Approximate integrals

\[ \int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(y_i) \Delta x \] (limit of Riemann sum)

where \( \Delta x = \frac{(b - a)}{n} \), \( x_0 = a \), \( x_1 = x_0 + \Delta x \), \( x_{k+1} = x_k + \Delta x \), \( x_n = b \), and \( x_{i-1} \leq y_i \leq x_i \).

Geometric idea: use rectangle with similar height to approximate irregular but almost rectangular shape.
Approximate Rules

Left endpoint rule ($L_n$): in $[x_{i-1}, x_i]$, choose $y_i = x_{i-1}$.

$$L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x.$$ 

Right endpoint rule ($R_n$): in $[x_{i-1}, x_i]$, choose $y_i = x_i$.

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x.$$ 

Midpoint rule ($M_n$): in $[x_{i-1}, x_i]$, choose $y_i = (x_{i-1} + x_i)/2$.

$$M_n = \sum_{i=1}^{n} f((x_{i-1} + x_i)/2)\Delta x.$$
Trapezoidal rule

Trapezoidal rule \( (T_n) \): \[ T_n = \sum_{i=1}^{n} \frac{f(x_{i-1}) + f(x_i)}{2} \Delta x = \frac{L_n + R_n}{2} \]; it is the average of the Left endpoint and Right endpoint rules.

The approximate unit is not a rectangle, but a trapezoid with a side connecting \((x_{i-1}, f(x_{i-1}))\) and \((x_i, f(x_i))\)

\[ T_n = \frac{1}{2} [f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)] \Delta x \]

**Example 1**: Consider \( \int_{0}^{1} \cos(x) \, dx \). Find \( L_4, R_4, M_4, \) and \( T_4 \).
Example 2: Consider \( \int_0^1 \cos(x) \, dx \). Determine \( L_4, R_4, M_4, \) and \( T_4 \) are over-estimate or under-estimate without solving them.

\( L_n \) and \( R_n \): is \( f(x) \) increasing or decreasing?
- \( f \) increasing: \( L_n \) under, \( R_n \) over
- \( f \) decreasing: \( L_n \) over, \( R_n \) under

\( M_n \) and \( T_n \): is \( f(x) \) concave up or concave down?
- \( f \) concave up: \( T_n \) over, \( M_n \) under
- \( f \) concave down: \( T_n \) under, \( M_n \) over

Example 3: Consider \( I = \int_0^1 \cos(x) \, dx \). List the numbers \( I, L_4, R_4, M_4, \) and \( T_4 \) in increasing order.
Error estimate: which approximation is good?

\[
\int_0^1 \cos(x) \, dx: \text{ True value}\,=\, \sin x\big|_0^1 = \sin 1 = .8415
\]

\[L_4 = .89455, \ R_4 = .779625, \ M_4 = .843675, \ T_4 = .8370625.\]

Error\,=\,|\text{True value}-\text{Approximation}|

\[E_L = 0.05305 \approx 0.05, \ E_R = 0.0618 \approx 0.06, \ E_M = 0.002175 \approx 0.002, \ E_T = 0.0034 \approx 0.003.\]

Conclusions:
1. Approximation is more accurate when you increase \( n \)
2. \( M_n \) and \( T_n \) are more accurate than \( L_n \) and \( R_n \)
The approximate unit is not a rectangle or a trapezoid, but a parabola connecting 
$(x_{i-1}, f(x_{i-1}))$, $(x_i, f(x_i))$, and $(x_{i-1} + x_i)/2, f((x_{i-1} + x_i)/2))$ (left endpoint, right end point and midpoint);

$$S_{2n} = \sum_{i=1}^{n} \frac{f(x_{i-1}) + 4f((x_{i-1} + x_i)/2) + f(x_i)}{6} \Delta x$$

$$S_n = \frac{1}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) \cdots 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)] \Delta x$$

Example 4 $\int_{0}^{1} \cos(x)dx$. Find $S_4$, and the error.

$E_S = 0.000021$

$E_L = 0.05305 \approx 0.05$, $E_R = 0.0618 \approx 0.06$,

$E_M = 0.002175 \approx 0.002$, $E_T = 0.0034 \approx 0.003$.

Summary
1. Approximation is more accurate when you increase $n$
2. $M_n$ and $T_n$ are more accurate than $L_n$ and $R_n$, but $S_n$ is the best, so in practice, use Simpson rule is most efficient

Error: $E_L, E_R \leq \frac{C}{n}$, $E_M, E_T \leq \frac{C}{n^2}$, and $E_S \leq \frac{C}{n^4}$.