# A Phase Plane Discussion of Convergence to Travelling Fronts for Nonlinear Diffusion

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**Abstract**

The paper is concerned with the asymptotic behavior as $t \to \infty$ of solutions $u(x,t)$ of the equation

$$u_t - u_{xx} - f(u) = 0, \qquad x \in (-\infty, \infty),$$

in the case $f(0) = f(1) = 0$, with $f(u)$ non-positive for $u(>0)$ sufficiently close to zero and $f(u)$ non-negative for $u(<1)$ sufficiently close to 1. This guarantees the uniqueness (but not the existence) of a travelling front solution $u = U(x - ct)$, $U(-\infty) = 0$, $U(\infty) = 1$, and it is shown in essence that solutions with monotonic initial data converge to a translate of this travelling front, if it exists, and to a "stacked" combination of travelling fronts if it does not. The approach is to use the monotonicity to take $u$ and $t$ as independent variables and $p = u_x$ as the dependent variable, and to apply ideas of sub- and super-solutions to the diffusion equation for $p$.

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## 1. Introduction

This paper is an alternative but at the same time independent account of the problems discussed by us in an earlier paper [2]. The results are of the same general nature, but they differ in certain details, and the proofs are quite different. To set the scene, we recall the main results from [2], but we do this quite briefly and refer the reader to [2] for further historical and bibliographical background. Two important papers which had not appeared when [2] was written, those by ROTHE [6] and UCHIYAMA [7], use an approach having certain affinities to that in the present paper. In [7] however the method is applied to the case when $f>0$ in the open interval of interest. In [6], convergence results are given both for this case and the case when $f$ has one sign change, as in [2]. The class of functions $f$ examined here does not include those which are positive throughout the open interval, but is more general than the class studied in [2] and in case ($\beta$) of [6].

We are concerned with the pure initial value problem for the nonlinear diffusion equation

$$u_t - u_{xx} - f(u) = 0 \qquad (-\infty < x < \infty, t > 0), \tag{1.1}$$

the initial value being

$$u(x,0) = \phi(x) \qquad (-\infty < x < \infty). \tag{1.2}$$

The question of interest is the behavior as $t \to \infty$ of the solution $u(x,t)$, and, in particular, under what circumstances does it (or does it not) tend to a travelling front solution, a travelling front being a solution of (1.1) of the special form $u = U(x-ct)$ for some $c$ (the velocity), with the limits $U(\pm\infty)$ existing and being unequal. As in [2] we will adopt the normalization that $f \in C^1$ with $f(0) = f(1) = 0$, so that $u \equiv 0$ and $u \equiv 1$ are particular solutions of (1.1), and we take $U(-\infty) = 0$, $U(+\infty) = 1$. With these assumptions on $f$, it is a standard result that if $\phi$ is piecewise continuous and $0 \leqq \phi(x) \leqq 1$ for all $x$, then there exists one and only one bounded classical solution $u(x,t)$ of (1.1), (1.2), and $0 \leqq u(x,t) \leqq 1$ for all $x, t$. We shall always make these assumptions on $\phi$ and $f$, and shall be concerned only with this unique bounded solution.

Our main object in [2] was to show that, under minimal assumptions on $\phi$, when $f'(0) < 0, f'(1) < 0$, the solution converges uniformly to one of several types of travelling front configurations. The three principal results in [2] are the following.

**Theorem A** (Theorem 3.1 of [2]). *Let* $f \in C^1[0,1]$ *satisfy* $f(0) = f(1) = 0$, $f'(0) < 0$, $f'(1) < 0$, $f(u) < 0$ *for* $0 < u < \alpha_0$, *and* $f(u) > 0$ *for* $\alpha_1 < u < 1$, *where* $0 < \alpha_0 \leqq \alpha_1 < 1$.

*Assume there exists a travelling front solution $U$ of* (1.1) *with speed $c$, and let $\phi$ satisfy* $0 \leqq \phi \leqq 1$, *and*

$$\limsup_{x \to -\infty} \phi(x) < \alpha_0, \qquad \liminf_{x \to \infty} \phi(x) > \alpha_1. \tag{1.3}$$

*Then for some constants $z_0$, $K$ and $\omega$, the last two positive, the solution $u(x,t)$ of* (1.1), (1.2) *satisfies*

$$|u(x,t) - U(x - ct - z_0)| < K e^{-\omega t}.$$

The existence of a travelling front is not guaranteed by the above conditions on $f$, although sufficient conditions for its existence are contained in §2 of [2]. We shall be returning to this point later, but if $f$ does satisfy these sufficient conditions, then of course the existence assumption in the statement of Theorem

A can be dropped. A particularly important case is that of the degenerate Nagumo equation, in which $\alpha_0 = \alpha_1$ and a travelling front does exist. Notice also that Theorem A certainly implies the uniqueness of the travelling front (modulo translation).

Theorem A asserts that a solution which vaguely resembles a front at some initial time will develop uniformly into such a front as $t \to \infty$. "Vaguely resembles" here means simply that the solution is less than $\alpha_0$ far to the left and greater than $\alpha_1$ far to the right. Of course, if the words "left" and "right" are interchanged in this statement, the same conclusion holds; the front will then face right rather than left, and will travel in the opposite direction. This corresponds to the fact that the equation (1.1) is invariant under the transformation $x \leftrightarrow -x$, but an increasing function of $x$ becomes decreasing and a positive speed becomes a negative one.

There are also situations in which the solution will develop into a pair of such fronts, moving in opposite directions. This is the point of the second theorem.

**Theorem B** (Theorem 3.2 of [2]). *Let $f$ satisfy the hypotheses of Theorem* A, *and in addition suppose that*

$$\int_0^1 f(u)\,du > 0. \tag{1.4}$$

*Let $\phi$ satisfy $0 \leqq \phi \leqq 1$, and*

$$\limsup_{|x| \to \infty} \phi(x) < \alpha_0, \qquad \phi(x) > \alpha_1 + \eta \quad \text{for } |x| < L, \tag{1.5}$$

*where $\eta$ and L are some positive numbers. Then if L is sufficiently large (depending on $\eta$ and $f$), we have for some constants $x_0, x_1, K$, and $\omega$ (the last two positive),*

$$\begin{aligned} |u(x,t) - U(x - ct - x_0)| &< K e^{-\omega t}, \qquad x < 0, \\ |u(x,t) - U(-x - ct - x_1)| &< K e^{-\omega t}, \qquad x > 0. \end{aligned} \tag{1.6}$$

The assumption (1.4) implies that a travelling front as we have defined it (i.e. an increasing function) has negative speed $c$ and so moves to the left. This is proved in (2.7) of [2], but it is an immediate consequence of multiplying by $U'$ and integrating, over $(-\infty, \infty)$, the equation

$$U'' + cU' + f(U) = 0$$

for the travelling front. The intuitive meaning of (1.6) is that the $x$-interval on which $u$ is near the value 1 is finite and is elongating in both directions with speed $|c|$. If the inequality in (1.4) is reversed, and appropriate changes in (1.5) are made, then an analogous convergence result is still obtained, but with the interval on which $u$ is near 0 elongating.

Finally, there is the case in which no travelling front exists with range $(0, 1)$. This can happen (in view of the existence of a travelling front for the degenerate Nagumo equation) only if $f$ has nore than one internal zero. To each triple of adjacent zeros with properties analogous to the zeros $(0, \alpha, 1)$ of Nagumo's equation, there of course corresponds a travelling front with characteristic speed and characteristic limits at $\pm\infty$. For simplicity consider the case of two adjacent triples of this type (thus five zeros in all), and a solution of (1.1) with range equal to the combined ranges of the two travelling fronts. Let $c_0$, $c_1$ be the two velocities, ordered by increasing $u$. If $c_0 < c_1$, we show in [2] that the solution will tend to split into two separate travelling fronts, becoming very flat for $u$ near the center zero of the five, and that there exists no simple travelling front with range from the first to the fifth zero. If $c_0 > c_1$, however, there exists a unique such travelling front: this corresponds to the fact that in this case a splitting as described above would be conceptually impossible. More generally and more precisely, we prove

**Theorem C** (Theorem 3.3 of [2]). *Let $f(u_i)=0$ and $f'(u_i)<0$, $i=1, 2, 3$, where $u_1<u_2<u_3$. Let there exist travelling fronts $U_1(x-c_1 t)$ and $U_2(x-c_2 t)$ with ranges $(u_1, u_2)$ and $(u_2, u_3)$ respectively. Assume $c_1<c_2$. Let $\alpha_1$ be the least zero of $f$ greater than $u_1$ and $\alpha_2$ the greatest zero less than $u_3$.*

*Suppose $u_1 \leqq \phi(x) \leqq u_3$, and*

$$\limsup_{x\to-\infty} \phi(x) < \alpha_1, \qquad \liminf_{x\to\infty} \phi(x) > \alpha_2.$$

*Then there exist constants $x_1, x_2$, $K$, and $\omega$, the last two positive, such that*

$$|u(x,t) - U_1(x-c_1 t - x_1) - U_2(x - c_2 t - x_2) + u_2| < K e^{-\omega t}.$$

This implies, in particular, that

$$\lim_{t\to\infty} u(\beta t, t) = \begin{cases} u_1 & \text{for } \beta < c_1, \\ u_2 & \text{for } c_1 < \beta < c_2, \\ u_3 & \text{for } c_2 < \beta. \end{cases}$$

The work in the present paper stems from two observations on these results. The first is that it would be a useful extension to be able to drop the restrictions

$$f'(0) < 0, \quad f'(1) < 0. \tag{1.7}$$

It is an immediate consequence of (1.7) that

$$\begin{aligned} f(u) &\leqq 0 \quad \text{for } u(\geqq 0) \quad \text{sufficiently near } 0, \\ f(u) &\geqq 0 \quad \text{for } u(\leqq 1) \quad \text{sufficiently near } 1; \end{aligned} \tag{1.8}$$

the conditions (1.8) are important because, as proved in Lemma 2.3 of [2], they guarantee that, if a travelling front exists, then it is unique (modulo translation). Convergence results must be easier to prove when there is a unique limit for the convergence, and so we retain (1.8) but drop (1.7). This allows us to consider, for example, the equation discussed by KANEL' [4] for the combustion of certain gases, in which $f(u) \equiv 0$ for $u \in (0, \alpha)$ and $f(u) > 0$ for $u \in (\alpha, 1)$.

The work in [2] depends crucially on the assumptions (1.7), and an alternative approach is therefore required. The second observation is therefore that the concept of monotonicity clearly has some significance in these results (it is proved in Lemma 2.1 of [2] that all travelling fronts are necessarily monotonic) and that, as is well-known, the monotonicity of $\phi$ in (1.2) implies that of the corresponding solution $u(x,t)$, as a function of $x$, for all $t>0$. (To prove this, differentiate (1.1) with respect to $x$ to obtain a diffusion equation for $u_x$ and apply the comparison theorem in §1 of [2] to this diffusion equation; with zero as a subsolution, this shows that $u_x \geqq 0$.) This monotonicity allows one to take as independent variables the pair $(u,t)$ instead of $(x,t)$, and to use $p=u_x$ as dependent variable. This transformation is discussed in §2.

As in [2], the principal tools used are comparison theorems for parabolic equations, although now the parabolic equation is one for $p$ in terms of $u,t$. The problem is somewhat complicated by the fact that this parabolic equation is degenerate, in that the coefficient of $p_{uu}$ is $p^2$, and the boundary conditions demand that $p$ vanishes at the end-points $(0,1)$ of the range for $u$. The necessary analysis to deal with this, and the statements and proofs of the comparison theorems, are given in §3.

A typical theorem that results from this approach is the following.

*Let $f\in C^1[0,1]$ satisfy, for some $\alpha\in(0,1)$,*

$$f(0)=f(1)=0,$$

$$f(u)\leqq 0 \textit{ for } u\in(0,\alpha), \qquad f(u)>0 \textit{ for } u\in(\alpha,1),$$

*and*

$$\int_0^1 f(u)\,du>0.$$

*Then there exists a travelling front solution $U(x-ct)$ of* (1.1), *unique modulo translation and necessarily monotonic.*

*Moreover, if $\phi\in C^1(-\infty,\infty)$ with*

$$\phi(-\infty)=0, \qquad \phi(+\infty)=1,$$

*and $\phi'(x)>0$ for all $x$, then there exists a function $\gamma\in C^1[0,\infty)$, with $\gamma'(t)\to 0$ as $t\to\infty$, such that*

$$|u(x,t)-U(x-ct-\gamma(t))|=o(1) \tag{1.9}$$

*uniformly in $x$ as $t\to\infty$, where $u$ is the solution of the initial value problem* (1.1)–(1.2) *corresponding to the initial function $\phi$.*

This result corresponds to Theorem A, in that the conditions imposed on $f$ are sufficient to guarantee the existence of a travelling front. It can be generalized (as in Theorem 4.4 in §4) to cover any $f$ which satisfies any of the sufficient conditions for the existence of a travelling front given in §2 of [2], or equivalently in Lemmas 2.5, 2.6 of the present paper. Theorem A in effect makes the existence of a travelling front in itself sufficient for convergence, but we cannot quite reach that degree of generality here. The result (1.9) is proved by first obtaining a convergence result in the transformed variables $(p,u,t)$ and then integrating back.

The convergence statement in (1.9) is much weaker than that in Theorem A. Not only do we not have exponential convergence, we do not even have uniform convergence to a specific travelling front. The solution takes up the correct "shape" asymptotically, and the correct speed, but since we know only that $\gamma'(t) \to 0$ as $t \to \infty$, and not that $\gamma(t)$ converges to a finite limit, we do not know that there is a specific limiting travelling front.

It is however possible to improve on (1.9), and to obtain convergence at an exponential rate to a specific travelling front, provided that one is prepared to make heavier assumptions on $\phi$. The essential point is that the initial function $\phi(x)$ should be, asymptotically in $x$ as $x \to +\infty$ (if $c>0$) or as $x \to \infty$ (if $c<0$), "sufficiently close" to a travelling front. This result is made precise and proved in § 11.

Theorem B has no corresponding result in the present paper, since its hypotheses preclude monotonicity of the initial function, but there are results corresponding to Theorem C in the case where there is no travelling front with range $(0, 1)$.

Most of the convergence theorems are stated in §4, and then proved in §§5-10. As already remarked, §11 contains more precise convergence results under heavier assumptions.

It should finally be remarked that, although in the various convergence theorems the initial function $\phi$ is monotonic with $\phi(-\infty)=0$, $\phi(+\infty)=1$, it is possible to extend the ideas and methods of the paper to initial functions $\phi$ which are not monotonic or for which the limiting values $\phi(\pm\infty)$ are not $(0, 1)$.

Thus if $\phi$ is monotonic but $\phi(-\infty)=\alpha>0$, then the corresponding initial function in the transformed variables $(p, u, t)$ is not defined for $0 \leqq u < \alpha$. It is however convenient to take it to be identically zero. This makes the problem more degenerate since the coefficient of $p_{uu}$ in the diffusion equation now vanishes initially, not merely at $u=0, 1$, but throughout an interval of values of $u$. In fact, however, this difficulty is faced even in the present paper, since in §3 we have to construct for comparison purposes subsolutions which vanish initially throughout an interval of values of $u$, and there is therefore no essential difficulty in discussing initial functions $\phi$ with $\phi(-\infty) \neq 0$. For simplicity, however, we have refrained from doing so.

To deal which initial functions $\phi$ which are not monotonic, we have to allow $p$, regarded as a function of $u$, to be multi-valued; this has been investigated by CHUEH [1]. Again we will not pursue the matter further in this paper.

## 2. The basic transformation

We introduce the transformation of the independent variables from $(x, t)$ to $(u, t)$ with a lemma the contents of which are well-known but usefully recalled in this form.

**Lemma 2.1.** *Let* $f \in C^1[0, 1]$, *with* $f(0)=f(1)=0$, *and suppose also that* $\phi \in C^1(-\infty, \infty)$, *with* $0 \leqq \phi \leqq 1$ *and* $\phi' > 0$. *Then the solution* $u$ *of* (1.1)-(1.2) *satisfies, for* $t>0$,

$$u_x(x, t) > 0 \quad \textit{for all finite } x \tag{2.1}$$

*and*

$$u_x(\pm\infty, t) = 0. \tag{2.2}$$

**Proof.** We have already remarked in the introduction that (2.1) is a consequence of a maximum, or comparison, principle applied to the diffusion equation for $u_x$, that is, to

$$(u_x)_t - (u_x)_{xx} - f'(u)\, u_x - 0 \tag{2.3}$$

with $u_x(x,0) > 0$. Strictly speaking, the comparison principle in its usual form (as, for example, in the comparison theorem enunciated in the introduction to [2]) applies to solutions which are classical, so that $u_{xt}$ and $u_{xxx}$ have to be continuous. In fact, if we assume merely that $f \in C^1$, we know that $u_t$ and $u_{xx}$ are continuous, but not that they are continuously differentiable, and (2.3) may hold only in the sense of distributions.

To obtain the required result in this case, we observe first that, for any fixed $\delta > 0$, both $u(x+\delta, t)$ and $u(x,t)$ satisfy (1.1). Furthermore the inequality $\phi' > 0$ implies that $u(x+\delta, 0) > u(x,0)$, and so comparison gives $u(x+\delta, t) \geqq u(x,t)$ for all $x$ and all $t > 0$. Since this is true for any $\delta > 0$, we must have $u_x(x,t) \geqq 0$ for all $x$ and all $t > 0$. To obtain strict inequality, set $u_x = v e^{-\alpha t}$, where the constant $\alpha$ is chosen so that $\alpha \geqq \sup\limits_{u \in [0,1]} |f'(u)|$. Then $v$ satisfies (perhaps only in the sense of distributions) the equation

$$v_t - v_{xx} = \{\alpha + f'(u)\}\, v$$

with $v(x,0) > 0$; if we use the usual Green's function to obtain an integral equation for $v$ (as we do in (2.4) below for $u$ itself), then we see that both integrals on the right hand side of this equation are non-negative and the first strictly positive, proving (2.1).

To prove (2.2), we use the integral representation for $u$, i.e.

$$u(x,t) = \int_{-\infty}^{\infty} \frac{e^{-(x-y)^2/4t}}{2\sqrt{\pi t}}\, \phi(y)\, dy + \int_0^t \int_{-\infty}^{\infty} \frac{e^{-(x-y)^2/4(t-s)}}{2\sqrt{\pi(t-s)}}\, f\{u(y,s)\}\, dy\, ds, \tag{2.4}$$

and we obtain an integral representation for $u_x$ by formal differentiation. To show that $u_x(\infty, t) = 0$ for $t > 0$, consider first the derivative of the first integral in (2.4), which, if we neglect unimportant factors, is

$$\int_{-\infty}^{\infty} (x-y)\, e^{-(x-y)^2/4t}\, \phi(y)\, dy. \tag{2.5}$$

We can suppose that $\phi(\infty)$ exists, since $\phi$ is assumed to be monotonic, and we can in fact suppose $\phi(\infty) = 0$. (If it is not, merely subtract an appropriate constant and note that from symmetry $\int_{-\infty}^{\infty} (x-y)\, e^{-(x-y)^2/4t}\, dy = 0$.) If $\phi(\infty) = 0$, it is a trivial exercise to show that (2.5) tends to zero as $x \to \infty$, since there is a significant contribution to the integral only when $x - y$ is not large, and there of course $\phi(y)$ is small.

The derivative of the second integral can be dealt with similarly, for we know from (2.1) that $u(y,s)$ is monotonic in $y$ for each $s$; so $u(\infty,s)$ and $f\{u(\infty,s)\}$ exist. This completes the proof of the lemma.

In view of Lemma 2.1, we are justified in taking $(u,t)$ as independent variables in place of $(x,t)$. The formal manipulations are easy. If $p=u_x$, then it is routine to verify that

$$\frac{\partial u}{\partial x}=1\bigg/\frac{\partial x}{\partial u},\qquad \frac{\partial u}{\partial t}=-\frac{\partial x}{\partial t}\bigg/\frac{\partial x}{\partial u},\qquad \frac{\partial t}{\partial x}=0,\qquad \frac{\partial t}{\partial t}=1,$$

the differentiations on the left of each equation being of the new variables with respect to the old, and vice versa on the right of each equation. Hence

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial p}{\partial x}=\frac{\partial p}{\partial u}\,\frac{\partial u}{\partial x}=p\frac{\partial p}{\partial u},$$
$$\frac{\partial^3 u}{\partial x^3}=p^2\frac{\partial^2 p}{\partial u^2}+p\left(\frac{\partial p}{\partial u}\right)^2,$$
$$\frac{\partial^2 u}{\partial x\,\partial t}=\frac{\partial p}{\partial u}\,\frac{\partial u}{\partial t}+\frac{\partial p}{\partial t}=\frac{\partial p}{\partial u}\left(\frac{\partial^2 u}{\partial x^2}+f(u)\right)+\frac{\partial p}{\partial t},$$

and if we differentiate (1.1) with respect to $x$ and substitute the above results, we have

$$p_t-p^2\{p_{uu}+(f/p)_u\}=0. \tag{2.6}$$

Again, the same remark as in Lemma 2.1 applies, that the differentiation of (1.1) may be possible only in the sense of distributions. If $f\in C^2$, then it is well-known that $u_{xt}$ and $u_{xxx}$ are continuous, and (2.6) then holds classically. If we have only $f\in C^1$, then (2.6) may hold only in the sense of distributions.

The boundary values $p(0,t)=p(1,t)=0$ are an immediate consequence of Lemma 2.1, and the initial function is

$$p(u,0)=\phi'(x)\equiv\Phi(u),$$

where $\Phi$ is continuous with $\Phi(u)>0$ for $0<u<1$.

We collect these facts in the following lemma.

**Lemma 2.2.** *Let $f\in C^1[0,1]$, with $f(0)=f(1)=0$, and suppose also that $\phi\in C^1(-\infty,\infty)$ with $0\leqq\phi\leqq 1$ and $\phi'>0$. Then there exists a positive solution of the initial value problem*

$$p_t=p^2\{p_{uu}+(f/p)_u\}\qquad (0<u<1,\,t>0), \tag{2.7}$$

*with*

$$p(0,t)=p(1,t)=0\qquad (t>0) \tag{2.8}$$

*and with*

$$p(u,0)=\Phi(u)>0\qquad (0<u<1), \tag{2.9}$$

*where*

$$\Phi(u)=\phi'\{\phi^{-1}(u)\},$$

$\phi^{-1}$ *being the function inverse to* $\phi$.

*Furthermore, if we are given a function* $\Phi(u)$ *that is continuous and positive for* $0<u<1$ *and such that*

$$\int_{0}^{1/2}\{\Phi(u)\}^{-1}\,du=\int_{1/2}^{1}\{\Phi(u)\}^{-1}\,du=\infty, \tag{2.10}$$

*then there exists a positive solution to the initial value problem* (2.7)–(2.9)

*Remark.* By a positive solution $p(u,t)$ of the initial value problem (2.7)–(2.9) we mean that $p(u,t)>0$ for $0<u<1$, $t>0$.

**Proof.** The proof of the first half of the lemma has already been given, it being remembered that (2.7) may be satisfied only in the sense of distributions. The second part is proved by observing that, in view of (2.10), the variable

$$x=\int_{1/2}^{u}\{\Phi(s)\}^{-1}\,ds\equiv\psi(u)$$

has the range $(-\infty,\infty)$ as $u$ traverses $(0,1)$. Thus if we set up the problem

$$u_t=u_{xx}+f(u)\qquad(-\infty<x<\infty, t>0) \tag{2.11}$$

with

$$u(x,0)=\psi^{-1}(x), \tag{2.12}$$

where $\psi^{-1}$ is the function inverse to $\psi$, then the solution of (2.11)–(2.12) leads by the first part of the theorem to a solution of the initial value problem (2.7)–(2.9).

There is a generalization of Lemma 2.2 which we will require in the sequel.

**Lemma 2.3.** *Let* $f\in C^1[0,1]$, *with* $f(0)=f(1)=0$, $f(u)\leqq 0$ *for* $u(\geqq 0)$ *sufficiently close to* 0, $f(u)\geqq 0$ *for* $u(\leqq 1)$ *sufficiently close to* 1. *Let* $A,B$ $(A<B)$ *be such that*

$$f(u)\leqq 0\quad \text{for } u\in[0,A],\qquad f(u)\geqq 0\quad \text{for } u\in[B,1].$$

*Suppose that* $\Phi(u)$ *is a given function, continuous and positive for* $A<u<B$, *and such that*

$$\int_{A}^{\frac{1}{2}(A+B)}\{\Phi(u)\}^{-1}\,du=\int_{\frac{1}{2}(A+B)}^{B}\{\Phi(u)\}^{-1}\,du=\infty. \tag{2.13}$$

*Then there exist functions* $a,b\in C^1[0,\infty)$ *with*

$$a \text{ non increasing},\qquad a(0)=A, a>0 \text{ unless } a\equiv 0$$

*and*

$$b \text{ nondecreasing},\qquad b(0)=B, b<1 \text{ unless } b\equiv 1,$$

*and a positive solution* $p(u,t)$ *to* (2.7) *over* $a(t)<u<b(t)$, $t>0$, *with*

$$p(a(t),t)=p(b(t),t)=0,$$
$$p(u,0)=\Phi(u)\quad \text{for } A<u<B.$$

**Proof.** This is based on the same principle as that of Lemma 2.2. We consider the initial value problem of (1.1) with initial function $u(x,0)=\phi(x)$, where

$$\phi^{-1}(u)=\int_{\frac{1}{2}(A+B)}^{u}\{\Phi(s)\}^{-1}\,ds.$$

In view of (2.13), we have

$$u(-\infty,0)=A, \qquad u(+\infty,0)=B.$$

If $u(x,t)$ is the solution of this initial value problem, then the function $p(u,t)=u_x(x,t)$ will satisfy the conclusions of the lemma. All that requires proof is that, if

$$u(-\infty,t)=a(t), \qquad u(+\infty,t)=b(t),$$

then $a(t)$, $b(t)$ have the required properties, and this can be deduced most easily from the Green's function formulation (2.4) for the solution. Thus, if we take the limit as $x\to\infty$ in (2.4), we obtain

$$a(t)=A+\int_0^t f\{a(s)\}\,ds,$$

so that certainly $a$ is continuously differentiable. Indeed, $a$ satisfies the equation

$$a'=f(a), \qquad \text{with } a(0)=A.$$

Since $f(a)\leqq 0$ for $0\leqq a\leqq A$, either $f(A)=0$, in which case $a(t)=A$ for all $t\geqq 0$, or $f(A)<0$, in which case a is initially strictly decreasing, and certainly always nonincreasing. Furthermore $f(0)=0$ implies $a\geqq 0$, and since $|f(a)|=O(a)$ for small $a$, we can integrate the inequality $a'=O(a)$ to show that $a(t)>0$ for all $t\geqq 0$ unless $A=0$. The arguments for $b$ are similar.

We close this section with some remarks about travelling front solutions. A travelling front solution of (1.1) appears as a stationary solution of (2.7), and so satisfies the equation

$$p_{uu}+(f/p)_u=0, \tag{2.14}$$

which can be integrated to give

$$p_u+f/p=-c, \tag{2.15}$$

where $c$ is readily identified as the speed of the front. Equations (2.14) and (2.15) have already been met in §2 of [2], where they are used to discuss the existence and uniqueness of travelling fronts.

The boundary conditions to be associated with (2.15) are

$$p(0)=p(1)=0, \tag{2.16}$$

and we are interested only in solutions positive in (0.1). Of course (2.15) and (2.16) are invariant under the transformation $p\leftrightarrow -p$, $c\leftrightarrow -c$ so that to any positive solution there is a negative one, corresponding to a travelling front facing in the opposite direction and travelling with an equal speed in the opposite direction. We recall the main result of [2] concerning the existence and uniqueness of travelling fronts.

**Lemma 2.4** (Corollary 2.3 of [2]). *Let $f$ satisfy the conditions of Lemma* 2.3. *Then there exists at most one solution which is positive in* $(0,1)$, *of the boundary value problem given by* (2.14), (2.16).

**Lemma 2.5** (Theorem 2.4 of [2]). *Let* $f \in C^1[0,1]$, *with* $f(0)=f(1)=0$. *For some* $\alpha \in (0,1)$, *suppose that one of the following assertions holds:*

(*a*) $f \leqq 0$ *in* $(0,\alpha)$, $f>0$ *in* $(\alpha,1)$, $\int_0^1 f(u)\,du>0$;

(*b*) $f<0$ *in* $(0,\alpha)$, $f \geqq 0$ *in* $(\alpha,1)$, $\int_0^1 f(u)\,du<0$;

(*c*) $f<0$ *in* $(0,\alpha)$, $f>0$ *in* $(\alpha,1)$.

*Then there exists one and (by Lemma* 2.4*) only one solution which is positive in* $(0,1)$, *of* (2.14) *and* (2.16).

Finally, to deal with functions $f$ which yield results similar to Theorem C in the introduction, suppose, as always, that $f \in C^1[0,1]$ with $f(0)=f(1)=0$. Then a closed interval $I \subset [0,1]$ is called *admissible* if $f$ vanishes at the end-points, $f \leqq 0$ near the left end-point, $f \geqq 0$ near the right end-point, and there exists a travelling front over $I$. (By a "travelling front over $[\alpha,\beta]$ with velocity $c$" we mean a solution of (2.15) with the given $c$, positive in $(\alpha,\beta)$, and vanishing at $\alpha$ and $\beta$.)

If we are given a decomposition of $[0,1]$ into non-overlapping adjacent admissible intervals $[0,1]=\bigcup_{j=1}^{m} I_j$, ordered from left to right (so that the right end-point of $I_j$ is the left end-point of $I_{j+1}$), and if $c_j$ is the velocity of the travelling front over $I_j$, then such a decomposition is called *minimal* if $c_j$ is nondecreasing in $j$, that is $c_{j+1} \geqq c_j$.

**Lemma 2.6** (Theorem 2.8 of [2]). *If there exists a decomposition of* $[0,1]$ *into admissible intervals, then there exists a unique minimal decomposition.*

The significance of minimal decomposition (as in Theorem C, where we are presented with a minimal decomposition) is that monotonic solutions of the diffusion equation with $(x,t)$ as independent variables split into a "stack" of travelling fronts, each with range in one of the intervals of the minimal decomposition and with its distinctive asymptotic speed and (at least when the $c_j$ are distinct) spreading away from each other; and correspondingly a positive solution of the diffusion equation with $(u,t)$ as independent variables will tend to a steady solution which is positive over each interval in the minimal decomposition but zero at the ends of these intervals. Precise statements of these results will be found in Theorems 4.3, 4.5.

## 3. Comparison theorems

**Lemma 3.1.** *Let $f$, $A$, $B$ satisfy the conditions of Lemma 2.3, and let $\Psi \in C^2(A,B) \cap C[A,B]$, with*

$$\begin{aligned} \Psi(A) = \Psi(B) = 0, \qquad \Psi > 0 \quad in\ (A,B), \\ \Psi'' + (f/\Psi)' \geqq 0 \quad in\ (A,B). \end{aligned} \tag{3.1}$$

*Let $A'$, $B'$ satisfy $0 \leqq A' \leqq A$, $B \leqq B' \leqq 1$, and suppose that $\Phi(u)$ is a given function, continuous and positive for $A' < u < B'$, such that*

$$\int_{A'}^{\frac{1}{2}(A'+B')} \{\Phi(u)\}^{-1}\,du = \int_{\frac{1}{2}(A'+B')}^{B'} \{\Phi(u)\}^{-1}\,du = \infty.$$

*Suppose also that $\Phi \geqq \Psi$ in $(A,B)$.*

*Then the solution $p$ of the initial value problem corresponding to the initial function $\Phi$, whose existence is guaranteed by Lemma 2.3, has the property that*

$$p(u,t) \geqq \Psi(u) \quad for\ A \leqq u \leqq B, \quad t > 0. \tag{3.2}$$

**Proof.** The initial value problem of this lemma corresponds to an initial value problem with $x$, $t$ as independent variables:

$$u_t - u_{xx} - f(u) = 0$$

with

$$u(x,0) = \phi(x).$$

Instead of solving this problem, let us consider a sequence of problems

$$u_t^{(n)} - u_{xx}^{(n)} - f_n(u^{(n)}) = \gamma_n(x)\, u_x^{(n)} \tag{3.3}$$

with

$$u^{(n)}(x,0) = \phi(x). \tag{3.4}$$

Here the functions $f_n$ make up a sequence converging uniformly to $f$ in $[0,1]$, with $f_n \in C^2[0,1]$ and $f_n(0) = f_n(1) = 0$,

$$\begin{aligned} f_n(u) < f(u) \quad &\text{for } u \text{ in } \left(0, \tfrac{1}{2}(A+B)\right), \\ f_n(u) > f(u) \quad &\text{for } u \text{ in } \left(\tfrac{1}{2}(A+B), 1\right); \end{aligned} \tag{3.5}$$

the functions $\gamma_n$ make up a sequence converging uniformly to 0 in $(-\infty, \infty)$, with $\gamma_n \in C^2(-\infty, \infty)$ and $\gamma_n' > 0$.

The initial value problem (3.3)–(3.4) has the usual unique classical solution, with $u_x^{(n)}(x,t) > 0$ for all $x, t$. By the arguments of §2, this corresponds to a positive solution $p^{(n)}$ of an initial value problem with $u, t$ as independent variables, namely

$$p_t^{(n)} - \{p^{(n)}\}^2 \{p_{uu}^{(n)} + (f_n/p^{(n)})_u\} = \gamma_n' \, p^{(n)} > 0$$

with

$$p^{(n)}(u,0) = \Phi(u).$$

The solution $p^{(n)}$ is classical, since both $f_n$ and $\gamma_n$ are sufficiently differentiable, and $p^{(n)}(u,t)$ is positive over a $u$-interval that does not decrease as $t$ increases.

We can then define a sequence of functions $\Psi_n$ by the equation

$$\Psi_n' + f_n/\Psi_n = \Psi' + f/\Psi \tag{3.6}$$

with

$$\Psi_n(\tfrac{1}{2}(A+B)) = \Psi(\tfrac{1}{2}(A+B)) - \varepsilon_n,$$

where $\{\varepsilon_n\}$ is a sequence of positive numbers with $\varepsilon_n \downarrow 0$ as $n \to \infty$. Then $\Psi_n < \Psi$ as long as both are positive; for, if we consider (as is sufficient) the interval $[\frac{1}{2}(A+B), 1]$, we see that, at the first point at which they meet, $\Psi_n$ meets $\Psi$ from below, so that $(\Psi_n - \Psi)' \geqq 0$, and this contradicts (3.5) and (3.6). It therefore follows that $\Psi_n$ is positive over $(A_n, B_n)$, where $A_n \geqq A$, $B_n \leqq B$, and $\Psi_n(A_n) = \Psi_n(B_n) = 0$. Clearly also $A_n \to A$ and $B_n \to B$ as $n \to \infty$, and $\Psi - \Psi_n$ is small (and positive) throughout $(A_n, B_n)$.

In fact, if $A$ is such that $f(u) \leqq 0$ for $u$ in $[0, A+\delta]$, for some $\delta > 0$, then we can assert that $A_n > A$, with a similar result for $B_n$ and $B$. For we can rewrite (3.6) as

$$\frac{d}{du}\left\{(\Psi_n - \Psi)\exp\left(-\int_{\frac{1}{2}(A+B)}^{u} \frac{f}{\Psi_n \Psi}\right)\right\} = \frac{f-f_n}{\Psi_n}\exp\left(-\int_{\frac{1}{2}(A+B)}^{u} \frac{f}{\Psi_n \Psi}\right).$$

The expression in braces $\{\ \}$ is negative at $\frac{1}{2}(A+B)$ and decreasing in $(\frac{1}{2}(A+B), B_n)$; it is therefore strictly negative as $u \to B_n$. Now with the assumptions on $f$, the exponential is bounded as $u \to B_n$, at least if $n$ is so large that $B_n > B - \delta$. Hence we cannot have $\Psi_n - \Psi \to 0$ as $i \to B_n$, and so accordingly we have $B_n < B$.

If we continue to suppose temporarily that $A$ and $B$ satisfy these extra assumptions, then we can apply the classical maximum principle argument to provide a comparison of $p^{(n)}$ and $\Psi_n$. Since $\Phi \geqq \Psi$ in $[A, B]$ and $\Psi > \Psi_n$ in $[A_n, B_n]$, we know that initially $p^{(n)} > \Psi_n$ in $[A_n, B_n]$. Suppose for contradiction that $p^{(n)} = \Psi_n$ first at $(u_0, t_0)$, where necessarily $A_n < u_0 < B_n$. Then, at $(u_0, t_0)$,

$$p_t^{(n)} \leqq 0,\ p^{(n)} = \Psi_n,\ p_u^{(n)} = \Psi_n',\ p_{uu}^{(n)} \geqq \Psi_n'', \tag{3.7}$$

while, from the equations for $p^{(n)}$, $\Psi_n$,

$$0 < p_t^{(n)} - \{p^{(n)}\}^2 \{p_{uu}^{(n)} + (f_n/p^{(n)})_u\} + \Psi_n^2 \{\Psi_n'' + (f_n/\Psi_n)'\} \leqq 0,$$

from (3.7). This gives the required contradiction, and we have

$$p^{(n)}(u,t) > \Psi_n(u) \quad \text{for } A_n \leqq u \leqq B_n, \quad t \geqq 0.$$

A limiting process as $n \to \infty$ then gives the final result of the lemma. Indeed for any fixed $t$ and uniformly in $x$, we have as $n \to \infty$,

$$u^{(n)}(x,t) \to u(x,t), \qquad u_x^{(n)}(x,t) \to u_x(x,t),$$

as can be seen by turning the equations for $u$ and $u^{(n)}$ into integral form and comparing them. Pick any $u_0$ with $A < u_0 < B$, so that $p(u_0, t) \neq 0$. Then, dropping dependence on $t$, define

$$\begin{array}{ll} x_0 & \text{by } u(x_0)=u_0, \\ u_0^{(n)} & \text{by } u^{(n)}(x_0)=u_0^{(n)}, \\ x^{(n)} & \text{by } u^{(n)}(x^{(n)})=u_0. \end{array}$$

The implicit function theorem implies that $x^{(n)} \to x_0$ as $n \to \infty$, and that

$$p^{(n)}(u_0)-p(u_0)=u_x^{(n)}(x^{(n)})-u_x(x_0)\to 0 \qquad \text{as } n\to\infty.$$

If $A$ and $B$ do not satisfy the extra assumptions, we can no longer assert that $\Phi > \Psi_n$, but we can then introduce a sequence of initial functions $\Phi_n$ with $\Phi_n > \Psi_n$, and with this extra complication achieve the desired result. The details do not require elaboration. Note that the introduction of such a sequence of functions $\Phi_n$ with $\Phi_n > \Psi$ would not be possible if $A=0$ or $B=1$, but in this case $A$ and $B$ satisfy the extra assumptions and the sequence $\{\Phi_n\}$ is unnecessary.

A second comparison theorem is proved by methods so similar that we content ourselves simply with its statement.

**Lemma 3.2.** *Let $f$, $A$, $B$, $A'$, $B'$, $\Phi$ be as in Lemma* 3.1, *but now let $\Psi(u)$ be a given function, continuous and positive for $A<u<B$, such that*

$$\int_A^{\frac{1}{2}(A+B)} \{\Psi(u)\}^{-1}\,du = \int_{\frac{1}{2}(A+B)}^{B} \{\Psi(u)\}^{-1}\,du = \infty.$$

*Suppose also that $\Phi \geqq \Psi$ in $(A,B)$.*

*Then if $p$, $q$ are the solutions of the initial value problem corresponding respectively to the initial functions $\Phi$, $\Psi$ we have*

$$p(u,t)\geqq q(u,t) \qquad \textit{for } a(t)\leqq u\leqq b(t), \qquad t>0,$$

*where the functions $a$, $b$ introduced in Lemma* 2.3 *define the interval in which $q$ is positive.*

Finally in this section we prove a result bearing on the increasing property of subsolutions.

**Lemma 3.3.** *Let $f$, $A$, $B$ be as in Lemma* 3.1, *and let $\Psi$ satisfy the conditons of both Lemma* 3.1 *and Lemma* 3.2. *Then if $q(u,t)$ is the solution of the initial value problem corresponding to the initial function $\Psi$, it has the property that pointwise in $u$ it is a nondecreasing function of $t$.*

*If $q(u,t)\to Q(u)$, say, as $t\to\infty$, then $Q$ is a positive solution of* (2.14) *over some interval $(A',B')$, where*

$$0\leqq A'\leqq A, \qquad B\leqq B'\leqq 1,$$

*and*

$$Q(A')=0, \qquad Q(B')=0.$$

**Proof.** By taking $\Phi=\Psi$ in Lemma 3.1, we immediately obtain

$$q(u,t)\geqq \Psi(u) \qquad \text{for } A\leqq u\leqq B, \qquad t>0.$$

Now use $q(u,t^*)$ and $\Psi(u)$ as comparable initial funtions in Lemma 3.2, for any given $t^*>0$. This gives

$$q(u,t+t^*)\geqq q(u,t) \qquad \text{for } a(t)\leqq u\leqq b(t), \qquad t>0,$$

from which the nondecreasing character of $q$ is apparent. Then $q(u,t)$ must converge to a limit as $t\to\infty$, say $Q(u)$, and it remains only to establish that $Q$ has the requisite properties.

We are concerned with $Q$ only where it is positive, i.e. in $(A',B')$. For any small fixed $\delta>0$ we know that $q(u,t)>0$ for $A'+\delta\leqq u\leqq B'-\delta$ and $t\geqq T$, say, and in view of the nondecreasing character of $q$ we know further that $q$ is bounded from zero in this range. Since $u_{xx}$ is bounded (for the corresponding solution with $x$, $t$ as independent variables) and $u_{xx}=q\,q_u$, it follows that $q_u$ (and similarly $q_t$ and $q_{uu}$) are bounded and equicontinuous for $A'+\delta\leqq u\leqq B'-\delta$ as $t\to\infty$. (We make the temporary assumption that $f\in C^2[0,1]$.) Hence $q_u, q_t, q_{uu}$ converge uniformly as $t\to\infty$, at least through some subsequence, to $Q', Q, Q''$, respectively. Taking the limit in the equation for $q$, we see that $Q$ satisfies (2.14), as required.

If $f$ is merely $C^1[0,1]$, then we can construct, by a procedure similar to that in Lemma 3.1, a sequence of functions $f_n\in C^2[0,1]$, with corresponding initial functions $\Psi_n$ and solutions $q^{(n)}$ of the initial value problem. Now, for any $T_1, T_2$ exceeding $T$, where $T$ is chosen as in the previous part of the proof, and for any $u, u_2$ in $[A'+\delta, B'-\delta]$, we have, at least if $n$ is sufficiently large,

$$\begin{aligned}&\int_{T_1}^{T_2}[\{q_u^{(n)}+(f_n/q^{(n)})\}_{u=u_2}-\{q_u^{(n)}+(f_n/q^{(n)})\}_{u=u_1}]\,dt\\ &\quad=\int_{T_1}^{T_2}\left\{\int_{u_1}^{u_2} q_t^{(n)}/\{q^{(n)}\}^2\,du\right\}dt=-\int_{u_1}^{u_2}\left\{\frac{1}{q^{(n)}(u,T_2)}-\frac{1}{q^{(n)}(u,T_1)}\right\}du.\end{aligned}$$

Taking the limit as $n\to\infty$, we can drop the sub- and super-scripts $n$; furthermore the condition $f\in C^1[0,1]$ allows us to conclude as before that $q(u,t), q_u(u,t)$ converge uniformly to $Q(u)$, $Q'(u)$, at least by a subsequence, as $t\to\infty$. Hence

$$\int_{T_1}^{T_2}[\{q_u+(f/q)\}_{u=u_2}-\{q_u+(f/q)\}_{u=u_1}]\,dt\to 0 \qquad \text{as } T_1, T_2\to\infty.$$

But the integrand is a non-negative continuous function of $t$. (The non-negativity arises in the limit as $n\to\infty$ from the same property for $q^{(n)}$, which in turn comes from the equation satisfied by $q^{(n)}$ and the fact that $q^{(n)}$ is a nondecreasing function of time.) Hence in the limit as $t\to\infty$ we must have

$$Q'+(f/Q)=\text{constant},$$

which gives the required result.

## 4. Statement of results

We first prove a convergence result with $(u,t)$ as independent variables. The easiest such result occurs in case there is no question about the existence of a travelling front, i.e. when we assume that $f$ satisfies one or other of the sets of conditions in Lemma 2.5. We shall in fact assume the conditions ($b$) of Lemma 2.5; the conditions ($a$) are equivalent under the transformation $u \leftrightarrow 1-u$, $f \leftrightarrow -f$; the conditions ($c$) are only a particular case of either ($a$) or ($b$) except that they allow the possibility $\int_0^1 f(u)\,du=0$. As a preliminary step, we prove a theorem which covers conditions ($c$) and also a particular case of conditions ($b$).

**Theorem 4.1.** *Let* $f\in C^1[0,1]$ *with* $f(0)=f(1)=0$. *For some* $\alpha\in(0,1)$, *suppose that* $f<0$ *in* $(0,\alpha)$, *and that in* $(\alpha,1)$ *either* $f>0$ *or* $f\equiv 0$. *Let $P$ be the unique positive solution of* (2.14) *and* (2.16) *in* $(0,1)$, *and let $p$ be the positive solution of the initial value problem* (2.7)–(2.9). *Then*

$$p(u,t)\to P(u) \qquad \text{as } t\to\infty,$$

*uniformly for* $u\in[0,1]$.

The proof of Theorem 4.1 is a matter of constructing suitable sub- and supersolutions, for which we have prepared the ground in the comparison theorems of §3. What we wish to do is, given the initial value problem (2.7)–(2.9) and *any* interval $(A,B)$ with $0<A<\alpha$, $\alpha<B<1$, to find a function $\underline{\Psi}$ satisfying the conditions on $\Psi$ in Lemma 3.3 and in addition lying below the initial function $\Phi$ for the initial value problem. For if such a $\underline{\Psi}$ can be found, then the corresponding solution $\underline{q}$ of the diffusion equation with $\underline{\Psi}$ as initial function increases with time to a solution $Q$ of (2.14), positive at least over $(A,B)$. By Lemma 3.2 the solution $p$ always lies above $\underline{q}$, and so in the limit lies above (or coincident with) $Q$. (A solution $Q$ of (2.14), positive over $(A,B)$ and with $Q(A)=Q(B)=0$, is necessarily unique, as can be shown by a repetition of the argument which proves Lemma 2.4.) If the choice of $A,B$ is arbitrary, subject only to $0<A<\alpha$, $\alpha<B<1$, then $p(u,t)$ (in the limit as $t\to\infty$) lies above (or coincident with) the solution $P(u)$ of (2.14) and (2.16) which is positive for $u\in(0,1)$.

Our final task is to find a function $\overline{\Psi}$ satisfying the conditions of Lemma 3.3, but with $A=0$, $B=1$ and the inequality (3.1) reversed. If, further, $\overline{\Psi}$ lies above the initial function $\Phi$, then theorems comparable to those in §3 show that the solution $\bar{q}$ of the diffusion equation with $\overline{\Psi}$ as initial function now decreases with time to a solution of (2.14) that is positive over $(0,1)$, and so necessarily to $P$. On the other hand, the solution $p$ always lies below $\bar{q}$ and so in the limit is, in view of what has already been said, coincident with $P$. This is the required result.

It is interesting that although the construction of such subsolutions $\underline{q}$ is possible under the conditions of Theorem 4.1, it is not necessarily possible if $f$ satisfies only the conditions of Lemma 2.5. We give an example to prove this in §6. Nevertheless, by use of a Lyapunov functional, we can dispense with the extra assumptions in Theorem 4.1 and so prove

**Theorem 4.2.** *Let $f$ satisfy one of the sets of conditions in Lemma* 2.5. *Then the conclusion of Theorem* 4.1. *holds.*

When there is no travelling front over $(0,1)$, we have a theorem based on Lemma 2.6.

**Theorem 4.3.** *Suppose that there exists a decomposition of* $[0,1]$ *into admissible intervals, and that in each of these intervals, say* $[a_i,b_i]$, $f$ *satisfies one or other of the sets of conditions in Lemma* 2.5, *with* $[a_i,b_i]$ *replacing* $[0,1]$. *Let the minimal decomposition be* $[0,1]=\bigcup_{j=1}^{m} I_j$, *and let* $P_j$ *be the unique positive solution of* (2.14) *over* $I_j$, *with* $P_j$ *vanishing at the end-points of* $I_j$. *If we define* $P$ *by*

$$P(u)=P_j(u) \quad \text{for } u\in I_j,$$

*then* $p(u,t)\to P(u)$ *as* $t\to\infty$, *uniformly for* $u\in[0,1]$.

These results have a corresponding interpretation when $x$, $t$ are taken as independent variables.

**Theorem 4.4.** *Let* $f$ *satisfy the conditions of Theorem* 4.2, *or of Theorem* 4.3 *where the minimal decomposition consists just of* $[0,1]$ *itself and* $P$ *is therefore positive over* $(0,1)$. *Let* $U$ *be the corresponding travelling front solution of* (1.1) *and let* $u(x,t)$ *be the solution of* (1.1)–(1.2) *corresponding to the solution* $p(u,t)$ *of* (2.7)–(2.9). *Then there exists a function* $\gamma\in C^1[0,\infty)$, *with* $\gamma'(t)\to 0$ *as* $t\to\infty$, *such that*

$$u(x,t)-U\bigl(x-ct-\gamma(t)\bigr)\to 0 \quad \text{as } t\to\infty,$$

*uniformly in* $x$.

There remains the case of Theorem 4.3 where the minimal decomposition of $[0,1]$ contains more than one subinterval. To be specific, let us suppose that the minimal decomposition is into two subintervals, $[0,\alpha]$, $[\alpha,1]$, and let the corresponding travelling fronts be $U_1(x-c_1t)$, $U_2(x-c_2t)$ with ranges $(0,\alpha)$, $(\alpha,1)$ respectively and $c_1\leqq c_2$. Then we have the following result.

**Theorem 4.5.** *Let* $f$ *satisfy the conditions of Theorem* 4.3 *with a minimal decomposition of* $[0,1]$ *into* $[0,\alpha]$ *and* $[\alpha,1]$, *and let corresponding travelling fronts be* $U_1(x-c_1t)$ *and* $U_2(x-c_2t)$ *where* $c_1\leqq c_2$. *Then there exist functions* $\gamma_i\in C^1[0,\infty)$, $i=1,2$, *such that* $\gamma_i'(t)\to 0$ *as* $t\to\infty$ *and*

$$u(x,t)-U_1\bigl(x-c_1t-\gamma_1(t)\bigr)-U_2\bigl(x-c_2t-\gamma_2(t)\bigr)+\alpha\to 0$$

*as* $t\to\infty$, *uniformly in* $x$. *In the particular case* $c_1=c_2$ *we can further assert that*

$$\gamma_2(t)-\gamma_1(t)\to\infty \quad \text{as } t\to\infty.$$

Under more restrictive assumptions on the initial function, we can improve Theorems 4.1–4.5 by giving estimates of the rates of decay to the limit. We leave a precise statement of this to §11.

## 5. Proof of Theorem 4.1

In view of the remarks following the statement of Theorem 4.1, we have to prove the existence of a function $\underline{\Psi}$, satisfying the conditions on $\Psi$ in Lemma 3.3 and with, in addition, $\underline{\Psi}\leqq\Phi$, where $\Phi$ is the given initial function in (2.9). Given any $A$ with $0<A<\alpha$, we start by choosing a function $w\in C^2[A,\alpha]$, positive in

$(A,\alpha)$ and with $w(A)=0$, and such that there exists a positive constant $k$ with

$$(f/w)' \geqq k > 0 \quad \text{in } (A,\alpha). \tag{5.1}$$

(To find such a function $w$, set $w=f/g$ and choose $g$ so that $g' \geqq k$, $g(A)=-\infty$, $g(\alpha)<0$.) The function $w$ may not be an immediate candidate for $\underline{\Psi}$, because we must have

$$\underline{\Psi}'' + (f/\underline{\Psi})' \geqq 0 \tag{5.2}$$

and

$$\underline{\Psi} \leqq \Phi;$$

but now replace $w$ by $\varepsilon w$, where $\varepsilon$ is a small positive constant. Since $w \in C^2[A,\alpha]$, certainly (5.1) implies that

$$(\varepsilon w)'' + (f/\varepsilon w)' \geqq 0 \quad \text{in } (A,\alpha) \tag{5.3}$$

if $\varepsilon$ is small enough, and this also ensures that $\varepsilon w \leqq \Phi$. A required $\underline{\Psi}$ has therefore been found which is positive in $(A,\alpha)$, for any given $A$ with $0<A<\alpha$. We note also that $(\varepsilon w)' + (f/\varepsilon w)$ is a nondecreasing function with a strictly negative limit as $u \to \alpha$, at least if $\varepsilon$ is sufficiently small. (Recall that $g(\alpha)<0$.)

We can in fact find a possible $\underline{\Psi}$ over an interval $(A,\alpha^*)$, for some $\alpha^*>\alpha$. This is an immediate consequence of the following lemma.

**Lemma 5.1.** *Let $f$ satisfy the conditions of Theorem* 4.1. *If a positive continuous function $\Psi$ has been found satisfying* (3.1) *over $(A,\alpha)$ with $\Psi(A)=\Psi(\alpha)=0$, and if*

$$\lim_{u\to\alpha} \{\Psi' + (f/\Psi)\} < 0,$$

*then, for $\alpha^*$ sufficiently close to $\alpha$ $(\alpha^*>\alpha)$, a positive continuous function $\Psi^*$ can be found satisfying* (3.1) *over $(A,\alpha^*)$ with $\Psi^*(A)=\Psi^*(\alpha^*)=0$. Furthermore, we can arrange that $\Psi^*$ is as small as we please in $(A,\alpha^*)$, and also $\Psi^*/\Psi^{*\prime}$ as small as we please at $\alpha$.*

The proof of this lemma will be left to the end of the section, but the construction of a suitable $\underline{\Psi}$ over $(A,B)$, for any $A$ with $0<A<\alpha$ and any $B$ with $\alpha<B<1$, is now almost immediate.

If we take first the case where $f>0$ in $(\alpha,1)$, then we can carry out the construction of a positive function $\Psi^{**}$ satisfying (3.1) over $(\alpha^{**},B)$, where $\alpha^{**}<\alpha$, and with $\Psi^{**}(\alpha^{**})=\Psi^{**}(B)=0$. This comes from the argument that we have already used for $(A,\alpha^*)$, together with the transformation $u \leftrightarrow 1-u$, $f \leftrightarrow -f$. If the first value of $u$ for which the functions $\Psi^*(u)$ and $\Psi^{**}(u)$ meet is $\beta$, say, then the continuous function $\underline{\Psi}$ defined by

$$\underline{\Psi} = \begin{cases} \Psi^* & \text{in } (A,\alpha), \\ \Psi^{**} & \text{in } (\beta,B) \end{cases}$$

has all the conditions required. There is the technicality that $\underline{\Psi}$ may not be twice continuously differentiable at $\beta$, having a jump increase in $\underline{\Psi}'$ there, but this does not invalidate the conclusions (or in any significant way the proof) of

Lemma 3.3; it is merely a reflection of the well-known fact, noted in the introduction to [2], that if we have two classical subsolutions, then their supremum is also a subsolution.

If now $f \equiv 0$ in $(\alpha, 1)$, choose $\alpha^*, \Psi^*$ from Lemma 5.1, and note that the condition $\Psi^{*\prime} + (f/\Psi^*) < 0$ at $\alpha$ implies in particular that $\Psi^{*\prime}(\alpha) < 0$. Our choice of $\Psi^{**}$ has to be made differently from before, and we ask now that it satisfy the requirements that, for any given $B$ with $\alpha < B < 1$,

$$\Psi^{**}(\alpha) = \Psi^*(\alpha), \quad \Psi^{*\prime}(\alpha) < \Psi^{**\prime}(\alpha) < 0, \quad \Psi^{**}(B) = 0,$$
$$\Psi^{**\prime} \quad \text{is negative increasing in } (\alpha, B).$$

There is no difficulty in satisfying these requirements, and such a $\Psi^{**}$ satisfies (3.1) in $(\alpha, B)$ since $f \equiv 0$ there, and moreover can clearly be chosen so that $\Psi^{**} \leqq \Phi$ in $(\alpha, B)$. Then

$$\underline{\Psi} = \begin{cases} \Psi^* & \text{in } (A, \alpha), \\ \Psi^{**} & \text{in } (\alpha, B) \end{cases}$$

gives the required $\underline{\Psi}$.

It remains to discuss supersolutions. This is in essence easier than the discussion of subsolutions because it is largely independent of $f$. Let $\bar{q}$ be the solution of the initial value problem consisting of the equation (2.7), the boundary conditions

$$\bar{q}(0, t) = \gamma_1(t), \qquad \bar{q}(1, t) = \gamma_2(t),$$

where $\gamma_1(t), \gamma_2(t)$ are positive decreasing functions of $t$, and the initial condition

$$\bar{q}(u, 0) = \bar{\Psi}(u) = K\{1 - (u - \tfrac{1}{2})^2\} \qquad (0 < u < 1), \tag{5.4}$$

where $K$ is a large positive constant. There is no difficulty in giving existence and uniqueness theorems for $\bar{q}$, for, with $\gamma_1, \gamma_2$ positive, the problem is not degenerate (see, for example, [5]). If we assume that $f \in C^2[0, 1]$ so that we can discuss classical solutions (the extension to $f \in C^1[0, 1]$ can be carried through on the lines of §3), then the usual arguments show that $\bar{q}$ is a nonincreasing function of time, since by direct substitution we find that

$$\bar{\Psi}'' + (f/\bar{\Psi})' < 0$$

in $(0, 1)$ if $K$ is sufficiently large, the size of $K$ depending only on $\|f\|_{C^1}$. Also the given initial function $\Phi$ clearly lies below $\bar{\Psi}$ if $K$ is sufficiently large, and so $p \leqq \bar{q}$ for all time. If $\gamma_1(t), \gamma_2(t) \to 0$ as $t \to \infty$, then $\bar{q}$ tends to the positive solution $P$ of (2.14) over $(0, 1)$, and this is the result we want. Thus the proof of Theorem 4.1 is complete once we have established Lemma 5.1.

**Proof of Lemma 5.1.** We have

$$\Psi' + (f/\Psi) \leqq -k^2,$$

say, in $(A, \alpha)$. Consider the solution of

$$y' + (f/y) = 0 \qquad \text{in } [\alpha - \delta, \alpha] \tag{5.5}$$

for which

$$y(\alpha-\delta)=\Psi(\alpha-\delta),$$

where $\delta$ is a positive constant chosen so that $y$ is as small as we need it to be in $[\alpha-\delta,\alpha]$. (In application this means that $y\leqq\Phi$ for the given initial function in the initial value problem. If it is assumed that already $\Psi\leqq\Phi$ in $[A,\alpha]$, then direct integration of (5.5) shows that we can certainly choose $\delta$ so that $y\leqq\Phi$.) With $\delta$ so chosen we note that (5.5) implies that $y'>0$ in $(\alpha-\delta,\alpha)$, and so $y(\alpha)>0$. If we let $y_\eta$ be the solution of

$$y_\eta'+(f/y_\eta)=-\eta \quad \text{in } [\alpha-\delta,\alpha]$$

for which

$$y_\eta(\alpha-\delta)=\Psi(\alpha-\delta),$$

then $y_\eta$ is pointwise a nonincreasing function of $\eta$ as $\eta$ increases, with $y_\eta\geqq y_2\geqq\Psi$ so long as $\eta\leqq k^2$; and if

$$\Psi'+f/\Psi=-l^2,$$

say, at $\alpha-\delta$, then $y_{l^2}(u)\leqq\Psi(u)$ for $u\geqq\alpha-\delta$, and so $y_{l^2}(u)$ vanishes at or before $u=\alpha$. Finally, $y_\eta$ can vanish only at $\alpha$, since if it vanishes first at $u_0<\alpha$, then we have $y_\eta'(u)\to\infty$ as $u\uparrow u_0$.

It follows therefore that there exists some first $\eta_0$ such that $y_{\eta_0}(\alpha)=0$, and we can choose $\eta(<\eta_0)$ so close to $\eta_0$ that $y_\eta(\alpha)/y_\eta'(\alpha)$ is as small as we please. But $u>\alpha$ implies (since now $f(u)\geqq 0$) that $y_\eta'(u)\leqq y_\eta'(\alpha)$. Hence $y_\eta$ must vanish for some $\alpha^*(>\alpha)$ close to $\alpha$. Then $\Psi^*$ given by

$$\Psi^*=\begin{cases}\Psi & \text{in } (A,\alpha-\delta),\\ y_\eta & \text{in } (\alpha-\delta,\alpha^*)\end{cases}$$

satisfies the conclusions of the lemma.

## 6. Nonexistence of subsolutions

The proof of Theorem 4.1 depends on the existence, given any $A,B$ with $0<A<\alpha$, $\alpha<B<1$, of a function $\Psi$ which is positive on $(A,B)$ with $\Psi(A)=\Psi(B)=0$, and satisfies there

$$\Psi''+(f/\Psi)'\geqq 0,$$

and further can be chosen to be arbitrarily small. While such a funtion $\Psi$ can be found if $f$ satisfies the conditions of Theorem 4.1, this is no longer necessarily true if $f$ satisfies merely one or the other of the sets of conditions in Lemma 2.5. It is the purpose of this section to provide an example of this; to do so we suppose that $f$ satisfies the conditions $(b)$ of Lemma 2.5. Thus $f<0$ in $(0,\alpha)$, $f\geqq 0$ in $(\alpha,1)$, and

$$\int_0^1 f(u)\,du<0.$$

Now suppose that, in $(\alpha, 1)$, $f \geqq k$ on some closed interval $I_1$ while $f=0$ on another closed interval $I_2$, where $I_1$ lies to the left of $I_2$ and both are of length $\frac{1}{4}$, say. Suppose also that we require $\Psi \leqq \varepsilon$, for some $\varepsilon < 0$.

If the required function $\Psi$ exists, then

$$g \equiv \Psi' + (f/\Psi)$$

is nondecreasing. On $I_1$,

$$\Psi' = g - (f/\Psi) \leqq g - (k/\varepsilon).$$

But if $\Psi' < -4\varepsilon$ throughout $I_1$, we could not also have $0 < \Psi \leqq \varepsilon$ on $I_1$. Hence at some point in $I_1$ we must have

$$g - (k/\varepsilon) \geqq -4\varepsilon, \qquad g \geqq (k/\varepsilon) - 4\varepsilon.$$

Since $g$ is nondecreasing, this inequality must also hold on $I_2$, where

$$\Psi' = g \geqq (k/\varepsilon) - 4\varepsilon.$$

Again, if $\Psi' > 4\varepsilon$ throughout $I_2$, we could not have $0 < \Psi \leqq \varepsilon$ throughout $I_2$. Hence

$$(k/\varepsilon) - 4\varepsilon \leqq 4\varepsilon, \qquad \varepsilon^2 \geqq k/8,$$

which is not necessarily true.

## 7. Proof of Theorem 4.2

We will assume that conditions $(b)$ of Lemma 2.5 hold. By the work in the earlier part of Theorem 4.1, including Lemma 5.1, we know that we can construct a funtion $\underline{\Psi}$, positive in $(A, \alpha^*)$, lying below the given initial function $\Phi$, satisfying

$$\underline{\Psi}'' + (f/\underline{\Psi})' > 0 \qquad \text{in } (A, \alpha^*),$$

and with $\underline{\Psi}(A) = \underline{\Psi}(\alpha^*) = 0$, where $A$ is any number with $0 < A < \alpha$, and $\alpha^*$ is some number with $\alpha^* > \alpha$. The solution $\underline{q}$ of (2.7)-(2.9) corresponding to the initial function $\underline{\Psi}$ then has the property that it increases with time to the unique positive solution $Q_\beta$ of

$$Q'' + (f/Q)' = 0$$

over $(0, \beta)$, with $Q_\beta(0) = Q_\beta(\beta) = 0$, for some $\beta \geqq \alpha^*$. Since $\Phi \geqq \underline{\Psi}$, we must have

$$\liminf_{t \to \infty} p(u, t) \geqq Q_\beta(u). \tag{7.1}$$

Now, for any $\beta$ in $[\alpha^*, 1]$, there exists the corresponding function $Q_\beta$, with

$$Q'_\beta + (f/Q_\beta) = -c_\beta,$$

say, and $Q_\beta(0)=Q_\beta(\beta)=0$. Further, $c_\beta>0$ since

$$\int_0^\beta f(u)\,du<0,$$

and $c_\beta$ is a strictly decreasing function of $\beta$ and $Q_\beta(u)$ pointwise in $u$ a strictly increasing function of $\beta$ (except at $u=0$). (These results are contained in Lemma 2.2 of [2].) Let $\beta^*$ be the largest value of $\beta$ for which (7.1) is valid. Thus

$$\liminf_{t\to\infty} p(u,t)\geqq Q_{\beta^*}(u),$$

but

$$\liminf_{t\to\infty} p(u,t)\ngeqq Q_\beta(u)$$

for any $\beta>\beta^*$.

If $\beta^*=1$, so that $\liminf p\geqq P$, then we are done. For we can construct a supersolution exactly as in Theorem 4.1 (we pointed out then that the construction of a supersolution was essentially independent of $f$) and so show that $\limsup p\leqq P$, from which the final result follows.

If $\beta^*<1$ (which we finally prove to be absurd), then we can show first that

$$\lim_{t\to\infty} p(\beta^*,t)=0. \tag{7.2}$$

Suppose for contradiction that (7.2) is not true, and consider the solution of

$$y'+(f/y)=-c$$

with

$$y(\beta^*-\delta)=Q_{\beta^*}(\beta^*-\delta),$$

where $\delta$ is a positive number chosen so that, for some arbitrarily large values of $t$,

$$p(u,t)>Q_{\beta^*}(u) \quad \text{for } u \quad \text{in } [\beta^*-2\delta,\beta^*]. \tag{7.3}$$

It is possible to find such a $\delta$ because $\limsup\limits_{t\to\infty} p(\beta^*,t)>0$ implies that, for some arbitrarily large values of $t$, $p(\beta^*,t)$ is bounded from zero and so has bounded derivatives. Then if $c(<c_{\beta^*})$ is close to $c_{\beta^*}$, $y$ is pointwise close to $Q_{\beta^*}$, at least so long as both are positive, with

$$y(u)\gtrless Q_{\beta^*}(u) \quad \text{according as } u\gtrless\beta^*-\delta.$$

Furthermore $y$ vanishes at $y_1, y_2$, say, where $y_1>0$ but is close to 0 and $y_2>\beta^*$ but is close to $\beta^*$, the strict inequalities being a consequence of the signs of $f$ and an argument employed in the proof of Lemma 3.1. It is thus clear that, in view of (7.3), we can choose $c$ sufficiently close to $c_{\beta^*}$ that, for some arbitrarily large $t$,

$$p(u,t)>y(u) \quad \text{for } u \quad \text{in } [\beta^*-2\delta, y_2],$$

and that, for all $t$ sufficiently large,

$$p(u,t)>y(u) \quad \text{for } u \quad \text{in } [y_1, \beta^*-2\delta],$$

since in this latter range $y < Q_{\beta^*}$ and $\liminf p \geqq Q_{\beta^*}$. But $p > y$ at any time implies that subsequently $p$ is never less than the solution of (2.7)–(2.9) with $y$ as initial function, which is a nondecreasing function of time. Hence certainly

$$\liminf_{t\to\infty} p(u,t) \geqq Q_{y_2}(u),$$

which contradicts the definition of $\beta^*$. Thus (7.2) is true.

It is now easy to establish that, in fact,

$$\lim_{t\to\infty} p(u,t) = Q_{\beta^*}(u) \quad \text{for } u \quad \text{in } [0,\beta^*]. \tag{7.4}$$

We already know that $\liminf p \geqq Q_{\beta^*}$; the complementary result $\limsup p \leqq Q_{\beta^*}$ follows by considering the auxiliary function $\bar{q}$, that is, the solution of (2.7) with the boundary conditions

$$\bar{q}(0,t) = \gamma_1(t), \qquad \bar{q}(\beta^*,t) = \gamma_2(t),$$

where $\gamma_1(t)$, $\gamma_2(t)$ are positive decreasing functions of $t$, and the same initial condition as in (5.4), although now over $(0,\beta^*)$. As in the proof of Theorem 4.1, $\bar{q}$ is a nonincreasing function of time, converging as $t \to \infty$ to $Q_{\beta^*}$ if $\gamma_1(t), \gamma_2(t) \to 0$. If further we arrange that $\gamma_2(t) \to 0$ more slowly than $p(\beta^*,t)$, then we will always have $p \leqq \bar{q}$, and so finally (7.4), as required.

From (7.4), it is immediate that, as $t \to \infty$,

$$p_u + (f/p) \to -c_{\beta^*} < 0 \quad \text{in } (0,\beta^*). \tag{7.5}$$

Having established (7.4), we can now obtain the requisite contradiction to $\beta^* < 1$; the argument differs depending upon whetther or not $f \equiv 0$ in $(\beta^*, 1)$.

Take first the case when $f \equiv 0$ in $(\beta^*, 1)$, and let $\gamma$ be any number with $\beta^* < \gamma < 1$. Consider as an initial function $\Phi_\gamma$, positive over $(A,\gamma)$ for some $A$ with $0 < A < \alpha$, and such that $\Phi_\gamma(A) = 0$ and

$$\Phi_\gamma(u) = Q_\gamma(u) \tag{7.6}$$

for $u (<\gamma)$ sufficiently close to $\gamma$. We shall also insist that $\Phi_\gamma \leqq \Phi$.

Now the condition (7.6) implies that the corresponding initial function $\phi_\gamma(x)$ (with $x, t$ as independent variables) is identical for $x$ sufficiently large with some translation of the travelling front that has range $(0,\gamma)$. Since $\phi_\gamma(-\infty) = A > 0$, we can in fact arrange that $\phi_\gamma$ lies entirely above some translate of the travelling front, and since this is so initially, the corresponding solution $u_\gamma(x,t)$ lies above this translate for all time.

But also, by the arguments used on $p$ itself, the inequality $\Phi_\gamma \leqq \Phi$ certainly implies that

$$\lim_{t\to\infty} p_\gamma(u,t) = Q_\beta(u) \quad \text{for } u \quad \text{in } [0,\beta],$$

for some $\beta \geqq \beta^*$, where $p_\gamma$ is the solution of (2.7)–(2.9) corresponding to the initial function $\Phi_\gamma$. Also, as in (7.5),

$$\frac{\partial p_\gamma}{\partial u} + \frac{f}{p_\gamma} \to -c_\beta \quad \text{in } (0,\beta)$$

as $t \to \infty$. This means that, with $x, t$ as independent variables, the corresponding solution $u_\gamma(x,t)$ has the property that it moves with an asymptotic speed $c_\beta$. More precisely, as is discussed in more detail in the proof of Theorem 4.4, if $x(t)$ is defined by

$$u_\gamma(x(t),t) = \delta,$$

for any fixed $\delta$ in $(0,\gamma)$, then $dx/dt \to c_\beta$. But $c_\beta \geqq c_{\beta^*} > c_\gamma$, so we have the contradiction that (as we saw before) $u_\gamma$ always lies above a travelling front moving with speed $c_\gamma$, while at the same time moving itself with a speed that ultimately exceeds $c_\gamma$.

This completes the discussion when $f \equiv 0$ in $[\beta^*, 1]$. If $f \not\equiv 0$, consider the functional

$$V(t) = \int_{\beta^*-\varepsilon}^{1-\varepsilon} \left(\frac{1}{2}p - \frac{1}{p}\int_{1-\varepsilon}^{u} f(s)\,ds\right) du,$$

where $\varepsilon$ is a small positive constant and $p$ is the solution of our initial value problem. Then

$$\begin{aligned} V' &= \int_{\beta^*-\varepsilon}^{1-\varepsilon} \left(\frac{1}{2}p_t + \frac{p_t}{p^2}\int_{1-\varepsilon}^{u} f(s)\,ds\right) du \\ &= \int_{\beta^*-\varepsilon}^{1-\varepsilon} \left(p_u + \frac{f}{p}\right)_u \left(\frac{1}{2}p^2 + \int_{1-\varepsilon}^{u} f(s)\,ds\right) du \\ &= \left[\left(p_u + \frac{f}{p}\right)\left(\frac{1}{2}p^2 + \int_{1-\varepsilon}^{u} f(s)\,ds\right)\right]_{\beta^*-\varepsilon}^{1-\varepsilon} - \int_{\beta^*-\varepsilon}^{1-\varepsilon} p\left(p_u + \frac{f}{p}\right)^2 du. \end{aligned}$$

In view of the known signs of $f$, we certainly have $V \geqq 0$. If $\varepsilon$ is small, then the integrated term at $1-\varepsilon$ is

$$\tfrac{1}{2}p(p\,p_u + f),$$

and since $p\,p_u$ is bounded (being $u_{xx}$) and $p(1-\varepsilon, t)$ is uniformly small for all $t$ if $\varepsilon$ is small, this integrated term is certainly small for small $\varepsilon$.

For large $t$, and any fixed $\varepsilon$, the contribution from the integrated term at the limit $\beta^* - \varepsilon$ is negative and not small, in view of (7.5) and the fact that

$$\int_{\beta^*}^{1} f\,du > 0.$$

Hence $V'(t)$ is bounded above by a negative constant for all $t$ sufficiently large, and this contradicts $V \geqq 0$. This final contradiction establishes that $\beta^* = 1$ and completes the proof of the theorem.

## 8. Proof of Theorem 4.3

We will restrict ourselves to the case where $f$ satisfies the conditions of Lemma 2.5 in each of two intervals $[0, A]$, $[A, 1]$. Extending the result to any number of intervals is a straightforward piece of induction. We thus have positive solutions $Q_1, Q_2$ of (2.14) and (2.16) over $(0, A)$, $(A, 1)$ respectively, with wave speeds $c_1, c_2$. We consider separately the two cases $c_1 \leqq c_2$ and $c_1 > c_2$.

First, by considering the interval $[0, A]$ on its own, we see by the arguments used in the proof of Theorem 4.2 that since the initial function is positive over $(0, A)$ (and indeed does not even vanish at $A$), we must have

$$\liminf_{t \to \infty} p(u,t) \geqq Q_1(u); \tag{8.1}$$

similarly

$$\liminf_{t \to \infty} p(u,t) \geqq Q_2(u). \tag{8.2}$$

In the case $c_1 \leqq c_2$, the decomposition of $[0,1]$ into $[0, A]$ and $[A, 1]$ is the minimal decomposition, and $Q_1, Q_2$ are identical with $P_1, P_2$ in the statement of the theorem. We can therefore assert that

$$\liminf_{t \to \infty} p(u,t) \geqq P(u),$$

where $P$ (as in the statement of the theorem) is defined by

$$P = \begin{cases} P_1 & \text{in } (0, A), \\ P_2 & \text{in } (A, 1). \end{cases}$$

We can also construct a supersolution as in Theorem 4.1. This supersolution tends to a limit which lies above (or coincident with) $P$ and which satisfies (2.14) where it is positive and (2.16); the only possible candidate when $c_1 \leqq c_2$ is $P$ itself. Thus

$$\limsup_{t \to \infty} p(u,t) \leqq P(u),$$

and the proof is complete for the case $c_1 \leqq c_2$.

From now on, therefore, we assume that $c_1 > c_2$. We still have (8.1)–(8.2), and we first need to establish that

$$p(A,t) \not\to 0 \quad \text{as } t \to \infty. \tag{8.3}$$

Suppose for contradiction that $p(A,t) \to 0$. Then we can construct a supersolution over $(0, A)$ by taking $\bar{q}$ to be the solution of (2.7) with boundary conditions

$$\bar{q}(0,t) = \gamma_1(t), \qquad \bar{q}(A,t) = \gamma_2(t),$$

where $\gamma_1(t)$, $\gamma_2(t)$ are positive decreasing functions of $t$, tending to zeros as $t \to \infty$ but (in the case of $\gamma_2(t)$) more slowly than $p(A,t)$. The initial function for $\bar{q}$ is the same as in (5.4). Then $\bar{q}$ decreases as $t \to \infty$ to $Q_1$, and so

$$\lim_{t \to \infty} p(u,t) = Q_1(u) \qquad \text{for } u \quad \text{in } [0, A],$$

which implies that

$$p_u + (f/p) \to -c_1 \qquad \text{in } (0, A). \tag{8.4}$$

Similarly,

or

$$p_u + (f/p) \to -c_2 \quad \text{in } (A, 1), \tag{8.5}$$

and if we interpret (8.4)–(8.5) with $x, t$ as independent variables, we get the same sort of contradiction as in Theorem 4.2; namely, values of $u$ less than $A$ are travelling with a higher speed than values of $u$ greater than $A$, since $c_1 > c_2$. This is not consistent with the continued monotonicity of $u(x,t)$ as a function of $x$. Hence (8.3) is true.

Now choose numbers $c_1^*, c_2^*$ with the properties that

$$c_1^* < c_1, \qquad c_2^* > c_2, \qquad c_1^* > c_2^*.$$

The solution $\Psi_2$ of the equation

$$\Psi' + (f/\Psi) = -c_2^*$$

with $\Psi_2(A) = 0$ has the property that $\Psi_2(u) < Q_2(u)$ for $u > A$, and indeed $\Psi_2$ vanishes at $u_2$ with $u_2 < 1$. (This follows from Lemma 2.2 of [2] and the argument already met in the proof of Lemma 3.1.) Similarly, the solution $\Psi_1$ of the equation

$$\Psi' + (f/\Psi) = -c_1^*$$

with $\Psi_1(A) = 0$ has the property that $\Psi_1(u) < Q_1(u)$ for $u < A$ and moreover vanishes at $u_1 > 0$. In view of (8.3) we can find values of $t$ arbitrarily large such that

$$p(u,t) > \Psi(u) \qquad \text{for} \quad u \text{ in } [u_1, u_2],$$

where

$$\Psi = \begin{cases} \Psi_1 \text{ in } (u_1, A), \\ \Psi_2 \text{ in } (A, u_2). \end{cases}$$

Let $\gamma$ be any number with $c_2^* < \gamma < c_1^*$. Let $\Psi_{\gamma 1}$ be the solution of

$$\Psi' + (f/\Psi) = -\gamma \tag{8.6}$$

for which

$$\Psi_{\gamma 1}(A - \eta) = \Psi_1(A - \eta),$$

$\eta$ being a small positive constant. Since $\gamma < c_1^*$, we see that

$$(\Psi_{\gamma 1} - \Psi_1) \exp\left\{ - \int_{A-\eta}^{u} f/\Psi_1 \Psi_{\gamma 1} \right\}$$

is strictly increasing from zero in $(A - \eta, A)$, and in view of the sign of $f$, the exponential is bounded. Hence $\Psi_{\gamma 1}(A) > \Psi_1(A) = 0$, and furthermore $\Psi_{\gamma 1}(A)$, which depends on $\gamma$, is a continuous decreasing function of $\gamma$ tending to zero as $\gamma \uparrow c_1^*$.

We can similarly take $\Psi_{\gamma_2}$ to be the solution of (8.6) for which $\Psi_{\gamma_2}(A+\eta) = \Psi_2(A+\eta)$, and then $\Psi_{\gamma_2}(A)$ is a positive continuous increasing function of $\gamma$ which tends to zero as $\gamma \downarrow c_2^*$. Hence we can find $\gamma$ for which $\Psi_{\gamma_1}(\mathrm{A}) = \Psi_{\gamma_2}(\mathrm{A})$, and for this $\gamma$ the function

$$\underline{\Psi} = \begin{cases} \Psi_1 \text{ in } (u_1, A-\eta), \\ \Psi_{\gamma_1} = \Psi_{\gamma_2} \text{ in } (A-\eta, A+\eta), \\ \Psi_2 \text{ in } (A+\eta, u_2) \end{cases}$$

satisfies (3.1) (with possible jump increases in $\underline{\Psi}'$) and lies below $p$ for some arbitrarily large $t$. (By virtue of (8.3), we can certainly take $\eta$ sufficiently small that $\Psi_{\gamma_1} = \Psi_{\gamma_2}$ lies below $p$ in $(A-\eta, A+\eta)$ for some arbitrarily large $t$.) Now the solution $\underline{q}$ of (2.7)–(2.9) with initial function $\underline{\Psi}$ increases with time to $P$, the positive solution of (2.14) and (2.16) over $(0,1)$; hence $\liminf p \geqq P$. But the usual choice of supersolution shows that $\limsup p \leqq P$, and the theorem is proved.

## 9. Proof of Theorem 4.4

Let us define $\gamma(t)$ by the requirement

$$u(x,t) = U\big(x - ct - \gamma(t)\big) = \tfrac{1}{2}. \tag{9.1}$$

Since $u, U$ are monotonic in $x$, this defines both $x$ and $\gamma$ as functions of $t$. Furthermore, since the second equality in (9.1) implies that

$$x - ct - \gamma(t) = \xi,$$

where $\xi$ is some fixed number, we have

$$u\big(\xi + ct + \gamma(t), t\big) = \tfrac{1}{2}.$$

Thus, differentiating with respect to $t$, we obtain

$$\{c + \gamma'(t)\}\, u_x(x,t) + u_t(x,t) = 0,$$

where $x = x(t)$ is given by (9.1), and so

$$\begin{aligned} c + \gamma' &= -u_t/u_x = -\{u_{xx} + f(u)\}/u_x \\ &= -\{p_u + (f/p)\} \to c \text{ as } t \to \infty. \end{aligned}$$

Thus $\gamma'(t) \to 0$ as $t \to \infty$.

Also,

$$u(x,t) - U\big(x - ct - \gamma(t)\big) = \int_{\xi + ct + \delta(t)}^{x} \{u_\sigma(\sigma,t) - U'\big(\sigma - ct - \gamma(t)\big)\}\, d\sigma.$$

Now make the change of variable, for any fixed $t$, from $x$ to $u$, so that $x = x(u,t)$, $\sigma = \sigma(v,t)$; we have

$$u(x,t)-U(x-ct-\gamma(t))=\int_{\frac{1}{2}}^{u}\frac{p(v,t)-P(U(\sigma-ct-\gamma(t)))}{p(v,t)}\,dv$$
$$=\int_{\frac{1}{2}}^{u}\{p(v,t)-P(v)+\{v-U(\sigma-ct-\gamma(t))\}\,P'(\theta)\}\frac{dv}{p(v,t)},$$

where $\theta$ lies between $v$ and $U(\sigma-ct-\gamma(t))$. This can now be regarded as an integral equation for the expression $u-U(x-ct-\gamma(t))$, where $x$ is a function of $u$ and $t$, and $t$ is regarded as a parameter. Since it follows from Theorems 4.2, 4.3 that $p(u,t)-P(u)$ tends to zero as $t\to\infty$ uniformly in $u$, and $p(u,t)$ is bounded from zero if $\eta\leqq u\leqq 1-\eta$, for any $\eta>0$, we see that, for this range of $u$,

$$|u(x,t)-U(x-ct-\gamma(t))|\leqq\varepsilon+K\int_{\frac{1}{2}}^{u}|v-U(\sigma-ct-\gamma(t))|\,dv$$

for any given $\varepsilon>0$ and some constant $K$ (depending on $\eta$), provided that $t$ is sufficiently large. Solution of this integral inequality by iteration, in the usual way, shows that

$$|u(x,t)-U(x-ct-y(t))|\leqq\varepsilon e^{\frac{1}{2}K}\quad\text{for}\quad\eta\leqq u\leqq 1-\eta. \tag{9.2}$$

In particular,

$$U(x-ct-\gamma(t))\leqq\varepsilon e^{\frac{1}{2}K}+\eta\quad\text{when}\quad u(x,t)=\eta.$$

Thus, from the monotonicity of $U$,

$$0<U(x-ct-\gamma(t))\leqq\varepsilon e^{\frac{1}{2}K}+\eta\quad\text{when}\quad 0<u(x,t)\leqq\eta,$$

so that

$$|u(x,t)-U(x-ct-\gamma(t))|\leqq\varepsilon e^{\frac{1}{2}K}+2\eta\quad\text{when}\quad 0<u(x,t)\leqq\eta. \tag{9.3}$$

A similar argument applies when $1-\eta\leqq u(x,t)<1$. Consequently (9.2) and (9.3) imply, when we take first $\eta$ as small as we please and then $\varepsilon$ the same way, that

$$u(x,t)-U(x-ct-\gamma(t))\to 0$$

as $t\to\infty$, uniformly in $x$. The proof of the theorem is therefore complete.

## 10. Proof of Theorem 4.5

The argument in the proof of Theorem 4.4 has to be just slightly modified. As there, we can show that

$$u(x,t)-U_1(x-c_1t-\gamma_1(t))\to 0$$

as $t\to\infty$, uniformly for $0<u\leqq\alpha-\eta$, for any fixed $\eta>0$, where $\gamma_1'(t)\to 0$ as $t\to\infty$; and similarly

$$u(x,t)-U_2(x-c_2t-\gamma_2(t))\to 0$$

as $t \to \infty$, uniformly for $\alpha+\eta \leqq u < 1$. Hence, given $\varepsilon > 0$, we have from the monotonicity of $U_1$, $U_2$ that, for $t$ sufficiently large,

$$\alpha > U_1(x - c_1 t - \gamma_1(t)) \geqq \alpha - \eta - \varepsilon \quad \text{for} \quad u \geqq \alpha - \eta, \tag{10.1}$$

$$\alpha < U_2(x - c_2 t - \gamma_2(t)) \leqq \alpha + \eta + \varepsilon \quad \text{for} \quad u \leqq \alpha + \eta. \tag{10.2}$$

Hence for $u \leqq \alpha - \eta$ we have

$$|u(x,t) - U_1(x - c_1 t - \gamma_1(t)) - U_2(x - c_2 t - \gamma_2(t)) + \alpha| < \eta + 2\varepsilon,$$

and similarly for other ranges of $u$, which gives the require conclusion.

In the particular case $c_1 = c_2$, we can further assert that $\gamma_2(\mathrm{t}) - \gamma_1(\mathrm{t}) \to \infty$ as $t \to \infty$. For if we define $x(t)$ by $u(x(t), t) = \alpha$, then (10.1)-(10.2) tells us that both $U_1(x(t) - c_1 t - \gamma_1(t))$ and $U_2(x(t) - c_1 t - \gamma_2(t))$ are close to $\alpha$ for large $t$. Since $U_1(\tau)$ is close to $\alpha$ only when $\tau$ is large and positive, while $U_2(\tau)$ is close to $\alpha$ only when $\tau$ is large and negative, the required result follows.

## 11. Estimates of rate of convergence

To obtain theorems that strengthen Theorems 4.1 – 4.5 by giving estimates of the rate of convergence, it seems to be necessary, when $f'(0) = 0$ or $f'(1) = 0$, to make further assumptions about the initial function $\phi$. We restrict ourselves for simplicity to the case where the minimal decomposition is $[0, 1]$ itself and prove the following theorem.

**Theorem 11.1.** *Let $f$ satisfy the conditions of Theorem 4.2, or of Theorem 4.3 where the minimal decomposition consists just of $[0, 1]$ itself. Let $u$ be the solution of (1.1)-(1.2) corresponding to the initial function $\phi$, where, as always, $\phi \in C^1(-\infty, \infty)$ with $\phi(-\infty) = 0$, $\phi(\infty) = 1$, $\phi' > 0$. Let $U$ be the travelling front solution, with speed $c$, and assume that*

$$\min\{c^2 - 4f'(0), c^2 - 4f'(1)\} > 0.$$

*If*

$$e^{\frac{1}{2}cx}\{\phi(x) - U(x - x_1)\} \in L^2(-\infty, \infty) \quad \textit{for some } x_1, \tag{11.1}$$

*then we have for some constants $x_0$, $\kappa$ and $\omega$, the last two positive,*

$$|u(x,t) - U(x - ct - x_0)| < K e^{-\omega t}, \tag{11.2}$$

*the result being uniform in $x$ if*

*either* $f'(0) \neq 0$ *and* $f'(1) \neq 0$
*or* $f'(0) \neq 0$ *and* $c > 0$
*or* $f'(1) \neq 0$ *and* $c < 0$;

*in all other cases, if $c < 0$ it is uniform for $x < (c + \varepsilon)t$ and if $c > 0$ it is uniform for $x > (c - \varepsilon)t$, for some $\varepsilon > 0$.*

**Remarks.** 1. The hypothesis (11.1) puts extra conditions on $\phi$ *ahead* of the wave; for (11.1) is a restriction only as $x \to -\infty$ if $c<0$ and as $x \to +\infty$ if $c>0$. Furthermore, if (11.1) is true for some $x_1$, then it is true for any $x_1$, this being a consequence of the asymptotic behavior of $U$ which is discussed in Remark 5 below. We could equivalently express (11.1) as

$$e^{\frac{1}{2}cx}\{\phi(x)-H(x)\}\in L^2(-\infty,\infty),$$

where $H$ is the Heaviside function.

2. If $f'(0)$, $f'(1)\neq 0$, the present theorem is a much weaker result than Theorem 3.1 in [2], because of the extra assumption (11.1). This arises because the present theorem, in both statement and proof, makes little distinction between the cases $f'(0), f'(1)\neq 0$ and $f'(0), f'(1)=0$. To recover Theorem 3.1 of [2], we should have to make explicit use of the assumptions $f'(0), f'(1)\neq 0$, essentially in the form of Lemma 4.1 of [2]. This assures us that, if $z=x-ct$ and $|z|\geqq \varepsilon t$, for any $\varepsilon>0$, then (11.2) holds. We have only, therefore, to establish (11.2) for $|z|<\varepsilon t$, and this can be done (as indeed it is done in the proof of Theorem 3.1 of [2]) essentially by carrying through the argument of the present theorem over a range of values of $x$ which (for any given $t$) is finite, so that the restriction (11.1) becomes irrelevant.

3. The function $f$ which corresponds to KANEL's equation for gas combustion [4], i.e. $f\equiv 0$ in $(0,\alpha)$, $f>0$ in $(\alpha,1)$, $f'(1)\neq 0$, is one of the cases which gives an exponential rate of convergence uniform for all $x$.

4. Even when the exponential rate of convergence is not uniform for all $x$, it is always uniform over the range in which the wave is actually being formed. Thus, if $z=x-ct$, and $u(x,t)=v(z,t)$, then

$$|v(z,t)-U(z-x_0)|<Ke^{-\omega t},$$

uniformly for $|z|<\varepsilon t$.

5. It is perhaps helpful to recall at this stage the asymptotic behavior of $U$. Linearizing the equation for $U$, namely

$$U''+cU'+f(U)=0, \tag{11.3}$$

about the constant solutions $U=0$ and $U=1$, we see that $U(z)\to 0,1$ as $z\to -\infty$, $+\infty$ at rates that are approximately

$$\exp\{\tfrac{1}{2}(-c+\sqrt{c^2-4f'(0)})\,z\}, \qquad \exp\{\tfrac{1}{2}(-c-\sqrt{c^2-4f'(1)})\,z\},$$

respectively. These rates are indeed exponential, provided that we do not have

$$\textit{either}\ f'(0)=0 \quad\text{and}\quad c\geqq 0 \quad \textit{or} \quad f'(1)=0 \quad\text{and}\quad c\leqq 0.$$

The non-exponential cases are therefore precisely the cases (for $c\neq 0$) in which we do not have the exponential rate of convergence uniform for all $x$. This is not coincidental.

**Proof.** This is a variant of the proof of the corresponding part of Theorem 3.1 of [2]. Let

$$z = x - ct, \qquad u(x,t) = v(z,t).$$

Set

$$h(z,t) = v(z,t) - U(z - \gamma(t) - \alpha(t)),$$

where $\gamma$(t), as in Theorem 4.4, is such that

$$v(z,t) - U(z - \gamma(t)) \to 0$$

as $t \to \infty$, uniformly in $z$, and $\alpha(t)$ is a continuously differentiable function to be chosen later.

We want a diffusion equation for $h$, namely

$$\begin{aligned} h_t &= h_{zz} + c\,h_z + f(v) - f(U) + (\alpha' + \gamma')\,U' \\ &= h_{zz} + c\,h_z + f'(U)\,h + (\alpha' + \gamma')\,U' + o(h). \end{aligned}$$

Setting $y = e^{\frac{1}{2}cz} h$, we obtain

$$y_t = y_{zz} - \{\tfrac{1}{4}c^2 - f'(U)\}\,y + (\alpha' + \gamma')\,e^{\frac{1}{2}cz}\,U' + o(y). \tag{11.4}$$

Now $y$, being initially in $L^2(-\infty,\infty)$, is, for $0 \leqq t \leqq T$, say, in the domain of the self-adjoint operator in $L^2(-\infty,\infty)$ given by

$$Ly = -y_{zz} + \{\tfrac{1}{4}c^2 - f'(U)\}\,y.$$

This is a standard result in the theory of evolution equations (see, for example, [3]), or equivalently it can be obtained from the integral equation for $y$ much as we obtained the results in Lemma 2.1. (For these arguments to apply, $f$ should be smoother than here assumed; but we can surmount this difficulty as unual by approximating $f$ in $C^1$ by a sequence of functions $f_n$ which are sufficiently differentiable, operating with $f_n$ throughout the proof of the theorem, and then finally letting $n \to \infty$. We omit the details.) The constant $T$ can be chosen arbitrarily large. This is a consequence of the deduction from (11.4) that

$$\frac{1}{2}\frac{d}{dt}\|y\|^2 = -(Ly, y) + (\alpha' + \gamma')(e^{\frac{1}{2}cz}U', y) + o(\|y\|^2), \tag{11.5}$$

where $\|\ldots\|$ denotes the norm in $L^2(-\infty,\infty)$, and $(\cdot,\cdot)$ the corresponding inner product, and where the notation $o(\|y\|^2)$ denotes a term which, given $\varepsilon > 0$, does not exceed $\varepsilon\|y\|^2$ in modulus for $t$ sufficiently large. By use of the Cauchy-Schwarz inequality, we can rewrite (11.5) as

$$\frac{1}{2}\frac{d}{dt}\|y\|^2 = -(Ly, y) + (\alpha' + \gamma')\,O(\|y\|) + o(\|y\|^2),$$

and since it is standard that the operator $L$has the spectrum bounded below, we have, for some constant $K$,

$$\frac{d}{dt}\|y\|^2 \leqq K\,\|y\|^2 + (\alpha' + \gamma')\,O(\|y\|).$$

Integration of this inequality for $\|y\|$ assures us that $y \in L^2(-\infty, \infty)$ for all time, and allows us to choose $T$ arbitrarily large.

We can in fact do much better than this by choosing $\alpha$ appropriately. We do this by requiring that $h$ is orthogonal to $e^{cz} U'$, i.e.

$$\int_{-\infty}^{\infty} e^{cz} h(z,t)\, U'(z-\gamma(t)-\alpha(t))\, dz = 0.$$

The work above justifies the convergence of this integral; and for any $t$ the equation is satisfied by one and only one value of $\alpha(t)$, as we see by rewriting it in the form

$$\int_{-\infty}^{\infty} e^{cw} \{v(w+\alpha(t),t) - U(w-\gamma(t))\}\, U'(w-\gamma(t))\, dw = 0$$

and noting that $v$ is monotonic in $\alpha$. Furthermore $\alpha(t)$ so defined is a continuously differentiable function of $t$.

We make this choice of $\alpha$, and note also that the operator $L$ is self-adjoint with a continuous spectrum to the right of $\min\{\frac{1}{4}c^2 - f'(0), \frac{1}{4}c^2 - f'(1)\}$, which is strictly positive by hypothesis, and a discrete spectrum to the left. Furthermore, we know by differentiating the equation (11.3) for $U$ that $e^{\frac{1}{2}cz} U'$ is an eigenfunction of $L$ corresponding to the eigenvalue 0; since this eigenfunction is of constant sign, 0 must be simple and the least eigenvalue, with all other eigenvalues strictly positive. Since $y$ is orthogonal to the eigenfunction $e^{\frac{1}{2}cz} U'$, we obtain from (11.5) that

$$\frac{1}{2}\frac{d}{dt}\|y\|^2 \leqq -M\|y\|^2$$

for some constant $M > 0$ which is independent of $t$ for $t$ sufficiently large. Hence

$$\|y\| = O(e^{-Mt}). \tag{11.6}$$

Finally, we can obtain an estimate for $y(z,t)$, with $t \geqq T$, say, by using the usual integral equation with $T$ as the initial time. This enables us to estimate $\|y(\cdot, T+1)\|_{C^0}$ in terms of $\|y(\cdot,\tau)\|_{L^2}$ for $T \leqq \tau \leqq T+1$. The estimate (11.6) then assures us that

$$\|y(\cdot,t)\|_{C^0} = O(e^{-Mt}),$$

the positive constant $M$ not necessarily being the same at each appearance, but always, of course, independent of $t$.

We also want to show that

$$\alpha' + \gamma' = O(e^{-Mt}),$$

and for this purpose we multiply (11.4) by $e^{\frac{1}{2}cz} U'$ and integrate over $(-\infty, \infty)$. Thus

$$(e^{\frac{1}{2}cz} U', y_t) = -(e^{\frac{1}{2}cz} U', Ly) + (\alpha' + \gamma')(e^{cz} U', U') + o\{(^{\frac{1}{2}cz} U', |y|)\}. \tag{11.7}$$

Differentiating the relation

$$(e^{\frac{1}{2}cz} U', y) = 0,$$

we obtain

$$(e^{\frac{1}{2}cz} U', y_t) = (\alpha' + \gamma')(e^{\frac{1}{2}cz} U'', y);$$

the scalar product on the right is seen to decay exponentially by use of the Cauchy-Schwarz inequality and (11.6). Also,

$$(e^{\frac{1}{2}cz} U', Ly) = \big(L(e^{\frac{1}{2}cz} U'), y\big) = 0,$$

and the remainder term in (11.7) also decays exponentially. Thus from (11.7) we have

$$\alpha' + \gamma' = O(e^{-Mt}).$$

Hence by integration, with a suitable choice of the constant $x_0$.

$$\alpha + \gamma = x_0 + O(e^{-Mt}). \tag{11.8}$$

The proof of the theorem is now virtually complete. We have, uniformly in $z$,

$$e^{\frac{1}{2}cz}\big\{v(z,t) - U\big(z - \gamma(t) - \alpha(t)\big)\big\} = O(e^{-Mt}).$$

If $c \leqq 0$ (the argument for $c > 0$ is similar), then

$$v(z,t) - U(z - \gamma(t) - \alpha(t)) = O(e^{-(M + \frac{1}{2}c\varepsilon)t}),$$

uniformly for $z < \varepsilon t$, where we choose $\varepsilon(>0)$ so small that $M + \frac{1}{2}c\varepsilon > 0$. Thus, for $z < \varepsilon t$, we can utilize (11.8) to obtain

$$|v(z,t) - U(z - x_0)| < Ke^{-\omega t}$$

for suitable choices of $K$ and $\omega$, and this is the required result. All that remains is to show that we can extend the range of uniformity to all $z$ if $f'(1) \neq 0$. But in this case

$$U(z) - 1 = O(e^{-Mz}),$$

say, as $z \to \infty$, and so, at $z = \varepsilon t$, we have

$$v(\varepsilon t, t) - 1 = O(e^{-Mt}).$$

In view of the monotonicity of $v$, we obtain

$$v(z,t) - 1 = O(e^{-Mt}),$$

uniformly for $z \geqq \varepsilon t$. Consequently

$$v(z,t) - U(z - x_0) = O(e^{-Mt}),$$

uniformly for $z \geqq \varepsilon t$, which completes the proof.

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