GLOBAL BIFURCATIONS OF PERIODIC ORBITS.

By J. C. ALEXANDER* and JAMES A. YORKE*

---

1. Statement of results. The purpose of this paper is to prove global versions of two results about bifurcation of periodic orbits from an equilibrium point—the Hopf bifurcation theorem for autonomous differential systems and the Liapunov center theorem for Hamiltonian systems. Our version of the first theorem is sufficiently general that the second follows from it as a corollary.

For our version of the Hopf bifurcation theorem we consider a parametrized autonomous differential system

$$\dot{x}=f(\lambda,x) \tag{E_λ}$$

defined on an n-dimensional C^1 manifold M (n finite). Here λ is a parameter ranging over some interval Λ of real numbers, and the parametrized cross section

$$f:\Lambda\times M\rightarrow \text{tangent bundle of } M$$

is continuous. Furthermore, we assume that for any $\lambda\in\Lambda$ and any initial value $x\in M$, the system (E_λ) has a unique solution for some future range of time. If x is a member of the boundary of M, the solution is assumed to remain in the boundary of M. We suppose for some x_0 in the interior of M that $f(\lambda,x_0)=0$ for all λ_0 in a neighborhood Λ_0 of some λ_0, so that x_0 is a stationary value of (E_λ) for all $\lambda\in\Lambda_0$.

The x-derivative of f at $x=x_0$,

$$L(\lambda)\equiv D_xf(\lambda,k)|_{x=x_0}=D_xf(\lambda,x_0),$$

is a linear endomorphism of the tangent space of M at x_0 [once a basis of that tangent space is fixed, $L(\lambda)$ may be considered an $(n\times n)$ matrix function of λ]; we require that it exist and be continuous in $\lambda\in\Lambda_0$.

Manuscript received September 16, 1975; revised April 15, 1977.

*Partially supported by NSF grants and (J.C.A.) by the OAS at the Centro de Investigación y Estudios Advancados del IPN, Mexico City.

American Journal of Mathematics, Vol. 100, No. 2, pp. 263–292
0002-9329/78/1001-0263 $01.50

If $L(\lambda)$ has no eigenvalue with real part zero, it is well-known that x_0 has a neighborhood containing no constant solutions or periodic solutions except $x=x_0$. Hence bifurcation can occur at (λ_0, x_0) only if $L(\lambda_0)$ has an eigenvalue with real part zero. P. M. Rabinowitz [18], generalizing work of M. A. Krasnosel'skii [13], investigated the case that zero is an eigenvalue of $L(\lambda_0)$. (See also [2], [17]). He assumed $M=R^n$ and $\Lambda_0=\Lambda=(-\infty,\infty)$ and showed that if the zero eigenvalues of $L(\lambda_0)$ have non-zero λ-derivatives at $\lambda=0$, and the determinant changes sign, there is a non-empty connected subset $\mathfrak{R}_1 \subset \Lambda \times (R^n - \{x_0\})$ with $(\lambda_0, x_0) \in \bar{\mathfrak{R}}_1$ such that $f(\lambda, x)=0$ for all $(\lambda, x) \in \mathfrak{R}_1$ and either $\mathfrak{R}_1$ is unbounded in $\Lambda \times R^n$ or there is some $\lambda_1 \neq \lambda_0$ with $(\lambda_1, x_0) \in \bar{\mathfrak{R}}_1$. That is, the family of stationary solutions either "goes to infinity" or returns to x_0 again, but for another value of the parameter. It is convenient to generalize this as follows (see [2]): drop the condition about eigenvalues having non-zero derivative at $\lambda=\lambda_0$; If the determinant changes sign as λ goes through λ_0, there exists global bifurcation. It is thus convenient (and a precursor to what we do later) to define a parity for L at λ_0; the parity is odd or even as the determinant changes or does not change sign at λ_0. Thus global bifurcation is guaranteed if the parity is odd.

In the present paper, we consider the case that $L(\lambda_0)$ is non-singular and has a pair of purely imaginary conjugate eigenvalues $\pm i\beta$. We assign a parity to the function L and the eigenvalue $i\beta$ of $L(\lambda_0)$; if the parity is odd, we obtain an analogous global result about bifurcation of non-constant periodic orbits. In previous papers, various researchers have imposed various additional technical assumptions and have shown that in a neighborhood of the stationary solution x_0 bifurcates into periodic orbits. That is, there exist non-constant periodic solutions of (E_λ) for λ sufficiently near λ_0, of period approximately $2\pi\beta^{-1}$, which collapse down to the stationary point x_0. E. Hopf [9] (see also the appendix of [15]) first considered the problem; since then results have been obtained by A. A. Androvov [3], N. N. Brushinskaya [4], N. Chafee [6, 7], R. Jost and E. Zehnder [10] and F. Takens [26]. All of these results are local; that is, they determine what happens only in a neighborhood of (λ_0, x_0).

There are also a multitude of articles about the implications of bifurcation vis-à-vis the various sciences. For general discussions of bifurcation, see [5], [12], [15], [17], [19], [25] and their bibliographies.

In order to state our result precisely, we collect our assumptions.

DATA 1.1. *Assume the following four hypotheses:*

(1) *A differential system* (E_λ) *is given satisfying the conditions of the first paragraph of this section.*

(2) *There exists a linear endomorphism* $L(\lambda)$ *of the tangent space of* M *at* x_0, *defined and continuous for* $\lambda \in \Lambda_0$, *such that for any* $\lambda_1 \in \Lambda_0$,

$$\exp^{-1}(f(\lambda, \exp v)) - L(\lambda)v = O(|v|)$$

as $(\lambda, v) \to (\lambda_1, 0)$.

Explanation: here v is a tangent vector to M at x_0, with length $|v|$ in some Riemannian metric of M, and exp is the exponential map from a neighborhood of the zero vector in the tangent space to M at x_0. The condition is independent of the choice of metric and uniquely defines L. It is certainly met if f is C^1 in a neighborhood of (λ_0, x_0). If M is an open subset of R^n and the metric is the Euclidean metric, the condition reads:

$$f(\lambda, v) - L(\lambda) \cdot v = O(|v|)$$

as $(\lambda, v) \to (\lambda_1, 0)$.

(3) *The endomorphism* $L(\lambda_0)$ *is non-singular and has a conjugate pair of purely imaginary eigenvalues* $\pm i\beta$.

In this case let Mult$(i\beta)$ be the ordered set $\{ik_1\beta, ik_2\beta, \ldots, ik_r\beta\}$ (with the k_j positive integers, $1 \leqslant k_1 \leqslant k_2 \leqslant \cdots \leqslant k_r$) of eigenvalues of $L(\lambda_0)$ which are positive integral multiples of $i\beta$, counted with multiplicity, and including $i\beta$. For λ sufficiently close to λ_0, there is a unique set Mult$_\lambda(i\beta)$ of eigenvalues close to the set Mult$(i\beta)$.

(4) *For* λ *near but not equal to* λ_0, *none of the eigenvalues in* Mult$_\lambda(i\beta)$ *has zero real part.*

Under these conditions, we can define the parity of $i\beta$. Let r^+ (r^-) be the number of elements in Mult$_\lambda(i\beta)$ with positive real part for $\lambda > \lambda_0$ $(\lambda < \lambda_0)$. Let $r = r^+ - r^-$. Thus r is the net number of changes of sign of the real parts of elements of Mult$_\lambda(i\beta)$ as λ passes through λ_0, and is called the *index* of $i\beta$ with respect to the function L. The *parity* of $i\beta$ with respect to L is the parity (even or odd) of r.

Before proceeding, let us consider this definition in the important special case that $i\beta$ is a simple eigenvalue and no other eigenvalue of $L(\lambda_0)$ is a positive integral multiple of $i\beta$. Then Mult$(i\beta) = \{i\beta\}$ and Mult$_\lambda(i\beta) = \{\alpha_\lambda + i\beta_\lambda\}$ with $\alpha_\lambda, \beta_\lambda$ real continuous functions of λ with $\alpha_{\lambda_0} = 0$. We see that the parity of $i\beta$ with respect to L is odd precisely when α_λ changes sign at λ_0.

We now recall some standard terminology. Let $G(\lambda, t, x)$ be the solution of (E_λ) at time $t \geqslant 0$, given the initial condition $x(0) = x$. That is, $G(\lambda, 0, x) = x$. We say x is *stationary* for (E_λ), or simply (x,λ) is stationary, if $G(\lambda,t,x)=x$ for all $t\geqslant 0$. If (x,λ) is not stationary and there exists $t\geqslant 0$ such that $g(\lambda,t,x)=x$, we say that (x,λ) is *periodic*. If (x,λ) is periodic, all positive t for which $g(\lambda,t,x)=x$ we call *periods*. The periods are a discrete set of positive numbers; the smallest positive period is called the *least period*. Let

$$\mathfrak{N}=\{(\lambda,t,x)\in\Lambda\times(0,\infty)\times M \mid G(\lambda,t,x)=x \text{ and } (x,\lambda) \text{ is periodic}\}.$$

Thus $\mathfrak{N}$ catalogues the parameter, period and initial condition of all non-stationary periodic solutions of (E_λ). We consider $\mathfrak{N}$ to be a topological subspace of $\Lambda\times[0,\infty)\times M$. Let $t_0=2\pi\beta^{-1}$.

THEOREM A. *Assume Data 1.1. Further suppose that the parity of $i\beta$ is odd.*

(1) *Then there exists a connected subset $\mathfrak{N}_0$ of $\mathfrak{N}\cup\{(\lambda_0,t_0,x_0)\}$ containing (λ_0,t_0,x_0) and at least one periodic solution. Moreover for some neighborhood $\mathfrak{M}$ of (λ_0,t_0,x_0) in $\Lambda\times[0,\infty)\times M$, if $(\lambda,t,x)\in\mathfrak{M}\cap\mathfrak{N}$, then for some positive integer $k=k(\lambda,t,x)$ such that $ik\beta\in\mathrm{Mult}(i\beta)$, the least period of $G(\lambda,\cdot,x)$ is $k^{-1}t$.*

(2) *In addition one or both of the following are satisfied:*

I. *$\mathfrak{N}_0$ is not contained in any compact subset of $\Lambda\times[0,\infty)\times M$,*
II. *There exists a point $(\bar\lambda,\bar t,\bar x)$ in $\overline{\mathfrak{N}}_0-\mathfrak{N}_0$.*

(3) *Furthermore, for any $(\bar\lambda,\bar t,\bar x)\in\overline{\mathfrak{N}}-\mathfrak{N}$, the solution $\bar x$ is stationary for $(E_{\bar\lambda})$. Also for any $\varepsilon>0$, there is a neighborhood U_ε of $(\bar\lambda,\bar t,\bar x)$ such that for any $(\lambda,t,x)\in U_\varepsilon\cap\mathfrak{N}$, all points of the orbit $G(\lambda,\cdot,x)$ are of distance less than ε from the point $\bar x$.*

Thus Theorem A(2.I) states that this connected family of orbits contains elements for λ arbitrarily close to the boundary of Λ, or contains elements of arbitrarily large period, or contains elements the orbits of which do not lie in any preassigned compact subset of M. As a caveat, we point out that saying the period is arbitrarily large is weaker than saying the least period is arbitrarily large, because all integral multiples of the least period are also periods, and the integral multiplier does not have to be continuous on $\mathfrak{N}_0$. The point is that $\mathfrak{N}_0$ is connected so that t "becomes large" continuously. Conclusion (3) states that the entire orbits of the periodic solutions $G(\lambda,\cdot,x)$ converge to a stationary solution, and so the diameters of these orbits go to zero as $(\bar\lambda,\bar t,\bar x)$ is approached.

Note that although our assumptions are essentially local, our conclusions are global. Indeed, our assumptions are somewhat less stringent than those of the previously mentioned local results, in that we do not require as much differentiability; neither do we require the remaining eigenvalues of $L(\lambda_0)$ to have non-zero real part. (On the other hand, we do not get regularity results.) Furthermore, so far as we know, all previous results have guaranteed bifurcation only for much more special sets Mult$(i\beta)$. Thus even as a local result, our theorem is new.

If we impose other reasonable local conditions on f, we can determine more precisely what happens near $(\bar{\lambda},\bar{t},\bar{x})$. Such is the purpose of our next results. For Proposition 1.2, we suppose that M is an open subset $\mathfrak{O}$ of R^n. Note however that the last part of Proposition 1.2 is valid for a general manifold.

PROPOSITION 1.2. *Let $(\bar{\lambda},\bar{t},\bar{x})\in\bar{\mathfrak{N}}-\mathfrak{N}$ with $\bar{x}\in\mathfrak{O}$. Suppose in some neighborhood U of $(\bar{\lambda},\bar{x})$ in $R\times\mathfrak{O}$ that f is Lipschitz in x with Lipschitz constant L. That is, for some norm $|\cdot|$ on R^n,*

$$|f(\lambda,x)-f(\lambda,y)|\leqslant L|x-y|$$

whenever $x,y\in U$. Then

$$\bar{t}\geqslant 4L^{-1}.$$

If the norm is the standard Euclidean norm, then in fact,

$$\bar{t}\geqslant 2\pi L^{-1}.$$

Suppose furthermore $D_xf(\lambda,x)$ exists and is continuous for all (λ,x) in some neighborhood of $(\bar{\lambda},\bar{x})$. Then $D_xf(\bar{\lambda},\bar{x})$ has a pair of purely imaginary eigenvalues $\pm i\beta$ with $\beta=2\pi k\bar{t}^{-1}$ for some integer $k\geqslant 1$.

PROPOSITION 1.3. *Suppose $(\bar{\lambda},\bar{t},0)\in\bar{\mathfrak{N}}-\mathfrak{N}$ with $\bar{t}\neq 0$, and suppose condition (2) of Data 1.1 is satisfied in some neighborhood of $\bar{\lambda}$. Then $L(\bar{\lambda})=D_xf(\bar{\lambda},0)$ has a purely imaginary eigenvalue $i\beta$, where $\beta=2\pi k\bar{t}^{-1}$ for some integer $k\geqslant 0$.*

We turn now to our global version of the Liapunov center theorem for a Hamiltonian system. For a general discussion of the classical theorem, see [24, Section 16]. We will work with a Hamiltonian system in Euclidean space, leaving the general formulation for a system on a symplectic manifold to the reader. Let $\mathfrak{O}\in R^{2m}$ be open and contain the origin, and let $H:\mathfrak{O}\rightarrow R$ be a C^2 function (the Hamiltonian). The Hamiltonian system associated with H is

$$\dot{x}=J\,\mathrm{grad}\,H(x), \tag{H}$$

where

$$J=\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

with I the identity $m \times m$ matrix. We want to investigate periodic solutions near a constant solution 0, so we assume $\operatorname{grad} H(0)=0$. Let Hess denote the Hessian matrix:

$$\text{Hess} = D^2H(x)\big|_{x=0} = \left[\left. \frac{\partial^2 H(x)}{\partial x_i \, \partial x_j} \right|_{x=0} \right]_{i,j=1}^{2m}.$$

Note that Hess is symmetric.

THEOREM B. *Assume the above data. Suppose in addition that* Hess *is non-singular and has a conjugate pair of purely imaginary eigenvalues* $\pm i\beta$. *Suppose the number of eigenvalues of* Hess *(counted with multiplicity) that are positive integral multiples of* $i\beta$ *is odd.*

Then there is a connected subset $\mathfrak{P} \subset R \times \mathfrak{O}$ *with* $(2\pi\beta^{-1}, 0) \in \mathfrak{P}$ *such that for each* $(\gamma, x) \in \mathfrak{P}$ *other than* $(2\pi\beta^{-1}, 0)$, *the solution of* (H) *through* x *is non-constant and periodic with a period of* γ. *There is a neighborhood* $\mathfrak{M}$ *of* $(2\pi\beta^{-1}, 0)$ *in* $R \times \mathfrak{O}$ *such that if* $(\gamma, x) \in \mathfrak{M} \cap \mathfrak{P}$, *then for some positive integer* $k = k(\gamma, x)$ *such that* $ik\beta$ *is an eigenvalue of* Hess, *the least period of the solution through* x *is* $k^{-1}\gamma$.

Furthermore $\mathfrak{P}$ *satisfies at least one of the following properties*:

I_H. $\mathfrak{P}$ *is unbounded in* $R \times \mathfrak{O}$; *that is,* $\mathfrak{P}$ *does not lie in any compact subset of* $R \times \mathfrak{O}$,

II_H. *There exists some* $(\bar{\gamma}, \bar{x})$ *with* $\bar{x}$ *a stationary solution. Also* $(\bar{\gamma}, \bar{x}) \neq (2\pi\beta^{-1}, 0)$ *and* $\bar{\gamma} \neq 0$. *If* $\bar{x} = 0$, *then* $\bar{\gamma} = 2\pi \bar{k} \bar{\beta}^{-1}$ *for some integer* $\bar{k} > 0$ *and some purely imaginary eigenvalue* $i\bar{\beta}$ *of* Hess.

This result is actually a corollary of Theorem A. We consider the one-parameter system

$$\dot{x} = (J + \lambda) \operatorname{grad} H(x) \qquad (H_\lambda)$$

and show it satisfies the hypotheses of Theorem A for $\lambda_0 = 0$. Furthermore there are no non-constant periodic solutions of (H_λ) if $\lambda \neq 0$, so any results about the non-constant periodic solutions of (H_λ) apply directly to (H). As a result, we do not repeat the full statements of Propositions 1.2, 1.3 for the present case. In particular, however, the smoothness assumptions of the propositions are satisfied for (H_λ). We might point out that the bifurcating solution is extended globally for some cases by a classical argument in [24, pp. 150–151].

2. Examples. In this section, we construct a number of examples to illustrate aspects of the theorems.

Example 2.1. Let

$$\psi(\lambda, r): R \times [0, \infty) \to R$$

be a continuous function. Consider the two-dimensional parametrized system which in polar coordinates is written

$$\dot{\theta} = 1,$$
$$\dot{r} = r\psi(\lambda, r).$$

In rectangular coordinates (x_1, x_2), it may be written

$$\dot{x}_1 = \psi(\lambda, r)x_1 - x_2,$$
$$\dot{x}_2 = x_1 + \psi(\lambda, r)x_2.$$

Here $M = R^2$. The origin is a stationary point for all λ, and if $\psi(\lambda_0, r_0) = 0$ there is a periodic orbit for the parameter value λ_0 which is the circle of radius r_0 and center 0. In this case the period is 2π. From the polar representation, it is easy to see that these are the only periodic orbits.

It is straightforward to calculate

$$L(\lambda) = \begin{pmatrix} \psi(\lambda, 0) & -1 \\ 1 & \psi(\lambda, 0) \end{pmatrix},$$

and so $L(\lambda_0)$ has a pair of purely imaginary eigenvalues if and only if $\psi(\lambda_0, 0) = 0$ (whence $\beta = 1$). In this case $\mathrm{Mult}_\lambda(i\beta) = \alpha_\lambda + i\beta_\lambda$ with $\beta_\lambda \equiv 1$, $\alpha_\lambda = \psi(\lambda, 0)$ for all λ. Accordingly, this system satisfies condition 4 of the data if and only if $\psi(\lambda, 0)$ is non-zero for λ near but not equal to λ_0. In this case, the parity is odd if and only if $\psi(\lambda, 0)$ changes sign at λ_0. Condition 3 is satisfied trivially, and condition 2 is easily checked.

Thus we have a large number of examples, one for each such function ψ. In Figure 1, a number of possibilities are illustrated. We have exhibited the zeros of ψ, i.e., the periodic solutions. Thus from the point $(\lambda_1, 0)$, bifurcates a family of periodic solutions which contain orbits of arbitrarily large r. From $(\lambda_2, 0)$ bifurcates a family of periodic solutions which converges back to the stationary

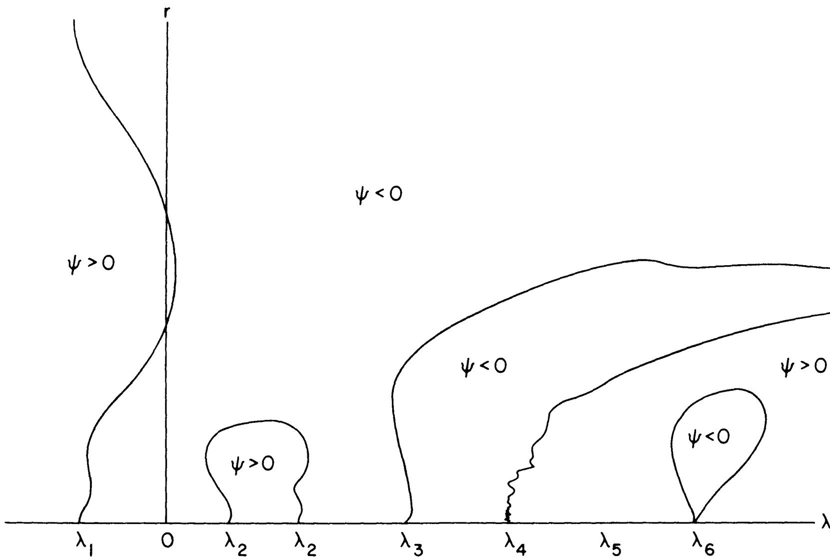

FIGURE 1.

point $r=0$ for another value of λ, viz. λ_2'. From λ_3, there is a family which contains elements defined for arbitrarily large λ. By choosing ψ complicated, the bifurcating family can be made complicated, as at λ_4. At λ_4, the bifurcation is multiple, as described in [6]. Finally, we note that if the parity is even, there need not be any bifurcation, for we could define ψ positive except at one point, as at λ_5, and then there is no bifurcation. Another possibility is λ_6, where α_λ does not change sign. There is bifurcation, but part (2) of Theorem A is not fulfilled.

Example 2.2. In this example the family of periodic orbits is bounded in λ, t, x and closes up to a stationary solution different from $x=0$. For such an example, it is necessary that $n \geqslant 3$.

Consider the system

$$\dot{x}_1 = \lambda x_1 - x_2,$$
$$\dot{x}_2 = x_1 + \lambda x_2,$$
$$\dot{x}_3 = x_1^2 + x_2^2 + (x_3 - 1)^2 - 1,$$

defined on all of R^3. First we show that non-stationary periodic solutions can occur only when $\lambda = 0$. For any solution $x(t)$, let

$$\rho(t) = [x_1(t)]^2 + [x_2(t)]^2.$$

By direct calculation, we find that $\dot{\rho}(t) = 2\lambda\rho(t)$. Hence ρ is monotone for any λ. Thus if $x(t)$ is a periodic solution, it must be that $\lambda\rho \equiv 0$; i.e., either $\lambda = 0$ or $\rho(t) \equiv 0$. In the latter case, we have the stationary solutions $(x_1, x_2, x_3) = (0,0,0)$ or $(0,0,2)$. Hence non-stationary periodic solutions occur only when $\lambda = 0$.

We now determine these non-stationary periodic solutions. We claim all the orbits are circles lying on the sphere $x_1^2 + x_2^2 + (x_3 - 1)^2 = 1$ as illustrated in Figure 2. At any point (x_1, x_2, x_3) on the sphere, the system reduces to

$$\dot{x}_1 = -x_2,$$
$$\dot{x}_2 = x_1,$$
$$\dot{x}_3 = 0,$$

and so the circles of intersection of the sphere with planes perpendicular to the x_3-axis are periodic orbits. On the other hand, inside the sphere $\dot{x}_3 < 0$, so x_3 is strictly decreasing and there cannot be any periodic orbits inside the sphere. Similarly, outside the sphere $\dot{x}_3 > 0$, and there are no periodic orbits in that region. Thus we have determined all periodic solutions.

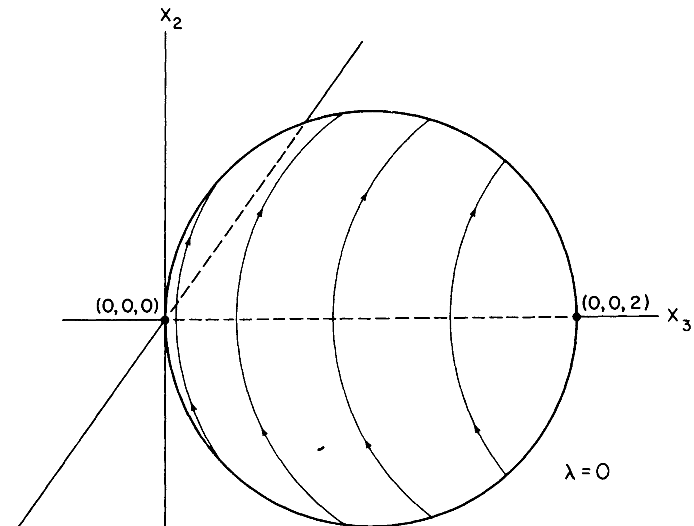

FIGURE 2.

By routine calculation, we find that

$$L(\lambda)=\begin{bmatrix}\lambda & -1 & 0\\ 1 & \lambda & 0\\ 0 & 0 & -2\end{bmatrix}$$

and that $\mathrm{Mult}_\lambda(i\beta)=\{\alpha_\lambda+i\beta\}$, where $\alpha_\lambda=\lambda$ for all λ. Also, condition (2) is satisfied. Thus we indeed have an example as claimed. The reader might also find it instructive to check the content of Proposition 1.2 for this example.

Example 2.3. In both of the previous examples, the period of all non-stationary periodic solutions is 2π. This example shows that it is possible to have the periods on $\mathfrak{N}_0$ unbounded while λ and x are bounded. Furthermore, there are no stationary solutions except $x=0$.

The system is defined on R^4. Let S^3 denote the sphere $\{u\in R^4 | |u|=1\}$. For some irrational real γ, let

$$g_0(u)=(u_2,-u_1,\gamma u_4,-\gamma u_3).$$

The equation

$$\dot u=g_0(u)$$

defines a dynamical system for which S^3 is an invariant set, since $\langle u,g_0(u)\rangle=0$. Also it is easily seen that this equation has precisely two periodic orbits, to wit,

$$\{u|u_1^2+u_2^2=1,\ u_3=u_4=0\}\quad\text{and}\quad\{u|u_1=u_2=0,\ u_3^2+u_4^2=1\}.$$

P. A. Schweitzer [23] has developed a method of construction which leads to a one-parameter family of differential equations

$$\dot u=g_r(u) \tag{2.1}$$

on S^3. This family is defined for $r\in[0,1]$ with g_0 as above; it is continuous in r, is C^1 in u, and is never 0. Furthermore (2.1) has exactly two periodic solutions for each $r\in[0,1)$ which depend continuously on r, and has no periodic solutions for $r=1$. Extend g_r for $r>1$ by letting $g_r=g_1$ if $r>1$. We will also assume that $g_r\equiv g_0$ for r close to 0.

Define

$$f(\lambda,x)=\begin{cases}\lambda x+|x|\,g_{|x|}(x/|x|) & \text{if } x\neq 0,\\ 0 & \text{if } x=0,\end{cases}$$

and consider the system

$$\dot{x} = f(\lambda, x). \tag{2.2}$$

Then $L(\lambda)\cdot x$ is defined and equals $\lambda x + g_0(x)$. Thus $L(\lambda)$ has the four eigenvalues $\lambda \pm i$, $\lambda \pm \gamma i$.

For any solution $x(t)$ of (2.2),

$$\frac{d}{dt}|x(t)|^2 = 2\pi |x(t)|^2,$$

where $|\cdot|$ is the Euclidean norm. Hence for $\lambda \neq 0$, $x \neq 0$, the absolute value $x(t)$ is strictly monotone and the solution cannot be periodic. If $\lambda = 0$ and $x(t) \neq 0$, then $x(t)$ satisfies (2.2) if and only if $u(t) = x(t)/|x(t)|$ satisfies (2.1) with $r = |x(t)|$. Hence $\mathfrak{R}$, cataloguing the periodic orbits, is a union of disjoint components which we denote by $\mathfrak{R}_{1,k}$ and $\mathfrak{R}_{\gamma,k}$. These are labeled so that $(0, 2\pi k, 0) \in \mathfrak{R}_{1,k}$ and $(0, 2\pi k\gamma^{-1}, 0) \in \mathfrak{R}_{\gamma,k}$. In particular, $\mathfrak{R}_{1,1}$ is the $\mathfrak{R}_0$ of Theorem A. Note that the t component of a point in $\mathfrak{R}_{1,1}$ is the period of the orbit. Since the λ and x components are bounded on $\mathfrak{R}_{1,1}$, the t component must be unbounded. This fact is also evident from the details of the construction.

Remark 2.4. One type of example which we do not have, and which would be very interesting, would satisfy conclusion (2.II) of Theorem A with $(\bar{\lambda}, \bar{t}, \bar{x}) = (\lambda_0, t_1, x_0)$ where $t_1 \neq t_0$.

Example 2.5. We give an example for Theorem B.

Consider the Hamiltonian system for a pendulum whose bob has unit mass and is allowed to swing freely in a unit circle in the plane. Let x_1 denote the angle of the bob from the center of the circle. Suppose x_1 is a multiple of 2π when the bob is at the lowest point, i.e., at minimum potential energy. Let $x_2 = \dot{x}_1$ denote the angular momentum. The total energy is $H(x_1, x_2) = -g\cos x_1 + \frac{1}{2}x_2^2$, where $g > 0$. The associated Hamiltonian system is

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -g\sin x_1. \end{aligned} \tag{2.3}$$

This system satisfies the hypotheses of Theorem B, since the eigenvalues of $D_x f(0)$ are $\pm i g^{1/2}$. It is easy to see that the solution of (2.3) through (x_1, x_2) is periodic and nonstationary if and only if $|H(x_1, x_2)| < g$. As $H \to g$ from below, the top of the arc of the pendulum approaches the vertical and the period $t \to \infty$.

Example 2.6. C. L. Siegel [24, pp. 109–110] exhibits an example of a Hamiltonian system with $\pm i$, $\pm 2i$ the (simple) eigenvalues of Hess, and lacking any periodic solutions of period 2π. It is defined on (x_1, x_2, y_1, y_2)-space with (x_1, y_1) and (x_2, y_2) the pairs of conjugate coordinates. The Hamiltonian is

$$H = \tfrac{1}{2}(x_1^2 + y_1^2) - x_2^2 - y_2^2 + x_1 y_1 + \tfrac{1}{2}(x_1^2 - y_1^2) y_2,$$

so that

$$\begin{aligned} \dot{x}_1 &= y_1 + x_1 x_2 - y_1 y_2, \\ \dot{x}_2 &= -2y_2 + \tfrac{1}{2}(x_1^2 - y_1^2), \\ \dot{y}_1 &= -x_1 - y_1 x_2 - x_1 y_2, \\ \dot{y}_2 &= 2x_2 - x_1 y_1. \end{aligned}$$

Clearly the origin is stationary and Hess has eigenvalues $\pm i$, $\pm 2i$. The solutions

$$\begin{aligned} x_1 &= y_1 = 0, \\ x_2 &= \alpha \cos 2t - \beta \sin 2t, \\ y_2 &= \alpha \sin 2t + \beta \cos 2t \end{aligned} \tag{2.4}$$

are a bifurcating system with period π. Note that since α, β are arbitrary, conclusion I_H of Theorem B obtains. On the other hand, if $p = x_1^2 + y_1^2$ and $q = x_2^2 + y_2^2$, then $\ddot{p} = 4pq + p^2$. Thus there are no periodic solutions if $p \not\equiv 0$, and (2.4) are the only periodic solutions.

Example 2.7. Using the previous example, it is easy to construct examples illustrating phenomena for more complicated Mult$(i\beta)$. We exhibit one such. Let (E'_λ), defined on R^4, be (H_λ) where $H(x)$ is as in Example 2.6. Let (E''_λ) be defined on R^2 by the equations

$$\begin{aligned} \dot{\theta} &= 2, \\ \dot{r} &= r\lambda \end{aligned}$$

in polar coordinates (compare Example 2.1). Let (E_λ), defined on R^6, be the direct sum of (E'_λ) and (E''_λ). The origin is stationary for all λ, and $L(0)$ has eigenvalues $\pm i, \pm 2i, \pm 2i$. If $\beta = 1$, the parity of $i\beta$ with respect to L is odd. However the period of all non-constant periodic solutions is π. Thus the period of the bifurcating family is a proper divisor of $2\pi\beta^{-1}$.

3. Initial Reductions. In this section we show that without loss of generality we may consider the manifold M to be an open subset $\mathfrak{O}$ of Euclidean space. In that case, the domain of G will be an open subset of $\Lambda \times [0,\infty) \times \mathfrak{O}$. We also compute in this section the x-derivative of $G(\lambda,t,x)$ at $x=0$ for $\lambda \in \Lambda_0$.

To effect the first reduction, we embed M in some Euclidean space and extend the vector field f to some tubular neighborhood. Recall that M is C^1 of dimension n. First we must handle the boundary of M. We do this by attaching an external collar to M. Let B be the boundary of M; the collar is the space $B \times [0,1)$. Let $M^+ = M \cup_B (B \times [0,1))$; that is, $b \in B \subset M$ is identified with $(b,0) \in B \times [0,1)$. The space M^+ can be given the structure of a C^1 manifold. (Compare [16, p. 56].) The tangent space of $B \times [0,1)$ at a point (b,r) is $V_b \oplus V_r$ where V_b is the tangent space of B at b and V_r is one-dimensional. Because of the hypothesis that solutions of (E_λ) starting on B never leave B, we have that $f(\lambda,b) \in V_b$ for all $b \in B$. Extend f defined on $\Lambda \times M$ to f^+ defined on $\Lambda \times M^+$ by

$$f^+(\lambda,x) = f(\lambda,x) \qquad \text{for} \quad x \in M,$$
$$f^+(\lambda,(b,r)) = f(\lambda,b) + r \in V_b \oplus V_r \qquad \text{for} \quad (b,r) \in B \times [0,1).$$

That is, on the collar, the vector field f^+ is given an outward component. Although f^+ is not C^1, it is still the case that (E_λ^+) has unique solutions. There are no periodic or stationary solutions of (E_λ^+) on the collar, and any statement about such solutions on M^+ applies directly to M. In sum, we may assume without loss of generality that M has no boundary.

A C^1 manifold M without boundary may be embedded in some Euclidean space R^N as a C^1 submanifold with x_0 at the origin [16, 2.10]. Furthermore, it admits a C^1 tubular neighborhood [16, 5.5]. That is, there is an open neighborhood $\mathfrak{O}$ of M in R^N and a C^1 retraction $\rho: \mathfrak{O} \to M$. Moreover, $\rho: \mathfrak{O} \to M$ admits the structure of a C^1 fiber bundle with disks as fibers. That is, for any $x \in M$, the fiber $D_x = \rho^{-1}(x)$ is C^1 diffeomorphic to a disk of dimension $N-n$ and there is a local product structure on $\mathfrak{O}$. We may suppose that the bundle $\rho: \mathfrak{O} \to M$ is endowed with a C^1 metric so that each disk D_x has radius one. We may give $\mathfrak{O}$ polar "coordinates." That is, we may represent $v \in \mathfrak{O}$ as $v = (x,r,\theta)$ with $x = \rho(v) \in M$, with $r = |v|$ the radial coordinate $(0 \leqslant r < 1)$, and with θ the "angular" coordinate in the associated sphere bundle. Define a system $f_\theta(\lambda,v)$ on $\mathfrak{O}$ by

$$\dot{x} = f(\lambda,\rho v),$$
$$\dot{r} = r,$$
$$\dot{\theta} = 0.$$

That is, on each disk, the vector field f_θ is given an outward component. Again it is clear there are no periodic or stationary solutions of the extended system except for $r=0$, i.e., on M itself.

Moreover, at $x_0 \in M \subset \mathfrak{O}$, the matrix

$$L_{\mathfrak{O}}(\lambda)=\begin{pmatrix} L(\lambda) & 0 \\ 0 & I \end{pmatrix}$$

(compare the calculations in Example 2.1), where I is the identity $N-n$ matrix. Thus $L_{\mathfrak{O}}(\lambda_0)$ has the same pure imaginary eigenvalue as $L(\lambda_0)$, and the index and parity of $i\beta$ with respect to $L_{\mathfrak{O}}$ are the same as with respect to L. Thus we may suppose without loss of generality that our original system (E_λ) is defined on the open subset $\mathfrak{O}$ of Euclidean space with $x_0=0$.

Finally on this point, since condition (2) of Data 1.1 is independent of the metric, we may suppose the metric on $\mathfrak{O}$ is the standard Euclidean metric. This is convenient when we calculate the derivative of G.

For any $x \in \mathfrak{O}$, let $T(\lambda,x)$ be the least upper bound (possibly $+\infty$) of the times t for which $G(\lambda,t,x)$ can be defined. Recall that a function T is lower semi-continuous if $\liminf T(\lambda_k,x_k) \geqslant T(\lambda,x)$ for any (λ,x) is its domain and any sequence $(\lambda_k,x_k) \to (\lambda,x)$. Theorem V.2.1 (p. 94) of [8] states in part that T is lower semi-continuous. From this it is immediate that the set

$$\mathfrak{U}=\{(\lambda,t,x)\,|\,\lambda \in \Lambda,\ x \in \mathfrak{O},\ 0<t<T(\lambda,x)\}$$

is an open subset of $R^{n+2}=R \times R \times R^n$. The same theorem of [8] also states that on $\mathfrak{U}$, the function $G(\lambda,t,x)\colon \mathfrak{U} \to R^n$ is continuous.

We fix a vector-space basis of R^n and identify the tangent bundle of R^n at any point with R^n. Thus we may consider the x-derivative $D_xG(\lambda,t,0)$ a matrix-valued function. We prove the following about it.

LEMMA 3.1. *The matrix $D_xG(\lambda,t,0)$ exists in the sense of Condition (2) of Data 1.1 and is continuous in (λ,t), and for each $\xi \in R^n$ and λ in a neighborhood of λ_0, the orbit $y(t)=D_xG(\lambda,t,0)\cdot\xi$ satisfies*

$$\dot{y}=L(\lambda)\cdot y, \qquad y(0)=\xi.$$

Hence

$$D_xG(\lambda,t,0)\cdot\xi=\left[\exp(tL(\lambda))\right]\cdot\xi.$$

Proof. The last equation is immediate, since the differential equation for y is linear with constant coefficients [8, Chapter IV, Section 5]. We remark that if $f(\lambda,x)$ is C^1 in a neighborhood of the x-axis, the result is a classical theorem of G. Peano. (See [8, Theorem V.3.1].)

We define an auxilliary system with an extra parameter δ. Let

$$\varphi(\delta,\lambda,y)=\begin{cases} f(\lambda,\delta y)/\delta & \text{if } \delta\neq 0 \\ L(\lambda)\cdot y & \text{if } \delta=0 \end{cases}.$$

Note φ is defined on an open subset of $R\times R\times R^n$. Condition (2) of Data 1.1 guarantees that $g(\lambda,y)=L(\lambda)\cdot y-f(\lambda,y)$ satisfies $g(\lambda,\delta y)/\delta\to 0$ uniformly for bounded y. For $\delta\neq 0$,

$$\varphi(\delta,\lambda,y)=L(\lambda)\cdot y+g(\lambda,\delta y)/\delta,$$

and thus $\varphi(\delta,\lambda,y)\to L(\lambda_1)\cdot y_1$ as $(\delta,\lambda,y)\to(0,\lambda_1,y_1)$. Hence φ is continuous.

Consider the differential equation

$$\dot y=\varphi(\delta,\lambda,y),\qquad y(0)=\xi. \tag{*}$$

Define

$$\psi(\delta,\lambda,t,\xi)=\begin{cases} G(\lambda,t,\delta\xi)/\delta & \text{if } \delta\neq 0 \\ \text{the unique solution of} & \\ \quad \dot y=L(\lambda)\cdot y,\quad \dot y(0)=\xi & \text{if } \delta=0. \end{cases}$$

We claim that $\psi(\delta,\lambda,t,\xi)$ is a solution of (*). For $\delta=0$, the claim is trivial. For $\delta\neq 0$, we have

$$\begin{aligned}\frac{d}{dt}\psi(\delta,\lambda,t,\xi)&=\frac{d}{dt}G(\lambda,t,\delta\xi)/\delta\\ &=f(\lambda,G(\lambda,t,\delta\xi))/\delta\\ &=f(\lambda,\delta\psi(\delta,\lambda,t,\xi))/\delta\\ &=\varphi(\delta,\lambda,\psi(\delta,\lambda,t,\xi)).\end{aligned}$$

Also $\psi(\delta,\lambda,0,\xi)=\xi$. Thus the claim is established.

Also note that (*) has unique solutions—for $\delta=0$ because it is linear, and for $\delta\neq 0$ because (E_λ) has unique solutions. Hence the solution $\psi(\delta,\lambda,t,\xi)$ is continuous in its four variables [8, Theorem V.2.1]. Thus

$$\lim_{\delta\to 0} G(\lambda,t,\delta\xi)/\delta=D_xG(\lambda,t,0)\cdot\xi$$

exists and is the solution of (*) for $\delta=0$. This proves (3.1). □

Now define

$$F(\lambda,t,x)=G(\lambda,t,x)-x:\mathfrak{U}\to R^n.$$

The zeros of F are thus precisely the stationary and periodic orbits of the solutions of (E_λ). In particular, for $\lambda\in\Lambda_0$ we have $F(\lambda,t,0)=0$. Moreover, by the previous lemma, for $\lambda\in\Lambda_0$ the derivative $D_xF(\lambda,t,0)$ exists and equals $\exp(tL(\lambda))-I$. Recall that $L(\lambda_0)$ has a conjugate pair of purely imaginary non-zero eigenvalues $\pm i\beta$ and that $t_0=2\pi\beta^{-1}$. Also recall condition 4 of Data 1.1. Let k be a positive integer. Let

$$q_k(\lambda,t)=\exp(ktL(\lambda))-I.$$

LEMMA 3.2. *There exists a neighborhood U of (λ_0,t_0) in $\Lambda_0\times(0,\infty)\subset R\times R$ such that if $(\lambda,t)\in U$ is not (λ_0,t_0), then $q(\lambda,t)$ is non-singular.*

Proof. If $\theta_{1\lambda},\dots,\theta_{n\lambda}$ are the eigenvalues of $L(\lambda)$, then

$$e^{kt\theta_{1\lambda}}-1,\dots,e^{kt\theta_{n\lambda}}-1$$

are the eigenvalues of $q_k(\lambda,t)$. One of these eigenvalues is zero if and only if either one of the $\theta_{j\lambda}$ is zero (which cannot happen near λ_0 by hypothesis) or one of the eigenvalues $\theta_{j\lambda}$ is a pure imaginary $i\beta_j$ ($\beta_j>0$) and $k\beta_jt\equiv 0 \bmod 2\pi$. Since $t_0=2\pi\beta^{-1}$, we need worry only about elements of $\mathrm{Mult}_\lambda(i\beta)$. By condition (4) of Data 1.1, for t near but not equal to t_0, none of the eigenvalues $e^{kt\theta_{j\lambda}}-1$ is zero. Thus (3.2) is proved. □

We are now in a position to quote Theorem 1.1 of [2] for the case of a two-dimensional parameter. The two variables in our parameter are λ and t. The origin in [2] has clearly been translated to (λ_0,t_0). We need to consider a small circle S^1 in $U-\{(\lambda_0,t_0)\}$ and consider its image under q in the general linear group $GL(n)$. This determines an element γ of the fundamental group of $GL(n)$, which is a copy of the integers Z if $n=2$, and $Z/2Z$ if $n>2$. If γ is a generator, [2, Theorem 1.1] guarantees bifurcation of zeros of F. The next section is devoted to showing that if the parity of $i\beta$ with respect to L is odd, the element γ is a generator.

Remark 3.3. In the original version of the present paper, a full proof was presented of the special case of [2, Theorem 1.1] that is needed. Subsequently, that particular portion of the present proof was generalized and isolated, since it seems to be useful in other contexts. Quoting [2] has led to a considerable shortening and rearrangement of the present paper, and published references to specific sections of earlier versions may be in error.

4. The Element of the Fundamental Group. We have established the existence of a good neighborhood U (which we may suppose is a disk) of (λ_0, t_0) in the two-dimensional (λ, t) plane. We want to show that if the parity of $i\beta$ is odd, the generator of the fundamental group of $U - \{(\lambda_0, t_0)\}$ is mapped by $q_k(\lambda, t) = \exp(ktL(\lambda)) - I$ to a generator of the fundamental group of $\mathrm{GL}(n)$. Conversely, if the parity is even, the generator is mapped to zero. We proceed through three steps. First we take on the case $n=2$. Then we handle the case when n is arbitrary, but $\mathrm{Mult}(i\beta)$ has only the one element $i\beta$. This we do by finite-dimensional perturbation theory. Finally we handle the general case by a general position argument.

Case I: $n=2$. Let the unique element of $\mathrm{Mult}_\lambda(i\beta)$ be $\alpha_\lambda + i\beta_\lambda$ with $\alpha_\lambda, \beta_\lambda$ real continuous functions of λ. Note that α_λ is not zero if $\lambda \neq \lambda_0$.

We first observe that there is a basis e_1, e_2 of R^2 such that $L(\lambda_0)$ has the form

$$\begin{pmatrix} 0 & b_1 \\ -b_2 & 0 \end{pmatrix}$$

with $b_1 b_2 = \beta^2$ and $b_1, b_2 > 0$. Let e_1', e_2' be any basis, and with respect to this basis let $L(\lambda_0)$ have the form

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Since the eigenvalues of $L(\lambda_0)$ are $\pm i\beta$, the trace $a_{11} + a_{22}$ must be zero. If $a_{11} \neq 0$, we may suppose $a_{11} > 0$. In that case $a_{22} < 0$. The component of Le_1' in the e_1' direction is $a_{11} > 0$. The component of Le_2' in the e_2' direction is $a_{22} < 0$. By continuity, there is a vector in the first quadrant—call it e_1—such that the component of Le_1 in the e_1 direction is 0. Similarly, there is a vector e_2 in the second quadrant such that the component of Le_2 in the e_2 direction is 0. With respect to the e_1, e_2 basis, $L(\lambda_0)$ has the desired form except possibly $b_1, b_2 < 0$. In that case, we interchange e_1, e_2.

We choose a particular representative for a generator of the fundamental group of $U - \{(\lambda_0, t_0)\}$. As a set, it is a rectangle Q with vertical sides

$$|\lambda - \lambda_0| = \eta_1, \qquad |t - t_0| \leqslant \eta_2$$

and horizontal sides

$$|\lambda - \lambda_0| \leqslant \eta_1, \qquad |t - t_0| = \eta_2.$$

The positive numbers η_1, η_2 are chosen small enough so that Q lies in $U-\{(\lambda_0, t_0)\}$. We suppose that the point $p=(\lambda, t)$ traces out Q counterclockwise.

For notational convenience, let $E(\lambda, t)$ denote the matrix $\exp(ktL(\lambda))-I$. Note that

$$E(\lambda_0, t)\cdot e_1 = \begin{pmatrix} \cos(kt\beta)-1 \\ -\beta^{-1/2} b_1 \sin(kt\beta) \end{pmatrix}.$$

Therefore for $t<t_0$ $(t>t_0)$, the vector $E(\lambda_0, t)\cdot e_1$ is above (below) the horizontal ($=e_1$) axis. We may suppose that η_1 is chosen small enough that $E(\lambda, t)\cdot e_1$ is above (below) the horizontal axis when p is on the horizontal side $t=t_0-\eta_2$, $|\lambda-\lambda_0|\leqslant\eta_1$ $(t=t_0+\eta_2, |\lambda-\lambda_0|\leqslant\eta_1)$ of Q.

The eigenvalues of $E(\lambda, t)$ are $e^{kt(\alpha_\lambda \pm i\beta_\lambda)}-1$. Therefore the only points on the vertical sides of Q for which the eigenvalues of $E(\lambda, t)$ are real are the points $p_\pm=(\lambda_0\pm\eta_1, 2\pi/\beta_{\lambda_0\pm\eta_1})$. If the eigenvalues are not real, there is no one-dimensional invariant subspace, and in particular $E(\lambda, t)\cdot e_1$ is not horizontal. Therefore the points $p_\pm$ divide Q into two pieces; for $p\in Q$ below these points, $E(\lambda, t)\cdot e_1$ is above the horizontal, and for $p\in Q$ above these points, below the horizontal.

Now consider $E(\lambda, t)\cdot e_1$ for $(\lambda, t)=p_\pm$. It is evident that the eigenvalues of $E(\lambda, t)$ have the opposite (the same) sign if the parity of $i\beta$ is odd (even), i.e., if α_λ changes sign (does not) at λ_0. Therefore, as p traces out Q, the curve $E(\lambda, t)\cdot e_1$ winds around the origin ± 1 times if the parity of $i\beta$ is odd. This is precisely the property that characterizes the generator of the fundamental group of GL(2). Conversely, if the parity is even, $E(\lambda, t)\cdot e_1$ has winding number 0. Thus case I is done.

Case II: $\text{Mult}(i\beta)=\{i\beta\}$. Choose a basis for R^n so that the eigenspace of $\pm i\beta$ is spanned by the first two basis vectors and so that the sum of the other eigenspaces is spanned by the rest of the basis vectors. Recall that GL(n) is open in the space of all matrices. Accordingly, a fundamental system of neighborhoods of a matrix may be taken to be open balls. We claim there is such a neighborhood V_1 of $L(\lambda_0)$ and a map $m: V_1 \to \text{GL}(n)$ such that $mL(\lambda_0)=I$ and for all $L\in V_1$, the matrix $(mL)L(m{:})^{-1}$ is of the form

$$\begin{pmatrix} B_L & 0 \\ 0 & B_L' \end{pmatrix},$$

where B_L is a 2×2 matrix.

To establish this, we use the following basic fact from perturbation theory. (See for example [11, II, Section 5.1 and II, Section 5.8]. In the finite-dimensional case, a direct proof is not difficult.) For L in some neighborhood V_1 of $L(\lambda_0)$, there are unique (because $i\beta$ is a simple eigenvalue) subspaces P_L^2, P_L^{n-2} invariant under the action of L, which are near the $L(\lambda_0)$-invariant subspaces $R^2\times 0$, $0\times R^{n-2}$. Moreover the assignments $L\mapsto P_L^2$, $L\mapsto {}_L^{n-2}$ are continuous on V_1. Let mL be the linear map which on P_L^2 is the orthogonal projection to $R^2\times 0$, and on P_L^{n-2} is the orthogonal projection to $0\times R^{n-2}$. If V_1 is small enough, mL is an isomorphism for all $L\in V_1$. Clearly $(mL)L(mL)^{-1}$ is as claimed. We suppose U is small enough that if $(\lambda,t)\in U$, then $L(\lambda)\in V_1$.

Let $u_1\colon V_1\times[0,1]\to V_1$ be the linear retraction of V_1 to $L(\lambda_0)$. That is, $u_1(L,\tau)=(1-\tau)L+\tau L(\lambda_0)$. Let V_2 be an open-ball neighborhood of $B'_{L(\lambda_0)}$ in $\mathrm{GL}(n-2)$. We may suppose V_1 is small enough that if $L\in V_1$, then $B'_L\in V_2$. Let u_2: $V_2\times[0,1]\to V_2$ be the linear retraction of V_2 to $B'_{L(\lambda_0)}$. Let $E'(\lambda,t)$ denote the $(n-2)\times(n-2)$ matrix $\exp(ktB'_{L(\lambda)}-I)$. Since no eigenvalue of $B'_{L(\lambda_0)}$ is an integral multiple of $i\beta$, the matrix $E'(\lambda_0,t_0)$ is non-singular. Let V_3 be an open-ball neighborhood of $E'(\lambda_0,t_0)$ in $\mathrm{GL}(n-2)$. We may suppose that V_2 is small enough that $E'(V_2)\subset V_3$.

Let Q be the generator for the fundamental group of $U-\{(\lambda_0,t_0)\}$ used in case I. Let $\xi_0=q|Q\colon Q\to\mathrm{GL}(n)$; i.e., $\xi_0(\lambda,t)=E(\lambda,t)$. We define a homotopy of ξ_0 in $\mathrm{GL}(n)$ in four stages. At the end of the fourth, we will have reduced case II to case I. Define $\Xi_1\colon Q\times[0,1]\to\mathrm{GL}(n)$ by

$$\Xi_1(\lambda,t,\tau)=\exp\left\{kt\left[mu_1(L(\lambda),\tau)\right]\cdot L(\lambda)\cdot\left[mu_1(L(\lambda),\tau)\right]^{-1}\right\}-I.$$

The eigenvalues of $[mu_1(L(\lambda),\tau]\cdot \mathrm{L}(\lambda)\cdot[mu_1(L(\lambda),\tau)]^{-1}$ are the same as those of $L(\lambda)$; hence $\Xi_1(\lambda,t,\tau)$ is never singular. The map Ξ_1 is a homotopy between ξ_0 and ξ_1 where

$$\xi_1(\lambda,t,\tau)=\exp\left[kt\begin{pmatrix} B_{L(\lambda)} & 0\\ 0 & B'_{L(\lambda)}\end{pmatrix}\right]-I$$

$$=\begin{pmatrix}\exp(ktB_{L(\lambda)})-I & 0\\ 0 & \exp(ktB'_{L(\lambda)})-I\end{pmatrix}.$$

Define $\Xi_2\colon Q\times[0,1]\to\mathrm{GL}(n)$ by

$$\Xi_2(\lambda,t,\tau)=\begin{bmatrix}\exp(ktB_{L(\lambda)})-I & 0\\ 0 & \exp\left[ktu_2(B'_{L(\lambda)},\tau)\right]-I\end{bmatrix}.$$

This is a homotopy between ξ_1 and ξ_2, where

$$\xi_2(\lambda,t,\tau)=\begin{bmatrix}\exp(ktB_{L(\lambda)})-I & 0\\ 0 & \exp(ktB'_{L(\lambda_0)})-I\end{bmatrix}.$$

Define $\Xi_3 : Q\times[0,1]\to \mathrm{GL}(n)$ by

$$\Xi_3(\lambda,t,\tau)=\begin{bmatrix}\exp(ktB_{L(\lambda)})-I & 0\\ 0 & \exp(k[\tau t_0+(1-\tau)t]B'_{L(\lambda_0)})-I\end{bmatrix}.$$

This is a homotopy between ξ_2 and ξ_3, where

$$\xi_3(\lambda,t,\tau)=\begin{bmatrix}\exp(ktB_{L(\lambda)})-I & 0\\ 0 & \exp(kt_0B'_{L(\lambda_0)})-I\end{bmatrix}.$$

Finally, let $\rho:[0,1]\to\mathrm{GL}(n)$ be a path from $\exp(kt_0B'_{L(\lambda_0)})-I$ to

$$I^{\pm}=\begin{pmatrix}I_{n-3} & 0\\ 0 & \det E(\lambda_0,t_0)\end{pmatrix},$$

where I_{n-3} is the identity $(n-3)\times(n-3)$ matrix. Define Ξ_4 by

$$\Xi_4(\lambda,t,\tau)=\begin{pmatrix}\exp(ktB_{L(\lambda)})-I & 0\\ 0 & \rho(\tau)\end{pmatrix}.$$

This is a homotopy between ξ_3 and ξ_4, where

$$\xi_4(\lambda,t)=\begin{pmatrix}\exp(ktB_{L(\lambda)})-I & 0\\ 0 & I^{\pm}\end{pmatrix}.$$

It is well known that the inclusion of GL(2) in GL(n) induces a surjection on the fundamental groups. Therefore if $\exp(ktB_{L(\lambda)})-I$ represents a generator (represents 0) of the fundamental group of GL(2), then ξ_4, and hence ξ_0, represents a generator of the fundamental group of GL(n) (represents 0). By case I, we are done.

Case III: *General case*. Choose a basis of R^n so that the eigenspaces of the elements of Mult($i\beta$) are spanned by the first $2r$ basis vectors, where r is the cardinality of Mult($i\beta$) and so the sum of the other eigenspaces is spanned by the remaining basis vectors. As in case II, we can find a neighborhood V_1 of $L(\lambda_0)$ in GL(n) and a map $m: V_1 \to \mathrm{GL}(n)$ such that $mL(\lambda_0) = 1$ and for all $L \in V_1$, the matrix $(mL)L(mL)^{-1}$ is of the form

$$\begin{pmatrix} B_L & 0 \\ 0 & B'_L \end{pmatrix},$$

where now B_L is a $2r \times 2r$ matrix. Fix Q as before. As in case II, we can homotop $\xi_0: Q \to \mathrm{GL}(n)$ to $\xi_1: Q \to \mathrm{GL}(n)$, where

$$\xi_1(\lambda, t) = \begin{pmatrix} \exp(ktB_{L(\lambda)}) - I & 0 \\ 0 & \exp(ktB'_{L(\lambda)}) - I \end{pmatrix}.$$

Note that $E'(\lambda, t) = \exp(ktB'_{L(\lambda)}) - I$ is non-singular throughout U.

Next we alter the map $B_L: (\lambda_0 - \eta_1, \lambda_0 + \eta_1) \to \mathrm{GL}(2r)$. Let M denote the space of all real $2r \times 2r$ matrices. Consider the subsets C_1 and C_2 of M, where C_1 is the set of matrices with at least one purely imaginary eigenvalue and C_2 is the set of matrices with at least two pairs of conjugate pure imaginary eigenvalues. According to the "M-structure lemma" of [1, p. 99], the set C_1 is closed and is a finite union of submanifolds of codimension at least one. (We thank Alberto Verjovsky for suggesting we go this route.) By the same proof, C_2 can be shown to be closed and a finite union of submanifolds of codimension at least two. The map $B_L: (\lambda_0 - \eta_1, \lambda_0 + \eta_1) \to \mathrm{GL}(2r)$ can be approximated by a smooth map $B_1: (\lambda_0 - \eta_1, \lambda_0 + \eta_1) \to \mathrm{GL}(2r)$ with $B_1(\lambda_0 \pm \eta_1) = B_{L(\lambda_0 \pm \eta_1)}$. Then by successive applications of the Thom transversality theorem, the map B_1 can be approximated by a map $B_2: (\lambda_0 - \eta_1, \lambda_0 + \eta_1) \to \mathrm{GL}(2r)$ with $B_2(\lambda_0 \pm \eta_1) = B_1(\lambda_0 \pm \eta_1)$, and which is transversal to the manifolds that make up C_1 and C_2. Taking into account the codimensions of these manifolds, we see that there are only a finite number of λ_i $(i = 1, \ldots, K)$ of points in $(\lambda_0 - \eta_1, \lambda_0 + \eta_1)$ for which $B_2(\lambda_i)$ has a purely imaginary eigenvalue, and each such $B_2(\lambda_i)$ has precisely one pair of conjugate purely imaginary eigenvalues, which are denoted $\pm i\beta_i$.

We require that the approximation B_2 be close enough to B_L that:

(i) there exists a homotopy $H: (\lambda_0 - \eta_1, \lambda_0 + \eta_1) \times [0,1] \to \mathrm{GL}(2r)$ [relative to $(\lambda_0 \pm \eta_1)$] between B_L and B_2 such that $\exp(ktH(\lambda, \tau)) - I$ is non-singular for all $(\lambda, t) \in Q$ and all $\tau \in [0,1]$,

(ii) for each $i = 1, \ldots, K$ there is a positive integer k_i with $|\beta_i - k_i| < \beta - \eta_2$ (note that $\eta_2 < \beta$).

By (i), the map $\xi_1: Q \to \mathrm{GL}(n)$ is homotopic to $\xi_2 = E_2|Q$, where

$$E_2(\lambda,t) = \begin{bmatrix} \exp(ktB_2(\lambda)) - I & 0 \\ 0 & \exp(ktB'_{L(\lambda)}) - I \end{bmatrix}.$$

Choose $l_0,\ldots,l_K \in (\lambda_0 - \eta_1, \lambda_0 + \eta_1)$ so that

$$\lambda_0 - \eta_1 = l_0 < \lambda_1 < l_1 < \cdots < l_{K-1} < \lambda_K < l_K = \lambda_0 + \eta_1,$$

and let Q_i $(i = 1,\ldots,K)$ be the rectangle in the (λ, t) plane with horizontal sides

$$l_{i-1} \leqslant \lambda \leqslant l_i, \qquad |t - t_0| = \eta_2$$

and vertical sides

$$\lambda = l_{i-1}, \quad \lambda = l_i, \qquad |t - t_0| \leqslant \eta_2.$$

By (ii) above, there is exactly one point in the interior of Q_i in the plane for which E_2 is singular. Hence by case II, $E_2|Q_i: Q_i \to \mathrm{GL}(n)$ represents a generator of the fundamental group if and only if r_i, the index of $i\beta$ with respect to B_2, has odd parity. By adding homotopy classes, we find that $E_2|Q: Q \to \mathrm{GL}(n)$ represents a generator of the fundamental group if and only if $\Sigma_{i=1}^K r_i$ is odd. Let r_i^+ denote the number of eigenvalues of $B_2(l_i)$ with positive real part. Then $r_i = r_i^+ - r_{i-1}^+$ and $\Sigma_{i=0}^K r_i = r_K^+ - r_0^+$ is the index of $i\beta$ with respect to L. Thus we are done.

5. Finish of the Proof. Using the notation from the end of Section 3, we have from Section 4 that if the parity of $i\beta$ is odd, there is a connected set of zeros of $F(\lambda,t,x) = G(\lambda,t,x) - x : \mathfrak{U} \to M$ bifurcating from the point (λ_0, t_0, x_0). To show that these are periodic solutions, we must eliminate the possibility they are stationary. The remainder of Theorem A and the propositions concerns accumulation points of M. We establish these results through a series of lemmas. We continue with the assumption that M is an open set $\mathfrak{D}$ in Euclidean space, and $x_0 = 0$.

LEMMA 5.1. *There exists a neighborhood V of $(\lambda_0, 0)$ in $\Lambda \times \mathfrak{D}$ such that if $(\lambda, x) \in V$, $x \neq 0$, then x is not stationary for* (E_λ).

With this and some point-set topology, it is straightforward to establish part (2) of Theorem A.

Proof. Suppose there exists a sequence $(\lambda_i, x_i) \to (\lambda_0, 0)$ with $x_i \neq 0$ such that (λ_i, x_i) is stationary. Note that $v_i = x_i/|x_i| \in S^{n-1}$. We can assume, by choosing a subsequence if necessary, that the v_i converge to $v \in S^{n-1}$. Also choose $t \neq t_0$ so that (λ_0, t) is in the neighborhood U of Lemma 3.2. Then

$$D_xF(\lambda_0, t_0, 0)\cdot v = \lim \frac{F(\lambda_i, t_i, x_i)}{|x_i|} = 0.$$

Thus $D_xF(\lambda_0, t, 0)$ is singular. This contradicts Lemma 3.2. □

Let $(\bar{\lambda}, \bar{t}, \bar{x}) \in \bar{\mathfrak{N}} - \mathfrak{N}$ with $\bar{x} \in \mathfrak{D}$. Let $\{(\lambda_n, t_n, x_n)\} \subset \mathfrak{N}$ be a sequence converging to $(\bar{\lambda}, \bar{t}, \bar{x})$.

Lemma 5.2. *The solution $\bar{x}$ is stationary for* $(E_{\bar{\lambda}})$.

Proof. The solution G satisfies $G(\lambda_n, t_n, x_n) - x_n = 0$. Hence $G(\bar{\lambda}, \bar{t}, \bar{x}) - \bar{x} = 0$, since G is continuous. Suppose first $\bar{t} > 0$. Since $(\bar{\lambda}, \bar{t}, \bar{x})$ is not in $\mathfrak{N}$, it cannot be a non-stationary periodic solution, and so it must be a stationary solution. Now suppose $\bar{t} = 0$. Of course $t_n > 0$ by the definition of $\mathfrak{N}$. Let $t' > 0$, and let k_n be the smallest integer such that $t_n k_n \geq t'$. Then $t_n k_n \to t'$ as $n \to \infty$. Also $G(\lambda_n, t_n k_n, x_n) - x_n = 0$, so by continuity, $G(\bar{\lambda}, t', \bar{x}) - \bar{x} = 0$. Since this is true for all $t' > 0$, the result is proved. □

Define the diameter of the orbit through x for λ, when $(\lambda, x) \in \mathfrak{N}$, by

$$\operatorname{diam}(\lambda, x) = \max\{ |G(\lambda, t_1, x) - G(\lambda, t_2, x)| \}.$$

We now show that the orbit diameter tends to 0 as we approach $(\bar{\lambda}, \bar{x})$. Thus the orbits tend uniformly to $\bar{x}$ as n tends to ∞.

Lemma 5.3. *As $n \to \infty$, we have* $\operatorname{diam}(\lambda_n, x_n) \to 0$.

Proof. Choose $t_1^{(n)}, t_2^{(n)}$ such that

$$\operatorname{diam}(\lambda_n, x_n) = \left|G\left(\lambda_n, t_1^{(n)}, x_n\right) - G\left(\lambda_n, t_2^{(n)}, x_n\right)\right|.$$

However, note that $G(\lambda_n, t, x_n) \to \bar{x}$ uniformly for $t \in [0, t+1]$, since $G(\bar{\lambda}, t, \bar{x}) \equiv \bar{x}$. Also, for n sufficiently large, $t_1^{(n)}, t_2^{(n)}$ are in $[0, \bar{t}+1]$. Hence $G(\lambda_n, t_i^{(n)}, x_n) \to \bar{x}$ as $n \to \infty$ for $i = 1, 2$. Hence $\operatorname{diam}(\lambda_n, x_n) \to 0$. □

Lemma 5.4. *Suppose, in some neighborhood U of $(\bar{\lambda}, \bar{x})$ in $R \times \mathfrak{D}$, that f is Lipschitz with Lipschitz constant L. That is, for some norm $|\cdot|$ on R^n,*

$$|f(\lambda, x) - f(\lambda, y)| \leq L|x - y|$$

whenever $x, y \subset U$. Then

$$\bar{t} \geq 4L^{-1}.$$

If the norm is the standard Euclidean norm, then in fact

$$\bar{t} \geqslant 2\pi L^{-1}.$$

Proof. Lemma 5.3 implies, for n greater than some N, that the orbit through x_n lies entirely in U. It is shown in [14] that the period of this orbit is at least $4L^{-1}$. In [28] it is shown if $|\cdot|$ is the Euclidean norm that the period is at least $2\pi L^{-1}$. By continuity, the limiting period $\bar{t}$ also satisfies these inequalities. □

LEMMA 5.5. *Suppose $(\bar{\lambda}, \bar{t}, \bar{x}) = (\bar{\lambda}, \bar{t}, 0)$ and that condition (2) of Data 1.1 is satisfied in a neighborhood of $\bar{\lambda}$ in R. Then $L(\bar{\lambda}) = D_x f(\bar{\lambda}, 0)$ has a purely imaginary eigenvalue $i\bar{\beta}$, where $\bar{\beta} = 2\pi k \bar{t}^{-1}$ for some integer $k \geqslant 0$.*

Note that this lemma establishes Proposition 1.3.

Proof. To prove this lemma, it is sufficient to prove that the linearized system

$$\dot{y} = L(\bar{\lambda})\, y \tag{5.1}$$

has a non-zero periodic solution with $\bar{t}$ a multiple of its period. Define

$$g(\lambda, y) = f(\lambda, y) - D_x f(\lambda, x) \cdot y.$$

Let $r_n = \max_t |G(\lambda_n, t, x_n)|$ and $y_n(t) \equiv r_n^{-1} G(\lambda_n, t, x_n)$. Then y_n satisfies

$$\dot{y}_n(t) = r_n^{-1} f(\lambda_n, r_n y_n(t)) = L(\lambda_n)\, y_n + q_n(t),$$

where $q_n(t) = r_n^{-1} g(\lambda_n, r_n y_n(t))$. From condition (2) of Data 1.1 and the fact that $|y_n(t)| \leqslant 1$ for all n and t, we have

$$\begin{aligned} |q_n(t)| &\leqslant r_n^{-1} |g(\lambda_n, r_n y_n(t))| \\ &\leqslant |r_n y_n(t)|^{-1} |g(\lambda_n, r_n y_n(t))| \to 0 \end{aligned}$$

since by Lemma 5.3, the diameters of the orbits tend uniformly to 0 as $n \to \infty$.

By choosing an appropriate x_n on the orbit $\{G(\lambda_n, t_n, x_n)\}$, we may assume $r_n = |G(\lambda_n, 0, x_n)| = |x_n|$, which tends to 0 as $n \to \infty$. Furthermore, by replacing $\{(\lambda_n, t_n, x_n)\}$ with a subsequence if necessary, we may assume $r_n^{-1} x_n$ converges to a point y_0 of norm 1. Now $y_n(0) \to y_0$ and

$$L(\lambda_n)\, y + q_n(t) \to L(\bar{\lambda})\, y$$

as $n \to \infty$, uniformly on bounded y sets and uniformly in t. Thus letting $y_0(t)$ be the unique solution of (5.1) having $y(0)=y_0$, we have $y_n(t)\to y_0(t)$ as $n\to\infty$ uniformly for t in each compact t set [8, Theorem II.3.2]. Also, $y_n(t_n)\to y_0(\bar{t})$, so, as in the proof of Lemma 5.2, the solution $y_0(\cdot)$ is periodic (possibly stationary) and $\bar{t}$ is a multiple of its period. Notice that $y_0(\cdot)$ is non-zero, since $y\neq 0$. □

LEMMA 5.6. *Suppose $D_xf(\bar{\lambda},\bar{x})$ exists. Define*

$$g(\lambda,x)=f(\lambda,x)-D_xf(\bar{\lambda},\bar{x})[x-\bar{x}]. \tag{5.2}$$

Suppose that for each $\delta>0$ there exists a neighborhood U_δ of $(\bar{\lambda},\bar{x})$ in $R\times\mathfrak{D}$ such that

$$|g(\lambda,x)-g(\lambda,y)|\leqslant\delta|x-y| \quad \textit{for} \quad (\lambda,x),(\lambda,y)\in U_\delta. \tag{5.3}$$

Then L has a non-zero pair of eigenvalues $\pm i\bar{\beta}$ for $\bar{\beta}=2\pi k t^{-1}$, where k is an integer $\geqslant 1$.

Note that (5.3) is satisfied if f is C^1 is a neighborhood of $(\bar{\lambda},\bar{x})$, since

$$\begin{aligned}&|g(\lambda,x)-g(\lambda,y)|\\&\quad=|f(\lambda,x)-f(\lambda,y)-D_xf(\lambda,x)[x-y]|\\&\quad\leqslant|f(\lambda,x)-f(\lambda,y)-D_x(\lambda,y)[x-y]|+|D_x(\lambda,y)-D_x(\bar{\lambda},\bar{x})|\,|x-y|.\end{aligned}$$

Thus Lemma 5.6 establishes the last part of Proposition 1.2.

Proof. As in the proof of Lemma 5.5, we will show that the linearized system

$$\dot{y}=Ly \tag{5.4}$$

has a periodic solution with $\bar{t}$ a multiple of its period. We must also show that this solution is not stationary.

For any periodic function φ on R with $\varphi(0)=\varphi(p)$, we denote its average by

$$\langle\varphi\rangle=p^{-1}\int_0^p\varphi(s)\,ds.$$

In particular, for $x_n(t)=G(\lambda_n,t,x_n)$ define $\langle x_n\rangle=\langle x_n(t)\rangle$ and $\langle g_n\rangle=\langle g(\lambda_n,x_n(t))\rangle$ and let $r_n=\max|x_n(t)-\langle x_n\rangle|$. As in the proof of Lemma 5.5, we assume $r_n=|x_n(0)-\langle x_n\rangle|$. We study how x_n oscillates around $\langle x_n\rangle$. Let

$$y_n(t)=r_n^{-1}[x_n(t)-\langle x_n\rangle].$$

Again as in the proof of Lemma 5.5, we may assume that $y_n(0)$ converges to some y_0 with $|y_0|=1$. Now let $q_n(t)=r_n^{-1}[g(\lambda_n,x_p(t))-\langle g_n\rangle]$, so that

$$\begin{aligned}\dot{y}_n(t)&=r_n^{-1}\left[Lx_n(t)+g(\lambda,x_n(t))\right]\\&=Ly_n(t)+r_n^{-1}\left[g(\lambda,x_n(t))+\langle Lx_n\rangle\right].\end{aligned}\tag{5.5}$$

However, by definition $\langle y_n\rangle=0$ and by periodicity $\langle \dot{y}_n\rangle=0$, so averaging (5.5),

$$\begin{aligned}0&=\langle Ly_n(t)\rangle+r_n^{-1}\langle g(\lambda,x_n(t))+\langle Lx_n\rangle\rangle\\&=L\langle y_n(t)\rangle+r_n^{-1}(\langle g(\lambda,x_n(t))\rangle+\langle Lx_n\rangle)\\&=0+r_n^{-1}\langle g_n\rangle+r_n^{-1}L\langle x_n\rangle.\end{aligned}$$

Thus $L\langle x_n\rangle=-\langle g_n\rangle$. Replacing $L\langle x_n\rangle$ by $-\langle g_n\rangle$ in (5.5), we see that y_n satisfies

$$\dot{y}=Ly+q_n(t).$$

The proof can be completed following the final procedures of the proof of Lemma 5.5, once we have shown that $|q_n|\to 0$ uniformly. Our limiting periodic solution $y_0(t)$ cannot be stationary, since $\langle y_n(t)\rangle\to\langle y_0(t)\rangle$, but $\langle y_n(t)\rangle=0$, so $\langle y_0(t)\rangle=0$, while of course $y_0(0)=y_0\neq 0$.

Let

$$\delta_n=\sup\left\{\frac{|g(\lambda_n,x_n(t))-g(\lambda_n,x_n(s))|}{|x_n(t)-x_n(s)|}\,\middle|\,x_n(t)\neq x_n(s)\right\}.$$

Since the orbits of x_n converge to 0 as $n\to\infty$, the hypothesis (5.2) assures that $\delta_n\to 0$. Notice also that $|x_n(t)-x_n(s)|\leqslant 2r_n$. Hence

$$\begin{aligned}q_n(t)&=r_n^{-1}|g(\lambda_n,x_n(t))-\langle g_n\rangle|\\&=t_n^{-1}r_n^{-1}\int_0^{t_n}|g(\lambda_n,x_n(t))-g(\lambda_n,x_n(s))|\,ds\\&\leqslant t_n^{-1}\int_0^{t_n}2\delta_n\,ds=2\delta_n\to 0.\qquad\square\end{aligned}$$

There is one last fact to prove.

LEMMA 5.7. *There exists a neighborhood $\mathfrak{M}_0$ of $(\lambda_0,t_0,0)$ such that for $(\lambda,t,x)\in\mathfrak{N}\cap\mathfrak{M}_0$, the period of $G(\lambda,\cdot,x)$ is k^{-1}, where $k=k(\lambda,t,x)$ is a positive integer such that $ik\beta\in\mathrm{Mult}(i\beta)$.*

Proof. Define functions $\tau:\mathfrak{N}\to R^+$ (the positive reals) and $\mu:\mathfrak{N}\to Z^+$ (the positive integers) by $\tau(\lambda,t,x)=$the period of $G(\lambda,\cdot,x)$ and $\mu(\lambda,t,x)=t/\tau(\lambda,t,x)$. We claim that μ is bounded in a neighborhood of $(\lambda_0,t_0,0)$. For suppose there exists a sequence (λ_i,t_i,x_i) in $\mathfrak{N}$ converging to $(\lambda_0,t_0,0)$ with $\mu_i=\mu(\lambda_i,t_i,x_i)\to\infty$. Let $\tau_i=\tau(\lambda_i,t_i,x_i)$. Choose K greater than any κ with $i\kappa\beta\in\text{Mult}(i\beta)$. Then $(\lambda_i,[K^{-1}\mu_i]\tau_i,x_i)\to(\lambda_0,K^{-1}t_0,0)$, where $[K^{-1}\mu_i]$ denotes the largest integer in $K^{-1}\mu_i$. Applying Proposition 1.3, we find that $L(\lambda_0)$ has an eigenvalue $iK\beta$. This is a contradiction.

Choose a sequence (λ_i,t_i,x_i) in $\mathfrak{N}$ converging to $(\lambda_0,t_0,0)$ with $K=\mu(\lambda,t,x)$ constant. We find that $L(\lambda_0)$ has an eigenvalue $ik\beta$. Thus $ik\beta\in\text{Mult}(i\beta)$. □

QUESTION 5.8. Is $k=k(\lambda,t,x)$ independent of (λ,t,x) in a neighborhood of $(\lambda_0,t_0,0)$?

6. Bifurcation of Hamiltonian Systems. In this section we derive Theorem B as a corollary of Theorem A. We consider the one-parameter system

$$\dot{x}=(J+\lambda)\,\text{grad}\,H(x) \qquad (\text{H}_\lambda)$$

and prove the following two lemmas about (H_λ).

LEMMA 6.1. *If* $\lambda\neq 0$, *the system* (H_λ) *has no stationary periodic solutions.*

LEMMA 6.2. *The parity of a purely imaginary eigenvalue* $i\beta$ *of* M *defined from* (6.1) *below is the same as the parity of the number of eigenvalues of* M *that are positive integral multiples of* $i\beta$.

The rest of the data required for the application of Theorem A can be easily checked. We leave the details to the reader. We also leave to the reader the interpretation of the conclusions of Theorem A in the language of Theorem B. Note since

$$f(\lambda,x)=(J+\lambda)\,\text{grad}\,H(x)=(J+\lambda)Mx+O(|x|),$$

that

$$L(\lambda)=D_xf(\lambda,0)=(J+\lambda)M. \qquad (6.1)$$

Proof of Lemma 6.1. Let $x(t)$ be any solution of (H_λ) in $R\times\mathfrak{D}$ for $\lambda\neq 0$. Then

$$\frac{d}{dt}H(x(t))=\langle\text{grad}\,H(x),(J+\lambda)\,\text{grad}\,H(x)\rangle.$$

But $\langle y, Jy \rangle = 0$ for all y and in particular for $y = \operatorname{grad} H$. Therefore

$$\frac{d}{dt} H(x(t)) = \lambda |\operatorname{grad} H(x)|^2.$$

Hence $\lambda^{-1} \mathrm{H}(x(t))$ must be monotonic increasing, so $x(t)$ cannot be a periodic trajectory unless $\operatorname{grad} H(x(t)) \equiv 0$. But if $\operatorname{grad} H(x(0)) = 0$, then obviously $x(0)$ is a stationary point for (H_λ) and $x(t) = x(0)$. □

It is well known in the study of Hamiltonian systems that if σ is an eigenvalue of JM, then so is $-\sigma$. The following lemma extends this fact to $(J+\lambda)M$. We write σ^* for the complex conjugate of σ.

LEMMA 6.3. *Let σ and σ^* be eigenvalues of $(J+\lambda)M$. Then $-\sigma$ and $-\sigma^*$ are eigenvalues of $(J-\lambda)M$.*

This result allows σ and σ^* to be the same real eigenvalue.

Proof. Let Det(0) denote the $n \times n$ matrices with zero determinant. By definition, σ is an eigenvalue of $(J+\lambda)M$ if and only if

$$(J+\lambda)M - \sigma I \in \mathrm{Det}(0).$$

This implies

$$[(J+\lambda)M - \sigma]M^{-1} = J + \lambda I - \sigma M^{-1} \in \mathrm{Det}(0),$$

which implies

$$[J + \lambda I - \sigma M^{-1}]^T = -J + \lambda I - \sigma M^{-1} \in \mathrm{Det}(0),$$

which implies

$$-[-J + \lambda I - \sigma M^{-1}]M = (J-\lambda)M + \sigma I \in \mathrm{Det}(0).$$

Hence $-\sigma$ is an eigenvalue of $(J-\lambda)M$. Similarly, so is $-\sigma^*$. □

Proof of Lemma 6.2. Let Mult$(i\beta)$ with respect to (6.1) have κ elements. Choose $\lambda > 0$, and let $\mathrm{Mult}_\lambda(i\beta) = \{\alpha_1 + i\beta_1, \alpha_2 + i\beta_2, \ldots, \alpha_\kappa + i\beta_\kappa\}$. We first claim that no α_i is zero. If so, the linear system

$$\dot{x} = (J+\lambda)Mx$$

must have a non-stationary periodic solution. By letting $H(x) = x^T M x$ and $\mathfrak{D} = R^{2m}$, we have by Lemma 6.1 that $\lambda = 0$.

So we assume that $\alpha_1, \alpha_2, \ldots, \alpha_r > 0$ and $\alpha_{r+1}, \ldots, \alpha_\kappa < 0$. By Lemma 6.3, we have that $\mathrm{Mult}_{-\lambda}(i\beta) = -\alpha_1 + i\beta_1, -\alpha_2 + i\beta_2, \ldots, -\alpha_\kappa + i\beta_\kappa$. Hence the signature of $i\beta$ is $r - (\kappa - r) = 2r - \kappa$. This has the same parity as κ. □

Remark 6.4. If there are eigenvalues of Hess of the form $ik\beta$ with k an integer larger than one, the Hamiltonian system is said to be in resonance. Recently, local results have been proved which sometimes guarantee bifurcation with period approximately $2\pi\beta^{-1}$ when $i\beta$, $ik\beta$ are simple and no other eigenvalue is an integral multiple of $i\beta$. See [20], [22] and their bibliographies. Perhaps the methods of the present paper can be applied to this problem. Also A. Weinstein has proved bifurcation occurs if Hess is positive definite [27].

Our proof of Theorem B shows there is a strong connection between the Hopf theorem and the Liapunov theorem. D. Schmidt has recently given a more classical proof of the local Hopf theorem that is general enough to imply the local Liapunov theorem [21].

UNIVERSITY OF MARYLAND.

REFERENCES.

[1] R. Abraham and J. Robbins, *Transversal Mappings and Flows*, Benjamin, New York, 1967.

[2] J. C. Alexander, Bifurcation of zeros of parametrized functions, *J. Functional Anal.*, to be published.

[3] A. A. Andronov and C. E. Chackin, *Theory of Oscillations*, Princeton U. P., Princeton, N.J., 1949.

[4] N. N. Brushinskaya, Qualitative integration of a system of n differential equations in a region containing a singular point and a limit cycle, *Dokl. Adad. Nauk SSSR* **139** (1961), pp. 9–12 (in Russian), MR 26 (1963) #5212.

[5] J. A. Canavati A., Algunos aspectos de la teoría de la bifurcación y problemas de autovalores para operadores no lineales, *Miscelánea Mat.* **5** (1973), pp. 1–18.

[6] N. Chafee, The bifurcation of one or more closed periodic orbits from an equilibrium point of an autonomous differential equation, *J. Differential Equations* **4** (1968), pp. 661–679.

[7] ———, A bifurcation problem for a functional differential equation of finitely retarded type, *J. Math. Anal. Appl.* **35** (1971), pp. 312–348.

[8] P. Hartman, *Ordinary Differential Equations*, Wiley, New York, 1969.

[9] E. Hopf, Abzweigung einer periodischen Lösung von einer stationaren Lösung eines Differential systems, *Ber. Math. Phys. Kl. Sachs Akad. Wiss. Leipzig* **94** (1942), pp. 1–22.

[10] R. Jost and E. Zehnder, A generalization of the Hopf bifurcation theorem, *Helv. Phys. Acta* **45** (1972), 258–276.

[11] T. Kato, *Perturbation Theory for Linear Operators*, Springer, Berlin, 1966.

[12] J. B. Keller and S. Antman (Eds.), *Bifurcation Theory and Nonlinear Eigenvalue Problems*, Benjamin, New York, 1969.

[13] M. A. Krasnosel'skii, *Topological Methods in the Theory of Nonlinear Integral Equations*, Pergamon, New York, 1965.

[14] A. Lasota and J. A. Yorke, Bounds for periodic solutions of differential equations, *J. Differential Equations* **10** (1971), pp. 83–91.

[15] J. Marsden and M. McCracken, *The Hopf Bifurcation and Its Applications*, Appl. Math. Sci. # 19, Springer, Berlin, 1976.

[16] J. R. Munkres, *Elementary Differential Topology* (revised ed.), Princeton U. P., Princeton, N.J., 1966.

[17] L. Nirenberg, Topics in non-linear functional Analysis, N.Y.U. Lecture Notes, 1974.

[18] P. H. Rabinowitz, Some global results for non-linear eigenvalue problems, *J. Functional Analysis* **7** (1971), pp. 487–573.

[19] D. H. Sattinger, *Topics in Stability and Bifurcation Theory*, Lecture Notes in Mathematics # 300, Springer, Berlin, 1973.

[20] D. S. Schmidt, Periodic solutions from a resonant equilibrium of a Hamiltonian system, *Celestial Mech.* **9** (1974), pp. 81–103.

[21] ———, Hopf's bifurcation theorem and the center theorem of Liapunov, in Ref. [15].

[22] D. S. Schmidt and D. Sweet, A unifying theory in determining periodic families for Hamiltonian systems at resonance, *J. Differential Equations* **14** (1973), pp. 597–609.

[23] P. A. Schweitzer, Counterexamples to the Seifert conjecture and opening closed leaves of foliations, *Ann. of Math.* **100** (1974), pp. 386–400.

[24] C. L. Siegel and J. K. Moser, *Lectures on Celestial Mechanics*, Springer, Berlin, 1971.

[25] I. Stakgold, Branching of solutions of non-linear equations, *SIAM Rev.* **13** (1971), pp. 289–332.

[26] F. Takens, Singularities of vector fields, IHES, Publications Mathématiques, # 43 (1974), pp. 47–100.

[27] A. W. Weinstein, Normal modes for nonlinear Hamiltonian systems, *Invent. Math.* **20** (1973), pp. 47–57.

[28] J. A. Yorke, Periods of periodic solutions and the Lipschitz constant, *Proc. Amer. Math. Soc.* **22** (1969), pp. 509–512.