z_*)/\partial\nu < 0$ where $\nu$ is the outer normal of $B_{r1}(y_{**})$ at $z_*$. If $f(0) = 0$, then $c(x) = -f(u(x))/u(x)$ is continuous near $x = z_*$ since $f$ is $C^1$. Then $L_c u = -\Delta u + c(x)u = 0$ for $x \in B_r(y_*)$, $u \geq 0$ for $x \in \partial B_r(y_*)$, thus again from Hopf lemma in Theorem 2.5, $\partial u(z_*)/\partial\nu < 0$. But on the other hand, $\nabla u(z_*) = 0$ since $z_*$ is a local minimum. That reaches a contradiction. Therefore either $u > 0$ or $u \equiv 0$ in $\Omega$. If $u > 0$ in $\Omega$, then $\partial u(x)/\partial\nu < 0$ for all $x \in \partial\Omega$ can be proved in the same way as above since $\partial\Omega$ is $C^{2,\alpha}$ then each boundary point satisfies the interior ball condition. For the second part, $\lambda f(u_M) = -\Delta u(x_0) \geq 0$ as $x_0 \in \Omega$. If $f(u_M) = 0$, we can repeat the arguments above to show a contradiction. Hence $f(u_M) > 0$. □

The results remain true when $n = 1$ and $f(0) \geq 0$. But the first conclusion in Theorem 2.7 does not always hold if $f(0) < 0$. For example, $u(x) = \cos x + 1$ is a non-negative solution of

$$u'' + (u - 1) = 0, \quad x \in (-3\pi, 3\pi), \quad u(-3\pi) = u(3\pi) = 0. \tag{2.31}$$

But the solution has interior zeros at $x = \pm\pi$, and $u'(\pm 3\pi) = 0$. However it is not known whether a higher dimensional solution can have interior zeros even $f(0) < 0$. On the other hand, there exist higher dimensional positive solutions of (2.1) with zero normal derivative at some boundary points, see Shi and Shivaji [SS2].

## 2.4 Moving plane method

The Laplace operator $\Delta$ possesses a strong symmetry property. When the spatial domain is also symmetric, solutions of equation $\Delta u + \lambda f(u) = 0$ could inherit the symmetry. Using maximum principles and a moving plane method, one can show that every positive solution of (2.1) is symmetric if the domain is symmetric. First such result was proved by Serrin [Se1], and the general cases were proved in Gidas, Ni and Nirenberg [GNN1; GNN2].

The proof presented here is simplified through the application of maximum principle on a narrow domain.

Let $\Omega$ be a bounded domain, and let $u$ be a positive solution of (2.1) in $C^2(\Omega)\cap C^0(\overline{\Omega})$. Since $\Omega$ is a bounded domain, it must lie on one side of some hyperplane. Without loss of generality, we assume that $\Omega \subset \{(x', x_n) : x_n > k_0\}$ for some $k_0 \in \mathbf{R}$, where $x' = (x_1, \cdots, x_{n-1})$. Let $T_k$ be the hyperplane defined by $T_k = \{(x', x_n) : x_n = k\}$, and let $H_k$ be the half space $H_k = \{(x', x_n) : x_n < k\}$ for any $k \in \mathbf{R}$. We assume that $T_{k_0} \bigcap \partial\Omega \neq \emptyset$. Define $\Omega_k = H_k \bigcap \Omega$. We assume that $u(x)$ is extended to $\mathbf{R}^n$ by being zero outside of $\Omega$. We show that moving plane method can be used at such a "convex point" on the boundary.

**Theorem 2.8.** ***(Moving Plane Procedure)*** *Let $\Omega$ be a bounded domain with $C^{2,\alpha}$ boundary, and let $u$ be a positive solution of* (2.1) *in $C^2(\Omega) \cap C^0(\overline{\Omega})$. If $f$ is a locally Lipschitz continuous function, then there exists $k_1 > k_0$ such that for any solution positive $(\lambda, u)$ of* (2.1)*,*

$$\begin{aligned} &(x', 2k_1 - x_n) \in \Omega, \quad u(x', 2k_1 - x_n) > u(x', x_n), \\ \textit{and} \quad &\frac{\partial u}{\partial x_n}(x', x_n) > 0, \quad \textit{for all} \quad x = (x', x_n) \in \Omega_{k_1}. \end{aligned} \tag{2.32}$$

*At $k_2 = \sup\{k_1 > k_0 :$ (2.32) *holds*$\}$, either there exists $x \in \partial\Omega \bigcap \Sigma_{k_2}$ such that the reflection $\overline{x}$ of $x$ about $T_{k_2}$ is also on the boundary; or $T_{k_2}$ is orthogonal to $\partial\Omega$.*

***Proof.*** For $x = (x', x_n) \in \mathbf{R}^n$, we define

$$v_k(x', x_n) = u(x', 2k - x_n), \quad \text{and } w_k(x', x_n) = v_k(x', x_n) - u(x', x_n). \tag{2.33}$$

Then for $k \leq k_0$, $w_k(x', x_n) \geq 0$. We claim that there exists $\varepsilon = \varepsilon(\lambda) > 0$ such that for $k \in (k_0, k_0 + \varepsilon)$, $w_k(x) > 0$ for $x \in \Omega_k$. Indeed let $\Omega'_k$ be the reflection of $\Omega_k$ with respect to $T_k$, then for small $\varepsilon > 0$, $\Omega'_k \subset \Omega$ for $k \in (k_0, k_0 + \varepsilon)$ since $\partial\Omega$ is $C^{2,\alpha}$. Note that $w_k$ satisfies

$$-\Delta w = \lambda c_k(x', x_n) w, \; x \in \Omega_k, \tag{2.34}$$

where

$$c_k(x', x_n) = \frac{f(v_k(x', x_n)) - f(u(x', x_n))}{v_k(x', x_n) - u(x', x_n)}.$$

Then $c_k \in L^\infty(\Omega)$ since $f$ is Lipschitz continuous, and $||c_k||_{L^\infty(\Omega)}$ is independent of $k$. Thus there exists $\varepsilon = \varepsilon(\lambda) > 0$ such that when $k \in (k_0, k_0+\varepsilon)$, $|\Omega_k| < \delta$. Then for such $k$, $w_k(x) > 0$ for $x \in \Omega_k$ since $w_k(x) \geq 0$ for

$x \in \partial\Omega_k$ and Proposition 2.6. This also implies that $u(x) > 0$ for $x \in \Omega_k$ when $k \in (k_0, k_0 + \varepsilon)$. Therefore the moving plane process can be initiated. Moreover since $w_k$ satisfies (2.34), $w_k(x', k) = 0$ and $w_k > 0$ in $\Omega_k$, then from Hopf Lemma we have

$$-2\frac{\partial u}{\partial x_n}(x', k) = \frac{\partial w_k}{\partial x_n}(x', k) < 0, \quad \text{if } (x', k) \in \partial\Omega_k. \tag{2.35}$$

Thus we have proved that when $k \in (k_0, k_0 + \varepsilon)$, any positive solution $u$ of (2.1) satisfies $w_k(x) > 0$ and $\partial u/\partial x_n(x) > 0$ for $x \in \Omega_k$.

Next we define $k_2 = \sup\{k > k_0 : w_k(x) > 0, \partial u/\partial x_n(x) > 0, \text{ for } x \in \Omega_a, a \in (k_0, k)\}$. Then at $k = k_2$, one of the following happens:

(1) $w_{k_2}(x) \geq 0$ for $x \in \overline{\Omega_{k_2}}$, and $w_{k_2}(x) = 0$ at some $x \in \Omega_k$;
(2) $w_{k_2}(x) > 0$ for $x \in \Omega_{k_2}$, and $\partial u/\partial x_n(x) = 0$ at some $x \in T_{k_2} \bigcap \Omega$;
(3) There exists $x \in \partial\Omega \bigcap \Sigma_{k_2}$ such that the reflection $\overline{x}$ of $x$ about $T_{k_2}$ is also on the boundary; or
(4) $T_{k_2}$ is orthogonal to $\partial\Omega$.

The first and second case cannot happen from Theorem 2.5. Hence at $k = k_2$, either case 3 or 4 occurs. Note that either case only depends on the geometry of $D$, but not $\lambda$. Hence for any $\lambda > 0$, we can take any $k_1 \in (k_0, k_2)$ such that $w_{k_1}(x) > 0, \partial u/\partial x_n(x) > 0,$ for $x \in D_{k_1}$. □

Notice that we do not require smoothness of the domain for the moving plane procedure thanks to the maximum principle for the narrow domain. In Theorem 2.8, we show that, for $x \in \Omega_{k_1}$, $u$ is monotonic increasing along the inward normal direction $\tau_0 = (0, 0, \cdots, 1)$ at a "convex point" $x_0 \in \partial D$. If the domain $\Omega$ is convex, then the moving plane procedure can be performed at every boundary point. Hence a positive solution $u$ of (2.1) is increasing along the inward normal direction at least for some distance. This is useful in obtaining the *a priori* estimates of solutions, see de Figueiredo, Lions and Nussbaum [DLN]. In general partial convexity is sufficient for symmetry results. We recall that a set $\Omega$ is *Steiner symmetric* with respect to a hyperplane $T$ if for any $x \in \Omega$, the line segment connecting $x$ and the reflected point $x^*$ with respect to $T$ is contained in $\Omega$.

**Corollary 2.2.** *Let $\Omega$ be a bounded domain in $\mathbf{R}^n$ which is Steiner symmetric with respect to the hyperplane $x_n = 0$. Suppose that $u$ is a positive solution of* (2.1) *in $C^2(\Omega) \cap C^0(\overline{\Omega})$, and $f$ is a locally Lipschitz continuous function on $\mathbf{R}$, then $u$ is symmetric with respect to $x_n$ and $\partial u(x)/\partial x_n < 0$ for $x \in \Omega$ and $x_n > 0$.*

A ball domain is symmetric with respect to any hyperplane passing through the center, hence

**Corollary 2.3.** *Let* $B^n = \{x \in \mathbf{R}^n : |x| \leq 1\}$, *and* $n \geq 1$. *Assume that* $f$ *is a locally Lipschitz continuous function on* $\mathbf{R}$, *then any positive solution of*

$$\begin{cases} \Delta u + \lambda f(u) = 0, & \text{in} \quad B^n, \\ u = 0, & \text{on} \quad \partial B^n, \end{cases} \tag{2.36}$$

*is radially symmetric,* $u'(r) < 0$ *for* $r \in (0,1)$, *and satisfy*

$$\begin{cases} u'' + \dfrac{n-1}{r} u' + \lambda f(u) = 0, & r \in (0,1), \\ u'(0) = u(1) = 0. \end{cases} \tag{2.37}$$

Note that the assumption $u(x) > 0$ for $x \in B^n$ cannot be replaced by $u(x) \geq 0$ since $u(x) = 1 + \cos(x)$ is a solution of $u'' + u - 1 = 0$ in $(-3\pi, 3\pi)$, $u(\pm 3\pi) = 0$, but $u(\pm\pi) = 0$ and $u$ is not monotone in $(0,3)$. Recall that in Theorem 2.7, the strong maximum principle implies that $u > 0$ if given $u \geq 0$ and $f(0) \geq 0$, and Castro and Shivaji [CS3] proved that Corollary 2.3 still holds under $u \geq 0$ and $f(0) < 0$ for $B^n$ with $n \geq 2$. The sign-changing solutions of (2.36) are not necessarily radially symmetric. Indeed even for linear equation $\Delta u + \lambda u = 0$, there are non-radial solutions when $\lambda = \lambda_2$.

The symmetry property in general does not hold for even positive solutions of $\Delta u + f(x,u) = 0$. But for the linearized equation of (2.1):

$$\begin{cases} \Delta \psi + \lambda f'(u)\psi = -\mu\psi, & \text{in} \quad \Omega, \\ \psi = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{2.38}$$

we have the following result with a different way of application of maximum principle:

**Theorem 2.9.** *Let* $\Omega$ *be a bounded domain in* $\mathbf{R}^n$ *which is Steiner symmetric with respect to the hyperlane* $x_n = 0$. *Suppose that* $u$ *is a positive solution of* (2.1) *in* $C^2(\Omega) \cap C^0(\overline{\Omega})$, $f \in C^1(\mathbf{R}^+)$, *and* $v$ *is a solution of* (2.38). *then*

*(1) If* $\mu = 0$ *and* $f(0) \geq 0$, *then* $v$ *is symmetric with respect to* $x_n = 0$;
*(2) If* $\mu < 0$, *then* $v$ *is symmetric with respect to* $x_n = 0$.

***Proof.*** We first prove part (1). Define $\Omega_x^- = \{x = (x', x_n) \in \Omega : x_n < 0\}$. From Corollary 2.2, $\varphi(x) \equiv \partial u(x)/\partial x_n$ is anti-symmetric with respect to $x_n = 0$. Moreover $\varphi|_{\partial\Omega} \neq 0$ unless $x_n = 0$ from Hopf's lemma since

$f(0) \geq 0$. Thus $\varphi(x) > 0$ for $x \in \Omega_x^-$, $\varphi(x) \not\equiv 0$ for $x \in \partial\Omega_x^-$, and $L\varphi = 0$ where $Lw = \Delta w + \lambda f'(u)w$. From Proposition 2.5, the maximum principle holds for $L$ in $\Omega_x^-$. Let $w(x) = v(x', x_n) - v(x', -x_n)$. Then $w(x) = 0$ when $x \in \partial\Omega_x^-$, and $Lw = 0$ in $\Omega_x^-$ since $u(x', -x_n) = u(x', x_n)$. Therefore $w \equiv 0$ from the maximum principle, and $v$ is symmetric with respect to $x_n = 0$.

For part (2), suppose that $v$ is not symmetric about $x_n = 0$. Then $\widetilde{v}(x) = v(x', -x_n)$ is also an eigenfunction of $L$ with eigenvalue $\mu < 0$. In particular, $v_1 = v + \widetilde{v}$ and $v_2 = v - \widetilde{v}$ are both eigenfunctions of $L$ with eigenvalue $\mu$. Notice that $v_2$ satisfies $v_2 = 0$ on $\partial\Omega_x^-$, then $v_2$ is also an eigenfunction of $L$ on $\Omega_x^-$ with zero boundary condition and eigenvalue $\mu < 0$. But $\varphi(x) > 0$ for $x \in \Omega_x^-$ and $L\varphi = 0$ from the last paragraph. From Proposition 2.5, $\lambda_1(-\lambda f'(u), \Omega_x^-) \geq 0$, that is a contradiction. Therefore $v$ is symmetric about $x_1 = 0$. □

**Corollary 2.4.** *Suppose that $u$ is a positive solution of* (2.36) *(thus radially symmetric), and $v$ is a solution of* (2.38) *corresponding to $u$. Then $v$ is also radially symmetric if $\mu < 0$, $\mu = 0$ and $f(0) \geq 0$, or $\mu = 0$, $f(0) < 0$ but $\nabla u(x) \neq 0$ for $x \in \partial B^n$.*

***Proof.*** We only need to prove the last case. In the proof of Theorem 2.4, we need $\varphi|_{\partial B^n} \neq 0$. Since $u$ is radial, then $\varphi = \partial u/\partial x_i \neq 0$ unless $x_i = 0$. Then the proof follows as that in Theorem 2.4. □

Note that the eigenfunction $v$ in Corollary 2.4 is not necessarily positive nor decreasing along the radial direction if $\mu$ is not the principal eigenvalue. Also the result may not hold when $\mu = 0$ and $f(0) < 0$. Again consider the example $u'' + u - 1 = 0$, $x \in (-\pi, \pi)$, and $u(\pm\pi) = 0$. The positive solution $u(x) = 1 + \cos x$ does not satisfy the Hopf boundary lemma, and $u'(x) = -\sin x$ is a solution of the linearized equation $\psi'' + \psi = 0$, $x \in (-\pi, \pi)$, and $\psi(\pm\pi) = 0$. But $u'$ is not symmetric. This example has a connection with anti-maximum principle (Theorem 3.12), see Shi [S8].

The result in Corollary 2.4 was first proved by Lin and Ni [LN] using other methods. Theorem 2.9 is from Damascelli, Grossi, and Pacella [DGP], in which they attributed this proof to Nirenberg. Symmetry can be the result of many different qualitative properties of nonlinear problems, see Kawohl [Ka1; Ka2] for interesting summaries. For example, even if $u$ is a unstable radially symmetric solution of (2.36) so that $\mu_1(u) > 0$, the positive eigenfunction $\phi_1$ is always radially symmetric as the minimizer of the Rayleigh quotient.

## 2.5 Monotone methods

Maximum principles can be used to construct solutions of (2.1). It is called the monotone method. Here we describe a weak version of the method. Assume that $p > n$. A function $\phi \in W^{2,p}(\Omega)$ is called a *subsolution* of (2.1) if $\Delta\phi + \lambda f(\phi) \geq 0$ for almost all $x \in \Omega$ and $\phi \leq 0$ for almost all $x \in \partial\Omega$; and A function $\psi \in W^{2,p}(\Omega)$ is called a *supersolution* of (2.1) if $\Delta\psi + \lambda f(\psi) \leq 0$ for almost all $x \in \Omega$ and $\psi \geq 0$ for almost all $x \in \partial\Omega$. The following existence result is based on the maximum principle for weakly differentiable functions:

**Theorem 2.10.** *Suppose that $f$ is locally Lipschitz continuous, $\partial\Omega$ is $C^{2,\alpha}$ smooth, and there exist a subsolution $\phi$ and a supersolution $\psi$ to* (2.1) *such that $\psi(x) \geq \phi(x)$ for all $x \in \overline{\Omega}$. If there exists $K > 0$ such that*

$$f(u_1) - f(u_2) \geq -K(u_1 - u_2), \tag{2.39}$$

*for $u_1, u_2 \in [m, M]$ and $u_1 > u_2$, where $m = \min_{x\in\overline{\Omega}} \phi(x)$, and $M = \max_{x\in\overline{\Omega}} \psi(x)$, then there exist classical solutions $\overline{u}$ and $\underline{u}$ of* (2.1)*, such that $\psi(x) \geq \overline{u}(x) \geq \underline{u}(x) \geq \phi(x)$ for all $x \in \overline{\Omega}$. Moreover $\overline{u}$ is the maximum solution of* (2.1) *and $\underline{u}$ is the minimum solution, in the sense if there is a solution $w$ of* (2.1) *satisfying $\psi \geq w \geq \phi$, then $\overline{u} \geq w \geq \underline{u}$.*

***Proof.*** We redefine $f$ to be

$$\widetilde{f}(u) = \begin{cases} f(m), & u \leq m, \\ f(u), & u \in [m, M], \\ f(M), & u \geq M. \end{cases}$$

In the following we still use $f$ instead of $\widetilde{f}$ for convenience. Then (2.39) holds for all $u_1, u_2 \in \mathbf{R}$. Define an iteration sequence: $u_0 = \psi$

$$-\Delta u_{n+1} + \lambda K u_{n+1} = \lambda f(u_n) + \lambda K u_n, \quad u_{n+1} \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega). \tag{2.40}$$

The existence and uniqueness of the solution comes from $L^p$ theory of linear elliptic equations ([GT] Theorem 9.15). We also define another sequence $\{v_n\}$ using (2.40) with $v_0 = \phi$. Define $L_K = -\Delta + \lambda K$, and $g(u) = f(u) + Ku$. Since $g$ is increasing, then the maximum principle ([GT] Theorem 8.1), sub/suprsolution property and induction yield the relations:

$$u_0 \geq u_1 \geq u_2 \geq \cdots v_2 \geq v_1 \geq v_0.$$

Since $\{u_n\}$ is bounded in $W^{2,p}(\Omega)$ with $p > n$, from Sobolev embedding, it is also bounded in $C^\alpha(\overline{\Omega})$. From Hölder estimates and (2.40), the sequence

$\{u_n\}$ is bounded in $C_0^{2,\alpha}(\overline{\Omega})$. Hence a subsequence of $\{u_n\}$ converges to a limit $\overline{u}$ in $C_0^2(\overline{\Omega})$. From the equation (2.40), $\overline{u}$ is indeed a classical solution. Similarly we can obtain the minimal solution $\underline{u}$. □

Theorem 2.10 is based on Berestycki and Lions [BL1], see also Amann and Crandall [ACr], Dancer and Sweers [DSw] for related results. Classical monotone/comparison methods require more smooth sub/supersolutions, see Amann [A1], Cohen and Keller [CK], and Sattinger [Sa]. Existence of solution between sub and supersolutions can also be proved via variational methods, see Chang [Ch1] and Struwe [St]. We notice that although our initial sub/supersolution here may not be smooth, each term in the iteration sequence is smooth from Hölder estimates. The $W^{2,p}$ weak sub/superslutions are more convenient for gluing two sub/supersolutions together to obtain new sub/supersolutions:

**Proposition 2.7.** *Suppose that $\psi_1, \psi_2 \in W^{2,p}(\Omega)$ for $p > n$, $\Omega_1 = \{x \in \Omega : \psi_1 > \psi_2\}$ and $\Omega_2 = \{x \in \Omega : \psi_2 > \psi_1\}$ are smooth subdomains of $\Omega$.*

*(1) If $\psi_1, \psi_2$ are two subsolutions, then $\max\{\psi_1, \psi_2\}$ is also a subsolution.*
*(2) If $\psi_1, \psi_2$ are two supersolutions, then $\min\{\psi_1, \psi_2\}$ is also a supersolution.*

***Proof.*** One can show that a function $\psi$ is a subsolution if and only if for any $\xi \in C_0^\infty(\Omega)$ and $\xi \geq 0$,

$$\int_\Omega [\nabla\psi \cdot \nabla\xi - \lambda f(\psi)\xi]dx \leq 0. \tag{2.41}$$

Let $\psi = \max\{\psi_1, \psi_2\}$. For any $\xi \in C_0^\infty(\Omega)$ and $\xi \geq 0$,

$$\begin{aligned}
&\int_\Omega [\nabla\psi \cdot \nabla\xi - \lambda f(\psi)\xi]dx \\
&= \int_{\Omega_1} [\nabla\psi_1 \cdot \nabla\xi - \lambda f(\psi_1)\xi]dx + \int_{\Omega_2} [\nabla\psi_2 \cdot \nabla\xi - \lambda f(\psi_2)\xi]dx \\
&= -\int_{\Omega_1} [\Delta\psi_1 + \lambda f(\psi_1)]\xi dx - \int_{\Omega_2} [\Delta\psi_2 + \lambda f(\psi_2)]\xi dx \\
&\quad + \int_\Gamma \left(\frac{\partial\psi_1}{\partial\nu} - \frac{\partial\psi_2}{\partial\nu}\right)\xi ds.
\end{aligned}$$

Here $\Gamma = \partial\Omega_1 \backslash \partial\Omega$, and $\nu$ is outer normal direction pointing from $\Omega_1$ to $\Omega_2$. From the definition of $\psi$, $\partial(\psi_1 - \psi_2)/\partial\nu \leq 0$ for $x \in \Gamma$. On the other hand, $\Delta\psi_i + \lambda f(\psi_i) \geq 0$ for $x \in \Omega_i$ and $i = 1, 2$. Hence $\int_\Omega [\nabla\psi \cdot \nabla\xi - \lambda f(\psi)\xi]dx \leq 0$ and $\phi$ is a subsolution on $\Omega$. □

Proposition 2.7 here is from Berestycki and Lions [BL1], and it is still true if $\phi_i$ is only a sub/supersolution on $\Omega_i$ (local sub/supersolition). The solution obtained from monotone method is usually stable, or at least neutrally stable. We state the following stability result first proved by Sattinger ([Sa] page 992):

**Theorem 2.11.** *Suppose the conditions in Theorem 2.10 are satisfied, and* $(\underline{u}, \overline{u})$ *is the pair of minimum and maximum solutions of* (2.1). *Then* $\mu_1(\underline{u}) \geq 0$ *and* $\mu_1(\overline{u}) \geq 0$, *where* $\mu_1(\cdot)$ *is the principal eigenvalue of* (2.16).

***Proof.*** We only prove that $\mu_1(\overline{u}) \geq 0$ and the other is similar. Recall that $\overline{u}$ is the limit of $\{u_n\}$ defined by (2.40), and one can see each $u_n$ in the sequence is still a supersolution. Hence we have for $n \geq 1$,

$$\begin{cases} \Delta(u_n - \overline{u}) + \lambda[f(u_n) - f(\overline{u})] \leq 0, & \text{in} \quad \Omega, \\ u_n - \overline{u} = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{2.42}$$

and $u_n - \overline{u} \geq 0$ for all $x \in \Omega$. From the intermediate-value theorem, there exists $\xi_n(x)$ such that $u_n(x) \geq \xi_n(x) \geq \overline{u}(x)$ and $f(u_n(x)) - f(\overline{u}(x)) = f'(\xi_n(x))[u_n(x) - \overline{u}(x)]$ for $x \in \Omega$. Define $\psi_n = u_n/||u_n||_{W_0^{1,2}(\Omega)}$. Then $\psi_n$ satisfies

$$\begin{cases} \Delta\psi_n + \lambda f'(\xi_n)\psi_n \leq 0, & \text{in} \quad \Omega, \\ \psi_n = 0, & \text{on} \quad \partial\Omega. \end{cases} \tag{2.43}$$

Since $\{\psi_n\}$ is bounded in $W_0^{1,2}(\Omega)$, it follows that there exists a subsequence of $\{\psi_n\}$ (still denote by $\{\psi_n\}$) such that $\psi_n$ converges to some $\psi$ weakly in $W_0^{1,2}(\Omega)$, strongly in $L^2(\Omega)$, and almost everywhere for $x \in \Omega$ as $n \to \infty$. In particular, $\psi(x) \geq 0$ almost everywhere in $\Omega$. On the other hand $\xi_n \to \overline{u}$ uniformly for $x \in \overline{\Omega}$ as $n \to \infty$, hence (2.43) and the weak convergence imply that

$$\mu_1(\overline{u}) \int_\Omega \phi_1 \psi dx = \int_\Omega (\nabla\phi_1 \cdot \nabla\psi - \lambda f'(\overline{u})\phi_1\psi)\, dx \geq 0, \tag{2.44}$$

where $\phi_1 > 0$ is the principal eigenfunction corresponding to $\mu_1(\overline{u})$. This shows that $\mu_1(\overline{u}) \geq 0$. □

To conclude this section, we introduce a "Sweeping principle" about a family of sub(super)solutions. Roughly speaking, the sweeping principle states that if one function in a family of continuous subsolutions is smaller than a given supersolution, then each function in the subsolution family is also smaller than that supersolution. The first reference of this idea is McNabb [Mc], and it becomes well-known since Serrin [Se2]. Several versions of the principles can be found in Dancer [D4], Clément and Sweers [CSw], and Jang [J].

**Theorem 2.12.** *Suppose that* $w_t(x) \in W^{2,p}(\Omega)$, $(p > n)$, *are a family of subsolutions of* (2.1), $t \in [0,1]$, *such that the mapping* $(t, x) \mapsto w_t(x)$ *is continuous in* $[0,1] \times \overline{\Omega}$. *Suppose that* $u$ *is a supersolution of* (2.1), *no any* $w_t$ *is a solution of* (2.1) *for* $t \in [0,1]$, *and there exists* $t_0 \in [0,1]$ *such that* $u(x) \geq w_{t_0}(x)$ *for* $x \in \overline{\Omega}$. *Then for any* $t \in [0,1]$, $u(x) \geq w_t(x)$ *for* $x \in \overline{\Omega}$.

***Proof.*** Let $T = \{t \in [0,1] : u \geq w_t\}$. Then $T$ is not empty. Since $w_t$ is continuous, then $T$ is a closed subset of $[0,1]$. We show that $T$ is also an open subset of $[0,1]$. Suppose that $t \in T$, and we define $v = u - w_t$. Then $v$ satisfies $\Delta v + \lambda c(x) v \leq 0$ in $\Omega$, and $v \geq 0$ on $\partial\Omega$, where

$$c(x) = \begin{cases} \dfrac{f(u(x)) - f(w_t(x))}{u(x) - w_t(x)}, & u(x) > w_t(x), \\ f'(u(x)), & u(x) = w_t(x). \end{cases}$$

Then $v(x) \geq 0$ in $\overline{\Omega}$. From Theorem 2.5, $v(x) > 0$ in $\Omega$ or $v(x) \equiv 0$ in $\Omega$. But $w_t$ is not a solution of (2.1), $u(x)$ cannot be identical to $w_t$ and thus $v > 0$ in $\Omega$. Again from the continuity of $w_t$, for some small $\varepsilon > 0$, $w_s \leq u$ for $s \in (t - \varepsilon, t + \varepsilon) \cap [0,1]$. Hence $T$ is also open. Therefore $T = [0,1]$ and the theorem is proved. □

## 2.6 Bounds of solution set

Here we describe some general properties of the solution set of

$$\begin{cases} \Delta u + \lambda f(u) = 0, & \text{in} \quad \Omega, \\ u > 0, & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{2.45}$$

where $\Omega$ is a bounded smooth domain. We show that some properties of the nonlinear function $f(u)$ implies the nonexistence of solutions to (2.45), which also implies some necessary conditions on $f(u)$ for the existence of solutions to (2.45).

When $f$ is too large, there is no solution of (2.45) for large $\lambda$, and when $f$ is too small, there is no solution of (2.45) for small $\lambda$. More precisely, we have

**Proposition 2.8.** *If there exists* $a > 0$ *such that* $f(u) \geq au$ *for* $u \geq 0$, *then for* $\Lambda = \lambda_1/a$, *if* $\lambda > \Lambda$, (2.45) *has no solution. Similarly, if there exists*

*$a > 0$ such that $f(u) \le au$ for $u \ge 0$, then for $\Lambda = \lambda_1/a$, if $\lambda < \Lambda$, (2.45) has no solution.*

***Proof.*** Suppose that $f(u) \ge au$ for $u \ge 0$. Let $\phi_1$ be the positive eigenfunction associated with principal eigenvalue $\lambda_1$. Multiplying (2.45) by $\phi_1$, and integrating over $\Omega$, we have

$$\lambda_1 \int_\Omega \phi_1 \cdot u dx = \lambda \int_\Omega \phi_1 \cdot f(u) dx \ge \lambda a \int_\Omega \phi_1 \cdot u dx. \tag{2.46}$$

Thus if (2.45) has a solution, then $\lambda \le \lambda_1/a$. Suppose that $f(u) \le au$ for $u \ge 0$. If $u$ is a solution of (2.45), then from Poincaré inequality (2.13),

$$\lambda_1 \int_\Omega u^2 dx \le \int_\Omega |\nabla u|^2 dx = \lambda \int_\Omega u f(u) dx \le \lambda a \int_\Omega u^2 dx. \tag{2.47}$$

Thus if (2.45) has a solution, then $\lambda \ge \lambda_1/a$. □

Bounds of the maximum values of solutions to (2.45) can also be derived from the properties of the nonlinear function $f(u)$. A simple bound is given by the largest $u$ such that $f(u) > 0$. Let

$$M = \sup\{u \ge 0 : f(u) > 0\}. \tag{2.48}$$

Then (2.45) has no solution with $\max_{x\in\overline{\Omega}} u(x) \ge M$ from the maximum principle (see Theorem 2.7). For continuous $f$, it is necessary that $f(M) = 0$. For solution satisfying $0 < u < M$, another restriction is as follows

**Proposition 2.9.** *Suppose that $f \in C^1(\mathbf{R}^+)$. If (2.45) has a solution $u(x)$ and $u_M = \max_{x\in\overline{\Omega}} u(x) < M$, then*

$$f(u_M) > 0, \quad \text{and } F(u_M) = \int_0^{u_M} f(t)dt \ge 0. \tag{2.49}$$

***Proof.*** $f(u_M) > 0$ has been proved in Theorem 2.7. We prove the second part of (2.49). We first assume $\Omega = B_R$, a ball with radius $R > 0$. From the Gidas-Ni-Nirenberg theorem [GNN1] (Corollary 2.3), a positive solution in $B_R$ is radially symmetric, then the radial symmetric solution satisfies an ordinary differential equation:

$$\begin{cases} u'' + \dfrac{n-1}{r} u' + \lambda f(u) = 0, & r \in (0, R), \\ u'(0) = 0 = u(R), \ u(r) > 0, & u'(r) < 0, \ r \in (0, R). \end{cases} \tag{2.50}$$

Multiplying (2.50) by $u'$ and integrating it from 0 to $R$, we obtain

$$\frac{1}{2}[u'(R)]^2 + (n-1) \int_0^R \frac{[u'(r)]^2}{r} dr - \lambda F(u(0)) = 0.$$

Hence $F(u_M) = F(u(0)) \geq 0$. Indeed if $n > 1$, $F(u_M) > 0$ for ball domains.

For a general bounded domain $\Omega$, first we consider the case of $f(0) \geq 0$ and assume that (2.45) has a solution $u$ in $\Omega$ such that $F(u_M) < 0$. We modify $f$ in $(u_M, M]$ so that $f(u) < 0$ for $u > M$ and $F(u) < 0$ for $u \geq u_M$. This modification does not affect $u(x)$. There is a ball $B_R \supset \Omega$, and we define $v(x) = u(x)$ for $x \in \overline{\Omega}$ and $v(x) = 0$ for $x \in B_R \backslash \overline{\Omega}$. Since $f(0) \geq 0$, then $v$ is a subsolution of (2.45) with $\Omega = B_R$; on the other hand, $w(x) = M$ is a supersolution of (2.45) satisfying $w \geq v$. Therefore from Theorem 2.10, (2.45) with $\Omega = B_R$ has a solution $v_1$ satisfying $w \geq v_1 \geq v$. But for $v_M = \max_{x \in \overline{\Omega}} v_1(x) \geq u_M$, $F(v_M) < 0$ from our modification of $f$, which contradicts with the proof for the ball case.

For the case $f(0) < 0$, $v(x)$ is no longer a subsolution. We assume that (2.45) has a solution $u$ in $\Omega$ such that $F(u_M) < 0$. We define $\tilde{f} \in C^1(\mathbf{R}^+)$ so that $\tilde{f} \geq f$ for $u \geq 0$, $\tilde{f}(0) = 0$ and $\tilde{F}(u) = \int_0^u \tilde{f}(t)dt < 0$ for $u \geq u_M$. Then $0 = \Delta u + \lambda f(u) \leq \Delta u + \lambda \tilde{f}(u)$, hence $u$ is a subsolution of

$$\Delta v + \lambda \tilde{f}(v) = 0, \quad \text{in } \Omega, \quad v = 0, \quad \text{on } \partial\Omega, \tag{2.51}$$

and $v = M$ is a supersolution. Therefore from Theorem 2.10, (2.51) has a solution $v_1$ such that $M \geq v_1 \geq u$. But (2.51) has no positive solution from our proof for the $f(0) \geq 0$ case. □

The result in Proposition 2.9 was proved in Dancer and Schmitt [DSc], and Clément and Sweers [CSw]. Note that since $f(u_M) > 0$, one can define $M_1 = \min\{u > u_M : f(u) = 0\} \leq M$. Then (2.49) implies $F(M_1) > 0$. A stronger result that $F(u_M) > 0$ was proved in Sweers [Sw] for bounded smooth domain (or even some unbounded domains) with $n > 1$. When $n = 1$, $u(x) = 1 + \cos x$ is a positive solution of $u'' + u - 1 = 0$, $x \in (-\pi, \pi)$, and $u(-\pi) = u(\pi) = 0$, but $F(u_M) = F(u(0)) = 0$.

# Chapter 3

# Abstract Bifurcation Theory

As shown in Chapter 2, the partial differential equation (2.1) can be formulated to a functional equation:

$$F(\lambda, u) = 0, \tag{3.1}$$

where $F : \mathbf{R} \times X \to Y$, $X$ and $Y$ are Banach spaces, and $\lambda$ is a real value parameter. When $F$ is sufficiently smooth, we can often use the bifurcation theory based on differential calculus in Banach spaces. In this chapter, we prove several local bifurcation theorems based on implicit function theorem in Banach spaces. In particular we obtain infinite dimension version of basic bifurcation phenomena like saddle-node, transcritical and pitchfork described in Section 1.2. In this chapter, we always assume $X$ and $Y$ are Banach spaces.

## 3.1 Banach spaces and implicit function theorem

A *metric space* is a pair $(M, d)$ where $M$ is a set and $d : M \times M \to \mathbf{R}$ is a metric which satisfies for any $x, y, z \in M$: (i) $d(x, y) \geq 0$; (ii) $d(x, y) = 0$ if and only if $x = y$; (iii) $d(x, y) = d(y, x)$; and (iv) (triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$. A metric space $(M, d)$ is said to be *complete* if any Cauchy sequence $\{x_n\} \subset M$ has a limit in $M$.

A *normed vector space* (over real numbers) is a pair $(V, ||\cdot||)$ where $V$ is a linear vector space over real numbers and the norm $||\cdot|| : V \to \mathbf{R}$ is a function which satisfies for any $a \in \mathbf{R}$ and $x, y \in V$: (i) $||ax|| = |a| \cdot ||x||$; (ii) $||x|| \geq 0$, and $||x|| = 0$ if and only if $x = 0$ (the zero vector); and (iii) (triangle inequality) $||x + y|| \leq ||x|| + ||y||$. A normed vector space is a metric space with the metric $d(x, y) = ||x - y||$. A complete normed vector space is called a *Banach space* named after Stefan Banach (1892–1945).

An important tool of nonlinear analysis is the contraction mapping principle (or Banach fixed point theorem):

**Theorem 3.1. (Contraction mapping principle)** *Let $(M, d)$ be a nonempty completed metric space. Assume that $T : M \to M$ is a contraction mapping, that is, there exists $k \in (0, 1)$ such that for any $x, y \in M$,*

$$d(Tx, Ty) \leq k \cdot d(x, y). \tag{3.2}$$

*Then the mapping $T$ has a unique fixed point $x_*$ in $M$ such that $Tx_* = x_*$.*

***Proof.*** Choose any $x_0 \in M$, and define $x_n = Tx_{n-1}$ for $n \geq 1$. From (3.2) and the principle of mathematical induction, one obtain that $d(x_{n+1}, x_n) \leq k^n d(x_1, x_0)$. This in turn implies that $\{x_n\}$ is a Cauchy sequence in $M$ since $k < 1$, and thus $\{x_n\}$ has a limit $x_* \in M$ from the completeness of $(M, d)$. Note that $0 \leq d(x_{n+1}, Tx_*) = d(Tx_n, Tx_*) \leq kd(x_n, x_*)$. Then $d(x_{n+1}, Tx_*) \to 0$ as $n \to \infty$ since $d(x_n, x_*) \to 0$ as $n \to \infty$. Since the limit of $\{x_n\}$ is unique, hence $Tx_* = x_*$ and $x_*$ is a fixed point of $T$ in $M$. If there is another fixed point $y_* \in M$ of $T$, then $0 \leq d(x_*, y_*) = d(Tx_*, Ty_*) \leq k \cdot d(x_*, y_*)$. But $k < 1$ so $d(x_*, y_*) = 0$, which implies $x_* = y_*$ from the definition of metric. Therefore $T$ has a unique fixed point $x_*$, which is the limit of any iterated sequence $\{x_n\}$ defined as $x_n = Tx_{n-1}$ and any initial point $x_0 \in M$. □

The foundation of analytical bifurcation theory is the following implicit function theorem, which is a consequence of contraction mapping principle.

**Theorem 3.2. (Implicit function theorem)** *Let $X, Y$ and $Z$ be Banach spaces, and let $U \subset X \times Y$ be a neighborhood of $(\lambda_0, u_0)$. Let $F : U \to Z$ be a continuously differentiable mapping. Suppose that $F(\lambda_0, u_0) = 0$ and $F_u(\lambda_0, u_0)$ is an isomorphism, i.e. $F_u(\lambda_0, u_0)$ is one-to-one and onto, and $F_u^{-1}(\lambda_0, u_0) : Z \to Y$ is a linear bounded operator. Then there exists a neighborhood $A$ of $\lambda_0$ in $X$, and a neighborhood $B$ of $u_0$ in $Y$, such that for any $\lambda \in A$, there exists a unique $u(\lambda) \in B$ satisfying $F(\lambda, u(\lambda)) = 0$. Moreover $u(\cdot) : A \to B$ is continuously differentiable, and $u'(\lambda_0) : X \to Y$ is defined as $u'(\lambda_0)[\psi] = -[F_u(\lambda_0, u_0)]^{-1} \circ F_\lambda(\lambda_0, u_0)[\psi]$.*

***Proof.*** To solve $u$ in the equation $F(\lambda, u) = 0$, we look for the solution $(\mu, v)$ of $F(\lambda_0 + \mu, u_0 + v) = 0$. We notice that

$$F(\lambda_0 + \mu, u_0 + v) = F(\lambda_0, u_0) + F_u(\lambda_0, u_0)v + R(\mu, v),$$

where $R(\mu, v) = F(\lambda_0 + \mu, u_0 + v) - F_u(\lambda_0, u_0)v$ is the remainder term. Since $F_u(\lambda_0, u_0)$ is invertible, then $F(\lambda, u) = 0$ is equivalent to

$$v + [F_u(\lambda_0, u_0)]^{-1} R(\mu, v) = 0.$$

Define $G(\mu, v) = -[F_u(\lambda_0, u_0)]^{-1}R(\mu, v) = [F_u(\lambda_0, u_0)]^{-1}(F_u(\lambda_0, u_0)v - F(\lambda_0 + \mu, u_0 + v))$. Then solving $F(\lambda, u) = 0$ is equivalent to finding the fixed points of $G(\mu, v)$. We show that for $\mu$ close to 0, $G(\mu, \cdot)$ is a contraction mapping in a neighborhood of $v = 0$. In the following we denote $[F_u(\lambda_0, u_0)]^{-1}$ by $H$. Notice that $G(\mu, v) = v - HF(\lambda_0 + \mu, u_0 + v)$, then

$$\begin{aligned} &||G(\mu, v_1) - G(\mu, v_2)|| \\ = \ & ||v_1 - v_2 - H(F(\lambda_0 + \mu, u_0 + v_1) - F(\lambda_0 + \mu, u_0 + v_2))|| \\ = \ & \left|\left|(v_1 - v_2) - H\int_0^1 F_u(\lambda_0 + \mu, u_0 + tv_1 + (1-t)v_2)dt(v_1 - v_2)\right|\right|. \end{aligned} \tag{3.3}$$

Since $F_u(\lambda, u)$ is continuous near $(\lambda_0, u_0)$, then there exists a ball $B_X \subset X$ centered at $\mu = 0$ and a ball $B_Y \subset Y$ centered at $v = 0$ such that when $(\mu, v) \in B_X \times B_Y$,

$$||I - HF_u(\lambda_0 + \mu, u_0 + v)|| \le \frac{1}{2}. \tag{3.4}$$

In (3.4), the norm $|| \cdot ||$ is the operator norm in the Banach space $L(Y, Y)$, which consists all the linear bounded operators from $Y$ to $Y$. From (3.3) and (3.4), we find that $||G(\mu, v_1) - G(\mu, v_2)|| \le ||v_1 - v_2||/2$ for $\mu \in B_X$ and $v_1, v_2 \in B_Y$.

Next we show that for $\mu$ in a neighborhood of 0 and $v \in B_Y$, then $G(\mu, v) \in B_Y$. In fact, for $\mu \in B_X$, and $v \in B_Y$ with $B_X, B_Y$ defined above, we assume that $B_Y = \{y \in Y : ||y|| \le \delta\}$, then

$$\begin{aligned} ||G(\mu, v)|| &\le ||G(\mu, 0)|| + ||G(\mu, 0) - G(\mu, v)|| \\ &\le ||HF(\lambda_0 + \mu, u_0)|| + \frac{1}{2}||v||. \end{aligned} \tag{3.5}$$

From the continuity of $HF$ and that $HF(\lambda_0, u_0) = 0$, we can find a ball $B'_X \subset B_X$ with $0 \in B'_X$ so that when $\mu \in B'_X$, $||HF(\lambda_0 + \mu, u_0) - HF(\lambda_0, u_0)|| \le (1/4)\delta$. Hence $||G(\mu, v)|| \le (3/4)\delta$ and $G(\mu, v) \in B_Y$. Therefore when $\mu \in B'_X$, $G(\mu, \cdot) : B_Y \to B_Y$ is a contraction mapping, then from the contraction mapping principle (Theorem 3.1), there exists a unique $v(\mu) \in B_Y$ such that $G(\mu, v(\mu)) = v(\mu)$. Hence the existence and uniqueness of the solution to $F(\lambda, u) = 0$ for $\lambda \in A$ and $u \in B$ in the theorem follows by letting $A = \{\lambda_0 + \mu : \mu \in B'_X\}$ and $B = \{u_0 + v : v \in B_Y\}$.

To show that $u(\lambda) = u_0 + v(\mu)$ is continuous, we see that for $\mu_1, \mu_2 \in B'_X$,

$$\begin{aligned} &||v(\mu_1) - v(\mu_2)|| = ||G(\mu_1, v(\mu_1)) - G(\mu_2, v(\mu_2))|| \\ \le \ & ||G(\mu_1, v(\mu_1)) - G(\mu_1, v(\mu_2))|| + ||G(\mu_1, v(\mu_2)) - G(\mu_2, v(\mu_2))|| \\ \le \ & \frac{1}{2}||v(\mu_1) - v(\mu_2)|| \\ & + ||HF(\lambda_0 + \mu_1, u_0 + v(\mu_2)) - HF(\lambda_0 + \mu_2, u_0 + v(\mu_2))||, \end{aligned}$$

hence

$$\begin{aligned} &||v(\mu_1)-v(\mu_2)|| \\ &\le 2\left|\left|H\int_0^1 F_\mu(\lambda_0+t\mu_1+(1-t)\mu_2,u_0+v(\mu_2))dt(\mu_1-\mu_2)\right|\right|, \end{aligned} \tag{3.6}$$

and the continuity of $u(\lambda)$ follows from the continuity of $F_\mu$. To show the differentiability of $v(\mu)$, we notice that $G$ is continuously differentiable near $(0,0)$ since $F$ is assumed to be $C^1$, $G_\mu(\mu,v)=-H\circ F_\lambda(\lambda_0+\mu,u_0+v)$ and $G_v(\mu,v)=H\circ(F_u(\lambda_0,u_0)-F_u(\lambda_0+\mu,u_0+v))$. Hence for $\mu\in B'_X$ and $\psi\in X$ small, from the differentiability of $G$ and continuity of $F_u$, we have

$$\begin{aligned} &\left|\left|v(\mu+\psi)-v(\mu)+[F_u(\lambda_0,u_0)]^{-1}\circ F_\lambda(\lambda_0,u_0)[\psi]\right|\right| \\ =\ &\left|\left|G(\mu+\psi,v(\mu+\psi))-G(\mu,v(\mu))+H\circ F_\lambda(\lambda_0,u_0)[\psi]\right|\right| \\ =\ &\Bigg|\Bigg|G_\mu(\mu,v(\mu))[\psi]+G_v(\mu,v(\mu))[v(\mu+\psi)-v(\mu)] \\ &+o(||\psi||)+o(||v(\mu+\psi)-v(\mu)||)+H\circ F_\lambda(\lambda_0+\mu,u_0+v(\mu))[\psi]\Bigg|\Bigg| \\ =\ &o(||\psi||)+o(||v(\mu+\psi)-v(\mu)||)=o(||\psi||). \end{aligned}$$

Therefore $v:B'_X\to B_Y$ is differentiable with the Fréchet derivative $v'(\mu)=-[F_u(\lambda_0,u_0)]^{-1}\circ F_\lambda(\lambda_0+\mu,u_0+v(\mu))$, which is continuous in $\mu$ from the continuity of $F_\lambda$ and $v$. This proves that $u(\lambda)$ is $C^1$. □

Note that if we assume that $F$ is continuously differentiable in $u$, and only continuous in $\lambda$, then the result still holds, but $u(\lambda)$ is only continuous. Similarly, if $F$ is of class $C^k$, so is $u(\lambda)$; if $F$ is analytic, so is $u(\lambda)$. The implicit function theorem in Banach spaces can be found in many standard references, and the version here is close to the ones in Ambrosetti and Prodi [APo2], Crandall and Rabinowitz [CR1] and Deimling [De].

An important special case is when $X=\mathbf{R}$, then the implicit function theorem Theorem 3.2 implies that when the linearized operator $F_u(\lambda_0,u_0)$ is non-degenerate, then the set of solutions to $F(\lambda,u)=0$ near $(\lambda_0,u_0)$ is a $C^1$ curve $\{(\lambda,u(\lambda)):\lambda\in(\lambda_0-\varepsilon,\lambda_0+\varepsilon)\}$. We apply it to (2.1):

**Theorem 3.3. (Implicit function theorem)** *Suppose that $f\in C^1(\mathbf{R})$, and $(\lambda_0,u_0)\in\mathbf{R}\times C_0^{2,\alpha}(\overline{\Omega})$ is a solution of (2.1), such that the equation*

$$\begin{cases} \Delta w+\lambda f'(u)w=0, & \text{in} \quad \Omega, \\ w=0, & \text{on} \quad \partial\Omega, \end{cases} \tag{3.7}$$

*has only the trivial solution $w=0$, then there exists $\varepsilon>0$ such that for $\lambda\in(\lambda_0-\varepsilon,\lambda_0+\varepsilon)$, (2.1) has a unique solution $(\lambda,u(\lambda))$ near $(\lambda_0,u_0)$, and $\{(\lambda,u(\lambda)):\lambda\in(\lambda_0-\varepsilon,\lambda_0+\varepsilon)\}$ is a smooth curve.*

***Proof.*** Define $F(\lambda, u)$ as in (2.14). From Proposition 2.2, $F_u(\lambda_0, u_0)$ is a Fredholm operator of index zero. Since (3.7) has only the trivial solution, then $N(F_u(\lambda_0, u_0)) = \emptyset$ and $R(F_u(\lambda_0, u_0)) = Y$. From the open mapping theorem, $F_u(\lambda_0, u_0)$ is an isomorphism. Then the result follows from Theorem 3.2. □

**Example 3.1.** Consider

$$\begin{cases} \Delta u + \lambda e^u = 0, & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega. \end{cases} \tag{3.8}$$

This is the Gelfand's equation which arises from combustion theory, see Section 1.4. $(\lambda, u) = (0, 0)$ is a solution to the equation, and $F_u(0, 0)w = \Delta w$, which is invertible on $X = C_0^{2,\alpha}(\overline{\Omega})$ since $\Delta w = 0$ in $\Omega$ and $w = 0$ on $\Omega$ has only trivial solution. Then near $(0, 0)$, from the implicit function theorem, the solution set is of form $\{(\lambda, u(\lambda)) : |\lambda| < \varepsilon\}$, where $u(\lambda) = \lambda\psi + o(|\lambda|)$ and $\psi$ is the solution of

$$\begin{cases} \Delta \psi + 1 = 0, & \text{in} \quad \Omega, \\ \psi = 0, & \text{on} \quad \partial\Omega. \end{cases} \tag{3.9}$$

In particular, $u(\lambda)$ is positive when $\lambda \in (0, \varepsilon)$ since $\psi$ is positive from the maximum principle (Theorem 2.4), and $u(\lambda)$ is negative when $\lambda \in (-\varepsilon, 0)$.

This example can be stated in a more general form as a bifurcation theorem. Note that there is a unique solution for all $\lambda$ near $\lambda = 0$, but we can still think a branch of positive solutions emerging from $(0, 0)$.

**Theorem 3.4.** *Let $f \in C^1(\mathbf{R}^+)$ and $f(0) > 0$. Then all positive solutions of (2.1) near $(0, 0)$ have the form $\{(\lambda, \lambda w + z(\lambda)) : 0 < \lambda < \delta\}$ and some $\delta > 0$ and smooth $z : [0, \delta) \to C_0^{2,\alpha}(\overline{\Omega})$, where $w$ is the solution of*

$$\begin{cases} \Delta w + f(0) = 0, & \text{in} \quad \Omega, \\ w = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{3.10}$$

*and $z(0) = z'(0) = 0$.*

## 3.2 Bifurcations on $\mathbf{R}^1$

The implicit function theorem provides a tool to describe the solution set of a nonlinear problem in an infinite dimensional space when the linearized operator is invertible. When the linearized operator is not invertible, but with only a kernel of finite dimension and a range space of finite codimension, the analytic bifurcation picture still resembles the ones in finite dimensional case described in Section 1.2. For that purpose, we first establish a result for finite dimensional bifurcation problem, and this result also gives a unified approach to the usual bifurcations of type saddle-node, transcritical and pitchfork, thus it is also of independent interest.

**Theorem 3.5.** *Suppose that $(\lambda_0, y_0) \in \mathbf{R}^2$ and $U$ is a neighborhood of $(\lambda_0, y_0)$. Assume that $f : U \to \mathbf{R}$ is a $C^p$ function for $p \geq 1$, $f(\lambda_0, y_0) = 0$, and there is at most one critical point $(\lambda_0, y_0)$ of $f$ in $U$. Define $S$ to be the connected component of $\{(\lambda, y) \in U : f(\lambda, y) = 0\}$ which contains $(\lambda_0, y_0)$.*

*(1) If $\nabla f(\lambda_0, y_0) \neq 0$, then $S$ is a $C^p$ curve passing through $(\lambda_0, y_0)$.*
*(2) If $\nabla f(\lambda_0, y_0) = 0$, we assume in addition that $p \geq 2$, and the Hessian $H = \nabla^2 f(\lambda_0, y_0)$ is non-degenerate with eigenvalues $\lambda_1, \lambda_2 \neq 0$, then*
  *(a) when $\lambda_1 \lambda_2 > 0$, $(\lambda_0, y_0)$ is the unique zero point of $f(x, y) = 0$ near $(\lambda_0, y_0)$;*
  *(b) when $\lambda_1 \lambda_2 < 0$, there exist two $C^{p-1}$ curves $\{(\lambda_i(t), y_i(t)) : |t| \leq \delta\}$, $i = 1, 2$, such that $S$ consists of exactly the two curves near $(\lambda_0, y_0)$, $(\lambda_i(0), y_i(0)) = (\lambda_0, y_0)$. Moreover $t$ can be rescaled so that $(\eta, \tau) = (\lambda_i'(0), y_i'(0))$, $i = 1, 2$, are the two linear independent solutions of*

$$f_{\lambda\lambda}(\lambda_0, y_0)\eta^2 + 2f_{\lambda y}(\lambda_0, y_0)\eta\tau + f_{yy}(\lambda_0, y_0)\tau^2 = 0. \tag{3.11}$$

***Proof.*** Part (1) follows from the implicit function theorem (Theorem 3.2). Indeed, if $f_y(\lambda_0, y_0) \neq 0$, then $S$ is in form $\{(\lambda, y(\lambda)) : |\lambda - \lambda_0| < \varepsilon\}$, and if $f_\lambda(\lambda_0, y_0) \neq 0$, then $S$ is in form $\{(\lambda(y), y) : |y - y_0| < \varepsilon\}$. Part (2a) follows from standard multi-variable calculus since in this case, $(\lambda_0, y_0)$ is a strict local maximum or minimum point of $f(x, y)$. So we only need to prove (2b).

Consider the system of differential equations:

$$\lambda' = \frac{\partial f(\lambda, y)}{\partial y}, \quad y' = -\frac{\partial f(\lambda, y)}{\partial \lambda}, \quad (\lambda(0), y(0)) \in U. \tag{3.12}$$

Then (3.12) is a Hamiltonian system with potential function $f(\lambda, y)$, and $(\lambda_0, y_0)$ is the only equilibrium point of (3.12) in $U$. The Jacobian of (3.12) at $(\lambda_0, y_0)$ is

$$J = \begin{pmatrix} f_{\lambda y}(\lambda_0, y_0) & f_{yy}(\lambda_0, y_0) \\ -f_{\lambda\lambda}(\lambda_0, y_0) & -f_{\lambda y}(\lambda_0, y_0) \end{pmatrix}. \tag{3.13}$$

Since $Trace(J) = 0$ and $Det(J) = Det(H) < 0$, then $(\lambda_0, y_0)$ is a saddle type equilibrium of (3.12) and $J$ has eigenvalues $\pm k$ for some $k > 0$.

From the invariant manifold theory of differential equations, there exists a unique curve $\Gamma_s \subset U$ (the stable manifold) such that $\Gamma_s$ is invariant for (3.12) and for $(\lambda(0), y(0)) \in \Gamma_s$, $(\lambda(t), y(t)) \to (\lambda_0, y_0)$ as $t \to \infty$; and similarly the unstable manifold is another invariant curve $\Gamma_u$ for (3.12) and for $(\lambda(0), y(0)) \in \Gamma_u$, $(\lambda(t), y(t)) \to (\lambda_0, y_0)$ as $t \to -\infty$. Both $\Gamma_s$ and $\Gamma_u$ are $C^{p-1}$ one-dimensional manifold by the stable and unstable manifold theorem ([Pe] page 107). $f(\lambda, y) = 0$ for $(\lambda, y) \in \Gamma_s \cup \Gamma_u$ since $f(\lambda, y)$ is the Hamiltonian function of the system and $\Gamma_s \cup \Gamma_u \cup \{(\lambda_0, y_0)\}$ . On the other hand, for any $(\lambda, y) \notin \Gamma_s \cup \Gamma_u \cup \{(\lambda_0, y_0)\}$, $f(\lambda, y) \neq 0$ from the Morse lemma.

Finally we consider the tangential direction of $\Gamma_s$ and $\Gamma_u$. We denote the two curves by $(\lambda_i(t), y_i(t))$, with $i = 1, 2$. Then

$$f(\lambda_i(t), y_i(t)) = 0. \tag{3.14}$$

Differentiating (3.14) in $t$ twice, we obtain (we omit the subscript $i$ for $\lambda_i(t)$ and $y_i(t)$ in the equation)

$$\begin{aligned} &f_{\lambda\lambda}(\lambda(t), y(t))(\lambda'(t))^2 + 2f_{\lambda y}(\lambda(t), y(t))\lambda'(t)y'(t) + f_{yy}(\lambda(t), y(t))(y'(t))^2 \\ &+ f_\lambda(\lambda(t), y(t))\lambda''(t) + f_y(\lambda(t), y(t))y''(t) = 0. \end{aligned}$$

evaluating at $t = 0$ and $\nabla f(\lambda_0, y_0) = 0$, we obtain (3.11). □

Theorem 3.5 is proved in Liu, Wang and Shi [LSW], and see also Shi and Xie [SX] in which we show that $C^{p-1}$ is the optimal regularity of the two crossing curves. We remark that Theorem 3.5 can also be deduced from a more general Morse Lemma, see Kuiper [Ku] and Chang [Ch1] (Lemma 4.1 and Theorem 5.1), and a weaker result is proved in Nirenberg [Nir] Theorem 3.2.1, in which the crossing curves are shown to be $C^{p-2}$. Here we give an alternate proof using the invariant manifold theory.

In Theorem 3.5, if $f_y(\lambda_0, y_0) \neq 0$, then the $C^p$ curve can be parameterized by $\lambda$; if $f_\lambda(\lambda_0, y_0) \neq 0$, then the $C^p$ curve can be parameterized by $y$ and indeed we have the saddle-node bifurcation; and if we assume that $f(\lambda, y_0) \equiv 0$, $f_y(\lambda_0, y_0) = 0$, and $f_{\lambda y}(\lambda_0, y_0) \neq 0$, then we obtain transcritical or pitchfork bifurcations. In this sense, Theorem 3.5 gives a unified unfolding of singularity in $\mathbf{R}^2$ with codimension 2. Using the implicit function theorem in Banach spaces, we will establish similar bifurcation results in Banach spaces in the following sections.

## 3.3 Saddle-node bifurcation

From the implicit function theorem (Theorem 3.2), a necessary condition for bifurcation is that

$$F_u(\lambda_0, u_0) \text{ is not invertible.} \tag{3.15}$$

When (3.15) is satisfied, we call $(\lambda_0, u_0)$ a *degenerate solution* of $F(\lambda, u) = 0$. Here we discuss the case when the kernel of $F_u(\lambda_0, u_0)$ is nonempty, and in particular, we discuss the case that $\mu = 0$ is a simple eigenvalue of $F_u(\lambda_0, u_0)$, *i.e.*

**(F1)** $dim N(F_u(\lambda_0, u_0)) = codim R(F_u(\lambda_0, u_0)) = 1$, and $N(F_u(\lambda_0, u_0)) = span\{w_0\}$.

(F1) is equivalent to: $F_u(\lambda_0, u_0)$ has a one-dimension kernel, and it is a Fredholm operator with index zero. The range space $R(F_u(\lambda_0, u_0))$ is a subspace of $Y$ of co-dimension one, then there exists $l \in Y^*$ (the space of linear functionals on $Y$), such that

$$u \in R(F_u(\lambda_0, u_0)) \Leftrightarrow \langle l, u \rangle = 0, \tag{3.16}$$

where $\langle l, u \rangle$ is the duality relation between $Y^*$ and $Y$. In the following whenever (F1) is assumed, $l$ is the associated linear functional in $Y^*$.

**Theorem 3.6. (Saddle-node bifurcation theorem)** *Let $U$ be a neighborhood of $(\lambda_0, u_0)$ in $\mathbf{R} \times X$, and let $F : U \to Y$ be a continuously differentiable mapping. Assume that $F(\lambda_0, u_0) = 0$, $F$ satisfies (F1) at $(\lambda_0, u_0)$ and*

**(F2)** $F_\lambda(\lambda_0, u_0) \notin R(F_u(\lambda_0, u_0))$.

*(1) If $Z$ is a complement of $span\{w_0\}$ in $X$, then the solutions of $F(\lambda, u) = 0$ near $(\lambda_0, u_0)$ form a curve $\{(\lambda(s), u(s)) = (\lambda(s), u_0 + sw_0 + z(s)) : |s| < \delta\}$, where $s \mapsto (\lambda(s), z(s)) \in \mathbf{R} \times Z$ is a continuously differentiable function, $\lambda(0) = \lambda'(0) = 0$, and $z(0) = z'(0) = 0$.*
*(2) If $F$ is $k-$times continuously differentiable, so are $\lambda(s)$ and $z(s)$.*
*(3) If $F$ is $C^2$ in $u$, then*

$$\lambda''(0) = -\frac{\langle l, F_{uu}(\lambda_0, u_0)[w_0, w_0] \rangle}{\langle l, F_\lambda(\lambda_0, u_0) \rangle}, \tag{3.17}$$

*where $l \in Y^*$ satisfying $N(l) = R(F_u(\lambda_0, u_0))$.*

***Proof.*** Define $G : U_1 \times (U_2 \times Z_1) \to Y$ by

$$G(s, \lambda, z) = F(\lambda, u_0 + sw_0 + z), \tag{3.18}$$

where $U_1, U_2$ are neighborhoods of 0 in $\mathbf{R}$ and $Z_1$ is neighborhood of 0 in $Z$ so that the right hand side of (3.18) is well-defined ($(\lambda, u_0 + sw_0 + z) \in U$). Then $G$ has the same smoothness as $F$ and $G(0, \lambda_0, 0) = 0$. We claim that the partial derivative $G_{(\lambda,z)}(0, \lambda_0, 0) : \mathbf{R} \times Z \to Y$ is an isomorphism. We first show that $G_{(\lambda,z)}(0, \lambda_0, 0)$ is injective. Suppose that there exists $(\tau, \psi) \in \mathbf{R} \times Z$ such that $G_{(\lambda,z)}(0, \lambda_0, 0)[(\tau, \psi)] = 0$, then

$$\tau F_\lambda(\lambda_0, u_0) + F_u(\lambda_0, u_0)[\psi] = 0. \tag{3.19}$$

Applying $l$ to (3.19), we obtain

$$\tau \langle l, F_\lambda(\lambda_0, u_0)\rangle = 0. \tag{3.20}$$

Since $F_\lambda(\lambda_0, u_0) \notin R(F_u(\lambda_0, u_0))$, then $\tau = 0$, and $\psi = 0$ since $\psi \in Z$ and $N(F_u(\lambda_0, u_0)) = span\{w_0\}$. Next we show that $G_{(\lambda,z)}(0, \lambda_0, 0)$ is surjective. Let $\theta \in Y$. Applying $l$ to

$$\tau F_\lambda(\lambda_0, u_0) + F_u(\lambda_0, u_0)[\psi] = \theta, \tag{3.21}$$

we obtain

$$\tau = \frac{\langle l, \theta\rangle}{\langle l, F_\lambda(\lambda_0, u_0)\rangle}, \tag{3.22}$$

and $\psi = K[\theta - \tau F_\lambda(\lambda_0, u_0)]$, where $K$ is the inverse of $F_u(\lambda_0, u_0)|_Z$. Thus $(\tau, \psi)$ is uniquely determined by $\theta$. Since $G$ is continuous, $G_{(\lambda,z)}(0, \lambda_0, 0)$ is a bijection, then $[G_{(\lambda,z)}(0, \lambda_0, 0)]^{-1}$ is also continuous by the open mapping theorem of Banach. Hence $G_{(\lambda,z)}(0, \lambda_0, 0)$ is an isomorphism. By Theorem 3.2, the first two statements of theorem are true. And $(\lambda'(0), z'(0))$ is determined by

$$G_{(\lambda,z)}(0, \lambda_0, 0)[(\lambda'(0), z'(0))] = -G_s(0, \lambda_0, 0) = -F_u(\lambda_0, u_0)[w_0] = 0, \tag{3.23}$$

so $\lambda'(0) = 0$ and $z'(0) = 0$ from the injectivity of $G_{(\lambda,z)}(0, \lambda_0, 0)$.

Differentiating $F(\lambda(s), u(s)) = 0$ with respect to $s$ twice, we obtain

$$\begin{aligned} &\lambda''(s)F_\lambda + [\lambda'(s)]^2 F_{\lambda\lambda} + 2\lambda'(s)F_{\lambda u}[u'(s)] \\ &+F_{uu}[u'(s), u'(s)] + F_u[u''(s)] = 0. \end{aligned} \tag{3.24}$$

Let $s = 0$ in (3.24) and apply $l$ to (3.24), then we obtain (3.17). □

Theorem 3.6 first appeared in Crandall and Rabinowitz [CR2]. When

**(F4)** $F_{uu}(\lambda_0, u_0)[w_0, w_0] \notin R(F_u(\lambda_0, u_0))$, is satisfied, $\lambda''(0) \neq 0$, and the solution set $\{(\lambda(s), u(s)) : |s| < \delta\}$ is a parabola-like curve which reaches an extreme point at $(\lambda_0, u_0)$. The degenerate solution $(\lambda_0, u_0)$ in this case is called a *turning point* on the solution curve. When $\lambda''(0) > 0$, the bifurcation is *supercritical*; and when $\lambda''(0) < 0$, the bifurcation is *subcritical.* In such cases, the bifurcation diagrams of (3.1) can still be sketched as in Fig. 1.3. Similar to previous applications to (2.1), we have

**Theorem 3.7.** *Suppose that $f \in C^1(\mathbf{R}^+)$ and $(\lambda_*, u_*)$ is a positive solution of* (2.1) *which satisfies*

$$\frac{\partial u_*}{\partial \nu}(x) < 0 \quad \text{for all } x \in \partial\Omega. \tag{3.25}$$

*and suppose that the linearized equation* (3.7) *has a unique (up to scale) nontrivial solution $w$, which satisfies*

$$\int_\Omega f(u_*) w dx \neq 0. \tag{3.26}$$

*Then all the positive solutions of* (2.1) *near $(\lambda_*, u_*)$ have the form $(\lambda(s), u_* + sw + z(s))$ for $s \in (-\delta, \delta)$ and some $\delta > 0$, where $\lambda(0) = \lambda_*$, $\lambda'(0) = 0$, $z(0) = z'(0) = 0$. Moreover, if $f \in C^2(\mathbf{R}^+)$, then*

$$\lambda''(0) = -\frac{\lambda_* \int_\Omega f''(u_*) w^3(x) dx}{\int_\Omega f(u_*) w(x) dx}. \tag{3.27}$$

***Proof.*** The setting is similar to that of Theorem 3.3. Recall from the proof of Theorem 3.3, $F_u(\lambda_*, u_*) = \Delta + \lambda_* f'(u_*)$, which is a Fredholm operator with index 0. (F1) is satisfied from the assumption that the solution space of (2.1) is one-dimensional. From Proposition 2.1, $R(F_u(\lambda_*, u_*)) = \{\phi \in C^\alpha(\overline{\Omega}) : \int_\Omega \phi(x) w(x) dx = 0\}$. Thus (3.26) is equivalent to (F2), and stated results follow from Theorem 3.6 except the positivity of $u(s) = u_* + sw + z(s)$, which follows from (3.25). □

We also comment that at a bifurcation point described in Theorem 3.6, if $F \in C^3$ and $\lambda''(0) = 0$, then

$$\lambda'''(0) = -\frac{\langle l, F_{uuu}(\lambda_0, u_0)[w_0, w_0, w_0]\rangle + 3\langle l, F_{uu}(\lambda_0, u_0)[w_0, \theta]\rangle}{\langle l, F_\lambda(\lambda_0, u_0)\rangle}, \tag{3.28}$$

where $\theta \in Z$ is the solution of

$$F_{uu}(\lambda_0, u_0)[w_0, w_0] + F_u(\lambda_0, u_0)[\theta] = 0. \tag{3.29}$$

The solvability of (3.29) is equivalent to

**(F4′)** $F_{uu}(\lambda_0, u_0)[w_0, w_0] \in R(F_u(\lambda_0, u_0))$,

or $\lambda''(0) = 0$. In the case $\lambda'''(0) \neq 0$, a cusp type bifurcation occurs near the degenerate solution $(\lambda_0, u_0)$. More discussion can be found in Shi [S1].

## 3.4 Transcritical and pitchfork bifurcations

If there is a branch of trivial solutions $u = u_0$ for all $\lambda$, then nontrivial solutions can bifurcate from the trivial branch at a degenerate solution. Here is the theorem of *Bifurcation from a simple eigenvalue* by Crandall and Rabinowitz [CR1]:

**Theorem 3.8. (Transcritical and pitchfork bifurcation theorem)** *Let $U$ be a neighborhood of $(\lambda_0, u_0)$ in $\mathbf{R} \times X$, and let $F : U \to Y$ be a twice continuously differentiable mapping. Assume that $F(\lambda, u_0) = 0$ for $(\lambda, u_0) \in U$. At $(\lambda_0, u_0)$, $F$ satisfies (F1) and*

**(F3)** $F_{\lambda u}(\lambda_0, u_0)[w_0] \notin R(F_u(\lambda_0, u_0))$.

*Let $Z$ be any complement of span$\{w_0\}$ in $X$. Then the solution set of (3.1) near $(\lambda_0, u_0)$ consists precisely of the curves $u = u_0$ and $\{(\lambda(s), u(s)) : s \in I = (-\epsilon, \epsilon)\}$, where $\lambda : I \to \mathbf{R}$, $z : I \to Z$ are $C^1$ functions such that $u(s) = u_0 + sw_0 + sz(s)$, $\lambda(0) = \lambda_0$, $z(0) = 0$, and*

$$\lambda'(0) = -\frac{\langle l, F_{uu}(\lambda_0, u_0)[w_0, w_0]\rangle}{2\langle l, F_{\lambda u}(\lambda_0, u_0)[w_0]\rangle}, \tag{3.30}$$

*where $l \in Y^*$ satisfying $N(l) = R(F_u(\lambda_0, u_0))$. If $\lambda'(0) = 0$, and in addition $F \in C^3$ near $(\lambda_0, u_0)$, then*

$$\lambda''(0) = -\frac{\langle l, F_{uuu}(\lambda_0, u_0)[w_0, w_0, w_0]\rangle + 3\langle l, F_{uu}(\lambda_0, u_0)[w_0, \theta]\rangle}{3\langle l, F_{\lambda u}(\lambda_0, u_0)[w_0]\rangle}, \tag{3.31}$$

*where $\theta$ is the solution of (3.29).*

When $\lambda'(0) \neq 0$ (thus (F4) is satisfied), then a *transcritical bifurcation* occurs (see Fig. 1.4); if instead (F4$'$) is satisfied, then $\lambda'(0) = 0$, and a *pitchfork bifurcation* occurs at $(\lambda_0, u_0)$ if $\lambda'(0) = 0$ and $\lambda''(0) \neq 0$. We remark that in the original theorem of [CR1], under the weaker assumption that $F_{\lambda u}$ exists and continuous near $(\lambda_0, u_0)$ instead of $F$ being $C^2$, it was shown that same result holds but the curve of nontrivial solutions is only continuous not $C^1$.

Here we prove this important theorem as a consequence of a more general result based on Theorem 3.5. We assume $F$ satisfies (F1) at $(\lambda_0, u_0)$, then we have decompositions of $X$ and $Y$: $X = N(F_u(\lambda_0, u_0)) \oplus Z$ and $Y = R(F_u(\lambda_0, u_0)) \oplus Y_1$, where $Z$ is a complement of $N(F_u(\lambda_0, u_0))$ in $X$, and $Y_1$ is a complement of $R(F_u(\lambda_0, u_0))$. In particular, $F_u(\lambda_0, u_0)|_Z : Z \to R(F_u(\lambda_0, u_0))$ is an isomorphism. Since $R(F_u(\lambda_0, u_0))$ is codimension one, then there exists $l \in Y^*$ such that $R(F_u(\lambda_0,u_0)) = \{v \in Y : \langle l, v\rangle = 0\}$. Recall the condition (F2) in saddle-node bifurcation theorem, here we assume the opposite:

**(F2′)** $F_\lambda(\lambda_0,u_0) \in R(F_u(\lambda_0,u_0))$.

Then the equation

$$F_\lambda(\lambda_0,u_0) + F_u(\lambda_0,u_0)[v] = 0 \tag{3.32}$$

has a unique solution $v_1 \in Z$. The following "crossing curve bifurcation theorem" is proved in Liu, Shi and Wang [LSW]:

**Theorem 3.9.** *Let $U$ be a neighborhood of $(\lambda_0,u_0)$ in $\mathbf{R}\times X$, and let $F : U \to Y$ be a twice continuously differentiable mapping. Assume that $F(\lambda_0,u_0) = 0$, $F$ satisfies (F1) and (F2′) at $(\lambda_0,u_0)$. Let $X = N(F_u(\lambda_0,u_0))\oplus Z$ be a fixed splitting of $X$, let $v_1 \in Z$ be the unique solution of (3.32), and let $l \in Y^*$ such that $R(F_u(\lambda_0,u_0)) = \{v \in Y : \langle l,v\rangle = 0\}$. We assume that the matrix (all derivatives are evaluated at $(\lambda_0,u_0)$)*

$$H_0 \equiv \begin{pmatrix} \langle l, F_{\lambda\lambda} + 2F_{\lambda u}[v_1] + F_{uu}[v_1,v_1]\rangle & \langle l, F_{\lambda u}[w_0] + F_{uu}[w_0,v_1]\rangle \\ \langle l, F_{\lambda u}[w_0] + F_{uu}[w_0,v_1]\rangle & \langle l, F_{uu}[w_0,w_0]\rangle \end{pmatrix} \tag{3.33}$$

*is non-degenerate, i.e. $Det(H_0) \neq 0$.*

*(1) If $H_0$ is definite, i.e. $Det(H_0) > 0$, then the solution set of $F(\lambda,u) = 0$ near $(\lambda,u) = (\lambda_0,u_0)$ is the single point set $\{(\lambda_0,u_0)\}$.*

*(2) If $H_0$ is indefinite, i.e. $Det(H_0) < 0$, then the solution set of $F(\lambda,u) = 0$ near $(\lambda,u) = (\lambda_0,u_0)$ is the union of two intersecting $C^1$ curves, and the two curves are in form of $(\lambda_i(s),u_i(s)) = (\lambda_0 + \mu_i s + s\theta_i(s), u_0 + \eta_i s w_0 + s v_i(s))$, $i = 1,2$, where $s \in (-\delta,\delta)$ for some $\delta > 0$, $\theta_i(0) = 0$, $v_i(s) \in Z$, $v_i(0) = 0$ ($i = 1,2$), and $(\mu_i,\eta_i)$ ($i = 1,2$) are non-zero linear independent solutions of the equation*

$$\begin{aligned}&\langle l, F_{\lambda\lambda} + 2F_{\lambda u}[v_1] + F_{uu}[v_1,v_1]\rangle\mu^2 \\ &+ 2\langle l, F_{\lambda u}[w_0] + F_{uu}[w_0,v_1]\rangle\eta\mu + \langle l, F_{uu}[w_0,w_0]\rangle\eta^2 = 0.\end{aligned} \tag{3.34}$$

***Proof.*** We start by reducing the equation in infinite-dimensional space to a finite dimensional one by a Lyapunov-Schmidt process. We denote the projection from $Y$ into $R(F_u(\lambda_0,u_0))$ by $Q$. Then $F(\lambda,u) = 0$ is equivalent to

$$Q\circ F(\lambda,u) = 0, \quad \text{and } (I-Q)\circ F(\lambda,u) = 0. \tag{3.35}$$

We rewrite the first equation in form

$$G(\lambda, t, g) \equiv Q \circ F(\lambda, u_0 + tw_0 + g) = 0 \tag{3.36}$$

where $t \in \mathbf{R}$ and $g \in Z$. Calculation shows that $G_g(\lambda_0, 0, 0) = Q \circ F_u(\lambda_0, u_0)$ is an isomorphism from $Z$ to $R(F_u(\lambda_0, u_0))$. Then $g = g(\lambda, t)$ in (3.36) is uniquely solvable from the implicit function theorem Theorem 3.2 for $(\lambda, t)$ near $(\lambda_0, 0)$, and $g$ is $C^2$. Hence $u = u_0 + tw_0 + g(\lambda, t)$ is a solution to $F(\lambda, u) = 0$ if and only if $(I - Q) \circ F(\lambda, u_0 + tw_0 + g(\lambda, t)) = 0$. Since $R(F_u(\lambda_0, u_0))$ is co-dimensional one, hence it becomes the scalar equation $\langle l, F(\lambda, u_0 + tw_0 + g(\lambda, t))\rangle = 0$.

From arguments above we have

$$f_1(\lambda, t) \equiv Q \circ F(\lambda, u_0 + tw_0 + g(\lambda, t)) = 0, \tag{3.37}$$

for $(\lambda, t)$ near $(\lambda_0, 0)$. Differentiating $f_1$ and evaluating at $(\lambda, t) = (\lambda_0, 0)$, we obtain

$$0 = \nabla f_1 = (Q \circ (F_\lambda + F_u[g_\lambda]), Q \circ F_u[w_0 + g_t]). \tag{3.38}$$

Since $F_u[w_0] = 0$ and $g_t \in Z$, and $F_u(\lambda_0, u_0)|_Z$ is an isomorphism, then $g_t(\lambda_0, 0) = 0$. Similarly $g_\lambda \in Z$ and $F_\lambda \in R(F_u(\lambda_0, u_0))$ from (F2′), hence

$$F_\lambda(\lambda_0, u_0) + F_u(\lambda_0, u_0)[g_\lambda(\lambda_0, 0)] = 0. \tag{3.39}$$

Hence $g_\lambda(\lambda_0, 0) = v_1$ where $v_1$ is defined as in (3.32).

To prove the statement in Theorem 3.9, we apply Theorem 3.5 to

$$f(\lambda, t) = \langle l, F(\lambda, u_0 + tw_0 + g(\lambda, t))\rangle. \tag{3.40}$$

From the proofs above, $F(\lambda, u) = 0$ for $(\lambda, u)$ near $(\lambda_0, u_0)$ is equivalent to $f(\lambda, t) = 0$ for $(\lambda, t)$ near $(\lambda_0, 0)$. To apply Theorem 3.5, we claim that

$$\nabla f(\lambda_0, 0) = (f_\lambda, f_t) = 0, \text{ and } Hess(f) \text{ is non-degenerate.} \tag{3.41}$$

It is easy to see that

$$\begin{aligned}&\nabla f(\lambda_0, 0)\\ =&(\langle l, F_\lambda(\lambda_0, u_0) + F_u(\lambda_0, u_0)[g_\lambda(\lambda_0, 0)]\rangle, \langle l, F_u(\lambda_0, u_0)[w_0 + g_t(\lambda_0, 0)]\rangle).\end{aligned} \tag{3.42}$$

Thus $\nabla f(\lambda_0, 0) = 0$ from (3.32) and $g_t(\lambda_0, 0) = 0$. For the Hessian matrix, we have

$$Hess(f) = \begin{pmatrix} f_{\lambda\lambda} & f_{\lambda t} \\ f_{t\lambda} & f_{tt} \end{pmatrix}. \tag{3.43}$$

Here

$$\begin{aligned} f_{\lambda t}(\lambda_0,0) =& f_{t\lambda}(\lambda_0,0) \\ =& \langle l, F_{\lambda u}[w_0+g_t] + F_{uu}[w_0+g_t, g_\lambda] + F_u[g_{\lambda t}]\rangle \\ =& \langle l, F_{\lambda u}[w_0] + F_{uu}[w_0, v_1]\rangle, \end{aligned} \tag{3.44}$$

since $g_t = 0$. Next we have

$$\begin{aligned} f_{\lambda\lambda}(\lambda_0,0) =& \langle l, F_{\lambda\lambda} + 2F_{\lambda u}[g_\lambda] + F_{uu}[g_\lambda, g_\lambda] + F_u[g_{\lambda\lambda}]\rangle \\ =& \langle l, F_{\lambda\lambda} + 2F_{\lambda u}[v_1] + F_{uu}[v_1, v_1]\rangle. \end{aligned} \tag{3.45}$$

Finally,

$$f_{tt}(\lambda_0,0) = \langle l, F_{uu}[w_0+g_t, w_0+g_t] + F_u[g_{tt}]\rangle = \langle l, F_{uu}[w_0,w_0]\rangle. \tag{3.46}$$

In summary, from our calculation,

$$Hess(f) = \begin{pmatrix} \langle l, F_{\lambda\lambda} + 2F_{\lambda u}[v_1] + F_{uu}[v_1,v_1]\rangle & \langle l, F_{\lambda u}[w_0] + F_{uu}[w_0,v_1]\rangle \\ \langle l, F_{\lambda u}[w_0] + F_{uu}[w_0,v_1]\rangle & \langle l, F_{uu}[w_0,w_0]\rangle \end{pmatrix}.$$

Therefore from Theorem 3.5, we conclude that the solution set of $F(\lambda,u) = 0$ near $(\lambda,u) = (\lambda_0,u_0)$ is a pair of intersecting curves if the matrix in (3.4) is indefinite, or is a single point if it is definite.

Now we consider only the former case of two curves. We denote the two curves by $(\lambda_i(s), u_i(s)) = (\lambda_i(s), u_0 + t_i(s)w_0 + g(\lambda_i(s), t_i(s)))$, with $i = 1,2$. Then

$$F(\lambda_i(s), u_0 + t_i(s)w_0 + g(\lambda_i(s), t_i(s))) = 0. \tag{3.47}$$

From Theorem 3.5 the vectors $v_i = (\lambda_i'(0), t_i'(0))$ are the solutions of $v^T H_0 v = 0$, which are the solutions $(\mu,\eta)$ of (3.34). □

Now we show that Theorem 3.8 is a special case of Theorem 3.9. In fact, the assumption of $F(\lambda,u_0) \equiv 0$ implies that (F2′) is satisfied and $F_\lambda(\lambda_0,u_0) = F_{\lambda\lambda}(\lambda_0,u_0) = 0$, thus $v_1 = 0$ and $det(H_0) = -\langle l, F_{\lambda u}(\lambda_0, w_0)\rangle^2$. Hence the assumption (F3) implies that $det(H_0) \neq 0$ and $H_0$ is indefinite. The formula in (3.30) can be obtained from (3.34) as it becomes

$$2\langle l, F_{\lambda u}[w_0]\rangle \eta\mu + \langle l, F_{uu}[w_0,w_0]\rangle \eta^2 = 0. \tag{3.48}$$

We can choose one solution of (3.48) to be $(\mu,\eta) = (1,0)$ which corresponds to the line of trivial solutions, and the other solution to be $(\mu,\eta) = (\lambda'(0), 1)$ with $\lambda'(0)$ being the expression in (3.30). Finally (3.31) can be obtained with further calculations. Indeed let $(\lambda(s), u(s))$ be the nontrivial solution curve. Then by differentiating $F(\lambda(s), u(s)) = 0$ three times, evaluating at $s = 0$ and applying $l$, one can obtain (3.31).

The implicit function theorem (transversal curve), saddle-node bifurcation (turning curve), and transcritical/pitchfork bifurcation (two crossing curves) illustrate the impact of different levels of degeneracy of the nonlinear mapping on the structure of local solution sets. In transcritical/pitchfork bifurcation, one solution curve is presumed. Indeed this is not necessary and a bifurcation structure with two crossing curves can be completely described via the partial derivatives of the nonlinear mapping. We remark that in the original result in [CR1], a slightly weaker smoothness condition is imposed on $F$: $F$ is not necessarily $C^2$, but only the partial derivative $F_{\lambda u}$ exists. However to obtain (3.30) which indicates the direction of the bifurcation, more smoothness as ours is needed.

We illustrate the application of the transcritical and pitchfork bifurcation theorem by considering the diffusive logistic equation.

**Example 3.2.** We consider the following diffusive logistic equation:

$$\begin{cases} \Delta u + \lambda(u - u^p) = 0, & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{3.49}$$

where $p \geq 2$. For any $\lambda > 0$, $u = 0$ is always a solution to (3.49). We use Theorem 3.8 to analyze the bifurcation occurring at $\lambda = \lambda_1$. It is easy to verify that $F_u(\lambda, 0)w = \Delta w + \lambda w$, which is invertible if $\lambda \neq \lambda_i$. At $\lambda = \lambda_1$, $N(F_u(\lambda_1, 0)) = span\{\phi_1\}$, where $\phi_1 > 0$ is the principle eigenfunction. $R(F_u(\lambda_1, 0))$ is codimension one, and indeed $R(F_u(\lambda_1, 0)) = \{v \in Y : \int_\Omega \phi_1 v dx = 0\}$. Finally $F_{\lambda u}(\lambda_1, 0)[\phi_1] \notin R(F_u(\lambda_1, 0))$ since $\int_\Omega \phi_1 \cdot \phi_1 dx > 0$. Thus Theorem 3.8 can be applied to (3.49).

The calculation of $\lambda'(0)$ can be done by directly applying (3.30). But to illustrate the involved calculation, we calculate it directly. Differentiate (2.1) with respect to $s$ once and twice, we obtain

$$\Delta u_s + \lambda f'(u) u_s + \lambda_s f(u) = 0, \tag{3.50}$$

$$\Delta u_{ss} + \lambda f'(u) u_{ss} + 2\lambda_s f'(u) u_s + \lambda f''(u)(u_s)^2 + \lambda_{ss} f(u) = 0. \tag{3.51}$$

Set $s = 0$ we obtain

$$\Delta u_s(0) + \lambda_1 f'(0) u_s(0) = 0, \tag{3.52}$$

$$\Delta u_{ss}(0) + \lambda_1 f'(0) u_{ss}(0) + 2\lambda_s(0) f'(0) u_s(0) + \lambda_1 f''(0)[u_s(0)]^2 = 0. \tag{3.53}$$

By using integral by parts, and $u_s(0) = \phi_1$, we obtain

$$2\lambda_s(0) f'(0) \int_\Omega \phi_1^2(x) dx + \lambda_1 f''(0) \int_\Omega \phi_1^3(x) dx = 0, \tag{3.54}$$

and

$$\lambda'(0) = -\frac{\lambda_1 f''(0)\int_\Omega \phi_1^3(x)dx}{2f'(0)\int_\Omega \phi_1^2(x)dx}. \tag{3.55}$$

In particular, for $f(u) = u - u^2$ when $p = 2$, $\lambda'(0) > 0$ and a transcritical bifurcation occurs. Thus for $\lambda \in (\lambda_1, \lambda_1 + \varepsilon)$ (which corresponds to $s \in (0, \varepsilon_1)$ since $\lambda'(0) > 0$), (3.49) has a positive solution with form $s\phi_1 + o(|s|)$.

For $p > 2$, (3.55) shows that $\lambda'(0) = 0$, and the smoothness of $\lambda(s)$ depends on $p$. When $p \geq 3$, $f \in C^3$ near $u = 0$, thus we have

$$\lambda''(0) = -\frac{\lambda_1 f'''(0)\int_\Omega \phi_1^4(x)dx}{3f'(0)\int_\Omega \phi_1^2(x)dx}, \tag{3.56}$$

by applying (3.31), where $\theta = 0$ since $F_{uu}(\lambda_1, 0)[\phi_1, \phi_1] = \lambda_1 f''(0)\phi_1^2 = 0$. Hence when $p = 3$, a pitchfork bifurcation occurs and $\lambda''(0) > 0$. Indeed one can show that for any $p \geq 2$, (3.49) has no positive solutions when $\lambda < \lambda_1$, and all positive solutions near $(\lambda, u) = (\lambda_1, 0)$ are on the right hand side of $\lambda = \lambda_1$, hence the bifurcation of positive solutions is always supercritical in that sense.

In general, we have the following result regarding (2.1):

**Theorem 3.10.** *Let $f \in C^2(\mathbf{R}^+)$, $f(0) = 0$ and $f'(0) > 0$. Then $\lambda_0 = \lambda_1/f'(0)$ is a bifurcation point. All positive solutions of* (2.1) *near $(\lambda_0, 0)$ have the form $(\lambda(s), s\phi_1 + sz(s))$ with $z(s)$ being a $C^1$ function satisfying $z(0) = 0$ for $s \in (0, \delta)$ and some $\delta > 0$, $\lambda(0) = \lambda_0$ and*

$$\lambda'(0) = -\frac{\lambda_0 f''(0)\int_\Omega \phi_1^3(x)dx}{2f'(0)\int_\Omega \phi_1^2(x)dx}. \tag{3.57}$$

**Example 3.3.** An alternate of the logistic growth is the Allee effect (see Section 1.4). For instance, consider the diffusive population model with weak Allee effect (see [SS3]):

$$\begin{cases} \Delta u + \lambda u(1-u)(u+a) = 0, & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{3.58}$$

where $a > 0$. Again for any $\lambda > 0$, $u = 0$ is always a solution to (3.58). For $f(u) = u(1-u)(u+a)$, $f'(0) = a > 0$, hence $\lambda_0 = \lambda_1/a$ is a bifurcation point from Theorem 3.10. From (3.57), we obtain that

$$\lambda'(0) = -\frac{\lambda_0(1-a)\int_\Omega \phi_1^3(x)dx}{a\int_\Omega \phi_1^2(x)dx}. \tag{3.59}$$

Thus $\lambda'(0) > 0$ and the bifurcation of positive solutions is supercritical if $a > 1$, and $\lambda'(0) < 0$ and the bifurcation of positive solutions is subcritical (backward) if $0 < a < 1$. Later we shall show that the backward bifurcation indicates the nonuniqueness of positive solutions and it implies the bistability in the corresponding reaction-diffusion dynamics.

**Example 3.4.** A similar analysis can be done for Neumann boundary value problem, which we briefly discuss here (more can be found in Shi [S4]). Consider

$$\begin{cases} \Delta u + \lambda f(u) = 0, & \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0, & \text{on} \quad \partial\Omega. \end{cases} \tag{3.60}$$

Here we assume that $f \in C^3(\mathbf{R})$, $f(\beta) = 0$ for some $\beta > 0$, and $f'(\beta) > 0$; moreover $\lambda_k > 0$ is a simple eigenvalue of the linear problem

$$\begin{cases} \Delta \phi + \lambda \phi = 0, & \text{in} \quad \Omega, \\ \dfrac{\partial \phi}{\partial \nu} = 0, & \text{on} \quad \partial\Omega. \end{cases} \tag{3.61}$$

Here we still define $F(\lambda, u) = \Delta u + \lambda f(u)$, but $u \in X' = \{v \in C^{2,\alpha}(\overline{\Omega}) : \partial v / \partial \nu = 0 \text{ on } \partial\Omega\}$. Bifurcations occur along the line of trivial solutions $\{(\lambda, \beta) : \lambda > 0\}$. Similar to example above, $\lambda = \lambda_* = \lambda_k / f'(\beta)$ is a bifurcation point if $N(F_u(\lambda_*, \beta)) = span\{\phi_k\}$, and $codim R(F_u(\lambda_*, \beta)) = 1$ with $R(F_u(\lambda_*, \beta)) = \{v \in Y : \int_\Omega \phi_k v dx = 0\}$ where $Y = C^\alpha(\overline{\Omega})$; and also $F_{\lambda u}(\lambda_*, \beta)[\phi_k] \notin R(F_u(\lambda_*, \beta))$ since $\int_\Omega \phi_k \cdot \phi_k dx > 0$. Again Theorem 3.8 can be applied, and we obtain the existence of a solution curve $(\lambda(s), u(s))$ near $(\lambda_*, \beta)$.

Differentiating (3.60) with respect to $s$ twice, evaluating at $s = 0$, we get

$$\Delta u_{ss}(0) + \lambda(0) f'(\beta) u_{ss}(0) + 2\lambda'(0) f'(\beta) \phi_k + \lambda(0) f''(\beta) \phi_k^2 = 0, \tag{3.62}$$

$\partial_\nu u_{ss}(0) = 0$ on $\partial\Omega$. Using (3.62) and (3.61), we obtain

$$\lambda'(0) = -\frac{\lambda_* f''(\beta) \int_\Omega \phi_k^3 dx}{2 f'(\beta) \int_\Omega \phi_k^2 dx} = -\frac{\lambda_k f''(\beta) \int_\Omega \phi_k^3 dx}{2[f'(\beta)]^2 \int_\Omega \phi_k^2 dx}. \tag{3.63}$$

However it is very often (always true for one dimension) that $\int_\Omega \phi_k^3 dx = 0$, so it is necessary to compute $\lambda''(0)$ in that case. Then differentiating (3.60) further, and if $\lambda'(0) = 0$, then we get

$$\begin{aligned} &\Delta u_{sss}(0) + \lambda_* f'(\beta) u_{sss}(0) + 3\lambda''(0) f'(\beta) \phi_k \\ &+ \lambda_* f'''(\beta) \phi_k^3 + 3\lambda_* f''(\beta) \phi_k u_{ss}(0) = 0, \end{aligned} \tag{3.64}$$

and $\partial_\nu u_{sss}(0) = 0$ on $\partial\Omega$, where $u_{ss}(0)$ satisfies (3.62) with $\lambda'(0) = 0$. Then

$$\begin{aligned}\lambda''(0) &= -\frac{\lambda_* f'''(\beta)\int_\Omega \phi_k^4 dx + 3\lambda_* f''(\beta)\int_\Omega \phi_k^2 u_{ss}(0)dx}{3f'(\beta)\int_\Omega \phi_k^2 dx}\\ &= -\frac{\lambda_k f'''(\beta)\int_\Omega \phi_k^4 dx + 3\lambda_k f''(\beta)\int_\Omega \phi_k^2 u_{ss}(0)dx}{3[f'(\beta)]^2\int_\Omega \phi_k^2 dx}.\end{aligned} \tag{3.65}$$

In the special case of $\Omega = (0, a)$, $w(x) = \cos(k\pi x/a)$, so $\lambda'(0) = 0$. And $u_{ss}(0)$ is the solution of

$$v'' + \left(\frac{k\pi}{a}\right)^2 v + \frac{(k\pi)^2 f''(\beta)}{a^2 f'(\beta)}\cos^2\left(\frac{k\pi x}{a}\right) = 0, \quad v'(0) = v'(a) = 0. \tag{3.66}$$

From simple calculations, we have

$$u_{ss}(0) = \frac{f''(\beta)}{3f'(\beta)}[\cos^2(k\pi x/a) - 2].$$

Then by computation, we obtain

$$\lambda''(0) = -\left(\frac{k\pi}{a}\right)^2 \cdot \frac{3f'(\beta)f'''(\beta) - 5[f''(\beta)]^2}{12[f'(\beta)]^3}. \tag{3.67}$$

Thus $\lambda''(\alpha)$ is determined by the value of $3f'f''' - 5(f'')^2$ at $u = \beta$. A pitchfork bifurcation always occurs at $(\lambda_*, \beta)$ for one-dimensional Neumann boundary value problem, which is also indicated by the fact $u(x)$ and $u(a-x)$ are a pair of solutions for the same $\lambda$. We also remark that we exclude the case $\lambda_0 = 0$, the principal eigenvalue for the Neumann problem (3.61). In fact, all conditions in Theorem 3.8 are satisfied, and a bifurcation does occur, but the solution branch is a trivial one $\{(\lambda, u) = (0, u) : u \in \mathbf{R}\}$.

A parameter $\lambda$ is always involved in the bifurcation theorem above to indicate the change of solution set as the parameter evolves. Similar idea can be used to describe the solution set of nonlinear equation near a degenerate solution. Theorem 3.5 can also be used to prove such secondary bifurcation theorem which also generalizes the one in [CR1]:

**Theorem 3.11.** *Let $W$ and $Y$ be Banach spaces, $\Omega$ an open subset of $W$ and $G : \Omega \to Y$ be twice differentiable. Suppose*

*(1) $G(w_0) = 0$,*
*(2) $dim N(G'(w_0)) = 2$, $codim R(G'(w_0)) = 1$.*

*Then*

*(1) if for any $\phi(\neq 0) \in N(G'(w_0))$, $G''(w_0)[\phi,\phi] \notin R(G'(w_0))$, then the set of solutions to $G(w) = 0$ near $w = w_0$ is the singleton $\{w_0\}$;*

*(2) if there exists $\phi_1(\neq 0) \in N(G'(w_0))$ such that $G''(w_0)[\phi_1,\phi_1] \in R(G'(w_0))$, and there exists $\phi_2 \in N(G'(w_0))$ such that $G''(w_0)[\phi_1,\phi_2] \notin R(G'(w_0))$, then $w_0$ is a bifurcation point of $G(w) = 0$ and in some neighborhood of $w_0$, the totality of solutions of $G(w) = 0$ form two continuous curves intersecting only at $w_0$. Moreover the solution curves are in form of $w_0 + s\psi_i + s\theta_i(s)$, $s \in (-\delta,\delta)$, $\theta_i(0) = \theta_i'(0) = 0$, where $\psi_i$ $(i = 1, 2)$ are the two linear independent solutions of the equation $\langle l, G''(w_0)[\psi,\psi]\rangle = 0$.*

***Proof.*** Let $l \in Y^*$ such that $\langle l, y\rangle = 0$ if and only if $y \in R(G'(w_0))$. Then if for any $\phi(\neq 0) \in N(G'(w_0))$, $G''(w_0)[\phi,\phi] \notin R(G'(w_0))$, we must have $\langle l, G''(w_0)[\phi,\phi]\rangle > 0$ (or $< 0$) for any $\phi(\neq 0) \in N(G'(w_0))$. Without loss of generality, we assume $>$ holds. We assume that $W = span\{\phi_1\} \oplus X$ is a splitting of $W$, and we choose $\phi_2 \in X \cap N(G'(w_0))$ so that $\{\phi_1, \phi_2\}$ is a basis of $N(G'(w_0))$. Clearly $X \cap N(G'(w_0)) = span\{\phi_2\}$. Define $F : I \times X \to Y$ ($I \subset \mathbf{R}$ is an open interval containing 0)

$$F(\lambda, u) = G(w_0 + \lambda\phi_1 + u). \tag{3.68}$$

Then $F \in C^2$ and $F(0, 0) = 0$. We check $F$ satisfies (F1) and (F2$'$). It is easy to calculate

$$\begin{aligned} &F_\lambda(0,0) = G'(w_0)[\phi_1], \;\; F_{\lambda\lambda}(0,0) = G''(w_0)[\phi_1,\phi_1], \\ &F_u(0,0)[\psi] = G'(w_0)[\psi], \;\; F_{\lambda u}[\psi] = G''(w_0)[\phi_1,\psi], \\ &F_{uu}(0,0)[\psi,\theta] = G''(w_0)[\psi,\theta]. \end{aligned} \tag{3.69}$$

Then $N(F_u(0, 0)) = span\{\phi_2\}$ and $R(F_u(0, 0)) = R(G'(w_0))$, hence (F1) is satisfied. (F2$'$) is obvious since $F_\lambda(0, 0) = G'(w_0)[\phi_1] = 0$. From calculations above and Theorem 3.9, we have

$$H_1 = \begin{pmatrix} \langle l, G''(w_0)[\phi_1,\phi_1]\rangle & \langle l, G''(w_0)[\phi_1,\phi_2]\rangle \\ \langle l, G''(w_0)[\phi_1,\phi_2]\rangle & \langle l, G''(w_0)[\phi_2,\phi_2]\rangle \end{pmatrix}. \tag{3.70}$$

Since $\langle l, G''(w_0)[\phi,\phi]\rangle > 0$ for any $\phi(\neq 0) \in N(G'(w_0))$, then $Det(H_1) > 0$ since

$$\langle l, G''(w_0)[k_1\phi_1 + k_2\phi_2, k_1\phi_1 + k_2\phi_2]\rangle = \mathbf{k}H_1\mathbf{k}^T > 0,$$

which implies that $H_1$ is positively definite, and here $\mathbf{k} = (k_1, k_2) \in \mathbf{R}^2$ and $\mathbf{k}^T$ is the transpose of $\mathbf{k}$. We apply Theorem 3.9 part 1, then the result follows. For the second part, the calculation above remains true with $\phi_1$, $\phi_2$ satisfying the conditions in theorem if we choose a complement subspace $X$

to $span\{\phi_1\}$ so that $\phi_2 \in X$, and $\{\phi_1, \phi_2\}$ makes a base for $N(G'(w_0))$. But $det(H_1) < 0$ from the assumptions, hence we can apply part 2 of Theorem 3.9. For the solutions $(\mu_i, \eta_i)$ of (3.34), $(\mu_1, \eta_1) = (1, 0)$ is one solution since $\langle l, G''(w_0)[\phi_1, \phi_1]\rangle = 0$, and the other solution is given by $\eta_2 = 1$ and $\mu_2 = -\langle l, G''(w_0)[\phi_2, \phi_2]\rangle/(2\langle l, G''(w_0)[\phi_1, \phi_2]\rangle)$. Hence the two solution branches are in form of $w_0 + s\phi_1 + s\theta_1(s)$ and $w_0 + s(\mu_2\phi_1 + \phi_2) + s\theta_2(s)$, and it is verify that $\psi_1 = \phi_1$ and $\psi_2 = \mu_2\phi_1 + \phi_2$ are the two linear independent solutions of $\langle l, G''(w_0)[\psi, \psi]\rangle = 0$. □

As an application of Theorem 3.11 and the maximum principle, we prove an anti-maximum principle:

**Theorem 3.12.** *Recall the linear operator* $L_c : W_0^{2,p}(\Omega) \to L^p(\Omega)$ *by* $L_c(u) = -\Delta u + c(x)u$ *where* $c(x) \in L^\infty(\Omega)$ *and* $p > 2n$. *Consider*

$$L_c u - \lambda u = f,\ x \in \Omega,\ \ u = 0,\ x \in \partial\Omega, \tag{3.71}$$

*where* $f \in L^p(\Omega)$. *Let* $(\lambda_1(c), \phi_1)$ *be the principal eigen-pair in Proposition 2.4 such that* $\phi_1 > 0$ *in* $\Omega$. *Suppose that* $f > 0$ *in* $\Omega$.

*(1) If* $\lambda < \lambda_1(c)$, *then the unique solution* $u$ *of* (3.71) *satisfies*

$$u(x) > 0,\ \ x \in \Omega,\ \ \frac{\partial u}{\partial n}(x) < 0,\ \ x \in \partial\Omega; \tag{3.72}$$

*(2) There exists* $\delta_f > 0$ *which depends on* $f$, *such that if* $\lambda_1(c) < \lambda < \lambda_1(c) + \delta_f$, *then the unique solution* $u$ *of* (3.71) *satisfies*

$$u(x) < 0,\ \ x \in \Omega,\ \ \frac{\partial u}{\partial n}(x) > 0,\ \ x \in \partial\Omega. \tag{3.73}$$

***Proof.*** (3.71) is uniquely solvable when $\lambda \neq \lambda_i(c)$ from Fredholm alternatives in Section 2.2. The result in part (1) follows from the maximum principle (Theorem 2.6). Let $W = \mathbf{R} \times W_0^{2,p}(\Omega)$ and let $Y = L^p(\Omega)$. Define $G : W \to Y$ by

$$G(\lambda, u) \equiv L_c u - \lambda u - (\lambda - \lambda_1(c))^2 f. \tag{3.74}$$

$G(\lambda_1(c), 0) = 0$, and

$$\begin{aligned} &G'(\lambda, u)[(s, v)] = -su - 2s(\lambda - \lambda_1(c))f + L_c v - \lambda v, \\ &G''(\lambda, u)[(s, v), (r, z)] = -2srf - sz - rv. \end{aligned} \tag{3.75}$$

Then $N(G'(\lambda_1(c), 0)) = span\{(1, 0), (0, \phi_1)\}$ is two dimensional, and $R(G'(\lambda_1(c), 0)) = \{y \in Y : \int_\Omega y\phi_1 dx = 0\}$ is codimension one; $G''(\lambda_1(c), 0)$ $[(0, \phi_1), (0, \phi_1)] = 0 \in R(G'(\lambda_1(c), 0))$, and $G''(\lambda_1(c), 0)$ $[(1, 0), (0, \phi_1)]$ $= -\phi_1 \notin R(G'(\lambda_1(c), 0))$. Hence we can apply Theorem 3.11, and the solutions of $G(\lambda, u) = 0$ near $(\lambda_1(c), 0)$ form two crossing curves. The tangential directions of the curves are $\psi_1 = (0, \phi_1)$, and $\psi_2 = (1, k\phi_1)$ where $k = -\int_\Omega f\phi_1 dx / \int_\Omega \phi_1^2 dx$. The direction $\psi_1$ corresponds to the trivial solutions $\lambda = \lambda_1(c)$ and $u = s\phi_1$ for $s \in \mathbf{R}$; and $\psi_2$ gives non-trivial solutions: $\lambda(s) = \lambda_1(c)+s+o(|s|)$ and $u(s) = ks\phi_1+o(|s|)$. Since $k < 0$, then $u(s) < 0$ and the solution $u$ of (3.71) is $u = [\lambda - \lambda_1(c)]^{-2}u(s) < 0$ for small $s > 0$, which corresponds to $\lambda_1(c) < \lambda < \lambda_1(c) + \delta_f$. □

This is an interesting application of a nonlinear bifurcation theorem to a linear problem, see [S7] for a more general abstract result. Notice $f > 0$ can be replaced by $\int_\Omega f\phi_1 dx > 0$ and the same result holds. We will show in Chapter 5 that the anti-maximum principle is useful in constructing subsolutions in some semi-positone problems.

## 3.5 Global bifurcation

Bifurcation theorems in the last two sections are of local nature, as they only describe the structure of the solution set near the bifurcation point. Global bifurcation theorem gives information of the connected components of the solution set in the function spaces, and they are usually proved via topological tools such as Leray-Schauder degree theory. As preparation, we first review the concept of Leray-Schauder degree and a key topological lemma. No proof is given here, and the readers can consult standard references such as [Ch2; De].

Let $X$ be a Banach space, and let $U$ be an open bounded subset of $X$. Denote by $K(\overline{U})$ the set of compact operators from $\overline{U}$ to $X$, and define

$$\begin{aligned} M = \{(I-G, U, y) : \ & U \subset X \text{ open bounded }, G \in K(\overline{U}), \\ & \text{and } y \notin (I-G)(\partial U)\}. \end{aligned} \tag{3.76}$$

Then the Leray-Schauder degree $d : M \to \mathbf{Z}$ is a well-defined function, which satisfies the following properties:

(1) $d(I, U, y) = 1$ if $y \in U$, and $d(I, U, y) = 0$ if $y \notin \overline{U}$;
(2) (Additivity) $d(I-G, U, y) = d(I-G, U_1, y) + d(I-G, U_2, y)$ if $U_1$ and $U_2$ are disjoint open subsets of $U$ so that $y \notin (I-G)(\overline{U}\backslash(U_1 \bigcup U_2))$;
(3) (Homotopy invariance) Suppose that $h : [0,1] \times \overline{U} \to X$ is compact and $y : [0,1] \to X$ is continuous, and $y(t) \notin (I-G)(\partial U)$, then $D(t) = d(I - h(t,\cdot), U, y(t))$ is a constant independent of $t \in [0,1]$.
(4) (Existence) If $d(I-G, U, y) \neq 0$, then there exists $u \in U$ such that $u - G(u) = y$;

(5) If for $G_1, G_2 \in K(\overline{U})$, $G_1(u) = G_2(u)$ for any $u \in \partial U$, then $d(I - G_1, U, y) = d(I - G_2, U, y)$.

Let $L$ be a linear compact operator on $X$. From Riesz-Schauder theory, the set of eigenvalues of $L$ is at most countably many, and the only possible limit point is $\lambda = 0$. For any eigenvalue $\lambda$ of $L$, the subspace

$$X_\lambda = \bigcup_{n=1}^{\infty} \{u \in X : (L - \lambda I)^n u = 0\} \tag{3.77}$$

is finite dimensional, and $dim(X_\lambda)$ is the *algebraic multiplicity* of the eigenvalue $\lambda$. The *geometric multiplicity* of $\lambda$ is defined as $dim\{u \in X : (L - \lambda I)u = 0\}$. The Leray-Schauder degree of a nonlinear mapping can be calculated from the following facts:

(1) If $L$ is a linear compact operator on $X$, then $d(I - \lambda L, B_R(0), 0) = (-1)^\beta$, where $B_R(v)$ is a ball centered at $v$ with radius $R$, and $\beta$ is the sum of algebraic multiplicity of eigenvalues $\mu$ of $L$ satisfying $\lambda\mu > 1$.
(2) Suppose that $G \in K(\overline{U})$, $u_0 \in U$ and $R > 0$ such that $u_0$ is the unique solution satisfies $u - G(u) = 0$ in $B_R(u_0)$, then the derivative $G'(u_0) : X \to X$ is a linear compact operator; if $\lambda = 1$ is not an eigenvalue of $G'(u_0)$, then $d(I - G, B_R(u_0), 0) = d(I - G'(u_0), B_R(0), 0)$ for some sufficiently small $R > 0$ (this number is also called *fixed point index* of $u_0$ with respect to $G$).

We also recall the following topological lemma (proof can be found in [Ch2; De]):

**Lemma 3.1.** *Let $(M, d)$ be a compact metric space, and let $A$ and $B$ be close subsets of $M$ such that $A \bigcap B = \emptyset$. Then there exist compact subsets $M_A$ and $M_B$ of $M$ such that $M_A \bigcup M_B = M$, $M_A \bigcap M_B = \emptyset$, $M_A \supset A$, and $M_B \supset B$.*

Consider

$$F(\lambda, u) = u - \lambda Lu - H(\lambda, u), \tag{3.78}$$

where $L : X \to X$ is a linear compact operator, and $H(\lambda, u)$ is compact on $U \subset \mathbf{R} \times X$ such that $||H(\lambda, u)|| = o(||u||)$ near $u = 0$ uniformly on bounded $\lambda$ intervals. Note the conditions imply that $F_u(\lambda, 0) = I - \lambda L$, and if 0 is an eigenvalue of $F_u(\lambda_0, 0)$, then $\lambda_0^{-1}$ must be an eigenvalue of the linear operator $L$. We will say that $\lambda$ is a *characteristic value* of $L$, if $\lambda^{-1}$ is an eigenvalue of $L$. Define $S = \{(\lambda, u) \in U : F(\lambda, u) = 0, u \neq 0\}$.

We say $(\lambda_0, 0)$ is a *bifurcation point* for the equation (3.78) if $(\lambda_0, 0) \in \overline{S}$ ($\overline{S}$ is the closure of $S$).

**Theorem 3.13. (Krasnoselski-Rabinowitz Global Bifurcation Theorem)** *Let $X$ be a Banach space, and let $U$ be an open subset of $\mathbf{R} \times X$ containing $(\lambda_0, 0)$. Suppose that $L$ is a linear compact operator on $X$, and $H(\lambda, u) : \overline{U} \to X$ is a compact operator such that $||H(\lambda, u)|| = o(||u||)$ as $u \to 0$ uniformly for $\lambda$ in any bounded interval. If $1/\lambda_0$ is an eigenvalue of $L$ with odd algebraic multiplicity, then $(\lambda_0, 0)$ is a bifurcation point. Moreover if $C$ is the connected component of $\overline{S}$ which contains $(\lambda_0, 0)$, then one of the following holds:*

*(i) $C$ is unbounded in $U$;*
*(ii) $C \bigcap \partial U \neq \emptyset$; or*
*(iii) $C$ contains $(\lambda_i, 0) \neq (\lambda_0, 0)$, such that $\lambda_i^{-1}$ is also an eigenvalue of $L$.*

***Proof.*** First we prove that $(\lambda_0, 0)$ is a bifurcation point. Suppose not, then there exists a $R > 0$ such that in the region $O = \{(\lambda, u) : |\lambda - \lambda_0| \leq R, |u| \leq R\}$, the only solutions of $F(\lambda, u) = 0$ are $\{(\lambda, 0) : |\lambda - \lambda_0| \leq R\}$. We choose $\lambda_-, \lambda_+$ so that $\lambda_0 - R < \lambda_- < \lambda_0 < \lambda_+ < \lambda_0 + R$. From the homotopy invariance of the Leray-Schauder degree,

$$d(F(\lambda_-, \cdot), B_\rho(0), 0) = d(F(\lambda_+, \cdot), B_\rho(0), 0),$$

for any $\rho \in (0, R)$. For $\rho$ small enough, $d(F(\lambda_\pm, \cdot), B_\rho(0), 0) = d(I - \lambda_\pm L, B_\rho(0), 0)$. But on the other hand,

$$|d(I - \lambda_+ L, B_\rho(0), 0) - d(I - \lambda_- L, B_\rho(0), 0)| = 1, \tag{3.79}$$

since $\lambda_0^{-1}$ is the only eigenvalue of $L$ in between $\lambda_-^{-1}$ and $\lambda_+^{-1}$, and the algebraic multiplicity of $\lambda_0^{-1}$ is odd. That is a contradiction. Thus $(\lambda_0, 0)$ is a bifurcation point.

Next we assume the stated alternatives do not hold, then $C$ is bounded in $U$, $C \bigcap \partial U = \emptyset$, and $C \bigcap \{(\lambda, 0) \in U\} = \{(\lambda_0, 0)\}$. From the compactness of $L$ and $H$, $C$ is compact since it is bounded. Let $C_\varepsilon = \{(\lambda, u) \in U : dist((\lambda, u), C) < \varepsilon\}$. Let $A = C$ and $B = S \bigcap \partial C_\varepsilon$. From Lemma 3.1, there exists compact $M_A$ and $M_B$ such that $M_A \bigcap M_B = \emptyset$, $M_A \bigcup M_B = S \bigcap \overline{C_\varepsilon}$, $M_A \supset C$ and $M_B \supset S \bigcap \partial C_\varepsilon$. Hence there exists an open bounded $U_0 = M_A$ such that

$$C \subset U_0 \subset \overline{U_0} \subset U, \quad \text{and } \overline{S} \bigcap \partial U_0 = \emptyset. \tag{3.80}$$

Define $U_0(\lambda) = \{u \in X : (\lambda, u) \in U_0\}$ for $\lambda \in I$ where $I = \{\lambda \in \mathbf{R} : (\{\lambda\} \times X) \bigcap U_0 \neq \emptyset\}$. Then $D(\lambda) = d(F(\lambda, \cdot), U_0(\lambda), 0)$ is constant for

$\lambda \in I$ since $\overline{S}\bigcap \partial U_0 = \emptyset$ and the homotopy invariance of $d(F, \Omega, 0)$, where $d(F(\lambda, \cdot), \Omega, 0)$ is the Leray-Schauder degree.

Since $(\lambda_0, 0)$ is the only intersection of $C$ with the line $\{(\lambda, 0)\}$, $U_0$ can be chosen so that $U_0 \bigcap \{(\lambda, 0) \in U\} = [\lambda_0 - \delta, \lambda_0 + \delta] \times \{0\}$, and no any point $\lambda$ in $[\lambda_0 - 2\delta, \lambda_0 + 2\delta]$ satisfies that $\lambda^{-1}$ is an eigenvalue of $L$. We choose $\lambda_\pm$ which satisfy $\lambda_0 - \delta < \lambda_- < \lambda_0 < \lambda_+ < \lambda_0 + \delta$. We choose $\rho > 0$ small enough so that $F(\lambda, u) \neq 0$ for $\lambda \in [\lambda_+, \lambda_0 + 2\delta]$ and $u \in B_\rho(0)\backslash\{0\}$, and we also choose $\lambda^* > \lambda_0 + 2\delta$ such that $U_0(\lambda^*) = \emptyset$. From the homotopy invariance of the Leray-Schauder degree on $U_0 \backslash ([\lambda_+, \lambda^*] \times \overline{B}_\rho(0))$, we have

$$d(F(\lambda_+, \cdot), U_0(\lambda_+)\backslash \overline{B}_\rho(0), 0) = d(F(\lambda^*, \cdot), U_0(\lambda^*), 0) = 0. \tag{3.81}$$

For the same argument,

$$d(F(\lambda_-, \cdot), U_0(\lambda_-)\backslash \overline{B}_\rho(0), 0) = 0. \tag{3.82}$$

On the other hand, from the additivity of the Leray-Schauder degree,

$$D(\lambda_\pm) = d(F(\lambda_\pm, \cdot), U_0(\lambda_\pm)\backslash \overline{B}_\rho(0), 0) + d(F(\lambda_\pm, \cdot), B_\rho(0), 0). \tag{3.83}$$

Hence we obtain

$$d(F(\lambda_+, \cdot), B_\rho(0), 0) = d(F(\lambda_-, \cdot), B_\rho(0), 0). \tag{3.84}$$

For $\rho > 0$ small enough,

$$d(F(\lambda_\pm, \cdot), B_\rho(0), 0) = d(I - \lambda_\pm L, B_\rho(0), 0). \tag{3.85}$$

From the formula of Leray-Schauder degree of $I - \lambda L$, we have (3.79) again. But (3.79) is a contradiction with (3.85). Hence the alternatives in the theorem hold. □

Krasnoselski [Kr] proved that $(\lambda_0, 0)$ is a bifurcation point if $\lambda_0$ is a characteristic value of odd algebraic multiplicity, and the global bifurcation result in Theorem 3.13 was due to Rabinowitz [R1]. One can also prove similar result for a cone in $X$, see for example, Dancer [D1], Amann [A1] and Deimling [De]. We also mention that degree theory can also be defined for Fredholm operators between $X$ and $Y$, different Banach spaces, and global bifurcation theory can be established for nonlinear operators with linearized operators being Fredholm operator with index zero. For these development see Kielhöfer [Ki1; Ki3], Ize [I], Fitzpatrick, Pejsachowiz and Rabier [FPR; PR1; PR2], López-Gómez [LG], Shi and Wang [SWang2] for details.

We apply Theorem 3.13 to (2.1). Recall from Section 2.1 that $K = (-\Delta)^{-1} : C^\alpha(\overline{\Omega}) \to C_0^{2,\alpha}(\overline{\Omega})$ is well defined as $K(f) = u$ that $u \in C_0^{2,\alpha}(\overline{\Omega})$

such that $-\Delta u = f$ for any $f \in C^\alpha(\overline{\Omega})$. We apply $K$ to equation (2.1), and it becomes

$$G(\lambda, u) \equiv u - \lambda K f(u) = 0. \tag{3.86}$$

We set the domain of $G(\lambda, u)$ to be $\mathbf{R} \times C^\alpha(\overline{\Omega})$. Apparently if $u \in E \equiv C^\alpha(\overline{\Omega})$ satisfies (3.86), then $u$ is a classical solution of (2.1). Combining with the maximum principle, we prove the global bifurcation theorem for the positive solutions of (2.1).

**Theorem 3.14.** *Let $f \in C^1(\mathbf{R}^+)$, $f(0) = 0$ and $f'(0) > 0$. Then $\lambda_0 = \lambda_1/f'(0)$ is a bifurcation point. Let $E = C^\alpha(\overline{\Omega})$, and let $S = \{(\lambda, u) \in \mathbf{R}^+ \times E : G(\lambda, u) = 0, u \neq 0\}$, where $G$ is defined in* (3.86)*. Then there exists a a connected component $\mathcal{C}$ of $\overline{S}$ such that $(\lambda_0, 0) \in \mathcal{C}$. Moreover, let $E^+ = \{u \in E : u(x) \geq 0$ in $\Omega\}$. Then $\mathcal{C}_+ = \mathcal{C} \bigcap (\mathbf{R}^+ \times E^+)$ is unbounded.*

***Proof.*** First we extend $f$ to $\mathbf{R}$ by an odd extension $f(u) = -f(-u)$ for $u < 0$. Then $f \in C^1(\mathbf{R})$, and the operator $G : \mathbf{R}^+ \times E \to E$ can be written as

$$G(\lambda, u) = u - \lambda f'(0) K u - \lambda K(f(u) - f'(0)u). \tag{3.87}$$

Then $H(\lambda, u) = \lambda K(f(u) - f'(0)u)$ is a nonlinear compact operator from the compactness of $K$, and apparently for $\lambda$ in a bounded interval, $||H(\lambda, u)|| \to 0$ as $||u|| \to 0$ uniformly since $f \in C^1(\mathbf{R})$. When $\lambda = \lambda_0 \equiv \lambda_1/f'(0)$, $N(I - \lambda f'(0)K) \neq \emptyset$ where $\lambda_1 = \lambda_1(0)$ is the principal eigenvalue defined in (2.11). Since $K$ is symmetric, then the algebraic multiplicity is same as the geometric multiplicity, and it is $dim N(I - \lambda f'(0)K)$. From Theorem 2.6, the multiplicity of the principal eigenvalue is 1. Hence all conditions in Theorem 3.13 are satisfied, and there is a a connected component $\mathcal{C}$ of $\overline{S}$ such that $(\lambda_0, 0) \in \mathcal{C}$.

From the alternatives in Theorem 3.13, $\mathcal{C}$ is unbounded, or $\mathcal{C} \bigcap \partial(\mathbf{R}^+ \times E) \neq \emptyset$, or there is another $\lambda_*$ such that $(\lambda_*, 0) \in \mathcal{C}$ and $\lambda_* f'(0)$ is another eigenvalue of $-\Delta$. If $(\lambda_a, u_a) \in \mathcal{C} \bigcap \partial(\mathbf{R}^+ \times E)$, then $\lambda_a = 0$, hence $u_a = 0$ from the uniqueness of Laplace equation. But near $(\lambda, u) = (0, 0)$, the only solutions of (2.1) are $(\lambda, 0)$ for $\lambda > 0$ from the implicit function theorem (Theorem 3.3), while $\overline{S}$ only contains $(\lambda_*, 0)$ such that $\lambda_* f'(0)$ is an eigenvalue of $-\Delta$. That is a contradiction since $-\Delta$ has no eigenvalues approaching 0. For the remaining cases, we note that from Hölder estimates, $\overline{S} \subset \mathbf{R} \times C_0^{2,\alpha}(\overline{\Omega})$. Hence $\mathcal{C} \subset \mathbf{R} \times C_0^{2,\alpha}(\overline{\Omega})$.

Define $E_2^+ = \{u \in C_0^{2,\alpha}(\overline{\Omega}) : u(x) \geq 0 \text{ in } \Omega\}$. Then the interior $int(E_2^+)$ of $E_2^+$ is non empty, and indeed $int(E_2^+) = \{u \in E_2^+ :$

$u(x) > 0$ in $\Omega, \partial u(x)/\partial\nu < 0$ on $\partial\Omega\}$. Let $\mathcal{C}_+ = \mathcal{C}\bigcap(\mathbf{R}^+ \times E_2^+)$ and $\mathcal{C}_- = \mathcal{C}\bigcap(\mathbf{R}^+ \times (-E_2^+))$. From Theorem 3.10, $\mathcal{C}_+ \neq \emptyset$ and $\mathcal{C}_- \neq \emptyset$. We claim $\mathcal{C}_+\bigcap(\mathbf{R}^+ \times \partial E_2^+) = \{(\lambda_0, 0)\}$. In fact, if $(\lambda, u) \in \mathcal{C}_+$, then $\lambda > 0$ from the argument in last paragraph, and $u$ is a non-negative classical solution. From the maximum principle (Theorem 2.7), either $u > 0$ or $u \equiv 0$. But if $u > 0$, $(\lambda, u)$ is in the interior of $\mathbf{R}^+ \times \partial E^+$, that is contradiction. Hence $u \equiv 0$, and $\lambda f'(0)$ is an eigenvalue of $-\Delta$. Near the bifurcation point, all solutions have the form $(\lambda, s\phi_k)$ where $\phi_k$ is the corresponding eigenfunction. But $\phi_1$ is the only eigenfunction of one sign from Proposition 2.4, thus $\lambda f'(0) = \lambda_1$ and $\lambda = \lambda_0$. Similarly $\mathcal{C}_-\bigcap(\mathbf{R}^+ \times \partial(-E^+)) = \{(\lambda_0, 0)\}$.

Define $\mathcal{C}_1 = \mathcal{C}_+\bigcup\{(\lambda_0, 0)\}\bigcup\mathcal{C}_-$. Then $\mathcal{C}_1 \subset O$, where $O = (\mathbf{R}^+ \times E_2^+)\bigcup(\mathbf{R}^+ \times (-E_2^+))\bigcup O_1$ and $O_1 = \{(\lambda, u) : |\lambda - \lambda_0| + ||u|| \leq \varepsilon\}$ for some small $\varepsilon > 0$. Moreover, $\mathcal{C}_1\bigcap\partial O = \emptyset$ where $\partial O$ is the boundary of $O$. Since $\mathcal{C}$ is the connected component of $\overline{S}$, then $\mathcal{C} = \mathcal{C}_1$ from the definition of connected component since $\mathcal{C}_1 \subset \mathcal{C}$ and $\mathcal{C}_1 \subset int(O)$. As a consequence, $\mathcal{C}$ cannot contain another $(\lambda_*, 0)$ so that $\lambda_* f'(0)$ is another eigenvalue of $-\Delta$. Hence $\mathcal{C}$ is unbounded. From our definition, $f(u)$ is an odd function, then $\mathcal{C}_-$ and $\mathcal{C}_+$ are symmetric in the sense that if $(\lambda, u) \in \mathcal{C}_+$ then $(\lambda, -u) \in \mathcal{C}_-$. Therefore $\mathcal{C}_+$ is unbounded. $\square$

It is useful to remark that Theorem 3.14 implies that the global continuum $\mathcal{C}_+$ is also unbounded in $\mathbf{R}^+ \times C^0(\overline{\Omega})$. In fact, if $\mathcal{C}_+$ is bounded in $\mathbf{R}^+ \times C^0(\overline{\Omega})$, $\mathcal{C}_+$ have uniform $L^p$ norm for any $p > 1$, $f$ is $C^1$ and $f$ can be assumed as bounded, then $\mathcal{C}_+$ also satisfy uniform $L^p$ estimates thus uniformly bounded in $W^{2,p}$ norm. From Sobolev embedding theorem, $\mathcal{C}_+$ is also bounded in $C^\alpha$ norm where $\alpha < 2-(n/p)$, that contradicts Theorem 3.14. On the other hand, if one can establish uniform *a priori* estimates for the positive solutions of (2.1), *i.e.* given $\Lambda > 0$, for $\lambda \in [0, \Lambda]$, $||u||_\infty \leq K_\Lambda$ for any positive solution $(\lambda, u)$, then $\mathcal{C}_+$ can be extended to $\lambda = \infty$. That is, let $p_+$ be the projection of $\mathcal{C}_+$ onto the $\lambda$-axis, then $p_+ \supset (\lambda_0, \infty)$. In that case, the *a priori* estimates and global bifurcation theorem give the existence for not only large $\lambda$ but a continuum of solutions.

**Example 3.5.** First we continue the discussion started in Example 3.2. Consider

$$\begin{cases} \Delta u + \lambda(u - u^p) = 0, & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{3.88}$$

where $p \geq 2$. We have shown in Example 3.2 that $\lambda = \lambda_1$ is a bifurcation point, and the solution set near $(\lambda_1, 0)$ is a curve. Now applying Theorem

3.14, then the branch bifurcating from $(\lambda_1, 0)$ is indeed unbounded. Moreover the maximum principle (Theorem 2.7) implies that $\max_{x\in\overline{\Omega}} u(x) < 1$ for any positive solution $(\lambda, u)$. From remark above, this implies the existence of a positive solution $(\lambda, u)$ of (3.88) for any $\lambda \in (\lambda_1, \infty)$, and $(\lambda, u) \in \mathcal{C}_+$, the continuum emanating from $(\lambda_1, 0)$. In fact $p_+ = (\lambda_0, \infty)$ in this case since (3.88) has no positive solution when $\lambda \leq \lambda_1$ (see Proposition 2.8). In Section 5.2, we will return to this problem, and we will show that the positive solution of (3.88) is unique hence $\mathcal{C}_+$ is globally a smooth curve.

Next we look at an equation:

$$\begin{cases} \Delta u + \lambda(u + u^p) = 0, & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{3.89}$$

where $p > 1$. Similar to (4.33), a bifurcation occurs at $(\lambda_1, 0)$ and the global continuum $\mathcal{C}_+$ is unbounded. But we can show that (3.89) has no positive solution when $\lambda \geq \lambda_1$ (see Proposition 2.8). Hence $p_+ \subset (0, \lambda_1)$ is bounded, and the unboundedness of $\mathcal{C}_+$ implies that $||u||_\infty$ is not uniformly bounded for $(\lambda, u) \in \mathcal{C}_+$. Note that we do not assume $p < (n+2)/(n-2)$, as the bifurcation occurs for (3.89) for any $p > 1$. For the critical case $p = (n+2)/(n-2)$, (3.89) was studied in Brezis and Nirenberg [BN].

We also point out that if $f(0) > 0$, then the curve emanating from $(\lambda, u) = (0, 0)$ is also part of an unbounded continuum:

**Theorem 3.15.** *Let $f \in C^1(\mathbf{R}^+)$ and $f(0) > 0$. Assume that $S, E, E_+$ and $G$ are defined as in Theorem 3.14. Then there exists a a connected component $\mathcal{C}$ of $\overline{S}$ such that $(0, 0) \in \mathcal{C}$, and $\mathcal{C}_+ = \mathcal{C} \bigcap (\mathbf{R}^+ \times E^+)$ is unbounded.*

The proof is similar to that of Theorem 3.13 in spirit, and we omit the details here. It is also a special case of a result of Rabinowitz [R1] (Theorem 3.2).

## 3.6 Bifurcation from infinity

We say that $(\lambda_\infty, \infty)$ is a bifurcation point for $F(\lambda, u) = 0$ if there exists a sequence of solutions $(\lambda_k, u_k)$ to the equation such that $\lim_{k\to\infty} \lambda_k = \lambda_\infty$ and $\lim_{k\to\infty} ||u_k|| = \infty$, and we say that a *bifurcation from infinity* occurs at $\lambda = \lambda_\infty$. We recall the equation defined in(3.78) again:

$$F(\lambda, u) \equiv u - \lambda L u - H(\lambda, u) = 0. \tag{3.90}$$

Again we recall that $S = \{(\lambda, u) \in \mathbf{R} \times X : F(\lambda, u) = 0, u \neq 0\}$. The following result by Rabinowitz [R2] is in the same spirit as Theorem 3.13, but for bifurcation from infinity:

**Theorem 3.16.** *Suppose that $L$ is a compact operator on $X$, $H(\lambda, u)$ is a continuous operator on $\mathbf{R} \times X$, $H(\lambda, u) = o(||u||)$ at $u = \infty$ uniformly on bounded $\lambda$ intervals, and $||u||^2 H(\lambda, u/||u||^2)$ is compact. If $\lambda_0^{-1}$ is an eigenvalue of $L$ of odd algebraic multiplicity, then $(\lambda_0, \infty)$ is a bifurcation point, and there is a closed connected component $C_1$ of $\overline{S}$ which meets $(\lambda_0, \infty)$.*

*Moreover if $\Lambda \subset \mathbf{R}$ is an interval such that $\lambda_0^{-1}$ is the only eigenvalue of $L$ in $\Lambda$, and $M$ is a neighborhood of $(\lambda_0, \infty)$ whose projection on $\mathbf{R}$ lies in $\Lambda$ and whose projection on $X$ is bounded away from 0, then either (i) $S \backslash M$ is bounded in $\mathbf{R} \times X$ in which case $S \backslash M$ meets $\{(\lambda, 0) : \lambda \in \mathbf{R}\}$; or (ii) $S \backslash M$ is unbounded, and if $S \backslash M$ has a bounded projection on $\mathbf{R}$, then $S \backslash M$ meets $(\lambda_i, \infty)$ where $\lambda_i^{-1} (\neq \lambda_0^{-1})$ is an eigenvalue of $L$.*

***Proof.*** Let $w = u/||u||^2$. Then the equation (3.90) becomes

$$w - \lambda L w - ||w||^2 H(\lambda, w/||w||^2) = 0. \tag{3.91}$$

Define $Q(\lambda, w) = ||w||^2 H(\lambda, w/||w||^2)$ for any $w \neq 0$ and $Q(\lambda, w) = 0$ for $w = 0$. Then $Q : \mathbf{R} \times X \to X$ is continuous, $Q(\lambda, w) = o(||w||)$ as $w \to 0$ uniformly on bounded $\lambda$ interval, and $Q$ is compact from the assumptions. Now if $\lambda_0^{-1}$ is an eigenvalue of $L$ of odd algebraic multiplicity, then from Theorem 3.13, $(\lambda_0, 0)$ is a bifurcation point for (3.91), and consequently $(\lambda_0, \infty)$ is a bifurcation point for (3.90). Moreover (3.91) possesses a connected component $C_1$ of nontrivial solutions containing $(\lambda_0, 0)$, which is either unbounded, or $C_1$ contains $(\lambda_i, 0) \neq (\lambda_0, 0)$ such that $\lambda_1^{-1}$ is another eigenvalue of $L$. Then using the inversion $w \mapsto w/||w||^2 = u$, we obtain the results stated in the theorem. □

Theorem 3.16 has a global nature, but it does not specify the structure of the solution set near the infinity bifurcation point. As suggested by Theorem 3.8, the solution set is locally a smooth curve if 0 is a simple eigenvalue of $I - \lambda_0 L$ and related operator is also continuously differentiable. The following result is an easy application of Theorem 3.8.

**Theorem 3.17.** *Let $\lambda_0 (\neq 0) \in \mathbf{R}$ and let $F : \mathbf{R} \times X \to X$ be a continuously differentiable mapping such that $F(\lambda, u) = u - \lambda L u - H(\lambda, u)$, where $L$ is a continuous linear operator on $X$. We also assume that*

(1) $\lambda_0^{-1}$ *is an isolated point of the spectrum of* $L$ *such that* $dimN(I - \lambda_0 L) = codimR(I - \lambda_0 L) = 1$, $dimN(I - \lambda_0 L) = span\{w_0\}$, *and* $Lw_0 \notin R(I - \lambda_0 L)$;
(2) $||H(\lambda, u)||/||u|| \to 0$, *as* $||u|| \to \infty$, *uniformly for* $\lambda$ *near* $\lambda_0$;
(3) *Define* $Q(\lambda, w) = ||w||^2 H(\lambda, w/||w||^2)$ *for any* $w \neq 0$ *and* $Q(\lambda, w) = 0$ *for* $w = 0$. *Then* $Q$ *is continuously differentiable in* $\mathbf{R} \times X$, $Q_w(\lambda, 0) = 0$ *for* $\lambda \in \mathbf{R}$, *and* $Q_{\lambda w}$ *exists and is continuous near* $(\lambda_0, 0)$.

*Let* $Z$ *be any complement of* $span\{w_0\}$ *in* $X$. *Then the solution set of* $F(\lambda, u) = 0$ *near* $(\lambda_0, \infty)$ *consists precisely the curve* $\{(\lambda(s), u(s)) : |s| > \delta\}$, *where* $\lambda(s), u(s)$ *are continuous functions such that* $\lim_{|s|\to\infty} \lambda(s) = \lambda_0$, *and* $u(s) = sw_0 + sz(s)$ *with* $z(s) \in Z$ *and* $\lim_{|s|\to\infty} z(s) = 0$.

***Proof.*** We apply a weaker version of Theorem 3.8 (see remark after Theorem 3.8) to $\widetilde{F}(\lambda, w) = w - \lambda Lw - Q(\lambda, w)$. Then $\widetilde{F}$ is continuously differentiable, and $\widetilde{F}(\lambda, 0) \equiv 0$. From assumptions, $\widetilde{F}_u(\lambda, u)$, $\widetilde{F}_\lambda(\lambda, u)$ and $\widetilde{F}_{\lambda u}(\lambda, u)$ exist and are continuous near $(\lambda_0, 0)$. Moreover (F1) and (F3) are satisfied. Hence, from the weaker version of Theorem 3.8, the solution set of $\widetilde{F}(\lambda, w) = 0$ near $(\lambda_0, 0)$ are exactly the continuous curves $w = 0$ and $\{(\lambda(t), tw_0 + tz(t)) : |t| < \epsilon\}$. For $\epsilon$ small enough, $t \mapsto t/||tw_0 + tz(t)||^2 \equiv s(t)$ is one-to-one for $0 < |t| < \epsilon$ thus invertible, hence there exists $\delta > 0$ such that $\{(\lambda(t(s)), sw_0 + sz(t(s))) : |s| > \delta\}$ is a solution curve for $F(\lambda, u) = 0$ where $t(s)$ is the inverse of $s(t)$ defined above. Clearly $t(s) \to 0$ as $s \to \pm\infty$. □

The assumption that the nonlinear operator $Q$ is differentiable is necessary for the smoothness of the solution curve, and it requires the differentiability of the norm function in Banach space $X$. The condition that the norm function $u \mapsto ||u||$ is $C^1$ for $u \neq 0$ is not restrictive. We can assume that $X$ is based on $L^p(\Omega)$ for bounded domain $\Omega$, and it is well-known that the norm of $L^p(\Omega)$ is $C^1$ for $u \neq 0$ and $p \in (1, \infty)$. In general, Restrepo [Res] proved that a separable Banach space $X$ has an equivalent norm of class $C^1$ on $X \backslash \{0\}$ if and only if $X^*$ is separable. Hence if that is the case, we work with this equivalent norm from the beginning. See also [SWang2] for related results. We also notice that we do not need compactness assumptions in Theorem 3.17.

One can also prove a bifurcation from infinity at a simple eigenvalue theorem without differentiability assumption on the nonlinear operator, but only assuming some weaker conditions. Here we state such a result due to Dancer [D2], and the proof can be found in [D2]:

**Theorem 3.18.** *Let $\lambda_0(\neq 0) \in \mathbf{R}$ and let $F : \mathbf{R} \times X \to X$ be a continuous mapping such that $F(\lambda, u) = u - \lambda Lu - H(\lambda, u)$, where $L$ is a continuous linear operator on $X$. We also assume that*

*(1) $\lambda_0^{-1}$ is an isolated point of the spectrum of $L$ such that $dimN(I - \lambda_0 L) = codimR(I - \lambda_0 L) = 1$, $dimN(I - \lambda_0 L) = span\{w_0\}$, and $Lw_0 \notin R(I - \lambda_0 L)$;*
*(2) $||H(\lambda, u)||/||u|| \to 0$, as $||u|| \to \infty$, uniformly for $\lambda$ near $\lambda_0$;*
*(3) For any $\varepsilon > 0$, there exist $M, \gamma > 0$ such that, if $|\lambda_a - \lambda_0| \leq \gamma$ and $|\lambda_b - \lambda_0| \leq \gamma$, $u = \alpha w_0 + w$, $v = \alpha w_0 + z$ where $|\alpha| \geq M$, $||w|| \leq \gamma|\alpha|$, and $||z|| \leq \gamma|\alpha|$, then*

$$||H(\lambda_a, u) - H(\lambda_b, v)|| \leq \varepsilon[||u - v|| + (||u|| + ||v||)|\lambda_a - \lambda_b|].$$

*If $Z$ is a complement of $span\{w_0\}$ in $X$, then there exists $N > 0$ and continuous mappings $\lambda : \{s : |s| \geq N\} \to \mathbf{R}$ and $\psi : \{s : |s| \geq N\} \to Z$ such that $\lambda(s) \to \lambda_0$ and $||\psi(s)|| \to 0$ as $|s| \to \infty$ and $F(\lambda(s), sw_0 + s\psi(s)) = 0$. Moreover, there exist $C$ and $\rho > 0$ such that each solution $(\lambda, u)$ of $F(\lambda, u) = 0$ with $|\lambda - \lambda_0| \leq \rho$ and $||u|| \geq C$ has the above form.*

Bifurcation from infinity for asymptotically linear problems has been studied in many papers. Besides [R2; D2], we mention Arcoya and Gámez [AG], Ambrosetti and Hess [AH], Deimling [De], Kielhöfer [Ki3], Peitgen and Schmitt [PeS], Schmitt [Sch], Shi [S2], Stuart [Stu], and Toland [To].

We now apply the abstract theorem of bifurcation from infinity to the positive solutions of (2.1).

**Theorem 3.19.** *Suppose that $f \in C^1(\mathbf{R}^+)$, and $f_\infty = \lim_{u\to\infty} \dfrac{f(u)}{u} = \lim_{u\to\infty} f'(u) \in (0, \infty)$ and $\lambda_\infty = \dfrac{\lambda_1}{f_\infty}$. Then all positive solutions of (2.1) near $(\lambda_\infty, \infty)$ have the form $(\lambda(s), s\phi_1 + z(s))$ for $s \in (\delta, \infty)$ and some $\delta > 0$, $\lim_{|s|\to\infty} \lambda(s) = \lambda_\infty$, and $||z(s)|| = o(|s|)$ as $|s| \to \infty$.*

***Proof.*** We extend $f$ to $C^1(\mathbf{R})$ so that $\lim_{u\to-\infty} \dfrac{f(u)}{u} = \lim_{u\to-\infty} f'(u) = f_\infty$. Recall the operator defined in (3.86), which can be rewritten as

$$G(\lambda, u) = u - \lambda f_\infty Ku - \lambda K[f(u) - f_\infty u]. \tag{3.92}$$

Here we use $X = L^p(\Omega)$ for $p > n$. Then $\lambda_\infty^{-1}$ is a simple eigenvalue of $L = f_\infty K$, and $L\phi_1 \notin R(I - \lambda_\infty L)$ since $L\phi_1 = \lambda_\infty^{-1}\phi_1 > 0$. Also $||H(\lambda, u)|| = |\lambda| \cdot ||K(f(u) - f_\infty u)|| = o(||u||)$ for $||u|| \to \infty$ uniformly for $\lambda$ near $\lambda_\infty$ from the assumptions.

Define $Q(\lambda, w) = \lambda K[||w||^2 f(w/||w||^2) - f_\infty w]$ for $w \neq 0$, and $Q(\lambda, w) = 0$ for $w = 0$. Then apparently $Q$ is continuous for any $(\lambda, w) \in \mathbf{R} \times X$. For $w \neq 0$,

$$Q_w(\lambda, w)[\psi] = \lambda K \left[ 2||w|| \left[ f\left(\frac{w}{||w||^2}\right) - \frac{w}{||w||^2} f'\left(\frac{w}{||w||^2}\right)\right] ||w||'[\psi] \right] + \lambda K \left[ f'\left(\frac{w}{||w||^2}\right) - f_\infty \right] \psi,$$

where $||w||'[\psi]$ is the Frechét derivative of the norm function $||\cdot||$ at $w \neq 0$. Then as $w \to 0$, $Q_w(\lambda, w)[\psi] \to 0$ from assumptions, hence $Q_w(\lambda, w)$ is continuous at $w = 0$ for any $\lambda \in \mathbf{R}$ and $Q_w(\lambda, 0) = 0$. Also $Q_{\lambda w} = \lambda^{-1} Q_w$ also exists and is continuous. Now we can apply Theorem 3.17 to obtain the result stated. Notice that any solution on the portion of the solution set with large $s > 0$ must be positive since $\phi_1 > 0$ and $z(s) = o(|s|)$ as $s \to \infty$. □

To use the information from local bifurcation in determining the global bifurcation curve, it is important to know the bifurcation direction of solution curve at bifurcation point. Let $(\lambda(s), u(s))$, $s \in I$, be a curve of positive solutions to (2.1), where $I = (0, \delta)$, $(\delta, \infty)$ as in Theorem 3.10 or 3.19. Here we denote the bifurcation point by $(\lambda_*, u_*)$. In Theorem 3.10, if we assume that $f$ is $C^2$ at $u = 0$, then the bifurcation direction can be determined by (3.57). In the following result, we only assume that $f \in C^1$, thus the solution curve is only continuous.

**Definition 3.1.** If there is $\delta_0 > 0$, such that $\lambda(s) \geq \lambda_*$ for $s \in I$, then we say a *supercritical bifurcation* occurs at $(\lambda_*, u_*)$; Similarly, if $\lambda(s) \leq \lambda_*$ for $s \in I$, then we say a *subcritical bifurcation* occurs at $(\lambda_*, u_*)$.

**Proposition 3.1.** *Assume that $f \in C^1(\mathbf{R}^+)$.*

(1) *Suppose that $(\lambda_0, 0)$ is a bifurcation point where a bifurcation from the trivial solutions occurs, and $(\lambda(s), u(s))$, $s \in (0, \delta)$, is the curve of positive solutions in Theorem 3.10. If there exists $\delta_1 > 0$ such that $f(u)/u \geq f'(0)$ (resp. $\leq f'(0)$) in $[0, \delta_1]$, then the solution curve $(\lambda(s), u(s))$ is subcritical (resp. supercritical).*
(2) *Suppose that that $(\lambda_\infty, \infty)$ is a bifurcation point where a bifurcation from infinity occurs, and $(\lambda(s), u(s))$, $s \in (\delta, \infty)$, is the curve of positive solutions in Theorem 3.19. If $f(u)/u \leq f_\infty$ (resp. $\geq f_\infty$) for $u \in (0, \infty)$, then the solution curve $(\lambda(s), u(s))$ is supercritical (resp. subcritical).*

***Proof.*** (1) Let $\phi_1$ be the normalized positive eigenfunction corresponding to $\lambda_1 = \lambda_0 f'(0)$. Then $\phi_1$ and $u(s)$ satisfy

$$\Delta\phi_1 + \lambda_0 f'(0)\phi_1 = 0 \tag{3.93}$$

and

$$\begin{aligned} &\Delta u(s) + \lambda_0 f'(0)u(s) + (\lambda(s) - \lambda_0)f'(0)u(s) \\ &+\lambda(s)[f(u(s)) - f'(0)u(s)] = 0. \end{aligned} \tag{3.94}$$

By integration, we get

$$\begin{aligned} &(\lambda(s) - \lambda_0)f'(0)\int_\Omega u(s)\phi_1 dx \\ &+\lambda(s)\int_\Omega \left[\frac{f(u(s))}{u(s)} - f'(0)\right] u(s)\phi_1 dx = 0. \end{aligned} \tag{3.95}$$

By the regularity theory of elliptic equation, since $f \in C^1$, $u(s) \in C^{2,\alpha}(\overline{\Omega})$, then for $\delta_1 > 0$, $||u(s)||_{C^{2,\alpha}(\overline{\Omega})} \leq \delta_1$ when $s > 0$ is small enough. If $f(u)/u \geq f'(0)$ for $u \in [0, \delta_1]$, then for $s > 0$ is small enough, the second integral in (3.95) is positive, hence $\lambda(s) < \lambda_0$ for small $s > 0$. The case of $f(u)/u \leq f'(0)$ is similar.

(2) Similar to (3.95), we have

$$\begin{aligned} &(\lambda(s) - \lambda_\infty)f_\infty \int_\Omega u(s)\phi_1 dx \\ &+\lambda(s)\int_\Omega \left[\frac{f(u(s))}{u(s)} - f_\infty\right] u(s)\phi_1 dx = 0. \end{aligned} \tag{3.96}$$

If $f(u)/u \leq f_\infty$, the second integral in (3.96) is negative, hence $\lambda(s) > \lambda_\infty$ for all $s > 0$. The case of $f(u)/u \geq f_\infty$ is similar. □

**Example 3.6.** Consider the following modified logistic equation with a "saturated crowding effect":

$$\begin{cases} \Delta u + \lambda\left(u - \dfrac{ku^2}{m+u}\right) = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.97}$$

where $m > 0$ and $0 < k < 1$. From Theorem 3.10, $\lambda_0 = \lambda_1$ is a bifurcation point where positive solutions of (3.97) bifurcate from the line of trivial solutions. On the other hand, $\lambda_\infty = \lambda_1/(1-k)$ is a bifurcation point where positive solutions bifurcate from infinity. From Proposition 3.1, the bifurcation from $u = 0$ is supercritical, and the bifurcation from infinity is subcritical. Indeed (3.95) and (3.96) shows that (3.97) only has positive solutions when $\lambda_0 < \lambda < \lambda_\infty$, thus the global bifurcation theorem (Theorem 3.14) implies that the curve of positive solutions near $u = 0$ and the one near infinity connect to each other. In fact, the growth rate per capita $f(u)/u$ is strictly decreasing, hence we will show in Theorem 4.1 that (3.97) has a unique positive solution $u_\lambda$ if and only if $\lambda \in (\lambda_0, \lambda_\infty)$, and for any $s > 0$, there exists a unique $\lambda \in (\lambda_0, \lambda_\infty)$ such that $||u_\lambda||_\infty = s$.

In (2) of Proposition 3.1, we require $f(u)/u \leq f_\infty$ for all $u > 0$, which is a very strong condition. In the future applications, we need a result which only imposes conditions on $f$ near $\infty$. The following result is proved in Ambrosetti and Hess [AH]:

**Proposition 3.2.** *Suppose that $(\lambda_*, \infty)$ is a bifurcation point where a bifurcation from infinity occurs, $(\lambda(s), u(s))$, $s \in (\delta, \infty)$, is the curve of the positive solutions in Theorem 3.19. We assume that*

$$\liminf_{u\to\infty}[f(u) - f_\infty u] < 0, \text{ (resp. } \limsup_{u\to\infty}[f(u) - f_\infty u] > 0,) \tag{3.98}$$

*then $(\lambda(s), u(s))$ is supercritical (resp. subcritical).*

***Proof.*** Again we use (3.96). Since $u(s)/||u(s)||_\infty \to \phi_1$ almost everywhere in $\Omega$ as $\lambda \to \lambda_*$, thus $u(s)(x) \to \infty$ for almost all $x \in \Omega$. Therefore, let $c = \liminf_{u\to\infty}[f(u) - f_\infty u] < 0$, then by Fatou's Lemma,

$$\lim_{\lambda\to\lambda_*} \int_\Omega [f(u) - f_\infty u]\phi_1 dx \leq c \int_\Omega \phi_1 dx < 0. \tag{3.99}$$

Thus $(\lambda(s), u(s))$ is supercritical. □

## 3.7 Stability

The stability of the solution $(\lambda_0, u_0)$ of an equation $F(\lambda, u) = 0$ is determined by the linearization $F_u(\lambda_0, u_0)$: $X \to Y$. Very often it is related to a generalized eigenvalue problem. Let $B(X, Y)$ denote the set of bounded linear maps from $X$ into $Y$. Let $K \in B(X, Y)$. Then $\mu \in \mathbf{R}$ is a $K$-*simple* eigenvalue of $T$ if

$$dim N(T - \mu K) = codim R(T - \mu K) = 1, \tag{3.100}$$

and if $N(T - \mu K) = span\{w_0\}$, then

$$Kw_0 \notin R(T - \mu K). \tag{3.101}$$

First we prove a perturbation result for the $K$-simple eigenvalue due to Crandall and Rabinowitz [CR2] (Lemma 1.3):

**Theorem 3.20.** *Suppose that* $T_0$, $K \in B(X,Y)$ *and* $\mu_0$ *is a* $K$*-simple eigenvalue of* $T_0$. *Then there exists* $\delta > 0$ *such that if* $T \in B(X,Y)$ *and* $||T - T_0|| < \delta$, *then there exists a unique* $\mu(T) \in \mathbf{R}$ *satisfying* $|\mu(T) - \mu_0| < \delta$ *such that* $N(T - \mu(T)K) \neq \emptyset$, *and* $\mu(T)$ *is a* $K$*-simple eigenvalue of* $T$. *Moreover if* $N(T_0 - \mu_0 K) = span\{w_0\}$ *and* $Z$ *is a complement of* $span\{w_0\}$ *in* $X$, *then there exists a unique* $w(T) \in X$ *such that* $N(T - \mu(T)K) = span\{w(T)\}$, $w(T) - w_0 \in Z$, *and the map* $T \mapsto (\mu(T), w(T))$ *is analytic.*

***Proof.*** Define $g : B(X,Y) \times \mathbf{R} \times Z \to Y$ by

$$g(T, \mu, z) = T(w_0 + z) - \mu K(w_0 + z).$$

Then from assumption, $g(T_0, \mu_0, 0) = 0$ and $D_{(\mu,z)} g(T_0, \mu_0, 0)[\eta, y] = (T_0 - \mu_0 K)y - \eta K w_0$. It is easy to show that $D_{(\mu,z)} g(T_0, \mu_0, 0)$ is an isomorphism from $\mathbf{R} \times Z$ to $Y$. Thus from the implicit function theorem (Theorem 3.2), for $T$ in a neighborhood of $T_0$ in $B(X,Y)$, there exists a unique $(\mu(T), z(T))$ near $(\mu_0, 0)$ such that $g(T, \mu(T), z(T)) = 0$ and $\mu(T_0) = \mu_0$, $z(T_0) = 0$. Let $w(T) = w_0 + z(T)$. Then $(\mu(T), w(T))$ is a $K$-eigen-pair satisfying the existence part of the theorem.

Next we prove that if $||T - T_0||$ and $|\mu - \mu_0|$ are sufficiently small, and $(\mu, w)$ is an $K$-eigen-pair of $T$, then $\mu = \mu(T)$ and $w = k(w_0 + z(T))$ for $k \in \mathbf{R}$. In fact, we decompose $w = w_0 + z$, then $(T - \mu K)(w_0 + z) = 0$, or equivalently

$$(T_0 - \mu_0 K)z = (\mu - \mu_0)K(w_0 + z) - (T - T_0)(w_0 + z).$$

Since $(T_0 - \mu_0 K)|_Z$ is an isomorphism, then $(T_0 - \mu_0 K)|_Z^{-1} : R(T_0 - \mu_0 K) \to Z$ is bounded. Hence

$$||z|| \leq C_1(|\mu - \mu_0| C_2 + ||T - T_0||)(||w_0|| + ||z||),$$

where $C_1 = ||(T_0 - \mu_0 K)|_Z^{-1}||$, and $C_2 = ||K||$. In particular, if we choose $|\mu - \mu_0| < 1/(4C_1 C_2)$, and $||T - T_0|| < 1/(4C_1)$, then

$$||z|| \leq 2C_1(|\mu - \mu_0| C_2 + ||T - T_0||)||w_0||, \tag{3.102}$$

which implies that $z$ is in a neighborhood of $z = 0$ if $||T - T_0||$ and $|\mu - \mu_0|$ are sufficiently small. Therefore $(\mu, w_0 + z)$ is near $(\mu_0, w_0)$, and the uniqueness from implicit function theorem implies $(\mu, w_0 + z) = (\mu(T), w_0 + z(T))$. This also proves that $N(T - \mu(T)K)$ is one-dimensional. It is well-known that a small perturbation of a Fredholm operator is still Fredholm, and the Fredholm index is locally constant ([Kat] Theorem 5.31). Hence $dim(N(T - \mu(T)K)) = 1$ implies that $codim(R(T - \mu(T)K)) = 1$.

Finally we prove $K(w(T)) \notin R(T-\mu(T)K)$. Suppose this is not true, then there exists $y \in X$ such that $(T-\mu(T)K)y = K[w(T)]$ for some $T$ close to $T_0$. From the continuity of $w(T)$, we have $X = span\{w(T)\} \oplus Z$. Then $y = \beta w(T) + z_1$ for some $\beta \in \mathbf{R}$ and $z_1 \in Z$, and $z_1$ satisfies $(T-\mu(T)K)z_1 = K[w(T)]$. From the properties of $(\mu(T), w(T))$, we have

$$(T_0 - \mu_0 K)z_1 + [(T-T_0) - (\mu - \mu_0)K]z_1 = Kw_0 + K[z(T)]. \quad (3.103)$$

Define $l \in Y^*$ which satisfies $R(T_0 - \mu_0 K) = \{v \in Y : \langle l, v\rangle = 0\}$. Applying $l$ to (3.103), we have

$$\langle l, [(T-T_0) - (\mu - mu_0)K]z_1\rangle = \langle l, Kw_0\rangle + \langle l, K[z(T)]\rangle.$$

Since $z(T)$ satisfies the estimate (3.102), then we obtain

$$|\langle l, Kw_0\rangle| \leq C_3(|\mu - \mu_0|C_2 + ||T - T_0||)$$

for some $C_3 > 0$, which contradicts with $Kw_0 \notin R(T_0 - \mu_0 K)$. Thus $K[w(T)] \notin R(T-\mu(T)K)$. $\square$

Now we apply the perturbation result to consider the stability of solutions near a saddle-node bifurcation point in Theorem 3.6 (see [CR2] Theorem 3.6):

**Theorem 3.21.** *Let $F$, $Z$, $\lambda_0$, $u_0$ and $w_0$ be as in Theorem 3.6, and $(\lambda(s), u(s))$ be the solution curve in Theorem 3.6. Suppose that for $K \in B(X,Y)$, $\mu = 0$ is a $K$-simple eigenvalue of $F_u(\lambda_0, u_0)$. Then there exist $C^1$ functions $\mu : (-\epsilon, \epsilon) \to \mathbf{R}$, $w : (-\epsilon, \epsilon) \to X$ such that*

$$F_u(\lambda(s), u(s))w(s) = \mu(s)Kw(s) \quad \text{for } s \in (-\epsilon, \epsilon), \quad (3.104)$$

*and $w(0) = w_0$, $\mu(0) = 0$, $w(s) - w_0 \in Z$. Moreover, let $l \in Y^*$ satisfy $N(l) = R(F_u(\lambda_0, u_0))$, then near $s = 0$, $\langle l, Kw_0\rangle\mu(s)$ and $-\lambda'(s)\langle l, F_\lambda(\lambda_0, u_0)\rangle$ have the same zeros and, whenever $\lambda'(s) \neq 0$, the same sign. More precisely,*

$$\lim_{s\to 0} \frac{\mu(s)}{\lambda'(s)} = -\frac{\langle l, F_\lambda(\lambda_0, u_0)\rangle}{\langle l, Kw_0\rangle}. \quad (3.105)$$

***Proof.*** The existence of $(\mu(s), w(s))$ follows directly from Theorem 3.20. To derive (3.105), we differentiate $F(\lambda(s), u(s)) = 0$ to obtain

$$F_\lambda(\lambda(s), u(s))\lambda'(s) + F_u(\lambda(s), u(s))[u'(s)] = 0. \quad (3.106)$$

Then from (3.104) and (3.106), we obtain

$$F_u(\lambda(s), u(s))[w(s) - u'(s)] = \mu(s)Kw(s) + F_\lambda(\lambda(s), u(s))\lambda'(s). \quad (3.107)$$

$F_u(\lambda(s),u(s))|_Z$ is an isomorphism since $F$ is $C^1$ and $F_u(\lambda_0,u_0)|_Z$ is an isomorphism. Thus $w(s)-u'(s)\in Z$ implies that

$$||w(s)-u'(s)||\le C_4(|\lambda'(s)|+|\mu(s)|) \tag{3.108}$$

for some $C_4>0$. On the other hand, (3.107) can be rewritten as

$$\begin{aligned}&\mu(s)Kw_0+\lambda'(s)F_\lambda(\lambda_0,u_0)\\ =&F_u(\lambda_0,u_0)(w(s)-u'(s))+[F_u(\lambda(s),u(s))-F_u(\lambda_0,u_0)](w(s)-u'(s))\\ &+\mu(s)K(w_0-w(s))+\lambda'(s)[F_\lambda(\lambda_0,u_0)-F_\lambda(\lambda(s),u(s))].\end{aligned} \tag{3.109}$$

Applying $l$ to (3.109) and using (3.108), we obtain

$$|\mu(s)\langle l,Kw_0\rangle+\lambda'(s)\langle l,F_\lambda(\lambda_0,u_0)\rangle|\le o(1)(|\lambda'(s)|+|\mu(s)|),$$

then the desired conclusion is reached via algebraic observation: if $a,b\in\mathbf{R}$, $\theta\in(0,1)$, and $|a+b|\le\theta(|a|+|b|)$, then

$$ab\le 0 \quad\text{and}\quad \frac{1-\theta}{1+\theta}|b|\le|a|\le\frac{1+\theta}{1-\theta}|b|.$$

□

We also consider the stability of bifurcating solutions when bifurcation from simple eigenvalue occurs from a line of trivial solutions (Theorem 3.8) (see [CR2] Corollary 1.13 and Theorem 1.16):

**Theorem 3.22.** *Let $F$, $Z$, $\lambda_0$ and $w_0$ be as in Theorem 3.8, and $(\lambda(s),u(s))$ be the solution curve in Theorem 3.8. Suppose that for $K\in B(X,Y)$, $\mu=0$ is a $K$-simple eigenvalue of $F_u(\lambda_0,u_0)$. Then there exist $\epsilon>0$, $C^1$ functions $\gamma:(\lambda_0-\epsilon,\lambda_0+\epsilon)\to\mathbf{R}$, $\mu:(-\epsilon,\epsilon)\to\mathbf{R}$, $v:(\lambda_0-\epsilon,\lambda_0+\epsilon)\to X$, $w:(-\epsilon,\epsilon)\to X$ such that*

$$F_u(\lambda,0)v(\lambda)=\gamma(\lambda)Kv(\lambda) \qquad \text{for } \lambda\in(\lambda_0-\epsilon,\lambda_0+\epsilon), \tag{3.110}$$

$$F_u(\lambda(s),u(s))w(s)=\mu(s)Kw(s) \qquad \text{for } s\in(-\epsilon,\epsilon), \tag{3.111}$$

*and $\gamma(\lambda_0)=\mu(0)=0$, $v(\lambda_0)=w(0)=w_0$, and $v(\lambda)-w_0\in Z$, $w(s)-w_0\in Z$. Moreover, $\gamma'(\lambda_0)\ne 0$, and near $s=0$ the functions $\mu(s)$ and $-s\lambda'(s)\gamma'(\lambda_0)$ have the same zeros, and, whenever $\mu(s)\ne0$, the same sign. More precisely,*

$$\lim_{s\to0,\mu(s)\ne0}\frac{-s\lambda'(s)\gamma'(\lambda_0)}{\mu(s)}=1. \tag{3.112}$$

***Proof.*** The existence of $(\mu(s), w(s))$ and $(\gamma(\lambda), v(\lambda))$ follows directly from Theorem 3.20. Differentiating (3.110) and evaluating at $\lambda = \lambda_0$, we obtain

$$F_{\lambda u}(\lambda_0, u_0)[w_0] + F_u(\lambda_0, u_0)[v'(\lambda_0)] = \gamma'(\lambda_0) K w_0. \tag{3.113}$$

From (F3) $F_{\lambda u}(\lambda_0, u_0)[w_0] \notin R(F_u(\lambda_0, u_0))$, thus $\gamma'(\lambda_0) \neq 0$, and let $l \in Y^*$ satisfy $N(l) = R(F_u(\lambda_0, u_0))$, then

$$\gamma'(\lambda_0)\langle l, Kw_0\rangle = \langle l, F_{\lambda u}(\lambda_0, u_0)[w_0]\rangle. \tag{3.114}$$

On the other hand (3.106) and (3.107) hold, and since $F_\lambda(\lambda_0, u_0) = F_{\lambda\lambda}(\lambda_0, u_0) = 0$, then

$$F_\lambda(\lambda(s), u(s)) = sF_{\lambda u}(\lambda_0, u_0)[w_0] + o(s). \tag{3.115}$$

Hence similar to (3.109), we have

$$\begin{aligned} &\mu(s)Kw_0 + s\lambda'(s)F_{\lambda u}(\lambda_0, u_0)[w_0] \\ =&F_u(\lambda_0, u_0)(w(s) - u'(s)) + \mu(s)K(w_0 - w(s)) + \lambda'(s) \cdot o(s) \\ &+ [F_u(\lambda(s), u(s)) - F_u(\lambda_0, u_0)](w(s) - u'(s)). \end{aligned} \tag{3.116}$$

Now (3.115) and (3.107) implies that

$$||w(s) - u'(s)|| \leq C_5(|s\lambda'(s)| + |\mu(s)|) \tag{3.117}$$

for some $C_5 > 0$. Therefore applying $l$ to (3.116) and from (3.117), we obtain

$$|\mu(s)\langle l, Kw_0\rangle + s\lambda'(s)\langle l, F_{\lambda u}(\lambda_0, u_0)[w_0]\rangle| \leq o(1)(|s\lambda'(s)| + |\mu(s)|).$$

With (3.114), and using the same algebraic observation as in the end of proof of Theorem 3.21, we obtain (3.112) and other conclusions. □

Finally we introduce a stability result for the solutions bifurcating from infinity as in Theorem 3.17:

**Theorem 3.23.** *Let $F$, $L$, $H$, $Q$, $Z$, $\lambda_0$ and $w_0$ be as in Theorem 3.17, and $(\lambda(s), u(s))$ be the solution curve in Theorem 3.17. Suppose that for $K \in B(X, Y)$, $\mu = 0$ is a $K$-simple eigenvalue of $F_u(\lambda_0, \infty)$, where $F_u(\lambda, \infty) = I - \lambda L$. Then there exist $\epsilon > 0$, $C^1$ functions $\gamma : (\lambda_0 - \epsilon, \lambda_0 + \epsilon) \to \mathbf{R}$, $\mu : \{s : |s| \geq \delta\} \to \mathbf{R}$, $v : (\lambda_0 - \epsilon, \lambda_0 + \epsilon) \to X$, $w : \{s : |s| \geq \delta\} \to X$ such that*

$$F_u(\lambda, \infty)v(\lambda) = \gamma(\lambda)Kv(\lambda) \qquad \text{for } \lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon), \tag{3.118}$$

$$F_u(\lambda(s), u(s))w(s) = \mu(s)Kw(s) \qquad \text{for } s \in \{s : |s| \geq \delta\}, \tag{3.119}$$

*$\gamma(\lambda_0) = \lim_{|s|\to\infty} \mu(s) = 0$, $v(\lambda_0) = \lim_{|s|\to\infty} w(s) = w_0$, and $v(\lambda) - w_0 \in Z$, $w(s) - w_0 \in Z$. Moreover, $\gamma'(\lambda_0) \neq 0$, and near $s = \infty$ the functions $\mu(s)$ and $-s\lambda'(s)\gamma'(\lambda_0)$ have the same zeros, and, whenever $\mu(s) \neq 0$, the same sign. More precisely,*

$$\lim_{|s|\to\infty, \mu(s)\neq 0} \frac{-s\lambda'(s)\gamma'(\lambda_0)}{\mu(s)} = 1. \tag{3.120}$$

***Proof.*** The result follows directly from Theorem 3.22 and the proof of Theorem 3.17. □

We apply all the stability results to the solution sets of (2.1).

**Theorem 3.24.** *Let $f$, $\lambda_*$, $u_*$, $w$ be as in Theorem 3.7, and $(\lambda(s), u(s))$ be the solution curve in Theorem 3.7. Then for $s \in (-\delta, \delta)$, the linearized equation*

$$\begin{cases} \Delta\psi + \lambda(s) f'(u(s))\psi = -\mu\psi, & \text{in} \quad \Omega, \\ \psi = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{3.121}$$

*has exactly one eigenvalue $\mu(s)$ near 0, and*

$$\lim_{s\to 0} \frac{\mu(s)}{\lambda'(s)} = \frac{\int_\Omega f(u_*) w dx}{\int_\Omega w^2 dx}. \tag{3.122}$$

*In particular, if $w(x)$ can be chosen as positive, and $\int_\Omega f(u_*) w dx > 0$, then $(\lambda(s), u(s))$ is stable if $\lambda'(s) > 0$, and $(\lambda(s), u(s))$ is unstable with Morse index 1 if $\lambda'(s) < 0$.*

***Proof.*** The result is clear from Theorem 3.21, and $K : C_0^{2,\alpha}(\overline{\Omega}) \to C^\alpha(\overline{\Omega})$ is the injection mapping $K(u) = u$. If $w(x)$ can be chosen as positive, then $\mu(s)$ is the principal eigenvalue and $\mu(s) > 0$ implies that the solution is stable. □

In Lemma 4.3, we will show that when $w > 0$, then $\int_\Omega f(u_*) w dx > 0$ often holds. If $\mu(s)$ is not the principal eigenvalue, then Theorem 3.24 shows that the Morse index of the solutions on the curve changes by one when passing through the degenerate solution $(\lambda_*, u_*)$. More specific examples of Theorem 3.7 and Theorem 3.24 will be shown in later chapters.

For the bifurcation from the line of trivial solutions $u = 0$, we have

**Theorem 3.25.** *Let $f$, $\lambda_0$, $\phi_1$ be as in Theorem 3.10, and $(\lambda(s), u(s))$ be the nontrivial solution curve in Theorem 3.10. Then for $s \in (0, \delta)$, the linearized equation (3.121) has exactly one eigenvalue $\mu(s)$ near 0, and for $s \in (0, \delta)$, $(\lambda(s), u(s))$ is stable if $\lambda'(s) > 0$, and $(\lambda(s), u(s))$ is unstable with Morse index 1 if $\lambda'(s) < 0$.*

***Proof.*** The result is clear from Theorem 3.22, and $K : C_0^{2,\alpha}(\overline{\Omega}) \to C^{\alpha}(\overline{\Omega})$ is the injection mapping $K(u) = u$. Note that $\gamma(\lambda) = \lambda - \lambda_1$ here. □

In the case that $f$ is $C^2$, then $\lambda'(0)$ can be determined by (3.57) and the bifurcating solutions are stable (or unstable) if $\lambda'(0) > 0$ (or $\lambda'(0) < 0$). But the result in Theorem 3.25 still holds even if $f \in C^1$, and the turning direction of the curve is determined by Proposition 3.1. For the logistic type bifurcation in Example 3.2, $\lambda'(0) > 0$, thus the bifurcating solution $(\lambda(s), u(s))$ is stable for $s \in (0, \delta)$. On the other hand, for the weak Allee effect type equation in Example 3.3, $\lambda'(0) < 0$ when $0 < a < 1$, hence the bifurcating solution $(\lambda(s), u(s))$ is unstable with Morse index 1 for $s \in (0, \delta)$. Note that usually we only consider the stability for the positive solution which satisfies $s > 0$. The stability of the negative solution (with $s \in (-\delta, 0)$) can also be considered similarly, and it also satisfies the formula (3.112).

Finally for the bifurcation from infinity, we have

**Theorem 3.26.** *Let $f$, $\lambda_\infty$, $\phi_1$ be as in Theorem 3.19, and $(\lambda(s), u(s))$ be the nontrivial solution curve in Theorem 3.19. Then for $s \in (\delta, \infty)$, the linearized equation* (3.121) *has exactly one eigenvalue $\mu(s)$ near* 0*; for $s \in (\delta, \infty)$, $(\lambda(s), u(s))$ is stable if $\lambda'(s) > 0$, and $(\lambda(s), u(s))$ is unstable with Morse index* 1 *if $\lambda'(s) < 0$.*

# Bibliography

[AG] Arcoya, David; Gámez, José L., Bifurcation theory and related problems: anti-maximum principle and resonance. *Comm. Partial Differential Equations* **26** (2001), no. 9-10, 1879–1911.

[AAC] Alberti, Giovanni; Ambrosio, Luigi; Cabré, Xavier, On a long standing conjecture of E. De Giorgi: old and recent results: symmetry in 3D for general nonlinearities and a local minimality property. *Acta Appl. Math.* **65** (2001), no. 1-3, 9–33.

[Al] Alessandrini, Giovanni, Nodal lines of eigenfunctions of the fixed membrane problem in general convex domains. *Comment. Math. Helv.* **69** (1994), no. 1, 142–154.

[All] Allee, W. C., Animal Aggregations. A study in General Sociology. University of Chicago Press, Chicago. (1931).

[ANZ] Allegretto W., Nistri P., Zecca P., Positive solutions of elliptic nonpositone problems. *Differential Integral Equations* **5** (1992), no. 1, 95–101.

[AC] Allen, S. M., Cahn, J. W., A Microscope Theory for Antiphase Boundary Motion and its Application to Antiphase Domain Coarsening. *Acta Metall.* **27**, (1979) no. 6, 1085-1095.

[ACS] Ali, Ismael; Castro, Alfonso; Shivaji, R., Uniqueness and stability of nonnegative solutions for semipositone problems in a ball. *Proc. Amer. Math. Soc.* **117** (1993), no. 3, 775–782.

[A1] Amann, Herbert On the existence of positive solutions of nonlinear elliptic boundary value problems. *Indiana Univ. Math. J.* **21** (1971/72), 125–146.

[A2] Amann, Herbert, Multiple positive fixed points of asymptotically linear maps. *J. Functional Analysis* **17** (1974), 174–213.

[A3] Amann, Herbert, Nonlinear eigenvalue problems having precisely two solutions. *Math. Z.* **150** (1976), 27–37.

[A4] Amann, Herbert, Fixed point equations and nonlinear eigenvalue problems in ordered Banach space. *SIAM Review* **18** (1976), 620–709.

[ACr] Amann, Herbert; Crandall, Michael G. On some existence theorems for semi-linear elliptic equations. *Indiana Univ. Math. J.* **27** (1978), no. 5, 779–790.

[AL] Amann, Herbert; Laetsch, Theodore, Positive solutions of convex nonlinear eigenvalue problems. *Indiana Univ. Math. J.***25** (1976), no. 3, 259–270.

[AZ] Amann, H.; Zehnder, E., Nontrivial solutions for a class of nonresonance problems and applications to nonlinear differential equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **7** (1980), no. 4, 539–603.

[AmC] Ambrosio, Luigi; Cabré, Xavier, Entire solutions of semilinear elliptic equations in $\mathbf{R}^3$ and a conjecture of De Giorgi. *J. Amer. Math. Soc.* **13**, (2000), no. 4, 725–739.

[AAB] Ambrosetti, A.; Arcoya, D.; Buffoni, B., Positive solutions for some semi-positone problems via bifurcation theory. *Differential Integral Equations* **7** (1994), no. 3-4, 655–663

[ABC] Ambrosetti, Antonio; Brézis, Haim; Cerami, Giovanna, Combined effects of concave and convex nonlinearities in some elliptic problems. *J. Funct. Anal.* **122** (1994), no. 2, 519–543.

[AH] Ambrosetti, A.; Hess, P., Positive solutions of asymptotically linear elliptic eigenvalue problems. *J. Math. Anal. Appl.* **73** (1980), no. 2, 411–422.

[AM] Ambrosetti, Antonio; Mancini, Giovanni, Sharp nonuniqueness results for some nonlinear problems. *Nonlinear Anal.***3** (1979), no. 5, 635–645.

[APo1] Ambrosetti, A.; Prodi, G. On the inversion of some differentiable mappings with singularities between Banach spaces. *Ann. Mat. Pura Appl. (4)* **93** (1972), 231–246.

[APo2] Ambrosetti, Antonio; Prodi, Giovanni, A primer of nonlinear analysis. *Cambridge Studies in Advanced Mathematics*, **34**. Cambridge University Press, Cambridge, (1995).

[AR] Ambrosetti, Antonio; Rabinowitz, Paul H., Dual variational methods in critical point theory and applications. *J. Functional Analysis* **14** (1973), 349–381.

[An] Angenent, S. B. Uniqueness of the solution of a semilinear boundary value problem. *Math. Ann.* **272** (1985), no. 1, 129–138.

[Ant] Antman, Stuart S. Nonlinear problems of elasticity. Second edition. Applied Mathematical Sciences, 107. Springer, New York, 2005.

[APe] Atkinson, F. V.; Peletier, L. A., Emden Fowler equations involving critical exponents. *Nonlinear Anal.* **10** (1986), no. 8, 755–776.

[BL] Bahri, A.; Lions, P.-L., Solutions of superlinear elliptic equations and their Morse indices. *Comm. Pure Appl. Math.* **45** (1992), no. 9, 1205–1215.

[BDS] Bates, Peter W.; Dancer, E. Norman; Shi, Junping, Multi-spike stationary solutions of the Cahn-Hilliard equation in higher-dimensional and instability, *Adv. of Diff. Equa.*, **4**, no. 1, (1999), 1–69.

[BS] Bates, Peter W.; Shi, Junping, Existence and instability of spike layer solutions to singular perturbation problems. *Jour. Func. Anal.* **196**, (2002), no. 2, 429–482.

[BE] Bebernes, Jerrold; Eberly, David, Mathematical problems from combustion theory. *Applied Mathematical Sciences*, **83**. Springer-Verlag, New York, (1989).

[BCN1] Berestycki, H.; Caffarelli, L. A.; Nirenberg, L., Uniform estimates for regularization of free boundary problems. *Analysis and partial differential equations*, 567–619, *Lecture Notes in Pure and Appl. Math.*, **122**, Dekker, New York, (1990).

[BCN2] Berestycki, H.; Caffarelli, L. A.; Nirenberg, L., Symmetry for elliptic equations in a half space. *Boundary value problems for partial differential equations and applications*, 27–42, *RMA Res. Notes Appl. Math.*, **29**, Masson, Paris, (1993).

[BCN3] Berestycki, H.; Caffarelli, L. A.; Nirenberg, L. Inequalities for second-order elliptic equations with applications to unbounded domains. I. A celebration of John F. Nash, Jr. *Duke Math. J.* **81** (1996), no. 2, 467–494.

[BCN4] Berestycki, H.; Caffarelli, L. A.; Nirenberg, L., Monotonicity for elliptic equations in unbounded Lipschitz domains. *Comm. Pure Appl. Math.* **50**, (1997), no. 11, 1089–1111.

[BCN5] Berestycki, Henri; Caffarelli, Luis; Nirenberg, Louis, Further qualitative properties for elliptic equations in unbounded domains. Dedicated to Ennio De Giorgi. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, **25**, (1997), no. 1-2, 69–94.

[BL1] Berestycki, H.; Lions, P.-L. Some applications of the method of super and subsolutions. Bifurcation and nonlinear eigenvalue problems (Proc., Session, Univ. Paris XIII, Villetaneuse, 1978), pp. 16–41, *Lecture Notes in Math.*, **782**, Springer, Berlin, 1980.

[BL2] Berestycki, H.; Lions, P.-L., Nonlinear scalar field equations. I. Existence of a ground state. *Arch. Rational Mech. Anal.* **82** (1983), no. 4, 313–345.

[BL3] Berestycki, H.; Lions, P.-L., Nonlinear scalar field equations. II. Existence of infinitely many solutions. *Arch. Rational Mech. Anal.* **82** (1983), no. 4, 347–375.

[BLP] Berestycki, H.; Lions, P.-L.; Peletier, L. A. , An ODE approach to the existence of positive solutions for semilinear problems in $\mathbf{R}^N$. *Indiana Univ. Math. J.* **30** (1981), no. 1, 141–157.

[BeN] Berestycki, H.; Nirenberg, L. On the method of moving planes and the sliding method. *Bol. Soc. Brasil. Mat. (N.S.)* **22** (1991), no. 1, 1–37.

[BNV] Berestycki, H.; Nirenberg, L.; Varadhan, S.R.S., The principle eigenvalue and maximum principle for second-order elliptic operators in general domains. *Comm. Pure Appl. Math.*

[BeS] Berezin, F. A.; Shubin, M. A., *The Schrödinger equation.* Mathematics and its Applications(Soviet Series), **66**. Kluwer Academic Publishers Group, (1991).

[BH] Beverton, R.J.H.; S.J. Holt, On the dynamics of exploited fish populations. Fish. Invest. Minist. Agric. Fish. Food G.B. (2. Sea Fish.), 19, (1957).

[BK] Brezis, H.; Kamin, S, Sublinear elliptic equations in $\mathbf{R}^n$, *Manu. Math.*

**74** (1992), 87–106.

[BN] Brezis, Haim; Nirenberg, Louis, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. *Comm. Pure Appl. Math.* **36** (1983), no. 4, 437–477.

[BV] Brezis, Haim; Vazquez, Juan Luis, Blow-up solutions of some nonlinear elliptic problems. *Rev. Mat. Univ. Complut. Madrid* **10** (1997), no. 2, 443–469.

[Br] Brown, K. J., Curves of solutions through points of neutral stability. *Results Math.* **37** (2000), no. 3-4, 315–321.

[BCS] Brown, K. J.; Castro, Alfonso; Shivaji, R., Nonexistence of radially symmetric nonnegative solutions for a class of semi-positone problems. *Differential Integral Equations* **2** (1989), no. 4, 541–545.

[BIS] Brown, K. J.; Ibrahim, M. M. A.; Shivaji, R., $S$-shaped bifurcation curves. *Nonlinear Anal.* **5** (1981), no. 5, 475–486.

[BSh] Brown, K. J.; Shivaji, R. Instability of nonnegative solutions for a class of semipositone problems. *Proc. Amer. Math. Soc.* **112** (1991), no. 1, 121–124.

[BuN] Budd, C.; Norbury, J., Semilinear elliptic equations and supercritical growth. *J. Differential Equations* **68** (1987), no. 2, 169–197.

[Ca] Cabré, Xavier, Extremal solutions and instantaneous complete blow-up for ellitpic and parabolic problems. Preprint, (2004).

[CCa] Cabré, Xavier; Capella, Antonio, On the stability of radial solutions of semilinear elliptic equations in all of $\mathbf{R}^n$. *C. R. Math. Acad. Sci. Paris* 338, (2004), no. 10, 769–774.

[CGS] Caffarelli, Luis A.; Gidas, Basilis; Spruck, Joel, Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth. *Comm. Pure Appl. Math.* **42** (1989), no. 3, 271–297.

[CC4] Cantrell, Robert Stephen; Cosner, Chris, Spatial ecology via reaction-diffusion equation. *Wiley series in mathematical and computational biology*, John Wiley & Sons Ltd, (2003).

[CaH] Casten, Richard G.; Holland, Charles J., Instability results for reaction diffusion equations with Neumann boundary conditions. *J. Differential Equations* **27** (1978), no. 2, 266–273.

[CCo] Castro, Alfonso; Cossio, Jorge, Multiple solutions for a nonlinear Dirichlet problem. *SIAM J. Math. Anal.* **25** (1994), no. 6, 1554–1561.

[CCN1] Castro, Alfonso; Cossio, Jorge; Neuberger, John M. A sign-changing solution for a superlinear Dirichlet problem. *Rocky Mountain J. Math.* **27** (1997), no. 4, 1041–1053.

[CCN2] Castro, Alfonso; Cossio, Jorge; Neuberger, John M. A minmax principle, index of the critical point, and existence of sign-changing solutions to elliptic boundary value problems. *Electron. J. Differential Equations* (1998), No. 02, 18 pp. (electronic).

[CG] Castro, Alfonso; Sudhasree, Gadam, Uniqueness of stable and unstable positive solutions for semipositone problems. *Nonlinear Anal.* **22** (1994), no. 4, 425–429.

[CGS1] Castro, Alfonso; Garner, J. B. and Shivaji, R., Existence results for classes of sublinear semipositone problems. *Results Math.* **23** (1993), no. 3-4, 214–220.

[CGS2] Castro, Alfonso; Gadam, Sudhasree; Shivaji, R., Branches of radial solutions for semipositone problems. *J. Differential Equations* **120** (1995), no. 1, 30–45.

[CHS] Castro, Alfonso; Hassanpour, M.; Shivaji, R. Uniqueness of non-negative solutions for a semipositone problem with concave nonlinearity. *Comm. Partial Differential Equations* **20** (1995), no. 11-12, 1927–1936.

[CaL] Castro, Alfonso; Lazer, A. C., Critical point theory and the number of solutions of a nonlinear Dirichlet problem. *Ann. Mat. Pura Appl. (4)* **120** (1979), 113–137.

[CMS] Castro, Alfonso; Maya, C.; Shivaji, R., *Nonlinear eigenvalue problems with semipositone structure.* Proceedings of the Conference on Nonlinear Differential Equations (Coral Gables, FL, 1999), *Electron. J. Differ. Equ.*, **Conf., 5,** (2000), 33–49.

[CS1] Castro, Alfonso; Shivaji, R. Nonnegative solutions for a class of nonpositone problems. *Proc. Roy. Soc. Edinburgh Sect. A* **108** (1988), no. 3-4, 291–302.

[CS2] Castro, Alfonso; Shivaji, R. Non-negative solutions for a class of radially symmetric non-positone problems. *Proc. Amer. Math. Soc.* **106** (1989), no.3, 735–740.

[CS3] Castro, Alfonso; Shivaji, R., Nonnegative solutions to a semilinear Dirichlet problem in a ball are positive and radially symmetric. *Comm. Partial Differential Equations*, **14** (1989), no. 8-9, 1091–1100.

[CS4] Castro, Alfonso; Shivaji, R., Positive solutions for a concave semipositone Dirichlet problem. *Nonlinear Anal.* **31** (1998), no. 1-2, 91–98.

[Cha] Chandrasekhar, S., An introduction to the study of stellar structure, Dover, (1967).

[CI] Chafee, N.; Infante, E. F., A bifurcation problem for a nonlinear partial differential equation of parabolic type. *Applicable Anal.* **4** (1974/75), 17–37.

[Ch1] Chang, Kung-ching, Infinite-dimensional Morse theory and multiple solution problems. *Progress in Nonlinear Differential Equations and their Applications*, **6**. Birkhäuser, Boston, (1993).

[Ch2] Chang, Kung-Ching, Methods in nonlinear analysis. Springer Monographs in Mathematics. Springer-Verlag, Berlin, (2005).

[CH] Chow, Shui Nee; Hale, Jack K., Methods of bifurcation theory. Springer-Verlag, New York-Berlin, (1982).

[CLin] Chen, Chiun Chuan; Lin, Chang-Shou, Uniqueness of the ground state solutions of $\Delta u + f(u) = 0$ in $\mathbf{R}^n$, $n \geq 3$. *Comm. Partial Differential Equations* **16** (1991), no. 8-9, 1549–1572.

[CLi] Chen, Wen Xiong; Li, Congming, Classification of solutions of some nonlinear elliptic equations. *Duke Math. J.* **63** (1991), no. 3, 615–622.

[CT] Church, P. T.; Timourian, J. G., Global structure for nonlinear operators in differential and integral equations. I. Folds. II. Cusps.

*Topological nonlinear analysis, II* (Frascati, 1995), 109–160, 161–245, *Progr. Nonlinear Differential Equations Appl.*, **27**, Birkhäüser, Boston, (1997).

[CP] Clément, Ph.; Peletier, L. A., An anti-maximum principle for second-order elliptic operators. *J. Differential Equations*, **34**, (1979), no. 2, 218–229.

[CSw] Clément, Ph.; Sweers, Guido, Existence and multiplicity results for a semilinear elliptic eigenvalue problem. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **14**, (1987), 97–121.

[CJ] Clemons, C. B.; Jones, C. K. R. T., A geometric proof of the Kwong-McLeod uniqueness result. *SIAM J. Math. Anal.* **24** (1993), no. 2, 436–443.

[C1] Coffman, Charles V. On the positive solutions of boundary-value problems for a class of nonlinear differential equations. *J. Differential Equations* **3** (1967) 92–111.

[C2] Coffman, Charles V. Uniqueness of the ground state solution for $\Delta u - u + u^3 = 0$ and a variational characterization of other solutions. *Arch. Rational Mech. Anal.* **46** (1972), 81–95.

[CK] Cohen, D. S.; Keller, H. B., Some positive problems suggested by nonlinear heat generation, *J. Math. Mech.* **16** (1967), 1361–1376.

[CR1] Crandall, Michael G.; Rabinowitz, Paul H., Bifurcation from simple eigenvalues. *J. Functional Analysis*, **8** (1971), 321–340.

[CR2] Crandall, Michael G.; Rabinowitz, Paul H., Bifurcation, perturbation of simple eigenvalues and linearized stability. *Arch. Rational Mech. Anal.* **52** (1973), 161–180.

[CR3] Crandall, Michael G.; Rabinowitz, Paul H., Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems. *Arch. Rational Mech. Anal.* **58** (1975), no. 3, 207–218.

[CWS] Cui, Renhao; Wang, Yuwen; Shi, Junping, Uniqueness of positive solution for a class of semilinear elliptic systems. To appear in *Nonl. Anal.* (2007).

[Dal] Dalmasso, Robert, Existence and uniqueness of positive radial solutions for the Lane-Emden system. *Nonlinear Anal.* **57** (2004), 341–348.

[DGP] Damascelli, Lucio; Grossi, Massimo; Pacella, Filomena, Qualitative properties of positive solutions of semilinear elliptic equations in symmetric domains via the maximum principle. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **16** (1999), no. 5, 631–652.

[D1] Dancer, E. N., Global solution branches for positive mappings. *Arch. Rational Mech. Anal.* **52** (1973), 181–192.

[D2] Dancer, E. N., A note on bifurcation from infinity. *Quart. J. Math. Oxford Ser. (2)* **25** (1974), 81–84.

[D3] Dancer, E. N., On the structure of solutions of an equation in catalysis theory when a parameter is large. *J. Differential Equations* **37** (1980), no. 3, 404–437.

[D4] Dancer, E. N., On the number of positive solutions of weakly nonlinear elliptic equations when a parameter is large. *Proc. London Math. Soc.*

*(3)* **53** (1986), no. 3, 429–452.

[D5] Dancer, E. N., The effect of domain shape on the number of positive solutions of certain nonlinear equations. *J. Differential Equations* **74** (1988), no. 1, 120–156.

[D6] Dancer, E. N., Weakly nonlinear Dirichlet problems on long or thin domains. *Mem. Amer. Math. Soc.* **105** (1993), no. 501.

[D7] Dancer, E. N., On the uniqueness of the positive solution of a singularly perturbed problem. *Rocky Mountain J. Math.* **25** (1995), no. 3, 957–975.

[D8] Dancer, E. N., On positive solutions of some singularly perturbed problems where the nonlinearity changes sign. *Topol. Methods Nonlinear Anal.* **5** (1995), no. 1, 141–175.

[D9] Dancer, E. N., A note on asymptotic uniqueness for some nonlinearities which change sign. *Bull. Austral. Math. Soc.* **61** (2000), no. 2, 305–312.

[D10] Dancer, E. N. Infinitely many turning points for some supercritical problems. *Ann. Mat. Pura Appl. (4)* **178** (2000), 225–233.

[D11] Dancer, Edward Norman, New solutions of equations on $\mathbf{R}^n$. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **30** (2001), no. 3-4, 535–563 (2002).

[D12] Dancer, E. N., Stable and finite Morse index solutions on $\mathbf{R}^n$ or on bounded domains with small diffusion. *Trans. Amer. Math. Soc.* **357** (2005), no. 3, 1225–1243.

[D13] Dancer, E. N. Stable and finite Morse index solutions on $\mathbf{R}^n$ or on bounded domains with small diffusion. II. *Indiana Univ. Math. J.* **53** (2004), no. 1, 97–108.

[DSc] Dancer, E. N.; Schmitt, Klaus, On positive solutions of semilinear elliptic equations. *Proc. Amer. Math. Soc.* **101** (1987), no. 3, 445–452.

[DSh] Dancer, E. N; Shi, Junping, Uniqueness of positive solution to sublinear semipositone problem. *Bull. London Math. Soc.*, **38** (2006), no. 6, 1033–1044.

[DSw] Dancer, E. N.; Sweers, G. On the existence of a maximal weak solution for a semilinear elliptic equation. *Differential Integral Equations* **2** (1989), no. 4, 533–540.

[DW] Dancer, E. N.; Wei, Juncheng, On the profile of solutions with two sharp layers to a singularly perturbed semilinear Dirichlet problem. *Proc. Roy. Soc. Edinburgh Sect. A* **127** (1997), no. 4, 691–701.

[DFP] Dang, Ha; Fife, Paul C.; Peletier, L. A., Saddle solutions of the bistable diffusion equation. *Z. Angew. Math. Phys.*, **43**, (1992), no. 6, 984–998.

[Da] Davies, E. B., *Spectral theory and differential operators*. Cambridge Studies in Advanced Mathematics, **42**. Cambridge University Press, Cambridge, (1995).

[DLN] de Figueiredo, D. G.; Lions, P.-L.; Nussbaum, R. D., A priori estimates and existence of positive solutions of semilinear elliptic equations. *J. Math. Pures Appl. (9)* **61** (1982), no. 1, 41–63.

[DG] De Giorgi, E. Convergence problems for functionals and operators. *Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978)*, 131–188, Pitagora, Bologna, 1979.

[De] Deimling, Klaus, Nonlinear functional analysis. Springer-Verlag, Berlin-New York, (1985).

[Du1] Du, Yi Hong, The structure of the set of positive solutions of a class of nonlinear eigenvalue problems. (Chinese) *Acta Math. Sinica* **33** (1990), no. 4, 445–455.

[Du2] Du, Yihong, Exact multiplicity and S-shaped bifurcation curve for some semilinear elliptic problems from combustion theory. *SIAM J. Math. Anal.* **32** (2000), no. 4, 707–733.

[Du3] Du, Yihong, Order structure and topological methods in nonlinear partial differential equations. Vol. 1. Maximum principles and applications. Series in Partial Differential Equations and Applications, 2. World Scientific Publishing Co., Hackensack, NJ, 2006.

[DM] Du, Yihong; Ma, Li, Logistic type equations on $\mathbf{R}^N$ by a squeezing method involving boundary blow-up solutions. *J. London Math. Soc. (2)* **64** (2001), no. 1, 107–124.

[DL] Du, Yihong; Lou, Yuan, Proof of a conjecture for the perturbed Gelfand equation from combustion theory. *J. Differential Equations* **173** (2001), no. 2, 213–230.

[DSW] Duo, Jia; Shi, Junping; Wang, Yuwen, Structure of the solution set of semilinear elliptic equations with asymptotic linear nonlinearity. To appear in *Nonl. Anal. TMA*.

[Dy] Dyson, F.J., Ground-state energy of a finite system of charged particles. *J. Math. Phys.* **8** (1967), 1538–1545.

[Em] Emden, R., Gaskugeln: Anwendungen der mechanischen Wrmetheorie auf kosmologische und meteorologische Probleme. Teubner, Leipzig, Germany, 1907.

[ET1] Erbe, Lynn; Tang, Moxun, Structure of positive radial solutions of semilinear elliptic equations. *J. Differential Equations* **133** (1997), no. 2, 179–202.

[ET2] Erbe, Lynn; Tang, Moxun, Uniqueness theorems for positive radial solutions of quasilinear elliptic equations in a ball. *J. Differential Equations* **138** (1997), no. 2, 351–379.

[ET3] Erbe, Lynn; Tang, Moxun, Uniqueness of positive radial solutions of $\Delta u + K(|x|)\gamma(u) = 0$. *Differential Integral Equations* **11** (1998), no. 4, 663–678.

[ET4] Erbe, Lynn; Tang, Moxun, Uniqueness of positive radial solutions of $\Delta u + f(|x|, u) = 0$. *Differential Integral Equations* **11** (1998), no. 5, 725–743.

[E] Evans, Lawrence C., Partial differential equations. *Graduate Studies in Mathematics*, **19**. American Mathematical Society, Providence, (1998).

[Fa] Farina, Alberto, Symmetry for solutions of semilinear elliptic equations in $\mathbf{R^N}$ and related conjectures. *Ricerche Mat.* **48** (1999), suppl., 129–154.

[F3] Fife, Paul C., Mathematical aspects of reacting and diffusing systems. *Lecture Notes in Biomathematics*, **28**. Springer-Verlag, (1979).

[Fis] Fisher, R.A., The wave of advance of advantageous genes. *Ann. Eugenics*, **7**, (1937), 353–369.

[Fiz] FitzHugh, R., Impulses and Physiological States in Theoretical Models of Nerve Membrane. *Biophysical Journal* **1** (1961), 445–466.

[FPR] Fitzpatrick, P. M.; Pejsachowicz, J.; Rabier, P. J., The degree of proper $C^2$ Fredholm mappings. I. *J. Reine Angew. Math.* **427** (1992), 1–33.

[Fl] Flores, Isabel, A resonance phenomenon for ground states of an elliptic equation of Emden-Fowler type. *Jour. Diff. Equa.* **198** (2004), 1–15.

[Fo] Fowler, R. H., Further studies of Emden's and similar differential equations. *Quart. J. Math. (Oxford series)* **2**, (1931), 259–288.

[Fu] Fujita, Hiroshi, On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$. *J. Fac. Sci. Univ. Tokyo Sect. I* **13** (1966), 109–124.

[GP] Gardner, R.; Peletier, L. A., The set of positive solutions of semilinear equations in large balls. *Proc. Roy. Soc. Edinburgh Sect. A* **104** (1986), no. 1-2, 53–72.

[GM] Gazzola, Filippo; Malchiodi, Andrea, Some remarks on the equation $-\Delta u = \lambda(1+u)^p$ for varying $\lambda, p$ and varying domains. *Comm. Partial Differential Equations* **27** (2002), no. 3-4, 809–845.

[Ge] Gelfand, I. M. Some problems in the theory of quasilinear equations. *Amer. Math. Soc. Transl. (2)* **29** (1963), 295–381.

[GNN1] Gidas, B.; Ni, Wei Ming; Nirenberg, L., Symmetry and related properties via the maximum principle. *Comm. Math. Phys.* **68** (1979), no. 3, 209–243.

[GNN2] Gidas, B.; Ni, Wei Ming; Nirenberg, L., Symmetry of positive solutions of nonlinear elliptic equations in $\mathbf{R}^n$. *Mathematical analysis and applications, Part A*, pp. 369–402, *Adv. in Math. Suppl. Stud.*, **7a**, Academic Press, New York-London, 1981.

[GS1] Gidas, B.; Spruck, J. Global and local behavior of positive solutions of nonlinear elliptic equations. *Comm. Pure Appl. Math.* **34** (1981), no. 4, 525–598.

[GS2] Gidas, B.; Spruck, J. A priori bounds for positive solutions of nonlinear elliptic equations. *Comm. Partial Differential Equations* **6** (1981), no. 8, 883–901.

[GT] Gilbarg, David; Trudinger, Neil S., Elliptic partial differential equations of second order. Second edition. Springer-Verlag, Berlin-New York, (1983).

[GG] Ghoussoub, N.; Gui, C., On a conjecture of De Giorgi and some related problems. *Math. Ann.* **311** (1998), no. 3, 481–491.

[GS] Gray, Peter; Scott, S. K., Chemical oscillations and instabilities. Oxford Univ. Press, New York-Oxford, (1990).

[G] Guo, Zongming, On the symmetry of positive solutions of the Lane-

Emden equation with supercritical exponent. *Adv. Differential Equations* **7** (2002), no. 6, 641–666.

[Ha1] Hale, Jack K., Asymptotic behavior of dissipative systems. *Mathematical Surveys and Monographs*, **25**. American Mathematical Society, Providence, RI, (1988).

[HL] Han, Qing; Lin, Fanghua, Elliptic partial differential equations. *Courant Lecture Notes in Mathematics*, **1**. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, (1997).

[Has] Hassell, M. P., Density dependence in single-species populations. *J. Anim. Ecol.* **44**, (1975), 283–95.

[Her] Hernández, Jesús, Qualitative methods for nonlinear diffusion equations. Nonlinear diffusion problems (Montecatini Terme, 1985), 47–118, *Lecture Notes in Math.*, **1224**, Springer, Berlin-New York, (1986).

[Ha] Hartman, Philip, Ordinary differential equations. Birkhäuser, Boston, (1982).

[HM] Hastings, S. P.; McLeod, J. B., The number of solutions to an equation from catalysis. *Proc. Roy. Soc. Edinburgh Sect. A* **101** (1985), no. 1-2, 15–30.

[Hen] Hénon, M., Numerical experiments on the stability of spherical stellar systems. *Astronom. Astrophys.* **24** (1973), 229-238.

[He] Henry, Daniel, Geometric theory of semilinear parabolic equations. *Lecture Notes in Mathematics*, **840**. Springer-Verlag, Berlin-New York, (1981).

[Ho] Holling, C.S., The components of predation as revealed by a study of small-mammal predation on the European pine sawfly. *Canadian Entomologist* **91** (1959), 294–320.

[HK] Holzmann, Marco; Kielhöfer, Hansjörg, Uniqueness of global positive solution branches of nonlinear elliptic problems. *Math. Ann.* **300** (1994), no. 2, 221–241.

[HS] Hsu, Cheng-Hsing; Shih, Yi-wen, Solutions of semilinear elliptic equations with asymptotic linear nonlinearity. *Nonlinear Anal.*. **50** (2002), no. 2, 257–283.

[I] Ize, Jorge, Bifurcation theory for Fredholm operators. *Mem. Amer. Math. Soc.* **7** (1976), no. 174, 128pp.

[J] Jang, Jaeduck, On spike solutions of singularly perturbed semilinear Dirichlet problem. *J. Differential Equations* **114** (1994), no. 2, 370–395.

[JLZ] Jiang, Jifa; Liang, Xing; Zhao, Xiao-Qiang: Saddle-point behavior for monotone semiflows and reaction-diffusion models. *J. Differential Equations* **203** (2004), no. 2, 313–330.

[JS1] Jiang, Jifa; Shi, Junping: Dynamics of a reaction-diffusion system of autocatalytic chemical reaction. *Discrete Contin. Dyn. Syst. A* **21** (2008), no. 1, 245–258.

[JS2] Jiang, Jifa; Shi, Junping, Bistability dynamics in some structured ecological models. In *Spatial Ecology: a collection of essays*, CRC Press,

2009.
[JK] Jones, C.; Küpper, T. On the infinitely many solutions of a semilinear elliptic equation. *SIAM J. Math. Anal.* **17** (1986), no. 4, 803–835.
[JKP] Jones, C. K. R. T.; Küpper, T.; Plakties, H. A shooting argument with oscillation for semilinear elliptic radially symmetric equations. *Proc. Roy. Soc. Edinburgh Sect. A* **108** (1988), no. 1-2, 165–180.
[JL] Joseph, D. D.; Lundgren, T. S., Quasilinear Dirichlet problems driven by positive sources. *Arch. Rational Mech. Anal.* **49** (1972/73), 241–269.
[KKL] Kaper, Hans G.; Kwong, Man Kam; Li, Yi, Symmetry results for reaction-diffusion equations. *Differential Integral Equations* **6** (1993), no. 5, 1045–1056.
[Kar] Karls, Michael A. A, uniqueness result for certain semilinear elliptic equations. *Differential Integral Equations* **9** (1996), no. 5, 949–966.
[Ka1] Kawohl, Bernd, Symmetry or not? *Math. Intelligencer* **20** (1998), no. 2, 16–22.
[Ka2] Kawohl, B., Symmetrization—or how to prove symmetry of solutions to a PDE. Partial differential equations (Praha, 1998), 214–229, *Chapman & Hall/CRC Res. Notes Math.*, **406**, Chapman & Hall/CRC, Boca Raton, FL, 2000.
[Kat] Kato, Tosio, Perturbation theory for linear operators. Reprint of the 1980 edition. Classics in Mathematics. Springer-Verlag, Berlin, (1995).
[Ke] Keller, Herbert B. Positive solutions of some nonlinear eigenvalue problems. *J. Math. Mech.* **19** (1969/1970) 279–295.
[KC] Keller, Herbert B.; Cohen, Donald S. Some positone problems suggested by nonlinear heat generation. *J. Math. Mech.* **16** (1967) 1361–1376.
[Ki1] Kielhöfer, Hansjörg, Multiple eigenvalue bifurcation for Fredholm operators. *J. Reine Angew. Math.* **358** (1985), 104–124.
[Ki2] Kielhöfer, Hansjörg, A bifurcation theorem for potential operators. *J. Funct. Anal.* **77** (1988), no. 1, 1–8.
[Ki3] Kielhöfer, Hansjörg, Bifurcation theory. An introduction with applications to PDEs. *Applied Mathematical Sciences*, **156**. Springer-Verlag, New York, (2004).
[KPP] Kolmogoroff, A., Petrovsky, I, Piscounoff, N, Study of the diffusion equation with growth of the quantity of matter and its application to a biological problem. (French) *Moscow Univ. Bull. Math.* **1**, (1937), 1–25.
[Kol] Kolodner, Ignace I., Heavy rotating string—a nonlinear eigenvalue problem. *Comm. Pure Appl. Math.* **8** (1955), 395–408.
[Ko1] Korman, Philip, Solution curves for semilinear equations on a ball. *Proc. Amer. Math. Soc.* **125** (1997), no. 7, 1997–2005.
[Ko2] Korman, Philip, Uniqueness and exact multiplicity results for two classes of semilinear problems. *Nonlinear Anal.* **31** (1998), no. 7, 849–865.
[Ko3] Korman, Philip, Steady states and long time behavior of some con-

vective reaction-diffusion equations. *Funkcial. Ekvac.* **40** (1997), no. 2, 165–183.

[Ko4] Korman, Philip, Curves of sign-changing solutions for semilinear equations. *Nonlinear Anal.* **51** no. 5, (2002), 801–820.

[KL] Korman, Philip; Li, Yi, On the exactness of an S-shaped bifurcation curve. *Proc. Amer. Math. Soc.*, **127** (1999), no. 4, 1011–1020.

[KLO1] Korman, Philip; Li, Yi; Ouyang, Tiancheng, Exact multiplicity results for boundary value problems with nonlinearities generalising cubic. *Proc. Roy. Soc. Edinburgh Sect. A* **126** (1996), no. 3, 599–616.

[KLO2] Korman, Philip; Li, Yi; Ouyang, Tiancheng, An exact multiplicity result for a class of semilinear equations. *Comm. Partial Differential Equations* **22** (1997), no. 3-4, 661–684.

[KS1] Korman, Philip; Shi, Junping, Instability and exact multiplicity of solutions of semilinear equations. *Elec. Jour. Diff. Equa.*, **Conf. 05**, (2000), 311–322.

[KS2] Korman, Philip; Shi, Junping, New exact multiplicity results with an application to a population model. *Proc. Royal Soc. Edin. A*, **131** (2001), No. 5, 1167–1182.

[KS3] Korman, Philip; Shi, Junping, On Lane-Emden Type systems. *Disc. Cont. Dyna. Syst.* Proceedings of 5th AIMS International Conference on Dynamic Systems and Differential Equations, (2005), 510–517.

[Kr] Krasnosel'skii, M. A., *Topological methods in the theory of nonlinear integral equations.* A Pergamon Press Book The Macmillan Co., New York, (1964).

[Kw1] Kwong, Man Kam, Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbf{R}^n$. *Arch. Rational Mech. Anal.* **105** (1989), no. 3, 243–266.

[Kw2] Kwong, Man Kam, Uniqueness results for Emden-Fowler boundary value problems. *Nonlinear Anal.* **16** (1991), no. 5, 435–454.

[Kw3] Kwong, Man Kam, On the Kolodner-Coffman method for the uniqueness problem of Emden-Fowler BVP. *Z. Angew. Math. Phys.* **41** (1990), no. 1, 79–104.

[KwL] Kwong, Man Kam; Li, Yi, Uniqueness of radial solutions of semilinear elliptic equations. *Trans. Amer. Math. Soc.* **333** (1992), no. 1, 339–363.

[KMPT] Kwong, M. K.; McLeod, J. B.; Peletier, L. A.; Troy, W. C., On ground state solutions of $-\Delta u = u^p - u^q$. *J. Differential Equations* **95** (1992), no. 2, 218–239.

[KwZ] Kwong, Man Kam; Zhang, Li Qun, Uniqueness of the positive solution of $\Delta u + f(u) = 0$ in an annulus. *Differential Integral Equations* **4** (1991), no. 3, 583–599.

[Ku] Kuiper, Nicolaas H. $C^1$-equivalence of functions near isolated critical points. *Symposium on Infinite-Dimensional Topology*, 199–218. Ann. of Math. Studies, No. 69, Princeton Univ. Press, Princeton, N. J., 1972.

[La] Laetsch, Theodore, On the number of solutions of boundary value problems with convex nonlinearities. *J. Math. Anal. Appl.* **35** (1971) 389–404.

[Li] Li, Yi, Asymptotic behavior of positive solutions of equation $\Delta u + K(x)u^p = 0$ in $R^n$. *J. Differential Equations* **95** (1992), no. 2, 304–330.

[LiN] Li, Yi; Ni, Wei-Ming, Radial symmetry of positive solutions of nonlinear elliptic equations in $\mathbf{R}^n$. *Comm. Partial Differential Equations* **18** (1993), no. 5-6, 1043–1054.

[LiS] Lieb, Elliott H.; Solovej, Jan Philip, Ground state energy of the two-component charged Bose gas. *Comm. Math. Phys.* **252** (2004), no. 1-3, 485–534.

[LiY] Lieb, Elliott H.; Yau, Horng-Tzer, The Chandrasekhar theory of stellar collapse as the limit of quantum mechanics. *Comm. Math. Phys.* **112** (1987), no. 1, 147–174.

[Li1] Lin, Chang Shou, On the second eigenfunctions of the Laplacian in $R^2$. *Comm. Math. Phys.* **111** (1987), no. 2, 161–166.

[Li2] Lin, Chang Shou, Uniqueness of least energy solutions to a semilinear elliptic equation in $\mathbf{R}^2$. *Manuscripta Math.* **84** (1994), no. 1, 13–19.

[LN] Lin, Chang-Shou; Ni, Wei-Ming, A counterexample to the nodal domain conjecture and a related semilinear equation. *Proc. Amer. Math. Soc.* **102** (1988), no. 2, 271–277.

[L] Lions, P.-L., On the existence of positive solutions of semilinear elliptic equations. *SIAM Review* **24** (1982), no. 4, 441–467.

[LSW] Liu, Ping; Shi, Junping; Wang, Yuwen, Imperfect transcritical and pitchfork bifurcations. *Jour. Func. Anal.* **251** (2007), no. 2, 573–600.

[LG] López-Gómez, Julian, Spectral theory and nonlinear functional analysis. *Chapman & Hall/CRC Research Notes in Mathematics*, **426**. Chapman & Hall/CRC, Boca Raton, FL, (2001).

[Lo] Lotka, A. J., Undamped oscillations derived from the law of mass action. *J. Am. Chem. Soc.* **42**, (1920), 1595–1599.

[LAW] Ludwig, D.; Aronson, D. G.; Weinberger, H. F., Spatial patterning of the spruce budworm. *J. Math. Biol.* **8** (1979), no. 3, 217–258.

[Ma] Malthus, Thimas Robert, An Essay On The Principle Of Population. 1st edition, London: J. Johnson, (1798).

[M1] Matano, Hiroshi, Asymptotic behavior and stability of solutions of semilinear diffusion equations. *Publ. Res. Inst. Math. Sci.* **15** (1979), no. 2, 401–454.

[M2] Matano, Hiroshi, Nonincrease of the lap-number of a solution for a one-dimensional semilinear parabolic equation. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **29** (1982), no. 2, 401–441.

[M] McLeod, Kevin, Uniqueness of positive radial solutions of $\Delta u + f(u) = 0$ in $\mathbf{R}^n$. II. *Trans. Amer. Math. Soc.* **339** (1993), no. 2, 495–505.

[MS] McLeod, Kevin; Serrin, James, Uniqueness of positive radial solutions of $\Delta u + f(u) = 0$ in $\mathbf{R}^n$. *Arch. Rational Mech. Anal.* **99** (1987), no. 2, 115–145.

[Mc] McNabb, A. Comparison and existence theorems for multicomponent diffusion systems. *J. Math. Anal. Appl.* **3** (1961), 133–144.

[MP1] Merle, F.; Peletier, L. A., Asymptotic behaviour of positive solutions of elliptic equations with critical and supercritical growth. I. The radial case. *Arch. Rational Mech. Anal.* **112** (1990), no. 1, 1–19.

[MP2] Merle, F.; Peletier, L. A., Positive solutions of elliptic equations involving supercritical growth. *Proc. Roy. Soc. Edinburgh Sect. A* **118** (1991), no. 1-2, 49–62.

[MP3] Merle, F.; Peletier, L. A., Asymptotic behaviour of positive solutions of elliptic equations with critical and supercritical growth. II. The nonradial case. *J. Funct. Anal.* **105** (1992), no. 1, 1–41.

[MPS] Merle, F.; Peletier, L. A.; Serrin, J. A bifurcation problem at a singular limit. *Indiana Univ. Math. J.* **43** (1994), no. 2, 585–609.

[Mu] Murray, J. D., Mathematical biology. Third edition. I. An introduction. *Interdisciplinary Applied Mathematics*, **17**. Springer-Verlag, New York, 2002; II. Spatial models and biomedical applications. *Interdisciplinary Applied Mathematics*, **18**. Springer-Verlag, New York, 2003.

[NS] Nagasaki, Ken'ichi; Suzuki, Takashi, Spectral and related properties about the Emden-Fowler equation $-\Delta u = \lambda e^u$ on circular domains. *Math. Ann.* **299** (1994), no. 1, 1–15.

[Nag] Nagumo, J., Animoto, S., Yoshizawa, S., An active pulse transmission line simulating nerve axon. *Proc. Inst. Radio Engineers*, **50**, (1962), 2061–2070.

[Ni1] Ni, Wei-Ming, Uniqueness of solutions of nonlinear Dirichlet problems. *J. Differential Equations* **50** (1983), no. 2, 289–304.

[Ni2] Ni, Wei-Ming, Lectures on semilinear elliptic equations, National Tsing Hua University, Taiwan, (1989).

[Ni3] Ni, Wei-Ming, Diffusion, cross-diffusion, and their spike-layer steady states. *Notices Amer. Math. Soc.* **45** (1998), no. 1, 9–18.

[Ni4] Ni, Wei-Ming Qualitative properties of solutions to elliptic problems. Stationary partial differential equations. Vol. I, 157–233, Handb. Differ. Equ., North-Holland, Amsterdam, 2004.

[NN] Ni, Wei-Ming; Nussbaum, Roger D., Uniqueness and nonuniqueness for positive radial solutions of $\Delta u + f(u, r) = 0$. *Comm. Pure Appl. Math.* **38** (1985), no. 1, 67–108.

[NT1] Ni, Wei-Ming; Takagi, Izumi, On the shape of least-energy solutions to a semilinear Neumann problem. *Comm. Pure Appl. Math.*, **44**, (1991), no. 7, 819–851.

[NT2] Ni, Wei-Ming; Takagi, Izumi, Locating the peaks of least-energy solutions to a semilinear Neumann problem. *Duke Math. J.*, **70**, (1993), no. 2, 247–281.

[NTW] Ni, Wei-Ming; Takagi, Izumi; Wei, Juncheng, On the location and profile of spike-layer solutions to a singularly perturbed semilinear Dirichlet problem: intermediate solutions. *Duke Math. J.* **94** (1998), no. 3, 597–618.

[Nic] Nicholson, A.J., An outline of the dynamics of animal populations. *Aust. J. Zool.* **2** (1954), 9–65.

[Nir] Nirenberg, Louis, Topics in nonlinear functional analysis. *Courant Lecture Notes in Mathematics*, **6**. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, (2001).

[OL] Okubo, Akira; Levin, Simon, Diffusion and Ecological Problems Modern Perspectives. Second Edition. *Interdisciplinary Applied Mathematics*, **14**. Springer, Berlin-New York, (2002).

[OSS] Oruganti, Shobha, Shi, Junping and Shivaji, Ratnasingham, Diffusive Logistic equation with constant effort harvesting, I: Steady States. *Trans. Amer. Math. Soc.*, **354** (2002), no. 9, 3601–3619.

[O1] Ouyang, Tiancheng, On the positive solutions of semilinear equations $\Delta u + \lambda u - hu^p = 0$ on the compact manifolds. *Trans. Amer. Math. Soc.* **331** (1992), no. 2, 503–527.

[O2] Ouyang, Tiancheng, On the positive solutions of semilinear equations $\Delta u + \lambda u + hu^p = 0$ on compact manifolds. II. *Indiana Univ. Math. J.* **40** (1991), no. 3, 1083–1141.

[OS1] Ouyang, Tiancheng and Shi, Junping, Exact multiplicity of positive solutions for a class of semilinear problem. *Jour. Diff. Equa.* **146** (1998), no. 1, 121–156.

[OS2] Ouyang, Tiancheng and Shi, Junping, Exact multiplicity of positive solutions for a class of semilinear problem:II. *Jour. Diff. Equa.* **158**, (1999), no. 1, 94–151.

[OSY] Ouyang, Tiancheng; Shi, Junping; Yao, Miaoxin, Exact multiplicity of positive solutions for a singular elliptic equation. Submitted, (2002).

[PeS] Peitgen, H.-O.; Schmitt, K., Global analysis of two-parameter elliptic eigenvalue problems. *Trans. Amer. Math. Soc.* **283** (1984), no. 1, 57–95.

[PR1] Pejsachowicz, Jacobo; Rabier, Patrick J., A substitute for the Sard-Smale theorem in the $C^1$ case. *J. Anal. Math.* **76** (1998), 265–288.

[PR2] Pejsachowicz, Jacobo; Rabier, Patrick J., Degree theory for $C^1$ Fredholm mappings of index 0. *J. Anal. Math.* **76** (1998), 289–319.

[PS1] Peletier, L. A.; Serrin, James, Uniqueness of positive solutions of semilinear equations in $\mathbf{R}^n$. *Arch. Rational Mech. Anal.* **81** (1983), no. 2, 181–197.

[PS2] Peletier, L. A.; Serrin, James, Uniqueness of nonnegative solutions of semilinear equations in $\mathbf{R}^n$. *J. Differential Equations* **61** (1986), no. 3, 380–397.

[Pe] Perko, Lawrence, Differential equations and dynamical systems. First edition. *Texts in Applied Mathematics*, **7**. Springer-Verlag, New York, 1991.

[Po] Pohožaev, S. I., On the eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$. (Russian) *Dokl. Akad. Nauk SSSR* **165** (1965), 36–39. (English) *Sov. Math.* **5** (1965), 1408-1411.

[PW] Protter, Murray H.; Weinberger, Hans F. Maximum principles in differential equations.Prentice-Hall, Inc., Englewood Cliffs, N.J. (1967).

[PuS] Pucci, Patrizia; Serrin, James, A general variational identity. *Indiana Univ. Math. J.* **35** (1986), no. 3, 681–703.

[RS1] Rabier, P. J.; Stuart, C. A., Global bifurcation for quasilinear elliptic equations on $\mathbf{R}^N$. *Math. Z.* **237** (2001), no. 1, 85–124.

[RS2] Rabier, Patrick J.; Stuart, Charles A., Fredholm and properness properties of quasilinear elliptic operators on $\mathbf{R}^N$. *Math. Nachr.* **231** (2001), 129–168.

[R1] Rabinowitz, Paul H., Some global results for nonlinear eigenvalue problems. *J. Func. Anal.* **7** (1971), 487–513

[R2] Rabinowitz, Paul H., On bifurcation from infinity. *J. Diff. Equations* **14** (1973), 462–475.

[R3] Rabinowitz, Paul H., Minimax methods in critical point theory with applications to differential equations. *CBMS Regional Conference Series in Mathematics*, **65**. American Mathematical Society, Providence, (1986).

[R4] Rabinowitz, Paul H., Pairs of positive solutions of nonlinear elliptic partial differential equations. *Indiana Univ. Math. J.* **23** (1973/74), 173–186.

[Rau] Rauch, Jeffrey, Five problems: an introduction to the qualitative theory of partial differential equations, *Lecture Notes in Mathematics* **446**, Partial Differential Equations and Related Topics, (Jerome A. Goldstein, ed.), Springer-Verlag, Berlin, (1974), 355–369.

[Re] Rellich, Darstellung der Eigenwerte von $\Delta u + \lambda u = 0$ durch ein Randintegral. *Math. Z.*, **46** (1940), 635–636.

[Res] Restrepo, G., Differentiable norms in Banach spaces. *Bull. Amer. Math. Soc.* **70** (1964) 413–414.

[Ric] Ricker, W.E., Stock and recruitment. *J. Fish. Res. Bd Can.* **11** (1954), 559–623.

[Sa] Sattinger, D. H., *Monotone methods in nonlinear elliptic and parabolic boundary value problems. Indiana Univ. Math. J.*, **21**, (1971/72), 979–1000.

[Sc1] Schaaf, Renate, Global solution branches of two-point boundary value problems. *Lecture Notes in Mathematics*, **1458**. Springer-Verlag, Berlin, (1990).

[Sc2] Schaaf, Renate, Uniqueness for semilinear elliptic problems: supercritical growth and domain geometry. *Adv. Differential Equations* **5** (2000), no. 10-12, 1201–1220.

[Sch] Schmitt, Klaus, Positive solutions of semilinear elliptic boundary value problems. Topological methods in differential equations and inclusions (Montreal, PQ, 1994), 447–500, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 472, Kluwer Acad. Publ., Dordrecht, 1995.

[Se1] Serrin, James, A symmetry problem in potential theory. *Arch. Rational Mech. Anal.* **43** (1971), 304–318.

[Se2] Serrin, James, Nonlinear elliptic equations of second order. Lectures at AMS Symposium on Partial Differential Equations, Berkeley, 1971.

[SeZ] Serrin, James; Zou, Henghui, Classification of positive solutions of quasilinear elliptic equations. *Topol. Methods in Nonli. Analy.* **3** (1994), no. 1, 1–26.

[S1] Shi, Junping, Persistence and bifurcation of degenerate solutions, *Jour. Func. Anal.* **169**, (1999), No. 2, 494–531.

[S2] Shi, Junping, Blow-up Points of Solution Curves for a Semilinear Problem. *Topo. Meth. of Nonlinear Anal.*, **15**, (2000), No. 2, 251–266.

[S3] Shi, Junping, Exact multiplicity of solutions to Superlinear and Sublinear Problems. *Nonlinear Anal.* **50** (2002), no. 5, 665–687.

[S4] Shi, Junping, Global Bifurcation of Semilinear Neumann Boundary Problem. *Tran. Amer. Math. Soc.* **354** (2002), no. 8, 3117–3154.

[S5] Shi, Junping, Saddle solutions of the balanced bistable diffusion equation. *Comm. Pure Appl. Math.* **55** (2002), no. 7, 815–830.

[S6] Shi, Junping, Exact Multiplicity of positive solutions to superlinear problems. *Elec. Jour. Diff. Equa.*, **Conf 10**, 257–265, (2002).

[S7] Shi, Junping, A new proof of anti-maximum principle via a bifurcation approach. *Resultate der Mathematik*, **48** (2005), no. 1-2, 162–167.

[S8] Shi, Junping, A radially symmetric anti-maximum principle and applications to fishery management models. *Elec. Jour. Diff. Equa.* **2004**, No. 27, 1–13, (2004).

[SS1] Shi, Junping; Shivaji, Ratnasingham, Exact Multiplicity of solutions for classes of problems with concave-convex nonlinearity. *Disc. Cont. Dyna. Syst.*, **7**, (2001), no. 3, 559–571.

[SS2] Shi, Junping; Shivaji, Ratnasingham, Global bifurcation for concave semipositon problems. *Advances in Evolution Equations: Proceedings in honor of J.A.Goldstein's 60th birthday*, Edited by G.R. Goldstein, R. Nagel, and S. Romanelli, Marcel Dekker, Inc., New York, Basel, 385–398, (2003).

[SS3] Shi, Junping; Shivaji, Ratnasingham, Persistence in reaction diffusion models with weak Allee effect. *Jour. Math. Biol.* **52** (2006), no. 6, 807–829.

[SWa] Shi, Junping; Wang, Junping, Morse indices and Exact multiplicity of solutions to Semilinear Elliptic Problems. *Proc. of Amer. Math. Soc.* **127** (1999), 3685–3695.

[SWang1] Shi, Junping; Wang, Xuefeng, Hair-Triggered Instability of Radial Steady States, Spread and Extinction in Semilinear Heat Equations, *Journal of Differential Equations*, **231** (2006), no. 1, 235–251.

[SWang2] Shi, Junping; Wang, Xuefeng, Quasilinear Elliptic Systems with Nonlinear Boundary Conditions Have Zero Fredholm Index, and Application to Global Bifurcation Theory. Submitted, (2007).

[SX] Shi, Junping; Xie, Zhifu, Smoothness of crossing curves. Submitted, (2007).

[SY] Shi, Junping; Yao, Miaoxin, On a singular nonlinear semilinear elliptic problem. *Proc. Royal Soc. Edin. A*, **128**, (1998), no. 6, 1389–1401.

[Sk] Skellam, J. G., Random dispersal in theoritical populations. *Biometrika* **38**, (1951), 196–218.

[Sm] Smoller, Joel, Shock waves and reaction-diffusion equations. Second edition. *Grundlehren der Mathematischen Wissenschaften*, **258**. Springer-Verlag, New York, (1994).

[SmW1] Smoller, J.; Wasserman, A., Global bifurcation of steady-state solutions. *J. Differential Equations* **39** (1981), no. 2, 269–290.

[SmW2] Smoller, Joel A.; Wasserman, Arthur G., Existence, uniqueness, and nondegeneracy of positive solutions of semilinear elliptic equations. *Comm. Math. Phys.* **95** (1984), no. 2, 129–159.

[SmW3] Smoller, Joel; Wasserman, Arthur, Existence of positive solutions for semilinear elliptic equations in general domains. *Arch. Rational Mech. Anal.* **98** (1987), no. 3, 229–249.

[SmW4] Smoller, Joel A.; Wasserman, Arthur G., Symmetry-breaking for solutions of semilinear elliptic equations with general boundary conditions. *Comm. Math. Phys.* **105** (1986), no. 3, 415–441.

[SmW5] Smoller, Joel; Wasserman, Arthur, Symmetry-breaking for positive solutions of semilinear elliptic equations. *Arch. Rational Mech. Anal.* **95** (1986), no. 3, 217–225.

[SmW6] Smoller, J.; Wasserman, Arthur G., Bifurcation and symmetry-breaking. *Invent. Math.* **100** (1990), no. 1, 63–95.

[SmW7] Smoller, Joel; Wasserman, Arthur, Symmetry, degeneracy, and universality in semilinear elliptic equations. Infinitesimal symmetry-breaking. *J. Funct. Anal.* **89** (1990), no. 2, 364–409.

[SoY] So, Joseph W.-H.; Yang, Yuanjie, Dirichlet problem for the diffusive Nicholson's blowflies equation. *J. Differential Equations* **150** (1998), no. 2, 317–348.

[Sr] Srikanth, P. N., Uniqueness of solutions of nonlinear Dirichlet problems. *Differential Integral Equations* **6** (1993), no. 3, 663–670.

[Stra] Strauss, Walter A., Existence of solitary waves in higher dimensions. *Comm. Math. Phys.* **55** (1977), no. 2, 149–162.

[Str] Strogatz, Steven, Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering. Perseus Publishing, (2000).

[St] Struwe, Michael, Variational methods. Applications to nonlinear partial differential equations and Hamiltonian systems. Third edition. *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge.*, **34**. Springer-Verlag, Berlin, (2000).

[Stu] Stuart, C. A. Solutions of large norm for non-linear Sturm-Liouville problems. *Quart J. Math. Oxford Ser. (2)* **24** (1973), 129–139.

[Su] Suzuki, Takashi, Semilinear elliptic equations. *GAKUTO International Series. Mathematical Sciences and Applications*, **3**. Gakkōtosho Co., Ltd., Tokyo, (1994).

[Sw] Sweers, Guido, On the maximum of solutions for a semilinear elliptic problem. *Proc. Royal Soc. Edin. A*, **108** (1988), 357–370.

[Ta1] Tang, Moxun, Existence and uniqueness of fast decay entire solutions of quasilinear elliptic equations. *J. Differential Equations* **164** (2000), no. 1, 155–179.

[Ta2] Tang, Moxun, Uniqueness and global structure of positive radial solutions for quasilinear elliptic equations. *Comm. Partial Differential Equations* **26** (2001), no. 5-6, 909–938.

[Ta3] Tang, Moxun, Uniqueness of positive radial solutions for $\Delta u - u + u^p = 0$ on an annulus. *J. Differential Equations* **189** (2003), no. 1, 148–160.

[Ta4] Tang, Moxun, Exact multiplicity for semilinear elliptic Dirichlet problems involving concave and convex nonlinearities. *Proc. Royal Soc. Edin. A*, **133** (2003), no. 3, 705–717.

[To] Toland, J. F., Bifurcation and asymptotic bifurcation for non-compact non-symmetric gradient operators. *Proc. Roy. Soc. Edinburgh Sect. A* **73** (1975), 137–147.

[Tur] Turchin, Peter, Complex population dynamics: a theoretical/empirical synthesis. Princeton University Press, Princeton, (2003).

[Tu] Turing, A.M., The chemical basis of morphogenesis. *Philosophical Transaction of Royal Society of London*. **B237**, (1952), 37–72.

[V] Verhulst, P. F., Notice sur la loi que la population pursuit dans son accroissement. *Correspondance mathmatique et physique* 10, (1838), 113–121.

[W1] Wang, Shin-Hwa, A correction for a paper by J. Smoller and A. G. Wasserman: "Global bifurcation of steady-state solutions" [J. Differential Equations 39 (1981), no. 2, 269–290]. *J. Differential Equations* **77** (1989), no. 1, 199–202.

[W2] Wang, Shin Hwa, On **S**-shaped bifurcation curves. *Nonlinear Anal.* **22** (1994), no. 12, 1475–1485.

[WLe] Wang, Shin Hwa; Lee, Fu Ping, Bifurcation of an equation from catalysis theory. *Nonlinear Anal.* **23** (1994), no. 9, 1167–1187.

[WLo] Wang, Shin-Hwa; Long, Dau-Ming, An exact multiplicity theorem involving concave-convex nonlinearities and its application to stationary solutions of a singular diffusion problem. *Nonlinear Anal.* **44** (2001), no. 4, 469–486.

[We1] Wei, Juncheng, Exact multiplicity for some nonlinear elliptic equations in balls. *Proc. Amer. Math. Soc.* **125** (1997), no. 11, 3235–3242.

[We2] Wei, Juncheng, On the interior spike solutions for some singular perturbation problems. *Proc. Roy. Soc. Edinburgh Sect. A* **128** (1998), no. 4, 849–874.

[Wi] Wiggins, Stephen, Introduction to applied nonlinear dynamical systems and chaos. *Texts in Applied Mathematics*, **2**. Springer-Verlag, New York, (1990).

[WL] Wilhelmsson, Hans; Lazzaro, Enzo, Reaction-diffusion problems in the physics of hot plasmas. Series in plasma physics. Institute of Physics Publishing, Bristol and Philadelphia, 2001.

[Y] Yosida, Kosaku, Functional analysis. Fifth edition. *Grundlehren der Mathematischen Wissenschaften*, **Band 123**. Springer-Verlag, Berlin-New York, (1978).

[Z1] Zhang, Li Qun, Uniqueness of positive solutions of $\Delta u + u + u^p = 0$ in a ball. *Comm. Partial Differential Equations* **17** (1992), no. 7-8, 1141–1164.

[Z2] Zhang, Li Qun, Uniqueness of positive solutions of semilinear elliptic equations. *J. Differential Equations* **115** (1995), no. 1, 1–23.

[Zo1] Zou, Henghui, Slow decay and the Harnack inequality for positive solutions of $\Delta u + u^p = 0$ in $R^n$. *Differential Integral Equations* **8** (1995), no. 6, 1355–1368.

[Zo2] Zou, Henghui, Symmetry of positive solutions of $\Delta u + u^p = 0$ in $R^n$. *J. Differential Equations* **120** (1995), no. 1, 46–88.

[Zo3] Zou, Henghui, Symmetry of ground states of semilinear elliptic equations with mixed Sobolev growth. *Indiana Univ. Math Jour.* **45** (1996), no. 1, 221–240.