The Approach of Solutions of Nonlinear Diffusion Equations to Travelling Front Solutions

PAUL C. FIFE & J. B. MCLEOD

Abstract

The paper is concerned with the asymptotic behavior as $t \to \infty$ of solutions $u(x, t)$ of the equation

$$u_t - u_{xx} - f(u) = 0, \qquad x \in (-\infty, \infty),$$

in the case

$$f(0) = f(1) = 0, \qquad f'(0) < 0, \qquad f'(1) < 0.$$

Commonly, a travelling front solution $u = U(x - ct)$, $U(-\infty) = 0$, $U(\infty) = 1$, exists. The following types of global stability results for fronts and various combinations of them will be given.

1. Let $u(x, 0) = u_0(x)$ satisfy $0 \leqq u_0 \leqq 1$. Let

$$a_- = \limsup_{x \to -\infty} u_0(x), \qquad a_+ = \liminf_{x \to \infty} u_0(x).$$

Then u approaches a translate of U uniformly in x and exponentially in time, if a_- is not too far from 0, and a_+ not too far from 1.

2. Suppose $\int_0^1 f(u)\,du > 0$. If a_- and a_+ are not too far from 0, but u_0 exceeds a certain threshold level for a sufficiently large x-interval, then u approaches a pair of diverging travelling fronts.

3. Under certain circumstances, u approaches a "stacked" combination of wave fronts, with differing ranges.

1. Introduction

This paper is concerned with the pure initial value problem for the nonlinear diffusion equation

$$u_t - u_{xx} - f(u) = 0 \qquad (-\infty < x < \infty,\ t > 0), \tag{1.1}$$

the initial value being, say,

$$u(x, 0) = \varphi(x) \qquad (-\infty < x < \infty). \tag{1.2}$$

One of the central questions of interest for this problem is the behavior as $t \to \infty$ of the solution $u(x,t)$; in particular one would like to determine under what circumstances it does (or does not) tend to a travelling front solution. This problem has attracted an increasing amount of attention in recent years [1–5, 11–17, 21, 23]. We mention in particular the classic paper of KOLMOGOROV, PETROVSKIĬ & PISKUNOV [16], the extensions by KANEL' [14, 15], and the more recent work of ARONSON & WEINBERGER [1, 2]. These papers assume, as we do, that $f \in C^1$ with $f(0)=f(1)=0$, so that $u \equiv 0$ and $u \equiv 1$ are particular solutions of (1.1). A travelling front is a solution of (1.1) of the form $u=U(x-ct)$ for some c (the velocity), with the limits $U(\pm\infty)$ existing and unequal; for definiteness we take $U(-\infty)=0$ and $U(+\infty)=1$. With the above assumptions on f, it is a standard result that if φ is piecewise continuous and $0 \leqq \varphi(x) \leqq 1$, then there exists one and only one bounded classical solution $u(x,t)$ of (1.1–2), and $0 \leqq u(x,t) \leqq 1$ for all x, t. We shall always make these assumptions on φ and f, and shall be concerned only with this unique bounded solution.

A particular case of (1.1) was introduced by FISHER [9] to model the spread of advantageous genetic traits in a population. A mathematical treatment was given in [16], assuming

$$f(u)>0 \quad \text{for } u \in (0,1), \quad f'(0)>0, \quad f'(1)<0, \quad f'(u) \leqq f'(0).$$

It is shown there that if the initial function φ is chosen so that

$$\varphi(x) \equiv 0 \quad \text{for } x<0, \qquad \varphi(x) \equiv 1 \quad \text{for } x>0,$$

then it is indeed true in a certain sense that the solution of the initial value problem "tends" to a travelling front. Specifically, there exists a travelling front $U(x-ct)$ and a function $\psi(t)$ such that, as $t \to \infty$,

$$u(x,t) - U(x-ct-\psi(t)) \to 0 \quad \text{uniformly in } x, \tag{1.3}$$

and $\psi'(t) \to 0$. Because it is not true that $\psi(t)$ tends to a finite limit as $t \to \infty$, the solution u does not approach a travelling front uniformly in x; what does happen, however, and what (1.3) implies, is that the x-profile of the function u (monotone in x for each t) approaches that of the travelling front U.

In [14], KANEL' proves similar convergence results for the case

$$\begin{aligned} &f(u) \leqq 0 \quad \text{for } u \in [0,\alpha],\ \alpha<1,\\ &f(u)>0 \quad \text{for } u \in (\alpha,1). \end{aligned}$$

He also assumes $f'(1)<0$ and $\int_0^1 f(u)\,du>0$. This set of conditions includes the equation for combustion of certain gases, in which $f(u) \equiv 0$ for $u \in (0,\alpha)$, and also the important case in which $f(u)<0$ in $(0,\alpha)$.

The latter case, when f has exactly one intermediate zero in $(0,1)$, is called the "heterozygote inferior" case by ARONSON & WEINBERGER [1], reference being made to the genetical context envisaged by FISHER. But it is relevant in other contexts besides Fisher's. It serves to describe signal propagation along bistable transmission lines [19], and is a degenerate case of the FITZHUGH-NAGUMO model for the propagation of nerve pulses (see also [18]). Finally, this case is also very relevant to models for pattern formation and wave propagation in a diffusing and reacting medium [6, 7]. This bistable case of Fisher's equation, and its generalizations, are the principal objects of study in the present paper.

In his work, KANEL' allows φ to be more general than a step function (as in [16]), though he still requires it to be either monotone and 0 or 1 outside a finite interval, or a perturbation of a travelling front. The convergence statement is stronger than that in [16], in that $\psi=$constant.

ARONSON & WEINBERGER [1] introduce also the "heterozygote superior" case

$$f(u)>0 \quad \text{for } u\in(0,\alpha), \quad f(u)<0 \text{ in } (\alpha,1), \quad f'(0)>0, \quad f'(1)>0.$$

In relation to the travelling front question, they show that in each case mentioned above, there is a number $c^*>0$ with the property that every nonzero disturbance of the state $u\equiv 0$ which is initially confined to a half-line $x<x_0$ (so that $\varphi(x)=0$ for $x\geqq x_0$) and which exceeds some threshold value propagates with an asymptotic speed c^*, in the following sense:

$$\lim_{t\to\infty} u(x+ct,t)=0 \quad \text{for each } x \text{ and each } c>c^*,$$

and

$$\lim_{t\to\infty} u(x+ct,t)\geqq\alpha \quad \text{for each } x \text{ and each c}$$

with $0\leqq c<c^*$.

ROTHE [21], HOPPENSTEADT [12], MCKEAN [17], STOKES [23], and KAMETAKA [13] have recently taken another look at the case $f(u)>0$ for $u\in(0,1)$. Stokes, taking φ to be a step function or a sufficiently steep monotone function, improves the convergence result in [16] by showing that $\psi=$constant in the case $4f'(0)<(c^*)^2$. ROTHE, HOPPENSTEADT, and KAMETAKA show, among other things, that by prescribing the precise x-asymptotic behavior of φ ahead of the front, one can obtain uniform convergence to travelling fronts. MCKEAN applies probabilistic methods to the case $f(u)=u(1-u)$ to obtain similar results.

CHUEH [4] has treated the case when f is allowed to depend on u_x, and a travelling front represented by a saddle-saddle phase plane trajectory exists. He obtains convergence of the profile of u to that of the front.

Our main object in the present paper is to show, under minimal assumptions on φ, that when $f'(0)<0$, $f'(1)<0$ the solution converges uniformly to one of several types of travelling front configurations. A later paper will present convergence results for more general functions f.

Typical results obtained here for the bistable case are the following.

Let $f\in C^1[0,1]$ satisfy, for some $\alpha\in(0,1)$,

$$f(u)<0 \quad \textit{for } u\in(0,\alpha), \quad f(u)>0 \textit{ in } (\alpha,1), \quad f'(0)<0, \quad f'(1)<0.$$

By [14] there exists a unique (except for translation) monotone travelling front $U(x-ct)$. Suppose that $0\leqq\varphi(x)\leqq 1$ for all x, with

$$\liminf_{x\to\infty} \varphi(x) > \alpha, \qquad \limsup_{x\to-\infty} \varphi(x) < \alpha.$$

Then for some x_0 *the solution of the initial value problem approaches* $U(x-ct-x_0)$ *uniformly in* x *as* $t\to\infty$. *Further,* $c \geqq 0 (\leqq 0)$ *according as* $\int_0^1 f(u)\,du \leqq 0(\geqq 0)$, *and the rate at which the limit is approached is exponential.*

On the other hand, suppose that φ *is of bounded support (or more generally, that* $\limsup_{x\to\pm\infty} \varphi(x) < \alpha$) *and that* $\varphi(x) > \alpha + \eta$ *for some* $\eta > 0$ *and* $|x| < L$. *If* L *is large enough, depending on* η, *and* $\int_0^1 f(u)\,du > 0$, *then the solution develops (uniformly in* x) *into a pair of diverging travelling fronts*

$$U(x-ct-x_0) + U(-x-ct-x_1) - 1.$$

We also treat cases where f has more than one internal zero. To each triple of adjacent zeros of f with properties analogous to the zeros $(0, \alpha, 1)$ in the heterozygote inferior case, there of course corresponds a travelling front with characteristic speed and characteristic limits at $\pm\infty$. For simplicity consider the case of two adjacent triples of this type (thus five zeros in all), and a solution of (1.1) with range equal to the combined ranges of the two travelling fronts. Let c_0, c_1 be the two velocities, ordered by increasing u. If $c_0 < c_1$, we can show that the solution will tend to split into two separate travelling fronts, becoming very flat for u near the center zero of the five, and that there exists no single travelling front with range from the first to the fifth zero. If $c_0 > c_1$, however, there exists a unique such travelling front, and this corresponds to the fact that in this case a splitting as described above would be conceptually impossible. The solution will develop into the unique travelling front. The case $c_0 = c_1$ is one which we are unable to discuss by the methods of the present paper.

The principal tools used throughout the paper are a priori estimates and comparison theorems for parabolic equations. It may be well to state here the particular results of this type that we shall need. The indicated Schauder estimates can be found, for example, in [10, Theorem 4 of Chapter 7, and Theorem 5 of Chapter 3], and the comparison theorems in [20] with extensions in [1].

Let Q be a rectangle $[x_0, x_1] \times [t_0, t_1]$ in the (x,t) plane with $t_0 \geqq 0$ and with any of the x_0, x_1, t_1 either finite or infinite. Let the sides be of length $\geqq 2$. Corresponding to Q, let Q' be the smaller rectangle $[x_0+1, x_1-1] \times [t_0+1, t_1]$. For a function u, for which the derivatives appearing in (1.1) are defined and continuous in Q, let

$$|u|_0^Q \equiv \sup_{(x,t)\in Q} |u(x,t)|, \qquad |u|_1^Q \equiv |u|_0^Q + |u_x|_0^Q,$$
$$|u|_2^Q \equiv |u|_1^Q + |u_{xx}|_0^Q + |u_t|_0^Q.$$

Consequences of interior Schauder estimates: Let u *be a solution of* (1.1) *in* Q. *Then for some* $C > 0$, *independent of* u *and* Q, *we have*

$$|u|_1^{Q'} \leqq C(|f\circ u|_0^Q + |u|_0^Q), \tag{1.4}$$

(1.5) $$|u|_2^{Q'} \leqq C(|f \circ u|_1^Q + |u|_0^Q) \leqq C(|f' \circ u|_0^Q |u|_1^Q + |u|_0^Q),$$

(1.6) *the moduli of continuity of u_{xx} and u_t in Q' are subject to a bound depending only on $|f \circ u|_1^Q$ and $|u|_0^Q$.*

An immediate consequence is that the uniform boundedness of u in the half-space $\{t>0\}$ implies that of u_x, u_{xx}, and u_t in the half-space $\{t>1\}$. We shall use this boundedness property throughout the paper without further mention.

The comparison arguments we use are standard. Let N be the nonlinear differential operator, acting on functions of x and t, defined by

(1.7) $$Nu \equiv u_t - u_{xx} - cu_x - f(u).$$

Consider the initial value problem

(1.8) $$Nu = 0 \quad \text{for } (x,t) \in (-\infty, \infty) \times (0, \infty),$$

(1.9) $$u(x,0) = \psi(x).$$

A *regular subsolution* $\underline{u}(x,t)$ of (1.8–9) is a function defined and continuous in $(-\infty,\infty) \times [0,T)$, $T \leqq \infty$, for which the derivatives appearing in (1.7) are continuous in $(-\infty,\infty) \times (0,T)$, and satisfying

$$N\underline{u} \leqq 0 \quad \text{in } (-\infty,\infty) \times (0,T), \qquad \underline{u} \leqq \psi \quad \text{for } t=0.$$

A *subsolution* is defined to be a function of the form

$$\underline{u}(x,t) = \operatorname*{Max}_i \{\underline{u}_1(x,t), \dots, \underline{u}_n(x,t)\}$$

for some set $\{\underline{u}_i\}$ of regular subsolutions with common domains. *Supersolutions* are defined analogously.

Comparison Theorem. *Let $\underline{u}$ be a subsolution, and $\bar{u}$ a supersolution, of* (1.8–9). *Then $\underline{u}(x,t) \leqq \bar{u}(x,t)$ in $(-\infty,\infty) \times [0,T)$.*

In this theorem either $\underline{u}$ or $\bar{u}$ could, of course, be an exact solution.

The plan of the paper is as follows. In § 2, we review the existence and uniqueness of travelling front solutions, primarily for the case where $f(u) \leqq 0$ for u sufficiently small and positive and $f(u) \geqq 0$ for u sufficiently near 1. Many, but not all, of the results covered in this section are known and have appeared previously. In § 3, we state our precise results on uniform convergence. These are proved in § 4–6. Most of our results were previously announced in [8].

2. Existence and Uniqueness of Travelling Fronts

We assume throughout that $f \in C^1[0,1]$ and $f(0)=f(1)=0$. We first make the point that any travelling front with range $[0,1]$ is necessarily monotonic.

Lemma 2.1. *Any solution $u = U(x-ct)$ of* (1.1) *with $U \in [0,1]$, $U(-\infty)=0$, $U(\infty)=1$, necessarily satisfies $U'(z)>0$ for finite $z=x-ct$.*

Proof. Such a function $U(z)$ satisfies the ordinary differential equation

$$U''+cU'+f(U)=0 \tag{2.1}$$

and so corresponds to a trajectory in the (U,P) phase plane of the system

$$\frac{dU}{dz}=P, \tag{2.2}$$

$$\frac{dP}{dz}=-cP-f(U) \tag{2.3}$$

connecting the stationary points $(0,0)$ and $(1,0)$. This trajectory is a simple curve, since the differential equation (2.1) is of the second order, and it has the properties that it stays in the strip $0\leqq U\leqq 1$, is directed toward the right for $P>0$, and toward the left for $P<0$. Any simple curve with these properties must be such that $P\geqq 0$ throughout its length. If it contains a point $(U_0,0)$ with $U_0\in(0,1)$, then there would exist a travelling front $U(z)$ such that $U(0)=U_0$, $U'(0)=0$. Then $U''(0)\neq 0$, for otherwise $U\equiv U_0$ by uniqueness of solutions of (2.1). This means that P would change sign as the point $(U_0,0)$ is crossed, which we have seen to be impossible. Therefore $P=U'>0$ except at the endpoints. This completes the proof.

In view of Lemma 2.1, to any travelling front with range $[0,1]$ there corresponds a function $P(U)$ defined for $U\in[0,1]$, positive in $(0,1)$, zero at $U=0$ or 1, representing the derivative dU/dz. From (2.1), we see that it satisfies the equation

$$P'+\frac{f}{P}=-c \tag{2.4}$$

or, eliminating c,

$$P''+\left(\frac{f}{P}\right)'=0, \tag{2.5}$$

where c is the corresponding wave speed. Moreover P must satisfy the boundary conditions

$$P(0)=P(1)=0. \tag{2.6}$$

Conversely, given such a function P satisfying (2.4), (2.6), we may obtain a corresponding solution of (2.1) by integrating the equation

$$U'(z)=P(U), \qquad U(0)=\tfrac{1}{2}.$$

This equation may be solved for z in an interval (z_0,z_1) to obtain a monotone solution with $\lim_{z\downarrow z_0} U(z)=0$, $\lim_{z\uparrow z_1} U(z)=1$. To show that $u(x,t)=U(x-ct)$ is a travelling front as we have defined it, we have only to verify that $z_0=-\infty$, $z_1=\infty$.

Since $f(0)=0$, we have $|f(U)|<\beta U$ for some β. Let γ be a positive number such that $\frac{\beta}{\gamma}-c<\gamma$. Let S be the line $P=\gamma U$ in the (U,P) plane. If the graph of the given solution $P(U)$ touches S at a point in the first quadrant distinct from the origin, then at that point we have

$$P' = -c - \frac{f}{P} \leqq -c + \frac{\beta}{\gamma} < \gamma,$$

so that the trajectory immediately goes below S. This implies that, for some $\delta > 0$, either

$$\text{(i)} \qquad P(U) > \gamma U \quad \text{for } U \in (0, \delta), \quad \text{or}$$

$$\text{(ii)} \qquad P(U) < \gamma U \quad \text{for } U \in (0, \delta).$$

In the former case we have, from (2.4),

$$P'(U) = -c - \frac{f}{P} \leqq -c + \frac{\beta}{\gamma} < \gamma$$

so that $P(U) \leqq \gamma U$. Therefore (ii) must hold. But then

$$-z_0 = \int_0^{1/2} \frac{du}{P(u)} > \frac{1}{\gamma} \int_0^{1/2} \frac{du}{u} = \infty$$

so that $z_0 = -\infty$. Similarly, $z_1 = \infty$.

Hence if $f \in C^1[0,1]$, $f(0) = f(1) = 0$, there is a one-one correspondence between travelling fronts (modulo shifts in the independent variable z) and solutions of (2.4), (2.6) which are positive in $(0,1)$.

The form of the equation (2.4) makes it clear that for every solution-pair (P, c), there is a second pair $(-P, -c)$, so that our theory applies to monotone decreasing solutions of (2.1) as well.

Integration of (2.4) (after multiplication by P) yields

$$c \int_0^1 P(u)\, du = -\int_0^1 f(u)\, du,$$

so that, for a positive solution of (2.4–6), we have

$$\text{(2.7)} \qquad c \geqq 0 \, (\leqq 0) \text{ according as } \int_0^1 f(u)\, du \leqq 0 \, (\geqq 0).$$

(For a negative solution, the sign of c is the same as that of $\int_0^1 f du$.)

Lemma 2.2 (KANEL' [14]). *Let $f \in C^1[0,1]$ satisfy $f(0) = 0$ and $f(u) \leqq 0$ for small positive u. Let $P_i(U)$, $i = 1, 2$, be solutions of (2.4) with corresponding speeds c_i. Assume $P_i(0) = 0$ and $P_i(U) > 0$ for $U \in (0, U_0)$. Then for each $U \in (0, U_0]$ we have*

$$P_1(U) \gtreqqless P_2(U) \qquad \textit{according as } c_1 \gtreqqless c_2.$$

Proof. From (2.4), we have

$$P_1' - P_2' - \frac{f}{P_1 P_2}(P_1 - P_2) = -(c_1 - c_2),$$

so that

$$\frac{dF(U)}{dU} = -(c_1 - c_2)\exp \int_{U_0/2}^{U} (-f(t)/P_1(t)\,P_2(t))\,dt,$$

where

$$F(U) = (P_1 - P_2)\exp \int_{U_0/2}^{U} (-f(t)/P_1(t)\,P_2(t))\,dt.$$

As $U \downarrow 0$, we have $F(U) \to 0$ since $P_1 - P_2 \to 0$, and the exponential factor is bounded as $U \downarrow 0$ because of the sign of f. If $c_1 = c_2$, then $F(U)$, being constant, is zero, so that $P_1 \equiv P_2$. But if $c_1 > c_2$, F is strictly decreasing, so that $P_1 < P_2$ for $U > 0$.

In the remainder of this section, we shall usually assume that f satisfies the following conditions:

$$\begin{aligned} &f \in C^1[0,1], \quad \text{with } f(0) = f(1) = 0,\\ &f(u) \leqq 0 \quad \text{for } u \text{ sufficiently small},\\ &f(u) \geqq 0 \quad \text{for } u \text{ sufficiently near } 1. \end{aligned} \tag{2.8}$$

Corollary 2.3. *Let f satisfy* (2.8). *Then there exists at most one solution of* (2.5–6) *which is positive in* $(0,1)$.

Proof. Suppose there exist two; let them be those in Lemma 2.2, wherein $U_0 = 1$. The fact that $P_1(1) = P_2(1) = 0$ implies, by that lemma, that $c_1 = c_2$, and in turn that $P_1 \equiv P_2$.

Theorem 2.4. *Let $f \in C^1[0,1]$, and $f(0) = f(1) = 0$. For some $\alpha \in (0,1)$, suppose that one of the following assertions holds:*

(a) $f \leqq 0$ *in* $(0,\alpha)$; $f > 0$ *in* $(\alpha, 1)$; $\int_0^1 f(u)\,du > 0$;

(b) $f < 0$ *in* $(0,\alpha)$; $f \geqq 0$ *in* $(\alpha, 1)$; $\int_0^1 f(u)\,du < 0$;

(c) $f < 0$ *in* $(0,\alpha)$; $f > 0$ *in* $(\alpha, 1)$.

Then there exists one and (*by Corollary* 2.3) *only one solution of* (2.5–6) *which is positive in* $(0,1)$.

Remark. The theorem is in some sense best possible. For if we relax the restriction

$$\int_0^1 f(u)\,du > 0$$

in case (a) and consider instead

$$\int_0^1 f(u)\,du = 0$$

with $f = 0$ in $(0,\beta)$, say, where $0 < \beta < \alpha$, then the only possible solution is, by (2.7), also a solution of (2.4) for which $c = 0$; and since $f = 0$ in $(0,\beta)$, we have $P = 0$ in $(0,\beta)$, which shows a positive solution to be impossible.

Proof. This theorem (in case (a)) was proved in [14], [1, Theorem 4.2], and [2, Theorem 4.1]. Case (b) follows from case (a) by replacing U by $1-U$, and f by $-f$. Case (c) for $c \neq 0$ follows from the other two cases. For $c=0$, (2.4) can be integrated, and the result is the required solution.

Our object now is to extend this existence theorem to a wider class of functions f, still retaining the hypothesis (2.8). At the same time, we shall consider the possibility of solutions of (2.4) with internal zeros, which represent phase-plane images of "stacked" combinations of travelling fronts.

The following preliminary lemmas will be needed.

Lemma 2.5. *Let $f \in C^1[0,1]$ with $f(0)=0, f(1)=0$, and let there exist a solution $P_0(U)$ of (2.4), positive on $(0,\alpha)$, with $P_0(0)=0$ and "velocity" $c=c_0$.*

Then for any $c \leqq c_0$, there exists a solution $P(U)$ of (2.4) on $(0,\alpha)$ with $P(0)=0$ and $P(U) \geqq P_0(U)$. There exists a maximal such solution, which we denote by $P_c(U)$, so that for any other solution $\tilde{P}$ with the given c satisfying $\tilde{P}(0)=0$, and for U in the domain of $\tilde{P}$, we have $P_c(U) \geqq \tilde{P}(U)$. Moreover, $P_c(U)$ depends continuously on c for $c \leqq c_0$.

Proof. We follow the construction used in [2]. For $\nu>0$, $c \leqq c_0$, let $P_{c,\nu}(U)$ be the solution of the regular initial value problem

$$P' + \frac{f}{P} + c = 0, \qquad P(0) = \nu.$$

Clearly $P_{c,\nu}(U) > P_0(U)$ for $U \in [0,\alpha]$. Since $P_{c,\nu}(U)$ is monotone in ν, $P_c(U) \equiv \lim\limits_{\nu \downarrow 0} P_{c,\nu}(U)$ exists and satisfies $P_c(U) \geqq P_0(U)$.

Furthermore, by the monotone convergence theorem, P_c satisfies (2.4), and so is the required solution. If $\tilde{P}_c$ is another solution, clearly $P_{c,\nu} \geqq \tilde{P}_c$ for all U where the latter is defined; passing to the limit, we find that P_c is maximal. Its continuous dependence on c is proved as in [2, Proposition 4.5].

In the following, when we speak of a "travelling front over $[\alpha,\beta]$ with velocity c", we shall mean a solution of (2.4) with the given c which is positive in (α,β) and vanishes at α and β.

Lemma 2.6. *Let f satisfy (2.8). For $0<\alpha \leqq \beta < 1$, assume that there exist travelling fronts over $[0,1]$, $[0,\alpha]$, and $[\beta,1]$, with velocities $c_{01}, c_{0\alpha}$, and $c_{\beta 1}$ respectively. Then necessarily*

$$c_{0\alpha} > c_{01} > c_{\beta 1}. \tag{2.9}$$

Proof. We apply Lemma 2.2 with P_1 being the solution over $[0,1]$, P_2 the solution over $[0,\alpha]$, and $U_0=\alpha$. Since $P_{2(\alpha)}=0<P_1(\alpha)$, we have $c_{01}=c_1<c_2=c_{0\alpha}$. The other inequality in (2.9) is proved in a similar fashion.

Theorem 2.7. *Let $f \in C^1[0,1]$ with $f(0)=f(1)=0$, and let there exist a travelling front over $[0,\alpha]$ with velocity $c_{0\alpha}$, and one over $[\alpha,1]$ with velocity $c_{\alpha 1} < c_{0\alpha}$. Then there exists a travelling front over $[0,1]$ with velocity c_{01} satisfying*

$$c_{0\alpha} > c_{01} > c_{\alpha 1}.$$

Remark. For this theorem to hold it is not necessary that f satisfy (2.8). If it does, however, then Lemma 2.6 shows the inequality $c_{\alpha 1} < c_{0\alpha}$ to be a necessary as well as a sufficient condition for the existence of a travelling front over $[0, 1]$.

Proof. For all $c < c_{0\alpha}$, let $P_c(U)$ be the (maximal) solution of (2.4) guaranteed by Lemma 2.5, and let $g(c) = P_c(\alpha)$, $c < c_{0\alpha}$. It is continuous in c, and satisfies $\lim_{c \uparrow c_{0\alpha}} g(c) = 0$.

By the symmetrical argument, for each $c > c_{\alpha 1}$ there is a positive solution $\bar{P}_c(U)$ of (2.4) satisfying $\bar{P}_c(1) = 0$, with $h(c) = \bar{P}_c(\alpha)$ continuous, and $\lim_{c \downarrow c_{\alpha 1}} h(c) = 0$. Hence there is a solution $c = c_{01}$ of $g(c) = h(c)$. For this value of c, $\bar{P}_c$ is the continuation of P_c; this is therefore the required travelling front over $[0, 1]$.

Definition. A closed interval $I \subset [0, 1]$ is called *admissible* if f vanishes at the endpoints, $f \leqq 0$ in I near the left endpoint, $f \geqq 0$ in I near the right endpoint, and there exists a travelling front over I.

Suppose we have given a decomposition of $[0, 1]$ into nonoverlapping adjacent admissible intervals

$$[0, 1] = \bigcup_{j=1}^{m} I_j,$$

ordered from left to right (so that the right endpoint of I_j is the left endpoint of I_{j+1}). Let $\{c_j\}$ be the associated velocities of the travelling fronts over the I_j.

Definition. Such a decomposition is called *minimal* if c_j is nondecreasing in j: $c_{j+1} \geqq c_j$.

Theorem 2.8. *If there exists a decomposition of* $[0, 1]$ *into admissible intervals, then there exists a unique minimal decomposition.*

Remark. The significance of minimal decompositions will be seen in Theorem 3.3 and in a later paper. In fact, monotone solutions of (1.1) with range $[0, 1]$ will split into a "stack" of travelling fronts, each with range in one of the intervals of the minimal decomposition and with its distinctive asymptotic speed, and (at least when the c_j are distinct) spreading away from each other.

Proof. The existence of a minimal decomposition is trivial. In fact, if the original decomposition is not minimal, there will be two adjacent intervals I_1 and I_2, say, with associated velocities satisfying $c_1 > c_2$. By Theorem 2.7, we may combine them into a single admissible interval. Thus proceeding in a finite number of steps (since each step reduces by one the total number of intervals), we arrive at a minimal decomposition.

We now show that there cannot be two distinct minimal decompositions. Let two minimal decompositions be given. If they are distinct, there will be an interval of one, call it I, which overlaps at least two intervals of the other. Call the latter overlapping intervals $J_1, \ldots, J_q$, ordered from left to right, so that $I \subset \bigcup_{k=1}^{q} J_k$ and $I \cap J_k \neq \phi$, $1 \leqq k \leqq q$. The interval $I \cap J_1$, being a union of the original intervals, has a minimal decomposition $I \cap J_1 = \bigcup_{k=1}^{n} I'_k$, again ordering from left to right. Let the velocities associated with I, J_k, and I'_k be c, d_k, and c'_k respectively. By Lemma 2.6 we have $c'_1 > c$ and $c'_n \leqq d_1$. By minimality, $c'_1 \leqq c'_n$. Hence $c < d_1$. A similar argument shows that $c > d_q$. Hence $d_1 > d_q$. But this contradicts the minimality of the second decomposition and proves the theorem.

3. Uniform Convergence Results

Beginning with this section, we take up the question of the asymptotic behavior as $t \to \infty$ of solutions of the initial value problem (1.1–2). We deal with circumstances under which a solution approaches a travelling front, or a combination of fronts, uniformly in x and exponentially in t as $t \to \infty$. Conclusions to this effect, under minimal assumptions on φ, can be made when the travelling front or fronts concerned are over u-intervals at the endpoints of which $f'(u) < 0$. The basic result is the following.

Theorem 3.1. *Let $f \in C^1[0,1]$ satisfy*

$$\begin{aligned} &f(0)=f(1)=0, \ f'(0)<0, \ f'(1)<0, \\ &f(u)<0 \quad \textit{for } 0<u<\alpha_0, \\ &f(u)>0 \quad \textit{for } \alpha_1<u<1, \end{aligned}$$

where $0<\alpha_0 \leqq \alpha_1 < 1$.

Assume there exists a travelling front solution U of (1.1) with speed c, let φ satisfy $0 \leqq \varphi \leqq 1$, and suppose

$$\limsup_{x \to -\infty} \varphi(x) < \alpha_0, \qquad \liminf_{x \to \infty} \varphi(x) > \alpha_1. \tag{3.1}$$

Then for some constants z_0, K, and ω, the last two positive, the solution $u(x,t)$ of (1.1–2) satisfies

$$|u(x,t) - U(x-ct-z_0)| < K e^{-\omega t}. \tag{3.2}$$

Remark. It is clear from §2 that the existence of a travelling front is by no means guaranteed. Readily verifiable conditions on f were, however, given in that section which ensure its existence. If f satisfies these conditions, the existence assumption in the statement of Theorem 3.1 may of course be omitted. A particularly important case is that of the degenerate Nagumo's equation, in which $\alpha_0 = \alpha_1$. A travelling front does exist in this case.

Theorem 3.1 implies that a solution which vaguely resembles a front at some initial time will develop uniformly into such a front as $t \to \infty$. "Vaguely resembles" simply means that the solution is less than α_0 far to the left, and greater than α_1 far to the right. Of course, if the words "left" and "right" are interchanged in this statement, the same conclusion holds; the front will then face right rather than left, and will travel in the opposite direction.

There are also situations in which the solution will develop into a pair of such fronts, moving in opposite directions. That is the gist of the following result.

Theorem 3.2. *Let f satisfy the hypotheses of Theorem 3.1, and in addition*

$$\int_0^1 f(u)\,du > 0. \tag{3.3}$$

Let φ satisfy $0 \leqq \varphi \leqq 1$, and

$$\limsup_{|x|\to\infty} \varphi(x) < \alpha_0, \quad \varphi(x) > \alpha_1 + \eta \qquad \text{for } |x| < L, \tag{3.4}$$

where η and L are some positive numbers. Then if L is sufficiently large (depending on η and f), we have for some constants x_0, x_1, K, and ω (the last two positive),

$$|u(x,t) - U(x - ct - x_0)| < K e^{-\omega t}, \qquad x < 0, \tag{3.5a}$$

$$|u(x,t) - U(-x - ct - x_1)| < K e^{-\omega t}, \qquad x > 0. \tag{3.5b}$$

Note that (3.3) implies $c<0$. The intuitive meaning of (3.5) is that the x-interval on which u is near the value 1 is finite and is elongating in both directions, with speed $|c|$. If the inequality in (3.3) is reversed, and appropriate changes in (3.4) are made, then an analogous convergence result is still obtained. In the latter case, the interval on which u is near 0 will elongate.

Finally, we consider the possibility of the solution developing into a combination of fronts with different, but adjacent, ranges. As in § 2, we call them a stacked combination of fronts, and for simplicity treat only the case when there are two of them.

Theorem 3.3. *Let $f(u_i) = 0$ and $f'(u_i) < 0$, $i = 1, 2, 3$, where $u_1 < u_2 < u_3$. Let there exist travelling fronts $U_1(x - c_1 t)$ and $U_2(x - c_2 t)$ with ranges (u_1, u_2) and (u_2, u_3) respectively. Assume $c_1 < c_2$. Let α_1 be the least zero of f greater than u_1, and α_2 the greatest zero less than u_3. Suppose $u_1 \leqq \varphi(x) \leqq u_3$, and*

$$\limsup_{x\to-\infty} \varphi(x) < \alpha_1, \qquad \liminf_{x\to\infty} \varphi(x) > \alpha_2. \tag{3.6}$$

Then there exist constants x_1, x_2, K, and ω, the last two positive, such that

$$|u(x,t) - U_1(x - c_1 t - x_1) - U_2(x - c_2 t - x_2) + u_2| < K e^{-\omega t}. \tag{3.7}$$

Note that (3.7) implies, in particular, that

$$\lim_{t\to\infty} u(\beta t, t) = \begin{cases} u_1 & \text{for } \beta < c_1, \\ u_2 & \text{for } c_1 < \beta < c_2, \\ u_3 & \text{for } c_2 < \beta. \end{cases}$$

4. Proof of Theorem 3.1 (First Part)

In this section we establish the uniform convergence of $u(x,t) - U(x - ct - z_0)$ to zero as $t \to \infty$, the exponential nature of this convergence being deferred to § 5.

Several lemmas will be needed in the proofs of the theorems given in the previous section. Some arguments are easiest to give in terms of a moving coordinate system. For the purposes of Theorem 3.1, we set $z = x - ct$, and write the solution of (1.1–2) as

$$v(z,t)=u(x,t)=u(z+ct,t).$$

Our basic lemma in the following.

Lemma 4.1. *Under the assumptions of Theorem* 3.1, *there exist constants* z_1, z_2, q_0, *and* μ *(the last two positive), such that*

$$U(z-z_1)-q_0 e^{-\mu t} \leqq v(z,t) \leqq U(z-z_2)+q_0 e^{-\mu t}. \tag{4.1}$$

Proof. We prove only the left-hand inequality; the other is similar. The function v satisfies

$$N[v] \equiv v_t - v_{zz} - c v_z - f(v) = 0, \qquad z \in (-\infty, \infty),\ t>0, \tag{4.2a}$$

$$v(z,0)=\varphi(z). \tag{4.2b}$$

Functions $\xi(t)$ and $q(t)\,(q(t)$ positive) will be chosen so that

$$\underline{v}(z,t) \equiv \operatorname{Max}\left[0, U(z-\xi(t))-q(t)\right]$$

will be a subsolution.

First, let $q_0>0$ be any number such that $\alpha_1 < 1-q_0 < \liminf\limits_{z\to\infty} \varphi(z)$. Then take z^* so that $U(z-z^*)-q_0 \leqq \varphi(z)$ for all z. This is possible for sufficiently large positive z^* by virtue of (3.1). Let

$$\Phi(u,q)=\begin{cases} [f(u-q)-f(u)]/q, & q>0,\\ -f'(u), & q=0.\end{cases}$$

Then Φ is continuous for $q \geqq 0$, and for $0<q\leqq q_0$ we have $\alpha_1 < 1-q_0 \leqq 1-q < 1$, so that $\Phi(1,q)>0$. Also $\Phi(1,0)=-\mathrm{f}'(1)>0$. Thus for some $\mu>0$ we have $\Phi(1,q) \geqq 2\mu$ for $0 \leqq q \leqq q_0$. By continuity, there exists a $\delta>0$ such that $\Phi(u,q)\geqq\mu$ for $1-\delta \leqq u \leqq 1$, $0\leqq q \leqq q_0$. In this range, we have

$$f(u-q)-f(u) \geqq \mu q.$$

Setting $\zeta = z-\xi(t)$, and using the fact that

$$U''+cU'+f(U)=0, \tag{4.3}$$

we find that, if $\underline{v}>0$,

$$\begin{aligned} N[\underline{v}] &= -\xi'(t)\,U'(\zeta)-c\,U'(\zeta)-q'(t)-U''(\zeta)-f(U-q)\\ &= -\xi'(t)\,U'(\zeta)-q'(t)+f(U)-f(U-q). \end{aligned}$$

Thus when $U\in[1-\delta,1]$, $q\in[0,q_0]$,

$$N[\underline{v}] \leqq -\xi' U' - q' - \mu q \leqq -(q'+\mu q),$$

provided $\xi' \geqq 0$, since $U' \geqq 0$ (see Lemma 2.1). We choose $q(t)=q_0 e^{-\mu t}$, which results in $N[\underline{v}] \leqq 0$ when $1-\delta \leqq U \leqq 1$.

By possibly further reducing the size of μ and δ and using the same arguments, we may also be assured that $N[\underline{v}] \leqq 0$ whenever $0 \leqq U \leqq \delta$ and $U \geqq q$.

Now consider the intermediate values, $\delta \leqq U \leqq 1-\delta$. In this range we know that $U'(z) \geqq \beta$ for some $\beta > 0$, as shown in Lemma 2.1. Also, by the differentiability of f, we have $f(U) - f(U-q) \leqq \kappa q$ for some $\kappa > 0$. Thus

$$N[\underline{v}] \leqq -\beta \xi' - q' + \kappa q.$$

We now set

$$\xi'(t) = (-q' + \kappa q)/\beta = (\mu + \kappa) q/\beta > 0, \qquad \text{with } \xi(0) = z^*.$$

(Specifically,

$$\xi = z_1 + z_2 e^{-\mu t}, \tag{4.4a}$$

where

$$z_2 = -q_0(\mu+\kappa)/\mu\beta, \qquad z_1 = z^* - z_2.) \tag{4.4b}$$

Thus $\xi(t)$ is increasing and approaches a finite limit as $t \to \infty$. Then $N[\underline{v}] \leqq 0$ whenever $\underline{v} > 0$, and by our condition on z^*, $\underline{v}$ will be a subsolution. Thus

$$v(z,t) \geqq \underline{v}(z,t) \geqq U(z-z_1) - q(t) = U(z-z_1) - q_0 e^{-\mu t},$$

which completes the proof.

Lemma 4.2. *Under the assumptions of Theorem* 3.1, *there exists a function* $\omega(\varepsilon)$, *defined for small positive* ε, *such that* $\lim_{\varepsilon \downarrow 0} \omega(\varepsilon) = 0$, *with the property that, if* $0 \leqq \varphi \leqq 1$ *and* $|\varphi(z) - U(z-z_0)| < \varepsilon$ *for some* z_0, *then*

$$|v(z,t) - U(z-z_0)| < \omega(\varepsilon)$$

for all z *and all* $t > 0$.

Proof. In the proof of Lemma 4.1, we may take $q_0 = O(\varepsilon)$ and $|z^* - z_0| = O(\varepsilon)$. Hence also $|z_1 - z_0| = O(\varepsilon)$, $|z_2 - z_0| = O(\varepsilon)$, and the conclusion follows from the lemma.

Remark. Lemma 4.2 already yields the stability of travelling fronts in the C^0 norm, but Theorem 3.1 claims much more.

In the following development, it will be necessary to have asymptotic estimates for the derivatives of v.

Lemma 4.3. *Under the assumptions of Theorem* 3.1, *there exist positive constants* σ, μ, *and* C *with* $\sigma > |c|/2$, *such that*

$$|1 - v(z,t)|, \quad |v_z(z,t)|, \quad |v_{zz}(z,t)|, \quad |v_t(z,t)| < C(e^{(-\frac{1}{2}c-\sigma)z} + e^{-\mu t}), \qquad z > 0; \tag{4.5a}$$

$$|v(z,t)|, \quad |v_z(z,t)|, \quad |v_{zz}(z,t)|, \quad |v_t(z,t)| < C(e^{(-\frac{1}{2}c+\sigma)z} + e^{-\mu t}), \qquad z < 0. \tag{4.5b}$$

Proof. The wave front $U(z)$ approaches its limits exponentially; this is easily seen by linearizing (2.1) about the constant states $U=0$ and $U=1$. In fact, this analysis shows that $U(z)\to 1$ as $z\to\infty$ at the approximate rate

$$\exp\{\tfrac{1}{2}[-c-\sqrt{c^2-4f'(1)}]\,z\},$$

and so at an exponential rate faster than $\exp\{(-\frac{1}{2}c-\frac{1}{2}|c|)\,z\}$. A similar analysis holds as $z\to-\infty$. This, together with (4.1), establishes (4.5) for the undifferentiated function v. Since $|f(u)|<k|u|$ for u near 0 and $|f(u)|<k(1-u)$ for u near 1, we also have

$$|f(v(z,t))|\leqq C(e^{-\frac{1}{2}cz-\sigma|z|}+e^{-\mu t}) \qquad \text{for some } C>0.$$

From this and (1.4) it follows that (4.5) is satisfied for v_z. The same estimates for v_{zz} follow then from (1.5), and (4.2a) yields them for v_t. This completes the proof.

Lemma 4.4. *For each $\delta>0$ the "orbit" set*

$$\{v(\cdot,t)\colon\, t\geqq\delta\},$$

considered as a subset of $C^2(-\infty,\infty)$, is relatively compact.

Proof. We know from (1.4–6) that v, v_z, and v_{zz} are bounded and equicontinuous for $t\geqq\delta$. Let $\{t_n'\}$ be a given sequence. If there is a finite accumulation point t_∞, then the (uniform) continuity of v and its derivatives implies that $v(\cdot,t)$ approaches the limit $v(\cdot,t_\infty)$ "along a subsequence". So assume there is none. For any $K>0$, let $v_K(z,t)$ be the restriction of v to the set $|z|\leqq K$, $t\geqq\delta$. By Arzelà's theorem, for each $K=1,2,\ldots$, there is a subsequence $\{t_{n,K}\}$ such that the sequence $\{v_K(z,t_{n,K})\}$ converges in $C^2[-K,K]$. We may always, in fact, choose $\{t_{n,\,K+1}\}$ to be a subsequence of $\{t_{n,K}\}$. We then take a diagonal sequence, denoted by $\{t_n\}$, so that $\{v(z,t_n)\}$ converges uniformly on each interval $[-K,K]$ to a limit $w(z)$, the derivatives to order two converging to those of w.

Since v satisfies (4.5), we may pass to the limit as $t\to\infty$ thus showing that w satisfies (4.5) with $t=\infty$.

Given any $\varepsilon>0$, by Lemma 4.3 one may choose T and K such that for $k=0,1,2,$

$$|\partial_z^k(v(z,t)-w(z))|<\varepsilon \qquad \text{for } |z|>K,\ t>T.$$

One may also choose N so that $t_N>T$ and

$$|\partial_z^k(v(z,t_n)-w(z))|<\varepsilon \qquad \text{for } n>N,\ |z|\leqq K.$$

This proves that $\lim_{n\to\infty} v(z,t_n)=w(z)$ in $C^2(-\infty,\infty)$, and completes the proof of the lemma.

Lemma 4.5. *Under the assumptions of Theorem* 3.1, *there exists a value z_0 such that*

$$\lim_{t\to\infty}|v(z,t)-U(z-z_0)|=0,$$

uniformly in z.

Proof. Let $\varepsilon>0$ be a number satisfying $|c|\varepsilon<2\mu$, where μ is the constant in Lemma 4.3. Let w be a truncation of v in the following sense:

$$\begin{aligned} &w(z,t)=v(z,t) && \text{for } |z|\leqq\varepsilon t,\\ &w(z,t)=0 && \text{for } z\leqq-\varepsilon t-1,\\ &w(z,t)=1 && \text{for } z\geqq\varepsilon t+1, \end{aligned}$$

and w satisfies (4.5). It is clear from (4.5) that v may be smoothed in this manner so that the truncation w also satisfies (4.5).

We define the Lyapunov functional

$$V[w]=\int_{-\infty}^{\infty} e^{cz}\left[\tfrac{1}{2}w_z^2-F(w)+H(z)F(1)\right]dz,$$

where $H(z)$ is the Heaviside step function and $F(v)\equiv\int_0^v f(s)\,ds$. It clearly converges, as do the integrals below, because of the truncation. In fact, $V[w]$ is bounded independently of t. To see this, we use (4.5) to estimate it as follows:

$$\begin{aligned} |V[w]| &\leqq C_1\int_{-\varepsilon t-1}^{\varepsilon t+1} e^{cz}\left(e^{-cz-2\sigma|z|}+e^{-2\mu t}\right)dz\\ &\leqq C_2\int_0^{\varepsilon t}\left(e^{-2\sigma|z|}+e^{|c|z-2\mu t}\right)dz. \end{aligned}$$

Since $|c|\varepsilon-2\mu<0$, the right side is bounded for all time.

Setting $V(t)\equiv V[w(\cdot,t)]$, we have by integration by parts

$$\frac{\Delta V(t)}{\Delta t}\equiv\frac{V(t+\Delta t)-V(t)}{\Delta t}=-\int_{-\infty}^{\infty}\left\{\left(e^{cz}\frac{w_z(z,t)+w_z(z,t+\Delta t)}{2}\right)_z\frac{\Delta w}{\Delta t}+\frac{\Delta F(w)}{\Delta t}\right\}dz.$$

Passing to the limit as $\Delta t\to 0$ and using the uniform (in t) convergence of the integral, we see that $\dot V(t)=dV/dt$ exists and

$$\dot V(t)=-\int_{-\infty}^{\infty} e^{cz}\left(w_{zz}+cw_z+f(w)\right)w_t\,dz.$$

Letting $Q[w]\equiv\int_{-\infty}^{\infty} e^{cz}[w_{zz}+cw_z+f(w)]^2\,dz$, we calculate

$$\dot V(t)+Q[w](t)=-\int e^{cz}[w_{zz}+cw_z+f(w)]\,N[w]\,dz,$$

where N is given by (4.2a). Since $N[w]=0$ for $|z|\leqq\varepsilon t$ and w satisfies (4.5),

$$\begin{aligned} |\dot V(t)+Q[w](t)| &\leqq C_1\int_{\varepsilon t}^{\varepsilon t+1} e^{cz}\left(e^{(-\frac{1}{2}c-\sigma)z}+e^{-\mu t}\right)^2dz\\ &\leqq C_2\left(e^{-2\sigma\varepsilon t}+e^{(\varepsilon|c|-2\mu)t}\right). \end{aligned}$$

Again, since $\varepsilon|c|-2\mu<0$, we obtain

$$\lim_{t\to\infty}|\dot V(t)+Q[w](t)|=0. \tag{4.6}$$

Since $Q[w] \geqq 0$, it follows in particular that $\limsup_{t\to\infty} \dot{V}(t) \leqq 0$. We thus deduce the existence of a sequence $\{t_n\}$ with $t_n \to \infty$ such that

(4.7) $$\lim_{n\to\infty} \dot{V}(t_n) = 0;$$

(for otherwise $\limsup_{t\to\infty} \dot{V}(t) < 0$, implying that $V(t) \to -\infty$, whereas we know from above that $V(t)$ is bounded). Combining (4.6) and (4.7), we obtain

(4.8) $$\lim_{n\to\infty} Q[w](t_n) = 0.$$

By Lemma 4.4, there is a subsequence of $\{t_n\}$, call it $\{t'_n\}$, along which $v(\cdot, t'_n)$, and hence $w(\cdot, t'_n)$, converges in the norm of $C^2(-\infty, \infty)$ to a limit function $\tilde{v}(z)$. From this and (4.8), we obtain, for any finite interval I,

$$\int_I e^{cz}(w_{zz} + c w_z + f(w))^2|_{t=t'_n} dz \to \int_I e^{cz}(\tilde{v}_{zz} + c\tilde{v}_z + f(\tilde{v}))^2 dz = 0;$$

thus

$$\tilde{v}_{zz} + c\tilde{v}_z + f(\tilde{v}) \equiv 0.$$

We also have $\tilde{v}(-\infty) = 0$, $\tilde{v}(\infty) = 1$. Hence by the uniqueness of travelling fronts (Corollary 2.3) we have $\tilde{v}(z) = U(z - z_0)$ for some z_0. This establishes that $v(z, t'_n)$ approaches $U(z - z_0)$ in the sense of C^2 as $n \to \infty$.

To finish the proof of Lemma 4.5, we now merely apply Lemma 4.2, which indicates that once v is close to $U(z - z_0)$ for some t'_n, it remains close for all later time.

5. Proof of Theorem 3.1 (Conclusion)

Lemma 4.5 asserts the convergence of v to a travelling front; we now show that the rate is exponential. This conclusion can be obtained by appealing directly to a theorem of SATTINGER [22], the conditions of which are satisfied by virtue of Lemma 4.5. We give, however, an alternative proof which is in some ways simpler than Sattinger's, though more limited in scope.

Recalling the definition of $w(z, t)$ in the proof of Lemma 4.5, we set

$$h(z, t) \equiv w(z, t) - U(z - z_0 - \alpha(t)),$$

where z_0 is the constant in that lemma, and $\alpha(t)$ is chosen so that for large t, h is orthogonal to $e^{cz} U'$, that is,

(5.1) $$\int_{-\infty}^{\infty} e^{cz} h(z,t)\, U'(z - z_0 - \alpha(t)) dz = 0.$$

The existence of such an α, with $\alpha(\infty) = 0$, follows from the implicit function theorem. In fact, by Lemma 4.5 and estimates (4.5) (which also hold for w, U, and h), the left side of (5.1) vanishes at $\alpha = 0$, $t = \infty$. Furthermore its derivative with respect to α is

$$\int_{-\infty}^{\infty} e^{cz}(U'(z-z_0-\alpha))^2 dz - \int_{-\infty}^{\infty} e^{cz} h(z,t)\, U''(z-z_0-\alpha)\, dz,$$

which is nonzero at $\alpha=0$, $t=\infty$ because the right-hand integral then vanishes. The implicit function theorem also shows that α is continuously differentiable.

Theorem 3.1 will be proved by showing

(*i*) $|h(z,t)|<Ce^{-\nu t}$, and

(*ii*) $|\alpha(t)|<Ce^{-\nu t}$.

This will imply that w converges exponentially to $U(z-z_0)$. But we know from (4.5) and the definition of w that

$$|v(z,t)-w(z,t)|<Ce^{-\nu t}$$

for some (possibly different) positive ν. We shall thus obtain that v converges exponentially to $U(z-z_0)$, as desired.

To establish (i), we work with a diffusion equation for h. First we see from the definition of w that $w=v$ for $|z|<\varepsilon t$, and that w and its derivatives satisfy (4.5). We therefore have

$$w_t = w_{zz} + c w_z + f(w) + O(r),$$

where

$$r(z,t)=\begin{cases} 0, & |z|<\varepsilon t,\\ e^{-\frac{1}{2}cz-\sigma\varepsilon t}+e^{-\mu t}, & \varepsilon t\leqq |z|\leqq \varepsilon t+1,\\ 0, & |z|>\varepsilon t+1.\end{cases}$$

Therefore

$$\begin{aligned} h_t = w_t+\alpha' U' &= w_{zz}+c w_z+f(w)-U''-cU'-f(U)+\alpha' U'+O(r)\\ &= h_{zz}+c h_z+f'(U)h+\alpha' U'+O(h^2)+O(r).\end{aligned}$$

Setting $h=e^{-\frac{1}{2}cz}y$ yields

$$y_t = y_{zz} - \{\tfrac{1}{4}c^2-f'(U)\}\, y+\alpha' e^{\frac{1}{2}cz} U' + O(hy)+O(e^{\frac{1}{2}cz} r). \tag{5.2}$$

The linear operator L given by

$$Ly \equiv -y_{zz}+\{\tfrac{1}{4}c^2-f'(U)\}\, y,$$

with appropriate domain in $\mathscr{L}^2(-\infty,\infty)$, is self-adjoint with a continuous spectrum to the right of $\operatorname{Min}\{\frac{1}{4}c^2-f'(0), \frac{1}{4}c^2-f'(1)\}$, which is strictly positive, and a discrete spectrum to the left. Furthermore, we know by differentiating (2.1) that the eigenvalue 0 lies in this discrete spectrum with eigenfunction $e^{\frac{1}{2}cz}U'$. Since this eigenfunction is of constant sign, 0 must be simple and the least eigenvalue, with all other eigenvalues strictly positive. Let $\|\cdot\|$ denote the norm in $\mathscr{L}^2(-\infty,\infty)$. Clearly $e^{\frac{1}{2}cz}h=y$ lies in this space. Multiplying (5.2) by y and integrating over $(-\infty,\infty)$, we obtain, by virtue of (5.1),

$$\frac{1}{2}\frac{d}{dt}\|y\|^2 = (-Ly,y)+O(\|h^{\frac{1}{2}}y\|^2)+O(\|e^{\frac{1}{2}cz}r\|\,\|y\|).$$

Since y is orthogonal to the eigenfunction $e^{\frac{1}{2}cz}U'$ corresponding to the zero eigenvalue of L, the right side will in turn be

$$\leqq -M\|y\|^2 + C(\sup_z |h(z,t| \, \|y\|^2 + e^{-\sigma\varepsilon t}\|y\| + e^{(\frac{1}{2}|c|\varepsilon-\mu)t}\|y\|),$$

where M is a positive constant independent of t. Since $h \to 0$ uniformly as $t \to \infty$, and since $2\mu > |c|\varepsilon$, we have, finally,

$$\frac{1}{2}\frac{d}{dt}\|y\|^2 \leqq -\frac{M}{2}\|y\|^2 + O(e^{-Kt})$$

for large enough t and some $K > 0$. Integration of this inequality shows that

$$\|y\| \leqq Ce^{-\nu t} \tag{5.3}$$

for some $\nu > 0$.

At this point we need an interpolation lemma. Though somewhat standard, its proof will be given later for completeness.

Lemma 5.1. *Let $f \in C^1(\mathbb{R})$, and put $f_0 = \|f\|_{C^0}$, $f_1 = \|f\|_{C^1}$. Then*

$$f_0^3 \leqq \tfrac{3}{2} f_1 \int_{-\infty}^{\infty} f^2 \, dx.$$

We apply this to the function $y(\cdot, t)$. Since $\|y(\cdot, t)\|_{C^1}$ is bounded independently of t, estimate (5.3) and the above lemma imply

$$\|y(\cdot, t)\|_{C^0} = O(e^{-\nu t}). \tag{5.4}$$

For each $\delta > 0$, we see from (5.4) and the definition of y that

$$|h(z,t)| < Ce^{(\frac{1}{2}|c|\delta - \nu)t}$$

for $|z| < \delta t$. Let δ be such that $\frac{1}{2}|c|\delta - \nu < 0$. For $|z| > \delta t$, however, (4.5) yields

$$|h(z,t)| < Ce^{-\nu t}, \qquad \nu > 0. \tag{5.5}$$

Therefore (5.5) holds, in fact, for all z and all $t > 0$.

The proof of Theorem 3.1 will be complete if we only can show that

$$|\alpha(t)| = O(e^{-\nu t}).$$

For this purpose we multiply (5.2) by $e^{\frac{1}{2}cz}U'$ and integrate over $(-\infty, \infty)$ (the integrals converging because of the asymptotic behavior of U'). Thus

$$\begin{aligned}(e^{\frac{1}{2}cz}U', y_t) = &-(e^{\frac{1}{2}cz}U', Ly) + \alpha'(e^{cz}U', U') \\ &+ O((U', y^2)) + O(\|e^{\frac{1}{2}cz}r\| \, \|e^{\frac{1}{2}cz}U'\|).\end{aligned} \tag{5.6}$$

Differentiating the relation

$$(c^{\frac{1}{2}cz}U', y) = 0$$

we get

$$(e^{\frac{1}{2}cz}U', y_t) = \alpha'(e^{\frac{1}{2}cz}U'', y),$$

and the scalar product on the right is seen to decay exponentially by use of the Cauchy-Schwarz inequality and (5.3). Also

$$(e^{\frac{1}{2}cz}U', Ly) = \big(L(e^{\frac{1}{2}cz}U'), y\big) = 0,$$

and the remainder terms in (5.6) also decay exponentially. We can therefore conclude from (5.6) that

$$\alpha' = O(e^{-\nu t}),$$

and so $\alpha = O(e^{-\nu t})$. This completes the proof of Theorem 3.1.

Proof of Lemma 5.1. Given $\delta > 0$, let x_0 be such that $|f(x_0)| \geqq f_0 - \delta$. There is no loss of generality in supposing $f(x_0) > 0$, so that $f(x_0) \geqq f_0 - \delta$. Then

$$f(x) = f(x_0) + \int_{x_0}^{x} f'(\bar{x})\, d\bar{x} \geqq f_0 - \delta - |x - x_0| f_1,$$

for $|x - x_0| \leqq (f_0 - \delta)/f_1 \equiv l$. Thus

$$\int_{-\infty}^{\infty} f^2 dx \geqq \int_{x_0 - l}^{x_0 + l} f^2 dx \geqq \int_{x_0 - l}^{x_0 + l} (f_0 - \delta - |x - x_0| f_1)^2 dx = \tfrac{2}{3}(f_0 - \delta)^3 / f_1 .$$

Letting $\delta \to 0$ yields the assertion of the lemma.

6. Proofs of Theorems 3.2 and 3.3

The following is the basic lemma we shall need for Theorem 3.2.

Lemma 6.1. *Under the hypotheses of Theorem* 3.2 *there exist constants* z_1, z_2, q_0 *and* μ *(the last two positive) such that*

$$\begin{aligned} &U(x - ct - z_1) + U(-x - ct - z_1) - 1 - q_0 e^{-\mu t} \\ &\qquad \leqq u(x,t) \leqq U(x - ct - z_2) + U(-x - ct - z_2) - 1 + q_0 e^{-\mu t}. \end{aligned} \tag{6.1}$$

Proof. First note that (3.3) implies $c < 0$. The right-hand inequality of (6.1) then follows from the proof of Lemma 4.1. More precisely, that proof shows that

$$u(x,t) \leqq U(x - ct - z_2) + q_1 e^{-\mu_0 t}$$

for some z_2, q_1 and μ_0. The same argument applied to $u(-x, t)$ reveals as well that

$$u(x,t) \leqq U(-x - ct - z_2') + q_1' e^{-\mu_0 t}.$$

Since decreasing z_2 and z_2' and increasing q_1 and q_1' strengthens the inequality, we may assume $z_2 = z_2' < 0$, $q_1 = q_1'$. Hence

$$u(x,t) \leqq \operatorname{Min}\,[U(x - ct - z_2), U(-x - ct - z_2)] + q_1 e^{-\mu_0 t}. \tag{6.2}$$

If $x > 0$, then the monotonicity of U and its exponential rate of convergence to its limits at $\pm\infty$ imply

$$1 - U(x - ct - z_2) \leqq 1 - U(-ct - z_2) \leqq K e^{-\nu |ct + z_2|}$$

for some positive constants ν and K. Furthermore

$$U(x-ct-z_2) \geqq U(-x-ct-z_2) \quad \text{for } x>0.$$

Hence from (6.2),

$$\begin{aligned} u(x,t) &\leqq U(-x-ct-z_2)+q_1 e^{-\mu_0 t} \\ &\leqq U(-x-ct-z_2)+U(x-ct-z_2)-1+q_1 e^{-\mu_0 t}+K e^{-\nu|ct+z_2|} \\ &\leqq U(-x-ct-z_2)+U(x-ct-z_2)-1+q_0 e^{-\mu_0 t}, \end{aligned}$$

if we choose $q_0 > q_1$ and further require μ_0 to be small enough and $(-z_2)$ large enough. A similar argument may be used for the range $x<0$.

We next prove the left-hand inequality of (6.1). Let

$$\underline{u}(x,t)=U_+(x,t)+U_-(x,t)-1-q(t),$$

where $\zeta_+ = x-ct-\xi(t)$, $\zeta_- = -x-ct-\xi(t)$, $U_\pm(x,t)=U(\zeta_\pm)$, for some $q(t)>0$ and $\xi(t)<0$ (with $\xi'(t)>0$) to be determined. Then

$$\begin{aligned} \mathrm{N}\underline{u} &\equiv \underline{u}_t-\underline{u}_{xx}-f(\underline{u}) \\ &= -\xi'(t)\big(U'(\zeta_+)+U'(\zeta_-)\big)-\big(U''(\zeta_+)+U''(\zeta_-)\big) \\ &\quad -c\big(U'(\zeta_+)+U'(\zeta_-)\big)-q'(t)-f(U_+ + U_- -1-q). \end{aligned}$$

Since $U''+cU'+f(U)=0$ we have

$$\begin{aligned} N\underline{u} = &-\xi'(t)\big(U'(\xi_+)+U'(\zeta_-)\big)+f(U_+)+f(U_-) \\ &-f(U_+ + U_- -1-q)-q'(t). \end{aligned} \tag{6.3}$$

Let q_0' and q_2 be such that

$$\alpha_1 < 1-q_2 < 1-q_0' < \alpha_1+\eta,$$

and let δ be as in the proof of Lemma 4.1. As in that proof, we then see that for some $\mu_1>0$,

$$f(U_-)-f\big(U_- -(1-U_+ +q)\big) \leqq -\mu_1(1-U_+ +q)$$

for $1-\delta \leqq U_- \leqq 1$, $0 \leqq 1-U_+ +q \leqq q_2$. The latter inequality will hold if $0 \leqq q \leqq q_0'$, $x>0$, and $(-\xi)$ is sufficiently large, for then

$$1-U_+ +q \leqq 1-U(-\xi)+q_0' \leqq q_0'+Ke^{-\nu|\xi|} \leqq q_2.$$

We finally note that $U'(\zeta_\pm)>0$ and $f(U_+) \leqq b(1-U_+)$ for some $b>0$. Therefore we see from (6.3) that for $1-\delta \leqq U_- \leqq 1$, $0 \leqq q \leqq q_0'$, $x \geqq 0$, $(-\xi)$ sufficiently large, and $\xi'>0$,

$$\begin{aligned} N\underline{u} &\leqq -\mu_1(1-U_+ +q)+b(1-U_+)-q' = (b-\mu_1)(1-U_+)-\mu_1 q-q' \\ &\leqq bKe^{-\nu|\xi+ct|}-\mu_1 q-q'. \end{aligned}$$

Setting $q=q_0' e^{-\mu_2 t}$ for $0<\mu_2<\mu_1$, we obtain for the above range,

$$N\underline{u} \leqq bKe^{-\nu|\xi+ct|}-(\mu_1-\mu_2)q_0' e^{-\mu_2 t} \leqq 0,$$

provided $\mu_2 < \nu c$ and $(-\xi)$ is sufficiently large.

A similar argument holds for $0 \leqq U_- \leqq \delta$, $0 \leqq q \leqq q_0'$, $x \geqq 0$, provided that $\underline{u} \geqq 0$. Finally for $\delta \leqq U_- \leqq 1-\delta$, $x \geqq 0$, we have

$$U_+' + U_-' \geqq \beta > 0,$$
$$f(U_-) - f(U_+ + U_- - 1 - q) \leqq C(1 - U_+ + q),$$
$$f(U_+) \leqq b(1 - U_+) \leqq b K e^{-\nu|\xi + ct|},$$

so that from (6.3),

$$N\underline{u} \leqq -\beta \xi'(t) + (C+b) K e^{-\nu|\xi+ct|} + (C+\mu_2) q_0' e^{-\mu_2 t}.$$

We now choose $\xi(t)$ so that

$$-\beta \xi'(t) + (C+b) K e^{-\nu|c|t} + (C+\mu_2) q_0' e^{-\mu_2 t} = 0,$$

with $\xi(0) = \xi_0$ sufficiently large and negative. Then from the above we obtain $N\underline{u} \leqq 0$ for all (x,t) with $x \geqq 0$, $\underline{u}(x,t) > 0$. A similar argument shows that $N\underline{u} \leqq 0$ for $x \leqq 0$ as well.

Now $\mathrm{Max}[0, \underline{u}(x,t)]$ will be a subsolution if we can show that $\varphi(x) \geqq \underline{u}(x,0)$. But

$$\underline{u}(x,0) = U(x - \xi_0) + U(-x - \xi_0) - 1 - q_0' < 1 - q_0' < \alpha_1 + \eta \leqq \varphi(x)$$

for $|x| \leqq L$, and

$$\underline{u}(x,0) \leqq 0 \leqq \varphi(x)$$

for $|x| \geqq M$, for some M depending on ξ_0. Therefore if $L \geqq M$, we shall have $\underline{u}(x,0) \leqq \varphi(x)$ for all x.

With this condition on L, it now follows that

$$u(x,t) \geqq \underline{u}(x,t) \geqq U\big(x - ct - \xi(\infty)\big) + U\big(-x - ct - \xi(\infty)\big) - 1 - q_0' e^{-\mu_2 t}.$$

Now set $z_1 = \xi(\infty)$ and $\mu = \mathrm{Min}[\mu_2, \mu_0]$; this completes the proof.

Lemma 6.2. *Let f and φ satisfy the hypotheses of Theorem 3.2. There exist functions $\omega(\varepsilon)$ and $T(\varepsilon)$, defined for small positive ε and satisfying $\lim_{\varepsilon \downarrow 0} \omega(\varepsilon) = 0$, such that if*

$$|u(x, t_0) - U(x - ct_0 - x_0)| < \varepsilon \tag{6.4}$$

for some x_0, some $t_0 > T(\varepsilon)$, and all $x < 0$, then

$$|u(x,t) - U(x - ct - x_0)| < \omega(\varepsilon)$$

for all $t > t_0$, $x < 0$.

Proof. Consider the subsolution $\underline{v}(z,t)$ used in the proof of Lemma 4.1. We express it in the original coordinates as

$$\underline{u}(x,t) = \underline{v}(x - ct, t) = U\big(x - ct - \xi(t)\big) - q_0 e^{-\mu t}, \tag{6.5}$$

where $\xi = \xi_1 + \xi_2 e^{-\mu t}$. It was shown that if μ is sufficiently small (positive) and

$\xi_2 = A_\mu q_0$ for a certain constant A_μ depending only on μ (see (4.4)), then for arbitrary ξ_1 and q_0,

$$N\underline{u} \equiv \underline{u}_t - \underline{u}_{xx} - f(\underline{u}) \leqq 0.$$

We shall now use $\underline{u}$ (with appropriate ξ_1, q_0 and μ) as a comparison function in the region $x<0$, $t>t_0$. If we can show that $\underline{u} \leqq u$ on the boundary $\{x=0\} \cup \{t=t_0\}$, then it will follow that $\underline{u}(x,t) \leqq u(x,t)$ in the quarter-plane under consideration.

First, consider the portion $\{t=t_0\}$ of the boundary. From (6.4) we have

$$u(x,t_0) \geqq U(x-ct_0-x_0)-\varepsilon.$$

If we now set $q_0 = \varepsilon e^{\mu t_0}$, $\xi_2 = \varepsilon A_\mu e^{\mu t_0}$, and $\xi_1 = x_0 - \varepsilon A_\mu$, then

$$\underline{u}(x,t_0) = U(x-ct_0-x_0) - \varepsilon \leqq u(x,t_0).$$

Next, consider the portion $\{x=0\}$. From (6.1) and the exponential approach of $U(z)$ to its limits, we have, for some ν, M_1,

$$\begin{aligned} u(0,t) &\geqq 2U(-ct-z_1) - 1 - q_0' e^{-\mu' t} = 1 - q_0' e^{-\mu' t} - 2\big(1 - U(-ct-z_1)\big) \\ &\geqq 1 - q_0' e^{-\mu' t} - M_1 e^{-\nu|c|t}, \end{aligned}$$

the primes added to distinguish these constants from the constants q_0 and μ used in (6.5). On the other hand, for $t \geqq t_0$,

$$\underline{u}(0,t) = U\big(-ct-\xi(t)\big) - q_0 e^{-\mu t} < 1 - q_0 e^{-\mu t} = 1 - \varepsilon e^{-\mu(t-t_0)}.$$

Thus

$$u(0,t) - \underline{u}(0,t) \geqq \varepsilon e^{-\mu(t-t_0)} - M_1 e^{-\nu|c|t} - q_0' e^{-\mu' t}. \tag{6.6}$$

The constant μ can be taken as small as desired. We choose it so that $0<\mu<\mu'$, $\mu<\nu|c|$. Then from (6.6),

$$\begin{aligned} u(0,t) - \underline{u}(0,t) &\geqq \varepsilon e^{-\mu(t-t_0)} - (M_1 + q_0') e^{-\mu t} \\ &= \big(\varepsilon - (M_1 + q_0') e^{-\mu t_0}\big) e^{-\mu(t-t_0)} > 0 \end{aligned}$$

for sufficiently large t_0 (depending on ε).

This completes the comparison argument. We conclude that

$$\begin{aligned} u(x,t) \geqq \underline{u}(x,t) &= U\big(x-ct-\xi(t)\big) - \varepsilon e^{-\mu(t-t_0)} \\ &\geqq U(x-ct-x_0) - \omega(\varepsilon) \end{aligned}$$

for $t \geqq t_0$, $x<0$.

A similar argument can be used to show that $u(x,t) \leqq U(x-ct-x_0) + \omega(\varepsilon)$. This completes the proof of the lemma.

Proof of Theorem 3.2. We define the "left truncation"

$$u_l(x,t) = \begin{cases} u(x,t), & x<0, \\ 1 - \zeta(x)\big(1-u(x,t)\big), & x \geqq 0, \end{cases}$$

where $\zeta(x) \in C^\infty(-\infty,\infty)$, $\zeta(x) \equiv 1$ for $x \leqq 0$, $\zeta(x) \equiv 0$ for $x \geqq 1$, and

$$v_l(z,t) = u_l(x,t) = u_l(z+ct,t).$$

Then with the aid of Lemma 6.1 and essentially the same proof as in Lemma 4.3, we conclude that v_l satisfies (4.5). Hence (as in Lemma 4.4) the set $\{v_l(\cdot, t), t \geqq \delta\}$ is relatively compact in $C^2(-\infty, \infty)$.

Exactly as in Lemma 4.5, we next establish that

$$\lim_{t\to\infty} |v_l(z,t) - U(z-x_0)| = 0$$

for some x_0, uniformly in z. It is now trivial to extend the proof in § 5 to show that

$$|v_l(z,t) - U(z-x_0)| \leqq K e^{-\omega t},$$

which proves (3.5a). The symmetrical argument establishes (3.5b), completing the proof of Theorem 3.2.

The following lemmas lead to the proof of Theorem 3.3.

Lemma 6.3. *Under the hypotheses of Theorem* 3.3, *the following inequality holds for some numbers* a_1, a_2, q_0 *and* μ *(the last two positive):*

$$U_1(x-c_1t-a_1) - q_0 e^{-\mu t} \leqq u(x,t) \leqq U_2(x-c_2t-a_2) + q_0 e^{-\mu t}. \tag{6.7}$$

Proof. The left-hand inequality follows at once from the left-hand inequality of Lemma 4.1 applied to the u-interval (u_1, u_2). The right-hand inequality is proved similarly.

For simplicity, we assume from now on that $c_1 < 0 < c_2$. If this is not the case, we may use a moving coordinate frame to reduce the problem to one for which it is so.

As in the proof of Theorem 3.2 above, we define the left truncation

$$u_l(x,t) = \begin{cases} u(x,t), & x<0, \\ u_2 - \zeta(x)(u_2 - u(x,t)), & x \geqq 0, \end{cases}$$

and $v_l(z,t) = u_l(x,t) = u_l(z + c_1 t, t)$, where $z = x - c_1 t$.

Lemma 6.4. *For some numbers* a_1, a_3, t_0, q_0 *and* μ *(the last three positive),*

$$U_1(z-a_1) - q_0 e^{-\mu t} \leqq v_l(z,t) \leqq U_1(z-a_3) + q_0 e^{-\mu t} \tag{6.8}$$

for $t \geqq t_0$.

Proof. The left inequality follows directly from Lemma 6.3, and so we need consider only the right one. Let η be such that $\limsup_{x\to-\infty} \varphi(x) < \eta < \alpha_1$. For some constants X_0, γ, and k, to be determined below, let

$$V(x) = \begin{cases} \eta, & x \leqq X_0, \\ \eta + \gamma(x-X_0)^2, & x \geqq X_0, \end{cases}$$

and $\bar{u}(x,t) = \mathrm{Min}[u_3, V(x+kt)]$. First, it is clear that $V \geqq \varphi$ for large enough negative X_0.

For $V = \eta$, we have $\bar{u} = \eta$ and $N\bar{u} = -f(\eta) > 0$, since α_1 is the first zero of f greater than u_1.

For $\eta < V < u_3$, we have

$$N\bar{u} = kV' - V'' - f(V) = 2k\gamma\zeta - 2\gamma - f(V),$$

where $\zeta = x + kt - X_0$. But

$$f(V) \leqq f(\eta) + m(V-\eta)\,(\text{for } V \geqq \eta, \text{ some } m>0) = f(\eta) + m\gamma\zeta^2,$$

so that

$$N\bar{u} \geqq -f(\eta) - 2\gamma + 2k\gamma\zeta - m\gamma\zeta^2.$$

We first choose γ so small that $-f(\eta) - 2\gamma > 0$, then k so large that $2k\gamma\zeta - m\gamma\zeta^2 \geqq 0$ for ζ such that V is in the indicated range.

This shows that $\bar{u}$ is a supersolution, whence $u \leqq \bar{u}$. In particular, it follows that at each value of $t > 0$,

$$u(x,t) \leqq \eta < \alpha_1 \quad \text{for } (-x) \text{ large enough.} \tag{6.9}$$

Next since $U_2(z) \leqq u_2 + Ke^{-\omega_1|z|}$ for $z \leqq 0$, the right-hand inequality in (6.7) implies

$$u(x,t) \leqq u_2 + Ke^{-\omega_2 t} \tag{6.10}$$

for $x \leqq 1$.

We now consider the function

$$\bar{u}(x,t) = U_1\big(x - c_1 t + \xi(t)\big) + q_0 e^{-\mu_2 t}$$

in the domain $x \leqq 1$, $t \geqq t_0$. With appropriately chosen ξ, q_0, μ_2, and t_0, it will be a supersolution.

First of all, from the proof in Lemma 4.1, where a similar comparison function was used, we know that $N\bar{u} \geqq 0$ provided q_0 and μ_2 are sufficiently small and $\xi' = -\xi_1 e^{-\mu_2 t}$ for some appropriate ξ_1.

We shall show that $\bar{u}(x,t) \geqq u(x,t)$ for $t = t_0$ and/or $x = 1$. First, with t_0 to be specified later, we choose q_0 so that $q_0 e^{-\mu_2 t_0} = \eta$. Taking the constants K and ω_2 from (6.10), we note that

$$u_2 + Ke^{-\omega_2 t} \leqq U_1(1 - c_1 t) + \eta e^{\mu_2(t_0 - t)} \tag{6.11}$$

for sufficiently large $t_0, t \geqq t_0$, and sufficiently small μ_2, by virtue of the facts that $c_1 < 0$ and $U_1(z) \to u_2$ exponentially as $z \to \infty$. We choose t_0 and μ_2 so that (6.11) holds for $t \geqq t_0$, and also so that the last term in (6.10) satisfies

$$Ke^{-\omega_2 t_0} < \eta. \tag{6.12}$$

Next, we choose X so large that (from (6.9)) $u(x, t_0) \leqq \eta$ for $x \leqq -X$, and $\xi(t_0)$ so large that

$$\begin{aligned} &U_1\big(x - c_1 t_0 + \xi(t_0)\big) + q_0 e^{-\mu_2 t_0} \\ &\quad = U_1\big(x - c_1 t_0 + \xi(t_0)\big) + \eta \geqq u_2 + Ke^{-\omega_2 t_0}, \end{aligned} \tag{6.13}$$

for $x \geqq -X$. This is possible by virtue of (6.12) and the fact that $U_1(\infty) = u_2$.

For $t=t_0$, the relations (6.10) and (6.13) yield $u(x,t_0)\leqq\bar{u}(x,t_0)$. For $x=1$, (6.10), (6.11), and the fact that $\xi(t)>0$ imply $u(1,t)\leqq\bar{u}(1,t)$ for $t\geqq t_0$. By the maximum principle, we conclude that

$$u(x,t)\leqq\bar{u}(x,t)\leqq U_1(x-c_1t-a_3)+q_0e^{-\mu_2 t} \quad \text{for } x\leqq 1,\ t\geqq 0.$$

Since $u(x,t)=v_l(x-c_1t,t)$ for $x\leqq 0$, this establishes the right side of (6.8) for $z\leqq -c_1t=|c_1|t$. But for small μ and large t,

$$U_1(z-a_3)+q_0e^{-\mu t}>u_2\geqq v_l(z,t) \quad \text{for } z>|c_1|t.$$

Thus (6.8) is guaranteed by (if necessary) further reducing μ and increasing t_0. This completes the proof of the lemma.

Proof of Theorem 3.3. With inequality (6.8) at hand, it follows as in the proof of Theorem 3.1 that, for some x_1,

$$\lim_{t\to\infty}|v_l(z,t)-U_1(z-x_1)|=0,$$

uniformly in z. Moreover, using once again the argument in § 5, we find that

$$|v_l(z,t)-U_1(z-x_1)|\leqq Ke^{-\omega t},$$

and hence

$$|u(x,t)-U_1(x-c_1t-x_1)|\leqq Ke^{-\omega t} \tag{6.14}$$

for $x\leqq 0$. A similar argument using the right truncation yields

$$|u(x,t)-U_2(x-c_2t-x_2)|\leqq Ke^{-\omega t} \tag{6.15}$$

for $x\geqq 0$. Combining (6.14) and (6.15), we obtain (3.7), completing the proof.

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Department of Mathematics
University of Arizona
Tucson, Arizona
and
Wadham College
Oxford

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