Hopf bifurcations, and Some variations of diffusive logistic equation

JUNPING SHI 史峻平

College of William and Mary
Williamsburg, Virginia 23187

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Mississippi State University
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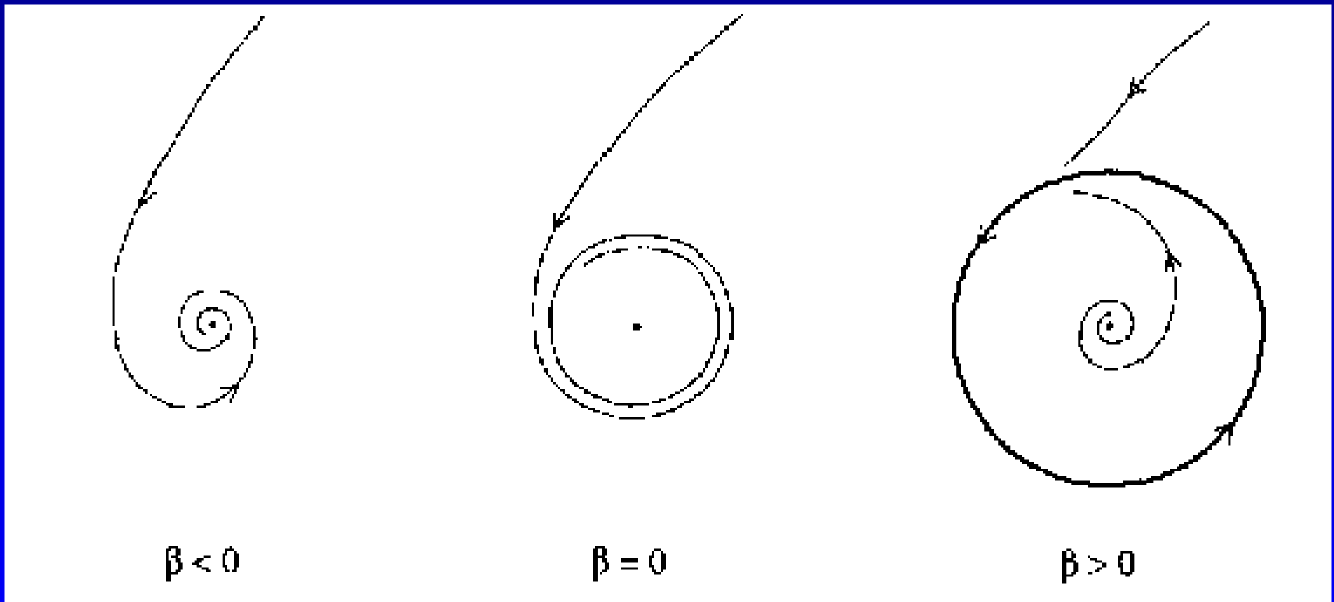
Hopf Bifurcation Theorem

Consider ODE $x' = f(\lambda, x)$, $\lambda \in \mathbf{R}$, $x \in \mathbf{R}^n$, and f is smooth.
(i) Suppose that for λ near λ_0 the system has a family of equilibria $x^0(\lambda)$.
(ii) Assume that its Jacobian matrix $A(\lambda) = f_x(\lambda, x^0(\lambda))$ has one pair of complex eigenvalues $\mu(\lambda) \pm i\omega(\lambda)$, $\mu(\lambda_0) = 0, \omega(\lambda_0) > 0$, and all other eigenvalues of $A(\lambda)$ have non-zero real parts for all λ near λ_0.

If $\mu'(\lambda_0) \neq 0$, then the system has a family of periodic solutions $(\lambda(s), x(s))$ for $s \in (0, \delta)$ with period $T(s)$, such that $\lambda(s) \to \lambda_0$, $T(s) \to 2\pi/\omega(\lambda_0)$, and $||x(s) - x^0(\lambda_0)|| \to 0$ as $s \to 0^+$.



Predator-prey system with functional response

$$\begin{cases} \dfrac{du}{dt} = u(a - bu) - c\phi(u)v, \\ \dfrac{dv}{dt} = -dv + f\phi(u)v. \end{cases}$$

$\phi(u)$: predator functional response

$\phi(u) = u$ (Lotka-Volterra)

$\phi(u) = \dfrac{u}{1 + mu}$ (Holling type II, m: the handling time of prey)

[Holling, 1959](Michaelis-Menton biochemical kinetics)

Biological work:

[Rosenzweig-MacArthur, *American Naturalist* 1963]

[Rosenzweig, *Science*, 1971] (Paradox of enrichment)

[May, *Science*, 1972] (Existence and uniqueness of limit cycle)

Basic analysis of the model

$$\frac{du}{dt} = u\,(1-u) - \frac{muv}{a+u}, \quad \frac{dv}{dt} = -dv + \frac{muv}{a+u}$$

Nullcline(isocline): $u = 0$, $v = \dfrac{(1-u)(a+u)}{m}$; $v = 0$, $d = \dfrac{mu}{a+u}$.

Solving $d = \dfrac{mu}{a+u}$, one have $u = \lambda \equiv \dfrac{ad}{m-d}$.

Equilibrium points: $(0,0)$, $(1,0)$, (λ, v_λ) where $v_\lambda = \dfrac{(1-\lambda)(a+\lambda)}{m}$

We take λ as a bifurcation parameter

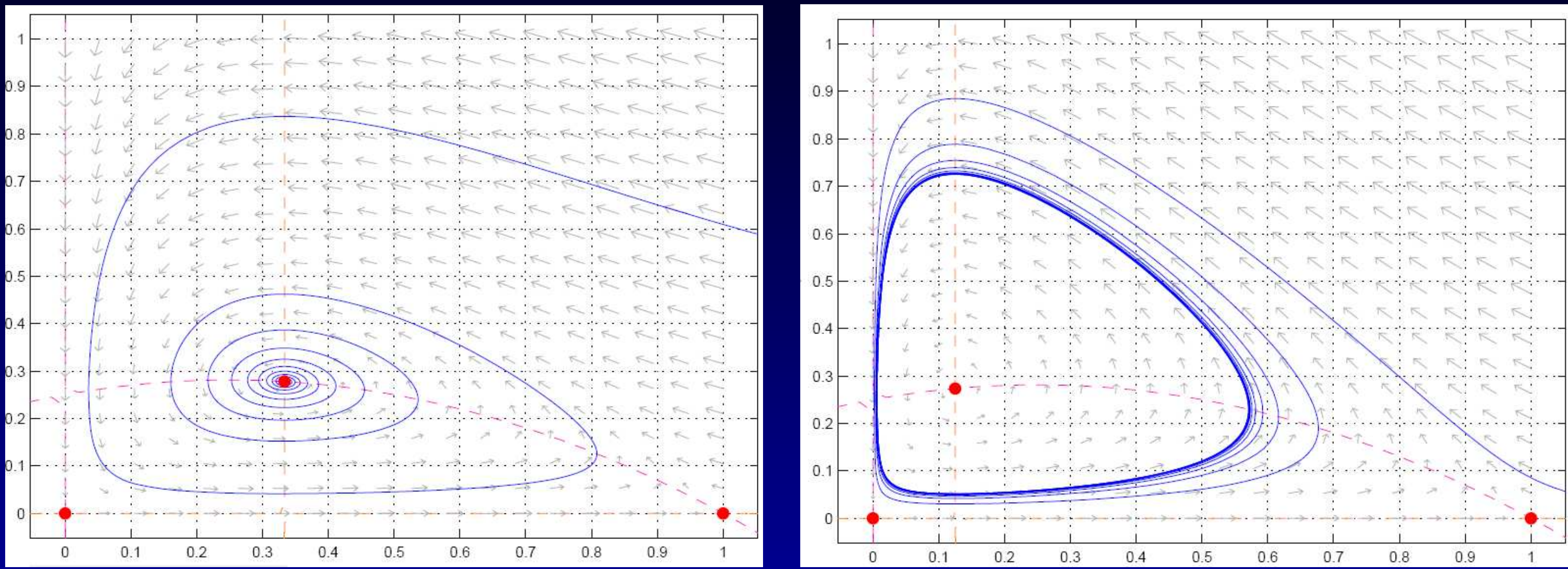
Case 1: $\lambda \geq 1$: $(1,0)$ is globally asymptotically stable

Case 2: $(1-a)/2 < \lambda < 1$: $(1,0)$ is a saddle, and (λ, v_λ) is a locally stable equilibrium

Case 3: $0 < \lambda < (1-a)/2$: $(1,0)$ is a saddle, and (λ, v_λ) is a locally unstable equilibrium

($\lambda = (1-a)/2$ is a Hopf bifurcation point)

Phase portrait



Left: $(1-a)/2 < \lambda < 1$: $(1,0)$ is a saddle, and (λ, v_λ) is a locally stable equilibrium

Right: $0 < \lambda < (1-a)/2$: $(1,0)$ is a saddle, and (λ, v_λ) is a locally unstable equilibrium; there exists a limit cycle

A subcritical Hopf bifurcation occurs.

Global stability

[Hsu, *Math. Biosci.*, 1978] (λ, v_λ) is globally asymptotically stable if $a \geq 1$, or $0 < a < 1$ and $1 - a \leq \lambda < 1$.

[Ardito et.al. *J. Math. Biol.*, 1995] (λ, v_λ) is asymptotically globally stable $(1-a)/2 < \lambda < 1 - a$.

[Cheng, *SIAM J. Math. Anal.*, 1981] If $0 < \lambda < (1-a)/2$, then (λ, v_λ) is unstable, and there is a unique periodic orbit which is globally asymptotically stable.

More on uniqueness of limit cycle:
[Zhang, 1986], [Kuang-Freedman, 1988], [Sugie et.al. 1997],
[Hsu et.al. 2001], [Xiao-Zhang, 2003]

Summary

$$\frac{du}{dt} = u(1-u) - \frac{muv}{a+u}, \quad \frac{dv}{dt} = -dv + \frac{muv}{a+u}$$

Nullcline(isocline): $u = 0$, $v = \dfrac{(1-u)(a+u)}{m}$; $v = 0$, $d = \dfrac{mu}{a+u}$.

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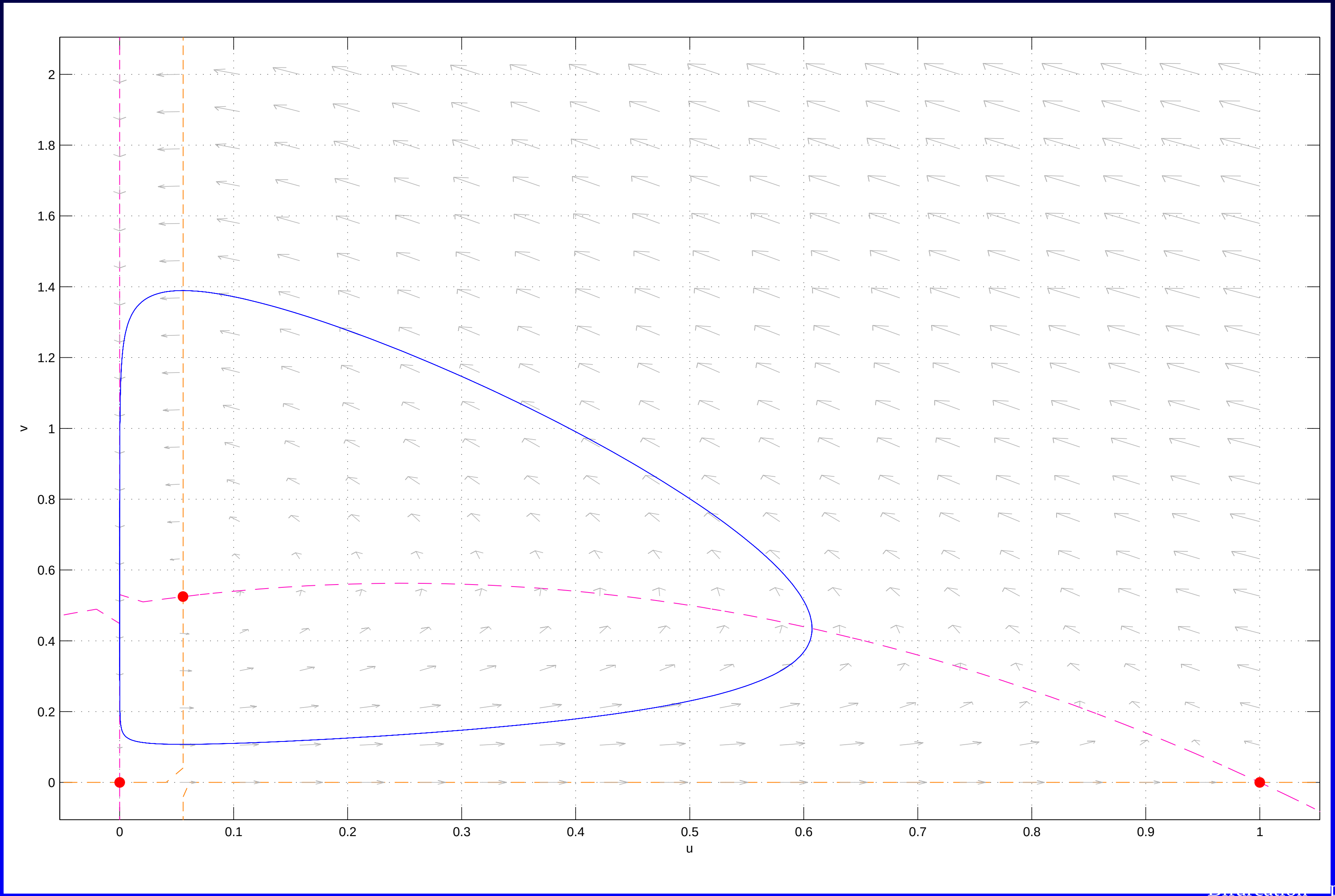
Case 2: $(1-a)/2 < \lambda < 1$: (λ, v_λ) is a globally asymptotically stable

Case 3: $0 < \lambda < (1-a)/2$: unique limit cycle is globally asymptotically stable

($\lambda = (1-a)/2$ is a Hopf bifurcation point)

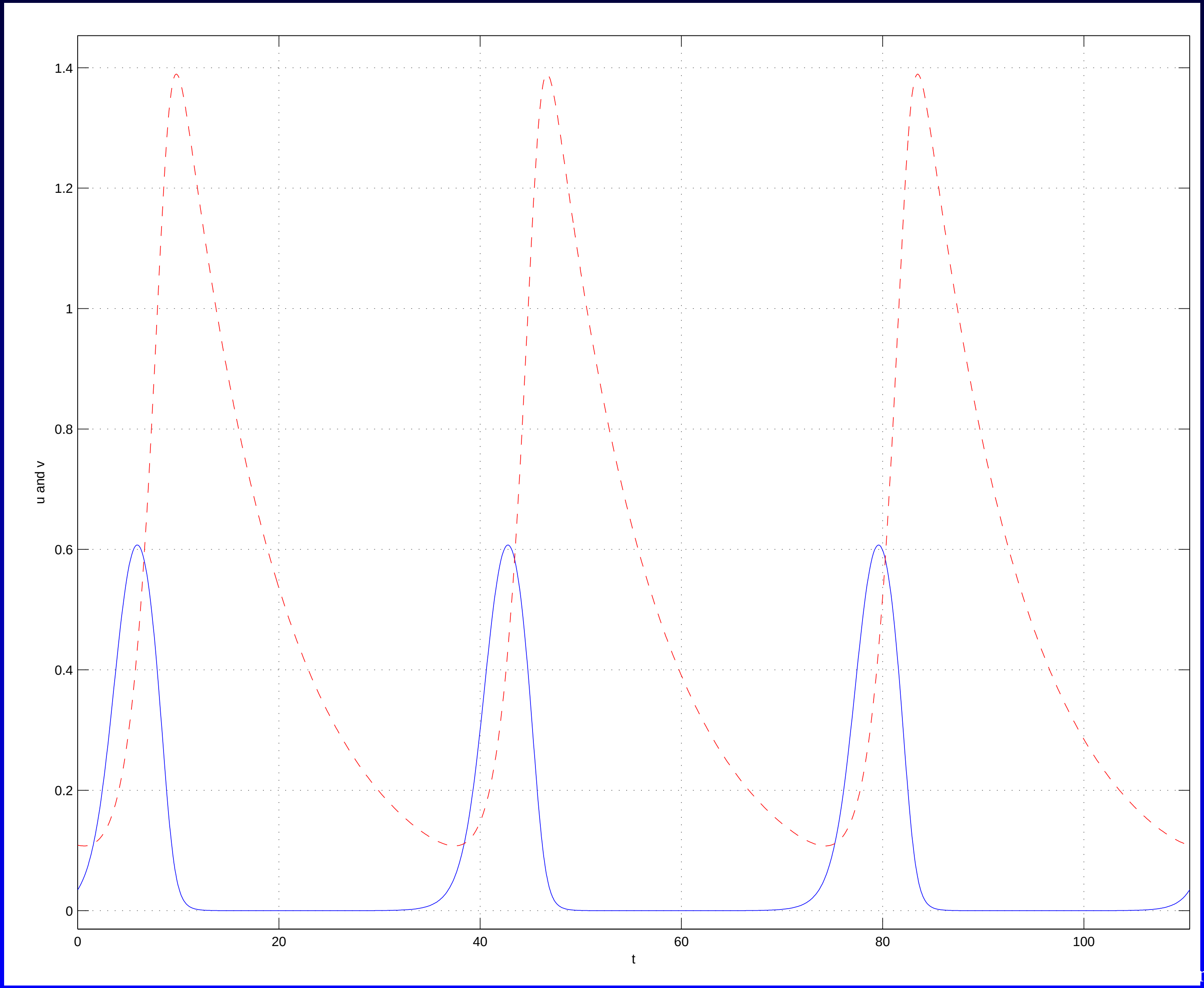
New result of this ODE

[Hsu-Shi, 2008] Relaxation oscillator profile of limit cycle in predator-prey system. Submitted. (Motivated by numerical observation)

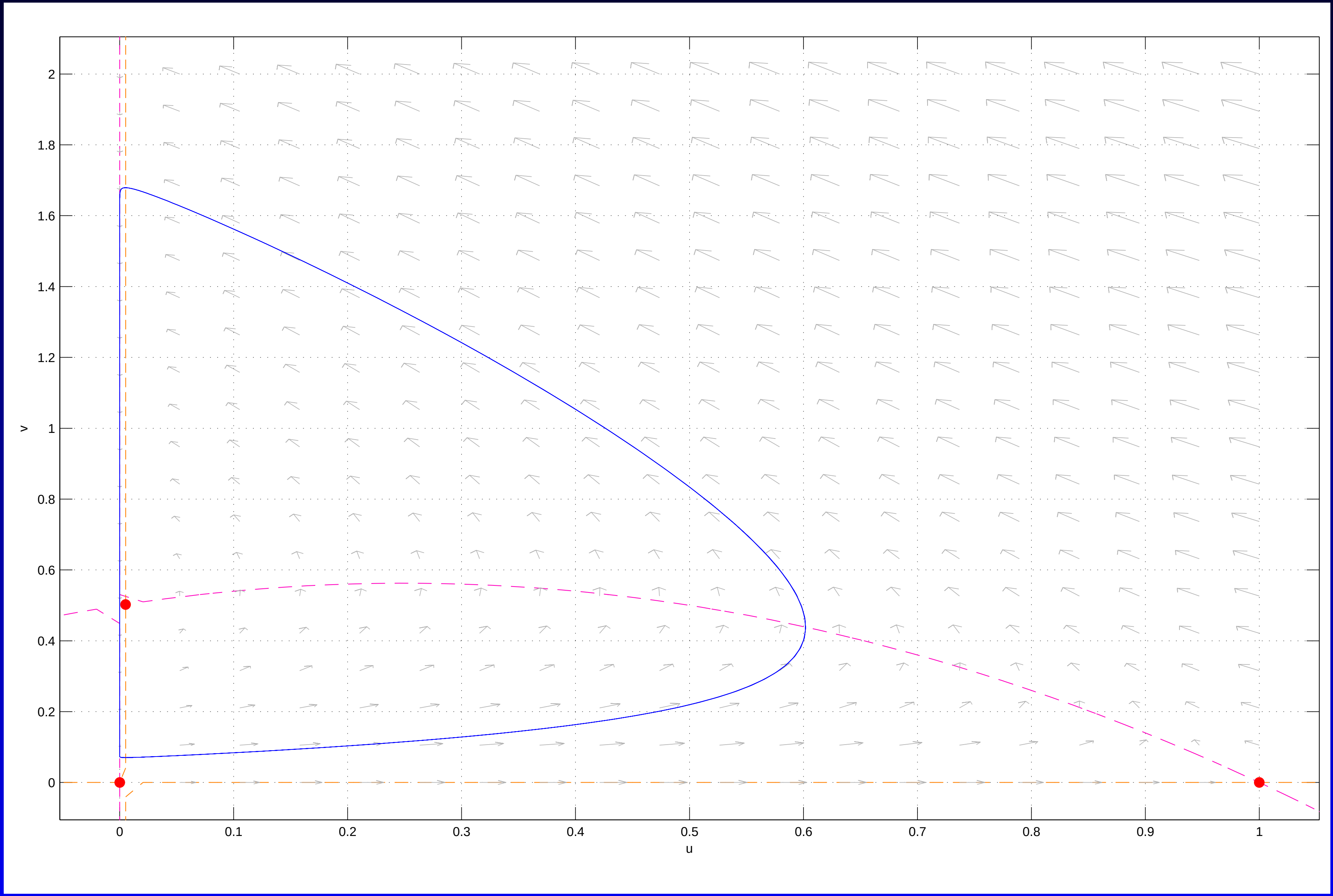


Graph of limit cycle

Parameters: $a = 0.5$, $m = 1$, $d = 0.1$, $\lambda = 1/18 \approx 0.056$, period $T \approx 37$.



Small d



Graph of limit cycle

Parameters: $a = 0.5$, $m = 1$, $d = 0.01$, $\lambda = 1/198 \approx 0.005$, period $T \approx 336$.

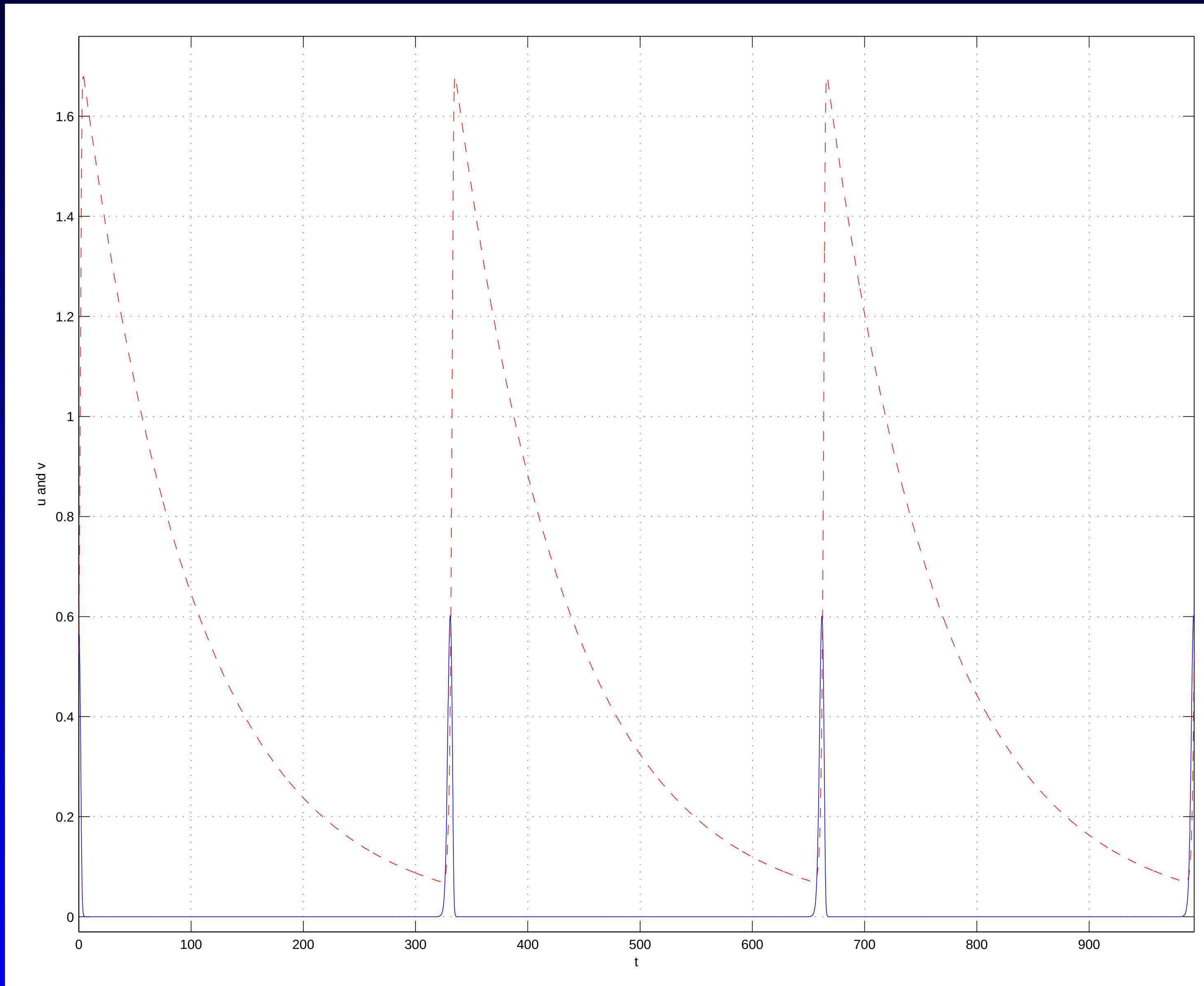
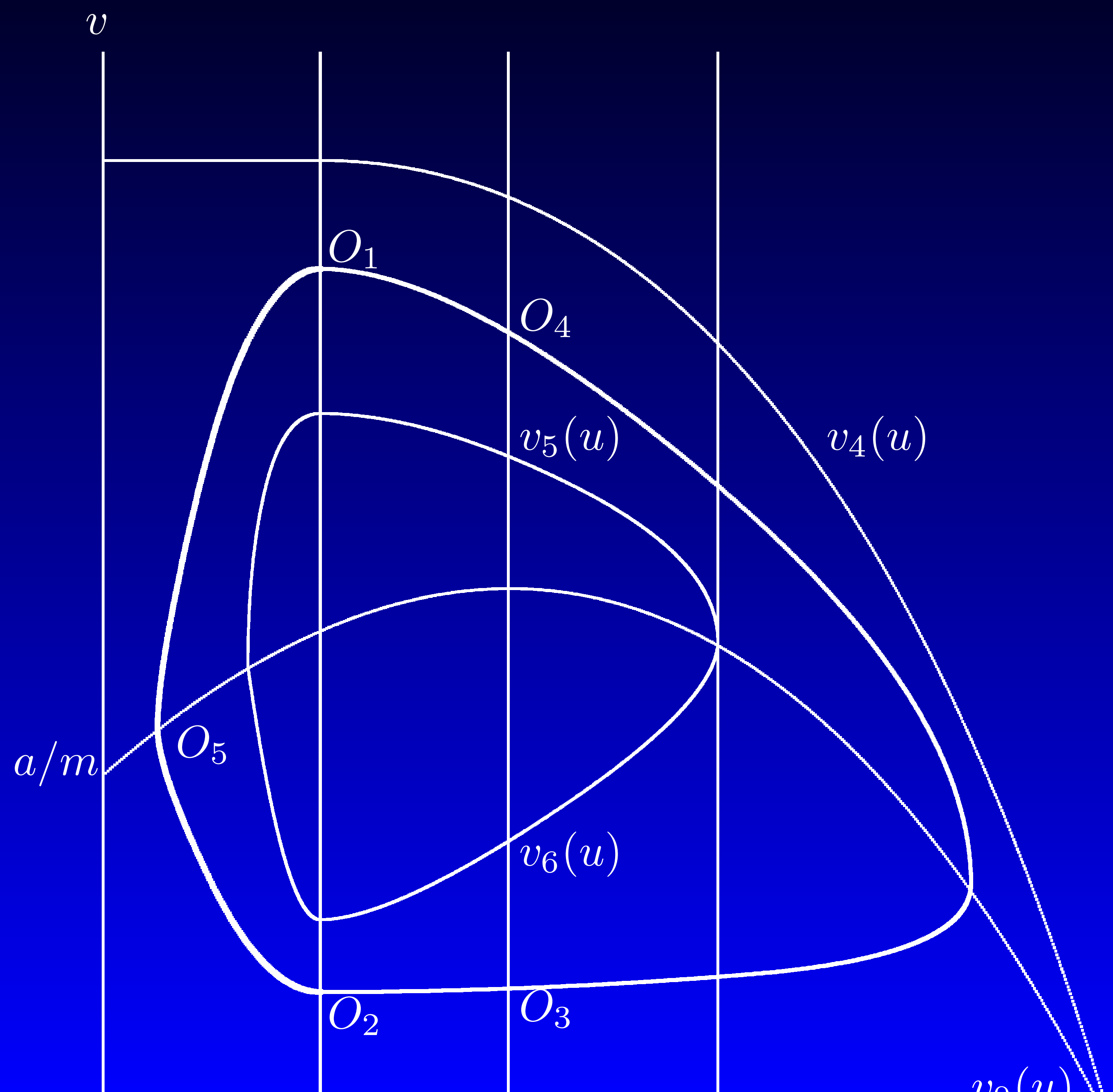


Illustration of limit cycle



Relaxation oscillation

Theorem 1[Hsu-Shi, 2008] If $0 < a < 1$ and $m > 0$ are fixed, and as $d \to 0$ (thus $\lambda \to 0$), then $C_1\lambda^{-1} \le T(O_1O_2) \le C_2\lambda^{-1}, T(O_2O_3)$, $T(O_4O_1) = O(|\ln\lambda|)$, and $T(O_3O_4) = O(1)$. In particular, the period $T \to \infty$ as $d \to 0$. The shape of the graph of the limit cycle is a relaxation oscillator.

Other known relaxation oscillators:
Van der Pol oscillator in electrical circuits employing vacuum tubes,
Fitzhugh-Nagumo oscillator in action potentials of neurons

Theorem 2[Hsu-Shi, 2008] If $m > d > 0$ are fixed, and as $a \to 0$ (thus $\lambda \to 0$), then $C_1\lambda^{-1} \le T(O_1O_2), T(O_3O_4) \le C_2\lambda^{-1}$, $T(O_2O_3) = O(|\ln\lambda|)$, and $T(O_4O_1) = O(1)$. In particular, the period $T \to \infty$ as $a \to 0$. The shape of the graph of the limit cycle is a nearly a Heaviside function.

Reaction-diffusion predator-prey model

$$\begin{cases} u_t - d_1 u_{xx} = u\left(1 - \dfrac{u}{k}\right) - \dfrac{muv}{u+1}, & x \in (0, \ell\pi),\ t > 0, \\ v_t - d_2 v_{xx} = -\theta v + \dfrac{muv}{u+1}, & x \in (0, \ell\pi),\ t > 0, \\ u_x(0,t) = v_x(0,t) = 0,\ u_x(\ell\pi, t) = v_x(\ell\pi, t) = 0, & t > 0, \\ u(x,0) = u_0(x) \geq 0,\ v(x,0) = v_0(x) \geq 0, & x \in (0, \ell\pi). \end{cases}$$

All bifurcations for ODE still occurs for PDE as spatial homogeneous solutions.

Case 1: $\lambda \geq k$: $(1,0)$ is globally asymptotically stable

Case 2: $k-1 < \lambda < k$: (λ, v_λ) is globally asymptotically stable

(when $(k-1)/2 < \lambda < k-1$, (λ, v_λ) is locally asymptotically stable)

Case 3: $0 < \lambda < (k-1)/2$: there is a spatial homogeneous periodic orbit

Does stability change with the addition of diffusion?

Determine the bifurcation points

Linearization at (λ, v_λ):

$$L(\lambda) := \begin{pmatrix} d_1 \dfrac{\partial^2}{\partial x^2} + \dfrac{\lambda(k-1-2\lambda)}{k(1+\lambda)} & -\theta \\ \dfrac{k-\lambda}{k(1+\lambda)} & d_2 \dfrac{\partial^2}{\partial x^2} \end{pmatrix},$$

$$\text{and } \; L_n(\lambda) := \begin{pmatrix} -\dfrac{d_1 n^2}{\ell^2} + \dfrac{\lambda(k-1-2\lambda)}{k(1+\lambda)} & -\theta \\ \dfrac{k-\lambda}{k(1+\lambda)} & -\dfrac{d_2 n^2}{\ell^2} \end{pmatrix}.$$

$$\begin{cases} T_n(\lambda) = \dfrac{\lambda(k-1-2\lambda)}{k(1+\lambda)} - \dfrac{(d_1+d_2)n^2}{\ell^2}, \\ D_n(\lambda) = \dfrac{\theta(k-\lambda)}{k(1+\lambda)} - \left[\dfrac{d_2\lambda(k-1-2\lambda)}{k(1+\lambda)}\right] \dfrac{n^2}{\ell^2} + \dfrac{d_1 d_2 n^4}{\ell^4}. \end{cases}$$

Existence of spatial non-homogeneous periodic orbits

Condition for Hopf bifurcation:

$T_n(\lambda_0) = 0, \quad D_n(\lambda_0) > 0, \quad$ and $\quad T_j(\lambda_0) \neq 0, \quad D_j(\lambda_0) \neq 0 \quad$ for $j \neq n$.

Theorem 3[Shi-Wei-Yi, 2008] Suppose $d_1, d_2, \theta > 0$ and $k > 1$ satisfy

$$\frac{d_1}{d_2} > \frac{\max h(\lambda)}{4\theta}, \quad \text{where} \quad h(\lambda) := \frac{\lambda^2(k-1-2\lambda)^2}{k(1+\lambda)(k-\lambda)}.$$

Then there exists $\ell_n > 0$, such that any ℓ in $(\ell_n, \ell_{n+1}]$, there exists $2n$ points $\lambda^H_{j,\pm}(\ell)$, $1 \leq j \leq n$, satisfying

$$0 < \lambda^H_{1,-}(\ell) < \lambda^H_{2,-}(\ell) < \cdots < \lambda^H_{2,+}(\ell) < \lambda^H_{1,+}(\ell) < \frac{k-1}{2},$$

such that the system undergoes a Hopf bifurcation at $\lambda = \lambda^H_{j,\pm}$, and the bifurcating periodic solution at $\lambda = \lambda^H_{j,\pm}$ is in form of

$(u, v) = (\lambda^H_{j,\pm}, v(\lambda^H_{j,\pm})) + s(a_0, b_0)\cos\left(\frac{jx}{\ell}\right)\cos(\omega_{j,\pm} t) + h.o.t.$

(There is no spatial non-homogeneous steady state solutions bifurcating for these parameters.)

More bifurcation: periodic orbits and steady states

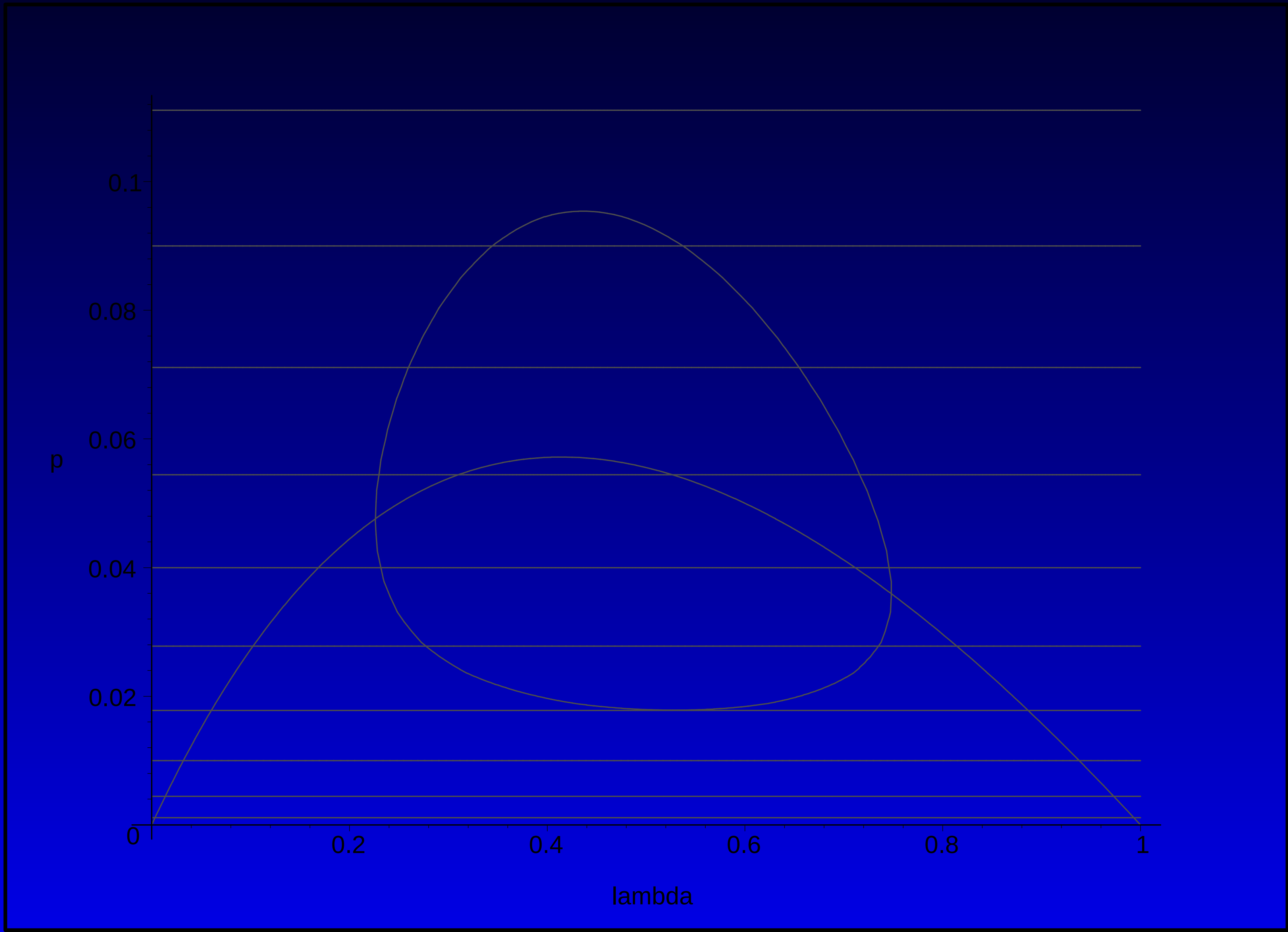
Theorem 4[Shi-Wei-Yi, 2008] [Peng-Shi, in preparation]

Suppose $d_1, d_2, \theta > 0$ and $k > 1$ satisfy

$$\frac{d_1}{d_2} < \frac{\max h(\lambda)}{4\theta}, \quad \text{where} \quad h(\lambda) := \frac{\lambda^2(k-1-2\lambda)^2}{k(1+\lambda)(k-\lambda)}.$$

Then there exist $\widetilde{\ell}_{n,\pm}$ such that if for each $\ell \in (\widetilde{\ell}_{n,+}, \widetilde{\ell}_{n,-})$ except a finite many exceptional ℓ, there exists exactly two points $\lambda^S_{n,\pm}$ such that a smooth curve $\Gamma_{n,\pm}$ of positive solutions of the system bifurcating from $(\lambda, u, v) = (\lambda^S_{n,\pm}, \lambda^S_{n,\pm}, v_{\lambda^S_{n,\pm}})$, and $\Gamma_{n,\pm}$ is contained in a global branch $\mathcal{C}_{n,\pm}$ of the positive solutions. Near $(\lambda, u, v) = (\lambda^S_{n,\pm}, \lambda^S_{n,\pm}, v_{\lambda^S_{n,\pm}})$, $\Gamma_{n,\pm} = \{(\lambda(s), u(s), v(s)) : s \in (-\epsilon, \epsilon)\}$, where $(u(s), v(s) = (\lambda^S_{n,\pm}, v_{\lambda^S_{n,\pm}}) + s(a_n, b_n)\cos(nx/\ell) + h.o.t.$. Moreover each $\mathcal{C}_{n,\pm}$ contains another $(\lambda^S_{j,\pm}, \lambda^S_{j,\pm}, v_{\lambda^S_{j,\pm}})$.

Bifurcation points: periodic orbits and steady states

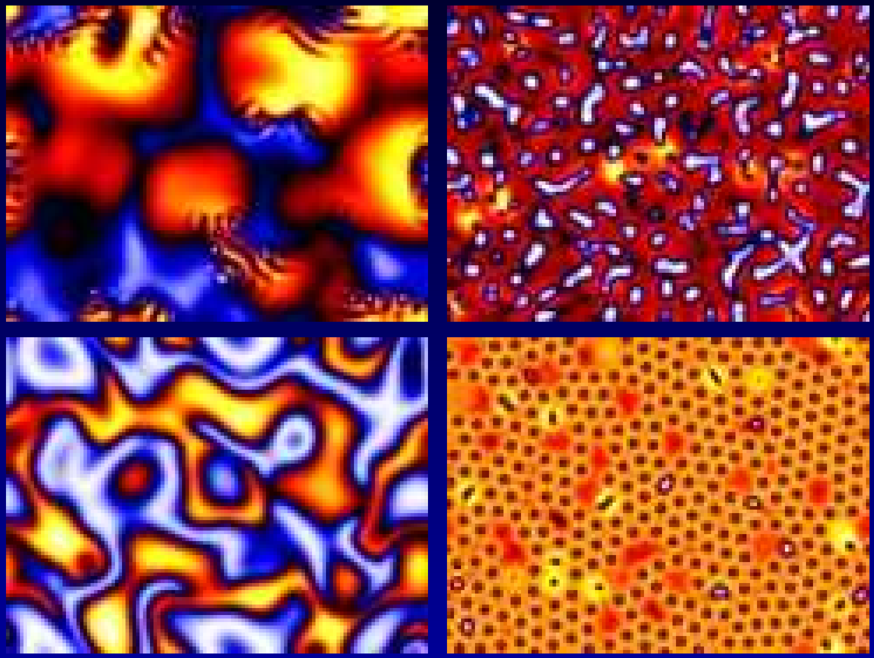


Bifurcation diagram: $d_1 = d_2 = 1$, $k = 3$, $\theta = 0.003$ and $\ell = 30$.

Remarks

1. We rigorously show the existence of spatial non-homogeneous periodic orbits in an autonomous and homogeneous reaction-diffusion system, which is rarely achieved previously. In general there are many Hopf and steady state bifurcations entangled for $\lambda \in (0, (k-1)/2)$, which indicates complicated spatiotemporal dynamics.
(Numerical evidence: Medvinsky, Li, et.al., Spatiotemporal complexity of plankton and fish dynamics. *SIAM Review* 2002.)
2. Near bifurcation points, these patterned solutions are all unstable, since (λ, v_λ) has lost the stability at $\lambda = (k-1)/2$ to spatial homogenous periodic orbit.
3. These bifurcations are not Turing bifurcations since the diffusion coefficients can be chosen arbitrarily.

Rich spatial patterns in diffusive predator-prey system



Patterns generated by diffusive predator-prey system

Our results show that the system does have many periodic solutions and steady state solutions.

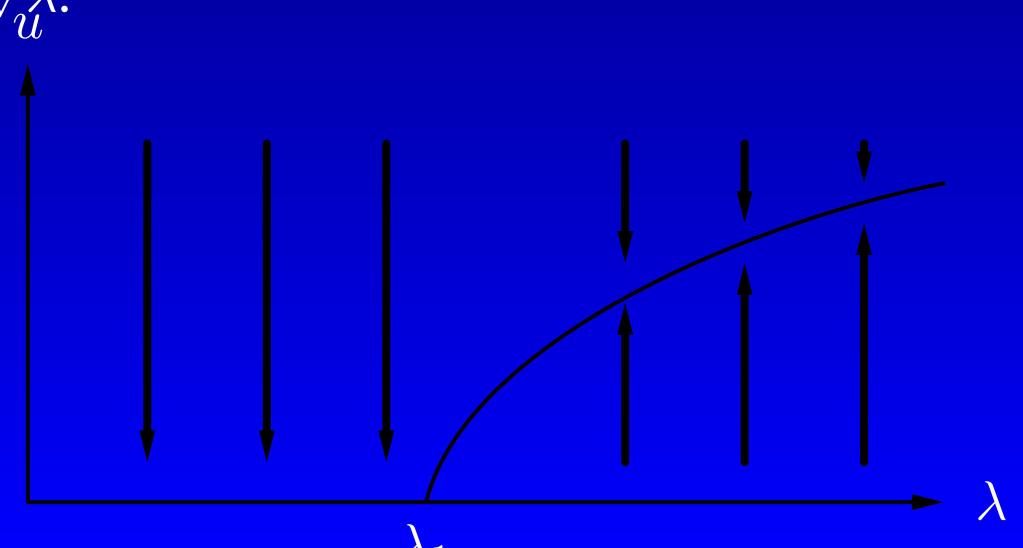
Diffusive logistic equation (Fisher-KPP equation)

[Fisher, 1937] [Kolmogoroff-Petrovsky-Piscounoff, 1937]

$$
\begin{cases}
\dfrac{\partial u}{\partial t} = \Delta u + \lambda u(1-u), & x \in \Omega, \quad t > 0, \\
u(x,t) = 0, & x \in \partial\Omega, \quad t > 0, \\
u(x,0) = u_0(x) \geq 0, & x \in \Omega.
\end{cases}
$$

Minimal patch size: for $\lambda > \lambda_1$, there is a unique steady state which is globally asymptotically stable.

Traveling wave: (1-D and $\Omega = \mathbf{R}$) traveling wave in form $u(x,t) = v(x - ct)$ exists for $c > 2\sqrt{\lambda}$.



Some ideas for the global stability

1. The uniqueness of positive steady state solution since $1 - u$ is a decreasing function.
2. The system is a monotone dynamical system which preserves the order of the solution. (If $u_1(x,0) \geq u_2(x,0)$, then $u_1(x,t) > u_2(x,t)$.)
3. The system is a gradient system with a Lyapunov functional $I(u) = (1/2)\int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega F(u)dx$, where $F(u) = \int_0^u s(1-s)ds$. Along a solution orbit $u(\cdot,t)$, $d/dt[I(u(\cdot,t))] \leq 0$. Thus the orbit must converge to the set of steady state solutions.

How much of these nice results are still true when more structure is added to the model?

An advection-Reaction-Diffusion equation

[Shi-Zeng, in preparation]

$$\begin{cases} u_t = u_{xx} + xu_x + Du(1-u), & x \in \mathbf{R},\ t > 0, \\ \lim\limits_{|x|\to\infty} u(x,t) = 0, & t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \mathbf{R}. \end{cases}$$

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- Advection-Reaction-Diffusion equation models chemical or biological reaction in a fluid flow with stirring
- Logistic growth rate models the autocatalytic chemical reaction: $A + B \to 2A$

The derivation of the model (1)

The spatiotemporal dynamics of interacting biological or chemical substances is governed by the system of reaction-advection-diffusion equations:

$$\frac{\partial C_i}{\partial t} + \mathbf{v} \cdot \nabla C_i = \mathrm{Da} f_i(C_1, \cdots, C_N) + Pe^{-1} \Delta C_i,$$

where $i = 1, \cdots, N$.

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where $i = 1, \cdots, N$. Here the equation is in dimensionless form, where Da and Pe are the Damköhler and the Péclet number respectively. The Damköhler number characterizes the ratio between the advective and the reaction time scales; and the Péclet number is a measure of the relative strength of the advective and diffusive transport.

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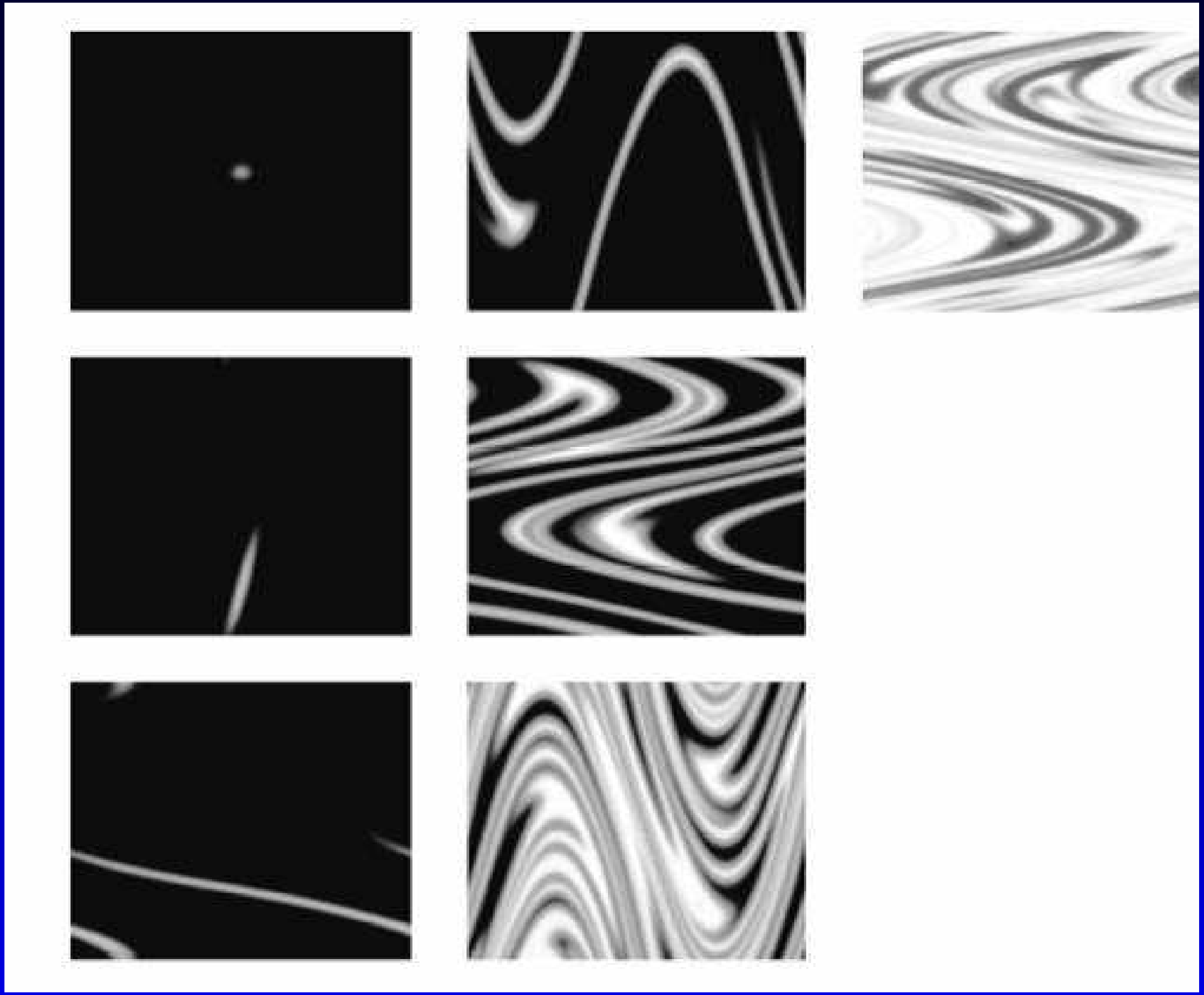
- The stirring process smoothes out the concentration of the advected tracer along the stretching direction, whilst enhancing the concentration gradients in the convergent direction.
- In the convergent direction we have the following one dimensional equation for the average profile of the filament representing the evolution of a transverse slice of the filament in a Lagrangian reference frame (following the motion of a fluid element):
$$\frac{\partial C}{\partial t} - x\frac{\partial C}{\partial x} = Df(C) + \frac{\partial^2 C}{\partial x^2}.$$

Numerical simulations: stretching of the patterns



Neufeld, et al, *Chaos*, Vol 12, 426-438, 2002.

A function space and a linear operator

The equation can be rewritten as
$u_t = e^{-x^2/2}(e^{x^2/2}u_x)_x + Du(1-u)$

$$\begin{aligned} W_\phi^{1,2}(\mathbf{R}) = &\{u : u \text{ and } u' \text{ are measurable in } \mathbf{R}, \\ &\int_{\mathbf{R}} e^{x^2/2}u^2 dx < \infty, \int_{\mathbf{R}} e^{x^2/2}(u')^2 dx < \infty\}. \end{aligned}$$

We define a linear operator $A\phi(x) = -e^{-x^2/2}\left(e^{x^2/2}\phi_x\right)_x$. A is a self-adjoint densely defined linear operator in $L^2_\phi(\mathbf{R})$, and the domain $D(A) = W_\phi^{1,2}(\mathbf{R}) \bigcap W_{\phi,loc}^{2,2}(\mathbf{R})$. The spectrum $\sigma(A)$ consists of simple eigenvalues $\lambda_k = k$, $k \in \mathbf{N}$, and the corresponding eigenfunction $\phi_k(x) = e^{-x^2/2}P_k(x)$ where P_k is a polynomial of degree $k-1$ which can be defined recursively. $\{\phi_k(x)\}$ is an orthonormal basis of $L^2_\phi(\mathbf{R})$.

Global stability

Theorem 5. Consider

$$\begin{cases} u_t = u_{xx} + xu_x + Du(1-u), & x \in \mathbf{R},\ t > 0, \\ \lim\limits_{|x|\to\infty} u(x,t) = 0, & t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \mathbf{R}. \end{cases} \tag{1}$$

1. When $D \leq 1$, (1) has only the trivial steady state solution $u \equiv 0$, and when $D > 1$, (1) has a unique positive steady state $U_D(x)$, which satisfies $U_D(-x) = U_D(x)$;
2. For any $D > 0$, $u_0 \in W_\phi^{1,2}(\mathbf{R})$ and $u_0 \geq (\not\equiv) 0$, (1) has a unique solution $u(x,t) > 0$ for $t > 0$ and $x \in \mathbf{R}$.
3. When $D \leq 1$, for any $u_0 \in W_\phi^{1,2}(\mathbf{R})$ and $u_0 \geq 0$, $u(x,t) \to 0$ in $W_\phi^{1,2}(\mathbf{R})$ as $t \to \infty$, and when $D > 1$, for any $u_0 \in W_\phi^{1,2}(\mathbf{R})$ and $u_0 \geq (\not\equiv) 0$, $u(x,t) \to U_D$ in $W_\phi^{1,2}(\mathbf{R})$ as $t \to \infty$.

Shape of steady state

Theorem 6. Let (D, U_D) be the unique steady state solution. Let x_D be the unique point in $\mathbf{R}^+$ such that $U_D(x_D) = 1/2$. Then as $D \to \infty$,

1. $U_D(x) \to 1$ uniformly for x in any compact subset P of $\mathbf{R}$;
2. $\lim_{D\to\infty} x_D D^{-1/2} = 2$;
3. $U_D(x_D + D^{-1/2}y) \to v(y)$ uniformly for any y in any compact subset of $\mathbf{R}$, where $v(y)$ is a solution of

$$v'' + cv' + v(1-v) = 0, \quad v'(y) < 0, \quad \lim_{y\to-\infty} v(y) = 1, \quad \lim_{y\to\infty} v(y) = 0,$$

where $c = 2$.

Hence the steady state U_D has a “plateau top” near $u = 1$ with width about $4\sqrt{D}$, and the sharp interface has the width $O(D^{-1/2})$ when D is large.

Comparison with classical results

Classical Fisher equation $u_t = u_{xx} + Du(1-u)$

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Critical patch size Assume a bounded spatial domain $(0, L)$ with zero boundary condition, then when $D < D_c = \pi^2/L^2$, $u(x,t) \to 0$, and when $D > D_c$, $u(x,t) \to w_D(x)$, which is the unique steady state of the boundary value problem.

connection with Theorem 5

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connection with Theorem 5

Asymptotic propagation speed [Kolmogoroff-Petrovsky-Piscounoff, 1937]
Assume spatial domain $\mathbf{R}$ with compact support initial value. Interfaces between $u = 0$ and $u = 1$ are developed, and they are moving toward $\pm\infty$ with asymptotic speed $c = 2\sqrt{D}$, which is the minimal speed or all possible traveling waves.
connection with Theorem 6

Numerical illustrations $D = 40$

- $u(x,0) = exp(-(x-2)^2) + exp(-(x+2)^2)$
  simulation
- $u(x,0) = 2exp(-2(x-2)^2) + exp(-(x+2)^2) + 3exp(-(x-5)^2)$
  simulation
- $u(x,0) = cos(\pi x/40)$ when $|x| < 20$ and $u(x,0) = 0$ when $|x| \geq 20$
  simulation
- $u(x,0) = exp(-(x-2)^2) + exp(-(x+2)^2)$ Fisher equation
  simulation

Effect of delay: diffusive Hutchingson equation

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \Delta u(x,t) + \lambda u(x,t)(1 - u(x,t-\tau)), & x \in \Omega, \ \ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, \ \ t > 0, \\ u(x,s) = \eta(x,s) \geq 0, & x \in \Omega, \ \ -\tau \leq s \leq 0. \end{cases}$$

It has the same unique steady state solution as diffusive logistic equation, but is it stable?

[Busenberg-Huang, 1996] (case of $n = 1$) The unique positive steady state u_λ may not be stable. For $\lambda > \lambda_1$ but near λ_1, u_λ is locally stable for small $\tau > 0$, but is unstable for large $\tau > 0$. A supercritical Hopf bifurcation occurs at a sequence of $\tau_n > 0$, and a periodic orbit exists for $\tau > \tau_0$.

Delay induced different Hopf bifurcation

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \Delta u(x,t) + \lambda u(x,t) f(u(x,t-\tau)), & x \in \Omega, \ \ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, \ \ t > 0, \\ u(x,s) = \eta(x,s) \geq 0, & x \in \Omega, \ \ -\tau \leq s \leq 0. \end{cases}$$

$f(u)$ is a decreasing function (Logistic growth rate)

[Su-Wei-Shi, in preparation] The unique positive steady state u_λ may not be stable. For $\lambda > \lambda_1$ but near λ_1, u_λ is locally stable for small $\tau > 0$, but is unstable for large $\tau > 0$. A Hopf bifurcation occurs at a sequence of $\tau_n > 0$. The bifurcation is always supercritical.


Thank You