Examples of stationary bifurcations: Turing bifurcation, fold bifurcation

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Summary of Bifurcation Theorems

Let $F : \mathbf{R} \times X \to Y$ be continuously differentiable. $F(\lambda_0, u_0) = 0$, F satisfies

(F1) $dim N(F_u(\lambda_0, u_0)) = codim R(F_u(\lambda_0, u_0)) = 1$, and

(F2) $F_\lambda(\lambda_0, u_0) \notin R(F_u(\lambda_0, u_0))$.

Then a saddle-node bifurcation occurs.

If F satisfies **(F1)**,

(F2') $F_\lambda(\lambda_0, u_0) \in R(F_u(\lambda_0, u_0))$,

and additional non-degeneracy condition on $D^2 F$

Then a crossing curve bifurcation occurs. (include pitchfork and transcritical bifurcations)

The bifurcation from trivial solutions is global if $F_u(\lambda, u)$ is always Fredholm

An equivalent form of **(F2)**: $F_u(\lambda, u_0)$ has a simple real eigenvalue $\gamma(\lambda)$ for λ near λ_0, continuously differentiable in λ, with $\gamma(\lambda_0) = 0$, and $\gamma'(\lambda_0) \neq 0$. All other eigenvalues of $F_u(\lambda, u_0)$ have non-zero real parts

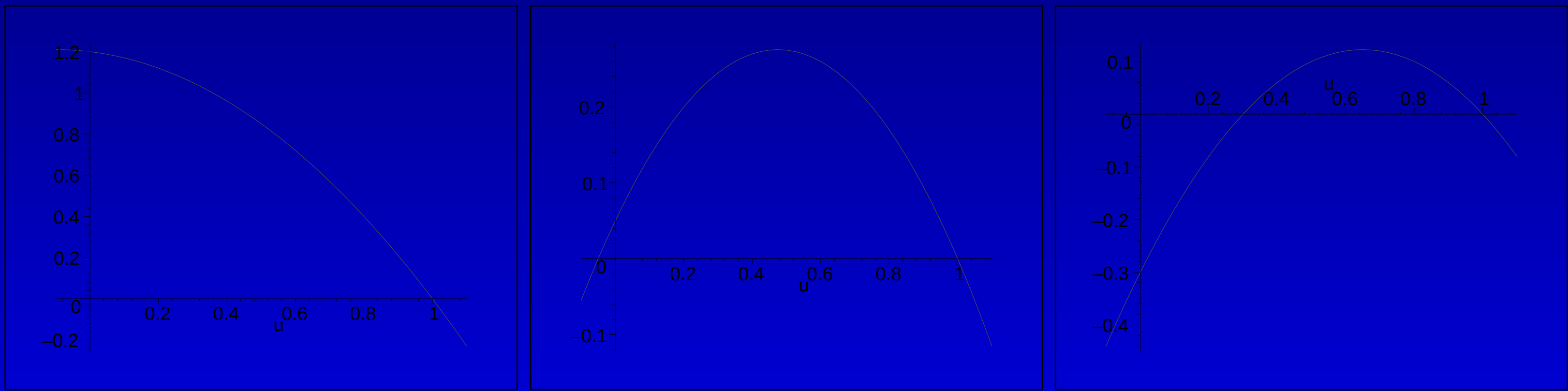
Single species: Logistic Model verses Allee effect

$$\begin{cases} \dfrac{\partial u}{\partial t} = \mathcal{D}\Delta u + uf(x,u), & x \in \Omega, \quad t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \Omega. \end{cases}$$

$u(x,t)$: population density at position x and time t

Ω: a bounded habitat, $u = 0$ on boundary $\partial\Omega$: hostile exterior environment

$f(x,u)$: heterogeneous growth rate per capita



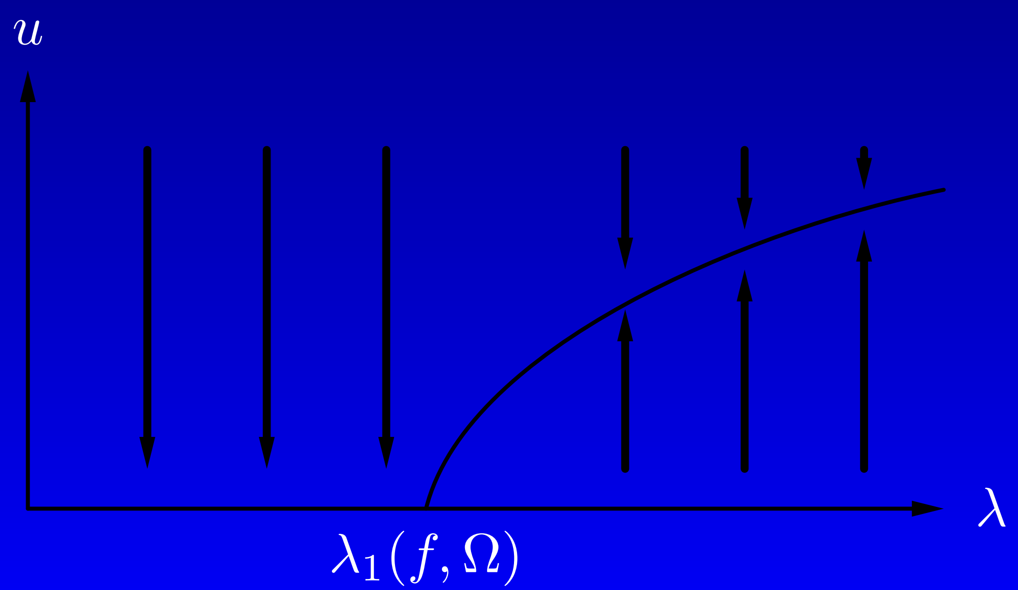
$f(x,u)$: (a) logistic; (b) weak Allee effect; (c) strong Allee effect.

Bifurcation problem

$$\Delta u + \lambda u f(x,u) = 0, \quad x \in \Omega, \quad u = 0, \quad x \in \partial\Omega.$$

$u = 0$ is always a solution for any $\lambda > 0$, $\lambda_1(f,\Omega)$ (minimal patch size) is the principal eigenvalue of $\Delta\psi + \lambda f(x,0)\psi = 0,\ x \in \Omega, \quad \psi = 0,\ x \in \partial\Omega$. We consider positive solutions only.

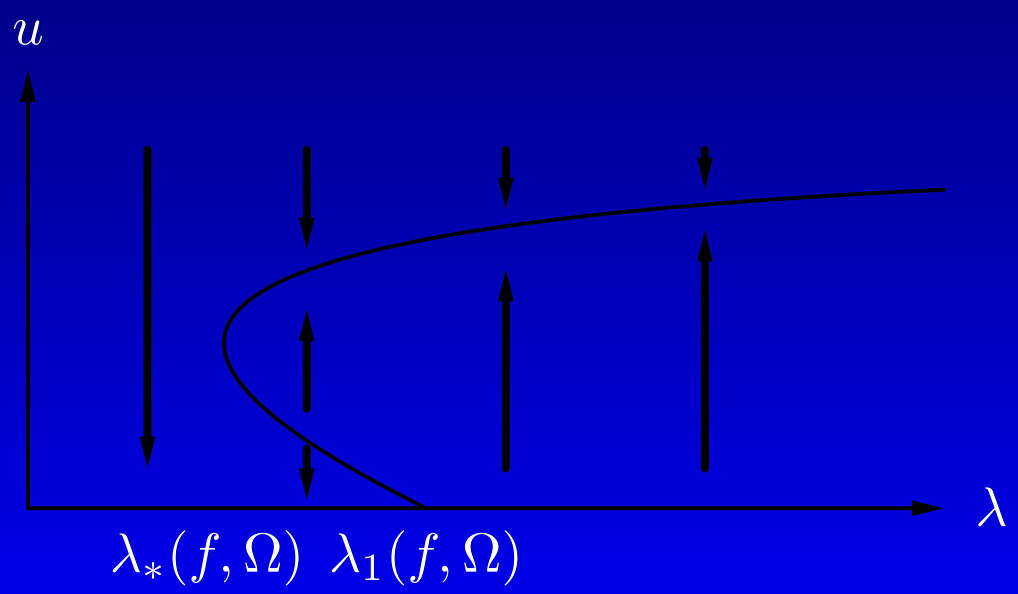
Logistic case: a supercritical transcritical bifurcation occurs at $\lambda_1(f,\Omega) > 0$; for $\lambda > \lambda_1$, there is a unique steady state which is globally stable.
[Cantrell-Cosner, 2003]



Weak Allee effect case

(A) a subcritical (backward) transcritical bifurcation occurs at $\lambda_1(f,\Omega) > 0$;
(B) for $\lambda \in (\lambda_*, \lambda_1)$, there are at least two steady state solutions (bistability);
(C) a saddle-node bifurcation occurs at λ_* (at least when Ω is a ball);
(D) for λ large, it is similar to logistic case. [Shi-Shivaji, 2006]

Allee effect caused by diffusion (ODE with weak Allee effect is similar to logistic case); danger of hysteresis.
[Jiang-Shi, 2008, in Book edited by Cantrell-Cosner-Ruan]



Strong Allee effect case

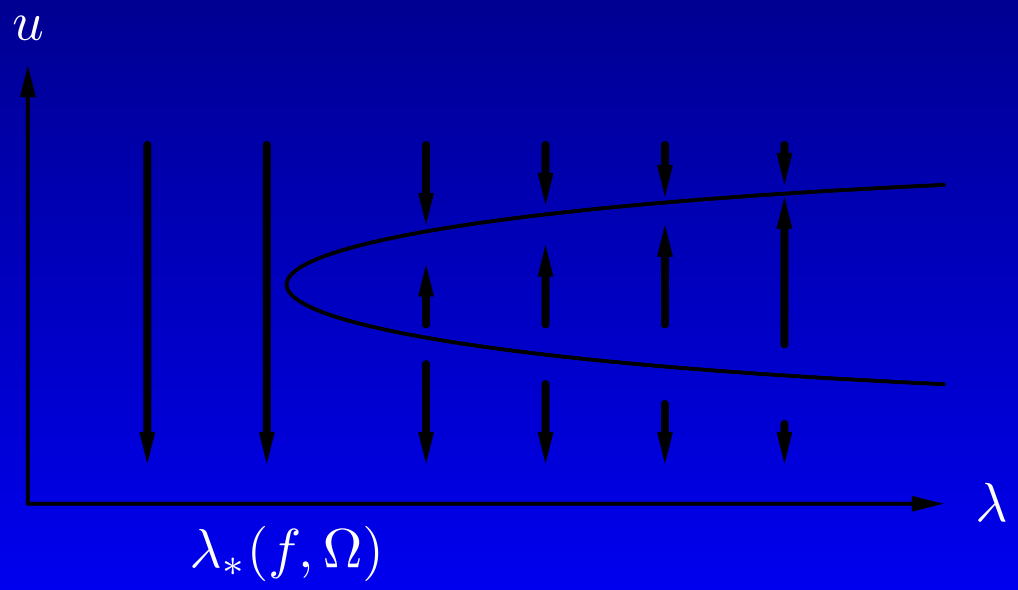
(A) $\lambda_1(f, \Omega) < 0$, $u = 0$ is always stable, and there is no bifurcation from $u = 0$;

(B) for $\lambda > \lambda_*$, there are at least two steady state solutions (bistability);

(C) a saddle-node bifurcation occurs at λ_* (at least when Ω is a ball);

(D) the basins of attraction of $u = 0$ and large stable steady state ("carrying-capacity") is a codimension-one manifold (surface in infinite dimensional space).

[Ouyang-Shi, 1998] [Jiang-Liang-Zhao, 2004], [Jiang-Shi, 2008]



Estimate of the breaking point λ_*

$$\Delta u + \lambda u(1-u)(u-M) = 0, \quad x \in \Omega, \quad u(x) = 0, \quad x \in \partial\Omega.$$

$0 < M < 1/2$ (if $1/2 < M < 1$, then there is no positive steady state)

Lower bound: $\lambda_* > \lambda_1 / f_*$, where $f_* = \max\limits_{u \in [0,1]} f(u)/u$

Upper bound: define $I(\lambda, u) = (1/2) \int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega F(u) dx$, then $\lambda_* < \lambda_0$, where $\min I(\lambda_0, u) < 0$ (which implies 0 is not the global minimum of the energy function $I(\lambda, u)$)

$n = 1, \Omega = (0, L)$.

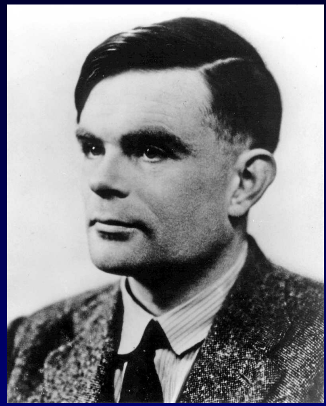
$$\frac{2\pi^2}{L^2(1+M)} < \lambda_* < \frac{48}{L^2(3-M)}.$$

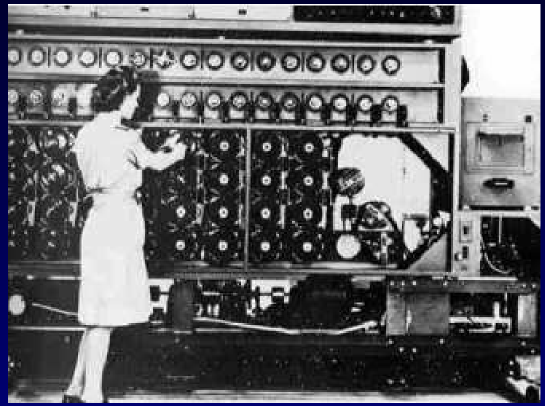
$L = 1$ and $M = 0.2$: $16.45 < \lambda_* < 17.14$

Numerical value of λ_*: $\lambda_* \approx 16.61$.

[Jiang-Shi, 2008]

Alan Turing (1912-1954)





- One of greatest scientists in 20th century
- Designer of Turing machine (a theoretical computer) in 1930s
- Designing electromechanical machine which breaks German U-boat Enigma, helping the battle of the Atlantic
- Initiate nonlinear theory of biological growth
  [Turing, 1952] The Chemical Basis of Morphogenesis.
  Philosophical transaction Royal Society of London Series B, **237**

http://www.turing.org.uk/

Turing's idea

ODE (1): $u' = f(u,v)$, $v' = g(u,v)$

Reaction-diffusion system (2): $u_t = d_1 \Delta u + f(u,v)$, $v_t = d_2 \Delta v + g(u,v)$

Here $u(x,t)$ and $v(x,t)$ are the density functions of two chemicals (morphogen) or species which interact or react

- A constant solution $u(t,x) = u_0$, $v(t,x) = v_0$ can be a stable solution of (1), but an unstable solution of (2). Thus the instability is induced by diffusion.
- On the other hand, there must be stable non-constant equilibrium solutions, or stable non-equilibrium behavior, which have more complicated spatial-temporal structure.

`http://en.wikipedia.org/wiki/Morphogen`

Turing bifurcation in 1-D problem

$$\begin{cases} u_t = D_u u_{xx} + \lambda f(u,v), & x \in (0,\pi),\ t>0, \\ v_t = D_v v_{xx} + \lambda g(u,v), & x \in (0,\pi),\ t>0, \\ u_x(t,0) = u_x(t,\pi) = v_x(t,0) = v_x(t,\pi) = 0, & t>0, \\ u(0,x) = u_0(x),\ v(0,x) = v_0(x), & x \in (0,\pi). \end{cases}$$

Equilibrium point: $f(u_0,v_0) = g(u_0,v_0) = 0$
Eigenvalue problem $\phi'' = \mu\phi$, $0 < x < \pi, \quad \phi'(0) = \phi'(\pi) = 0$
eigenvalue $\mu_k = -k^2$, eigenfunction $\phi_k(x) = \cos(kx)$.

Linearized equation:

$$L \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \phi_{xx} \\ d\psi_{xx} \end{pmatrix} + \lambda \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix}$$

Calculation of stability

Let $\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} \cos(kx)$, and $D = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$, then eigenvalues of L are determined by

$$(\lambda J - k^2 D) \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \lambda f_u - k^2 & \lambda f_v \\ \lambda g_u & \lambda g_v - k^2 d \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \mu \begin{pmatrix} A \\ B \end{pmatrix}$$

$$Tr(J - k^2 D) = \lambda(f_u + g_v) - k^2(1 + d),$$
$$Det(J - k^2 D) = \lambda^2(f_u g_v - f_v g_u) - k^2(df_u + g_v)\lambda + k^4 d$$

<u>Stable w.r.t. ODE</u>: $D_1 = f_u g_v - f_v g_u > 0$ and $f_u + g_v < 0$
Thus $Tr(J - k^2 D) < 0$, and we must have $Det(J - k^2 D) < 0$ if it is unstable w.r.t. R-D system

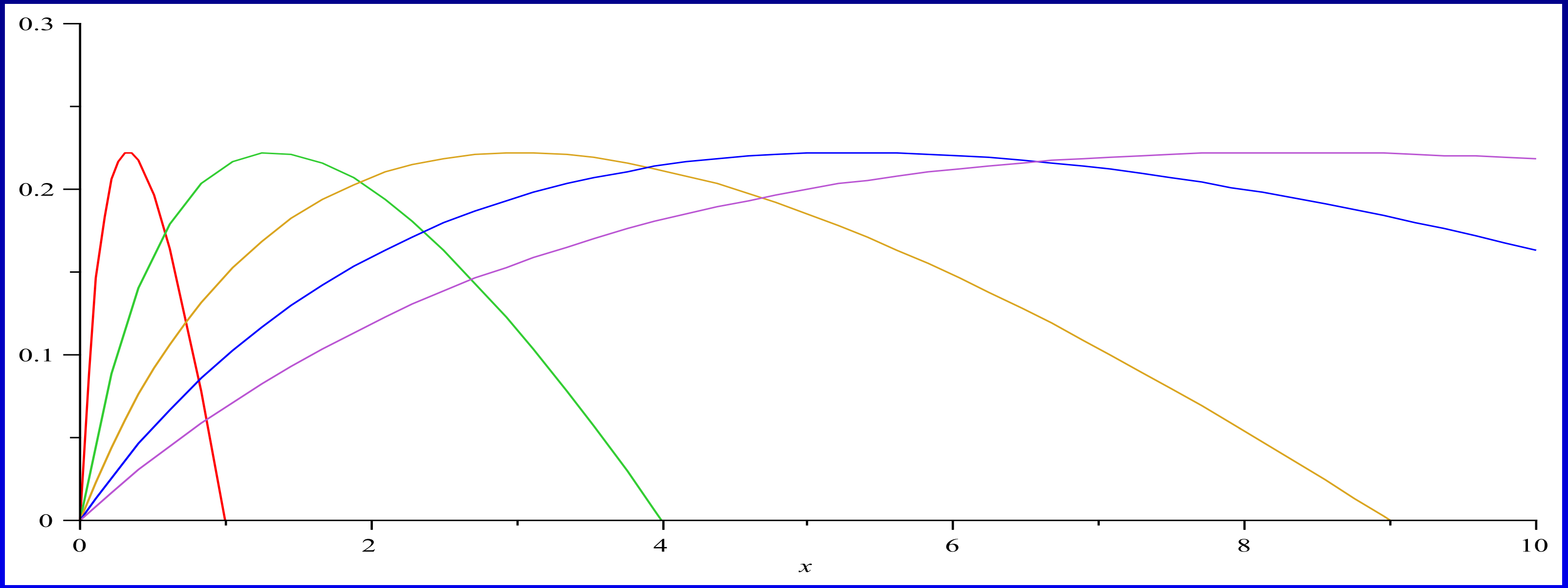
Condition for Turing instability: $f_u < 0$, $g_v > 0$, $0 < d < 1$,
$0 < d < \dfrac{\lambda[g_v k^2 - \lambda D_1]}{k^2(k^2 - \lambda f_u)} \equiv d_k(\lambda)$ (bifurcation point)

(artificial) Example

$$J = \begin{pmatrix} -3 & 2 \\ -4 & 2 \end{pmatrix}, \; f_u < 0, \; g_v > 0, \; f_u + g_v < 0, \; f_u g_v - f_v g_u > 0,$$

$$d_k(\lambda) = \frac{\lambda(2k^2 - 2\lambda)}{k^2(k^2 + 3\lambda)},$$

$$d_1(\lambda) = \frac{2\lambda(1-\lambda)}{1+3\lambda}, \; d_2(\lambda) = \frac{2\lambda(4-\lambda)}{4(4+3\lambda)}, \; d_3(\lambda) = \frac{2\lambda(9-\lambda)}{9(9+3\lambda)},$$



Horizontal axis: λ, vertical axis: d.

Global Turing Bifurcation

Theorem: Suppose that $f(u_0, v_0) = g(u_0, v_0) = 0$, and at (u_0, v_0),
(A) $f_u < 0$ (inhibitor), $g_v > 0$ (activator);
(B) $D_1 = f_u g_v - f_v g_u > 0$ and $f_u + g_v < 0$.
For fixed $\lambda > 0$, if $d_k(\lambda) \equiv \dfrac{\lambda[g_v k^2 - \lambda D_1]}{k^2(k^2 - \lambda f_u)} \neq d_j(\lambda)$ for any $j \neq k$, then
(i) $d = d_k$ is a bifurcation point where a continuum Σ of non-trivial solutions of

$$\begin{cases} u_{xx} + \lambda f(u,v) = 0, \quad d v_{xx} + \lambda g(u,v) = 0, \qquad x \in (0,\pi), \\ u_x(0) = u_x(\pi) = v_x(0) = v_x(\pi) = 0, \end{cases}$$

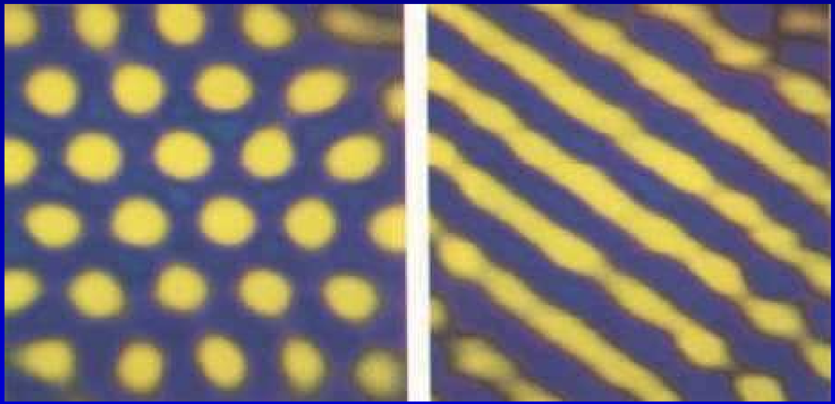
bifurcates from the line of trivial solutions (d, u_0, v_0);
(ii) The continuum Σ is either unbounded in the space of (d, u, v), or it connects to another $(d_j(\lambda), u_0, v_0)$;
(iii) Σ is locally a curve near $(d_k(\lambda), u_0, v_0)$ in form of $(d, u, v) = (d(s), u_0 + sA\cos(kx) + o(s), v_0 + sB\cos(kx) + o(s))$, $|s| < \delta$, and $d'(0) = 0$ thus the bifurcation is pitchfork type ($d''(0)$ can be computed in term of $D^3(f, g)$).

Turing patterns in real experiment:

Lengyel-Epstein CIMA chemical reaction

The first experimental evidence of Turing pattern was observed in 1990, nearly 40 years after Turing's prediction, by the Bordeaux group in France, on the chlorite-iodide-malonic acid-starch (CIMA) reaction in an open unstirred gel reactor. This observation represents a significant breakthrough for one of the most fundamental ideas in morphogenesis and biological pattern formation.

[Castets, et.al., 1990] Experimental evidence of a sustained Turing-type equilibrium chemical pattern. *Phys. Rev. Lett.* **64**.



Reaction-diffusion system for CIMA reaction

Lengyel and Epstein simplify the reaction into a system of two equations:

$$\begin{cases} u_t = \triangle u + a - u - \dfrac{4uv}{1+u^2}, & x \in \Omega, t > 0, \\ v_t = \sigma[c\Delta v + b(u - \dfrac{uv}{1+u^2})], & x \in \Omega, t > 0, \\ \partial_\nu u = \partial_\nu v = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x) > 0,\ v(x,0) = v_0(x) > 0, & x \in \Omega, \end{cases}$$

[Lengyel-Epstein, 1991] Modeling of Turing Structures in the Chlorite-Iodide-Malonic Acid-Starch Reaction System. *Science* **251**.

We consider $a = 5\alpha$, $\sigma = b = 1$, $c = d$ and $\Omega = (0, l\pi)$.

Steady state equation:

$$\begin{cases} u_{xx} + 5\alpha - u - \dfrac{4uv}{1+u^2} = 0, & x \in (0, l\pi), \\ dv_{xx} + u - \dfrac{uv}{1+u^2} = 0, & x \in (0, l\pi), \\ u_x(0) = u_x(l\pi) = v_x(0) = v_x(l\pi) = 0. \end{cases}$$

Bifurcation Analysis

Constant equilibrium: $(u_*, v_*) = (\alpha, 1+\alpha^2)$

Jacobian at (u_*, v_*): $J = \begin{pmatrix} \dfrac{3\alpha^2-5}{\alpha^2+1} & -\dfrac{4\alpha}{\alpha^2+1} \\ \dfrac{2\alpha^2}{\alpha^2+1} & -\dfrac{\alpha}{\alpha^2+1} \end{pmatrix}$.

Assume $0 < 3\alpha^2 - 5 < \alpha$

$f_u > 0$, $g_v < 0$, $D_1 = f_u g_v - f_v g_u > 0$ and $f_u + g_v < 0$.

Bifurcation points: $d_j = \dfrac{\alpha}{1+\alpha^2} \cdot \dfrac{5+\lambda_j}{\lambda_j(f_0 - \lambda_j)}$,
where $f_0 = \dfrac{3\alpha^2-5}{1+\alpha^2}$, and $\lambda_j = j^2/l^2$.

[Ni-Tang, 2005] Turing patterns in the Lengyel-Epstein system for the CIMA reaction. *Trans. Amer. Math. Soc.* **357**.
[Jang-Ni-Tang, 2004] Global bifurcation and structure of Turing patterns in the 1-D Lengyel-Epstein model. *J. Dynam. Differential Equations* **16**.

Global Turing Bifurcation for CIMA reaction

[Ni-Tang, 2005]:

(A) For $d > 0$ small, (u_*, v_*) is the only steady state solution;

(B) All non-negative steady state solution satisfies $0 < u(x) < 5\alpha$, $0 < v(x) < 1 + 25\alpha^2$.

[Jang-Ni-Tang, 2004]:

(C) Each connected component bifurcated from (d_j, u_*, v_*) is unbounded in the space of (d, u, v), and its projection over d-axis covers (d_j, ∞).

(D) For each $d \neq d_k$, there exists a non-constant solution.

More results for Lengyel-Epstein system: (Hopf bifurcation etc.)

[Yi-Wei-Shi, 2008] Diffusion-driven instability and bifurcation in the Lengyel-Epstein system. *Nonlinear Anal. Real World Appl.* **9**.

[Yi-Wei-Shi, to appear] Global asymptotical behavior of the Lengyel-Epstein reaction-diffusion system. *Appl. Math. Lett.*

Generalization to cross-diffusion system

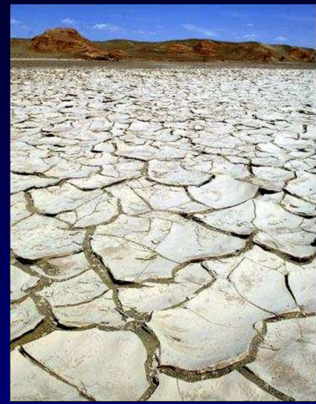
[Shi-Xie-Little, submitted] Cross-diffusion induced instability and stability in reaction-diffusion systems.

$$\begin{cases} u_t = d_{11}u_{xx} + d_{12}v_{xx} + \alpha f(u,v), & t > 0,\ x \in \mathbf{R}, \\ v_t = d_{21}u_{xx} + d_{22}v_{xx} + \alpha g(u,v), & t > 0,\ x \in \mathbf{R}, \end{cases}$$

Equilibrium point: $f(u_0, v_0) = g(u_0, v_0) = 0$

Following scenarios are possible:
(A) (u_0, v_0) is stable for ODE, still stable for (self)-diffusion system ($d_{12} = d_{21} = 0$), but it is unstable for cross-diffusion system ($d_{12}, d_{21} \neq 0$). (cross-diffusion induced instability)
(B) (u_0, v_0) is stable for ODE, unstable for (self)-diffusion system ($d_{12} = d_{21} = 0$) (Turing instability), but it is stable for cross-diffusion system ($d_{12}, d_{21} \neq 0$). (cross-diffusion induced stability)

Another water-limited ecosystem



[von Hardenberg, et.al. 2001] Diversity of Vegetation Patterns and Desertification. *Phys. Rev. Lett.* **87**, 198101.

$$n_t = \frac{\gamma w}{1+\sigma w} n - n^2 - \mu n + \Delta n,$$
$$w_t = p - (1-\rho n)w - w^2 n + \delta \Delta (w - \beta n) - v(w - \alpha n)_x,$$

We only consider the case when $v = 0$ (flat land).

$$n_t = \frac{\gamma w}{1+\sigma w} n - n^2 - \mu n + \Delta n,$$
$$w_t = p - (1-\rho n)w - w^2 n + \delta \Delta (w - \beta n).$$

Bifurcation with cross-diffusion effect

[Shi-Xie-Little, submitted]

(i) Suppose that $0 < \gamma - \mu\sigma < \dfrac{\sigma}{\rho}$, and $w > \rho$, then $(n_*, w_*) = (\dfrac{\gamma w}{1+\sigma w} - \mu, w)$ is an equilibrium point satisfying $1 - \rho n > 0$, and it is stable with respect to ODE dynamics.

(ii) (n_*, w_*) is still stable with diffusion added (but $\beta = 0$, no cross-diffusion)

(iii) If $\beta > \beta_0 \equiv \dfrac{(\delta n + pw^{-1} + wn + 2\sqrt{\delta Det(J)})(1+\sigma w)^2}{\delta \gamma n}$, then (n_*, w_*) is unstable. (cross-diffusion induced instability)

$\beta = \beta_0$ is a bifurcation point where strip patterned solutions bifurcate from spatial uniform equilibrium solution.

Last example: an ODE model

[Shi, EJDE, problem section 2006-1]

Consider the differential equation

$$\frac{du(t)}{dt} = u(t)[a - bu(t)] - h(t).$$

Here a and b are given positive constants, and $h(t)$ is a given function of period T, called harvesting function. Prove that this equation possesses at most two T-periodic solutions. If there are two, they do not intersect.

[Lazer, 1980] Qualitative studies of the solutions of the equation of population growth with harvesting, (Spanish) Mat. Ense nanza Univ. No. 17, (1980), 29-39.

Treat it as a bifurcation problem

Theorem Consider a bifurcation problem

$$\frac{du(t)}{dt} = u(t)[a - bu(t)] - \varepsilon h(t),$$

where $\varepsilon \geq 0$. Let $h(t)$ be a continuous function of period T such that $h(t) \geq 0$. Then there exists a $\varepsilon_0 > 0$ such that it has exactly two T-periodic solutions when $\varepsilon < \varepsilon_0$, exactly one T-periodic solution when $\varepsilon = \varepsilon_0$, and no T-periodic solution when $\varepsilon > \varepsilon_0$.

(i) Think the non-periodic case:

$u' = u(a - bu) - h, \quad h_0 = \dfrac{a^2}{4b}$

(ii) $\varepsilon_0 < \dfrac{a^2\overline{h}}{4b}$, where $\overline{h} = T^{-1}\int_0^T h(t)dt$. Hence the maximum sustainable yield with seasonal effect is smaller than the one without seasonal effect.

Sketch of the proof (1)

Define $F : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ by $F(\varepsilon, \xi) = z(\varepsilon, T, \xi) - \xi$, where z is the solution of $z' = f(\varepsilon, t, z), \;\; t > 0, \;\; z(0) = \xi$.
T-periodic solution is equivalent to $F(\varepsilon, \xi) = 0$.
Notice that $F_\xi(\varepsilon, \xi) = A(\varepsilon, T, \xi) - 1$, where A satisfies
$A' = (a - 2bz(t))A, \;\; A(0) = 1$, thus $A(t) = \exp(at - 2b \int_0^t z(s)ds)$.

(A) When $\varepsilon = 0$, there are exactly two "T-periodic orbits" $u(t) = 0$ and $u(t) = a/b$, and implicit function theorem implies that they persist for small $\varepsilon > 0$. There is no periodic solution when $\varepsilon > 0$ is large.
(B) Repeatedly applying implicit function theorem until a degenerate solution $(\varepsilon_*, u_*(0))$ is reached. At a degenerate solution, $F_\xi(\varepsilon, u_*(0)) = 0$.
(C) We apply saddle-node bifurcation theorem at degenerate solution $(\varepsilon_*, u_*(0))$.

(F1) $dimN(F_u(\lambda_0, u_0)) = codimR(F_u(\lambda_0, u_0)) = 1$, and
(F2) $F_\lambda(\lambda_0, u_0) \notin R(F_u(\lambda_0, u_0))$.

Sketch of the proof (2)

(D) Need to show that $F_\varepsilon(\varepsilon_*, u_*) \notin R(F_\xi(\varepsilon_*, u_*))$.
$F_\varepsilon(\varepsilon_*, u_*) = \partial z(\varepsilon, T, u_*(0))/\partial\varepsilon$ satisfies
$B' = (a - 2bu_*(t))B - h(t), \;\; t > 0, \;\; B(0) = 0$. Then
$B(t) = -A(t)\int_0^t [A(s)]^{-1} h(s)ds$, thus $B(T) < 0$.
(E) Near a degenerate solution (ε_*, u_*), the T-periodic solutions form a curve
$(\varepsilon(s), u(s))$ such that $\varepsilon(0) = \varepsilon_*$, $\varepsilon'(0) = 0$, and $\varepsilon''(0) = -\varepsilon_* \dfrac{F_{\xi\xi}(\varepsilon_*, u_*)}{F_\varepsilon(\varepsilon_*, u_*)}$.
$F_{\xi\xi}(\varepsilon_*, u_*) = C(T)$, and $C(t)$ satisfies
$C' = (a - 2bu_*(t))C - 2bA^2, \;\; C(0) = 0$, thus
$C(t) = -2bA(t)\int_0^t A(s)ds < 0$ hence $C(T) < 0$. So $\varepsilon''(0) < 0$.

Every critical point is a local maximum, so there is only one critical point!

More of such results for Reaction-diffusion models:
[Korman-Li-Ouyang, 1996, 1997] [Ouyang-Shi, 1998, 1999]
[Oruganti-Shi-Shivaji, 2002]


Thank You