Alternative Stable States of Native Oyster Reefs in Chesapeake Bay

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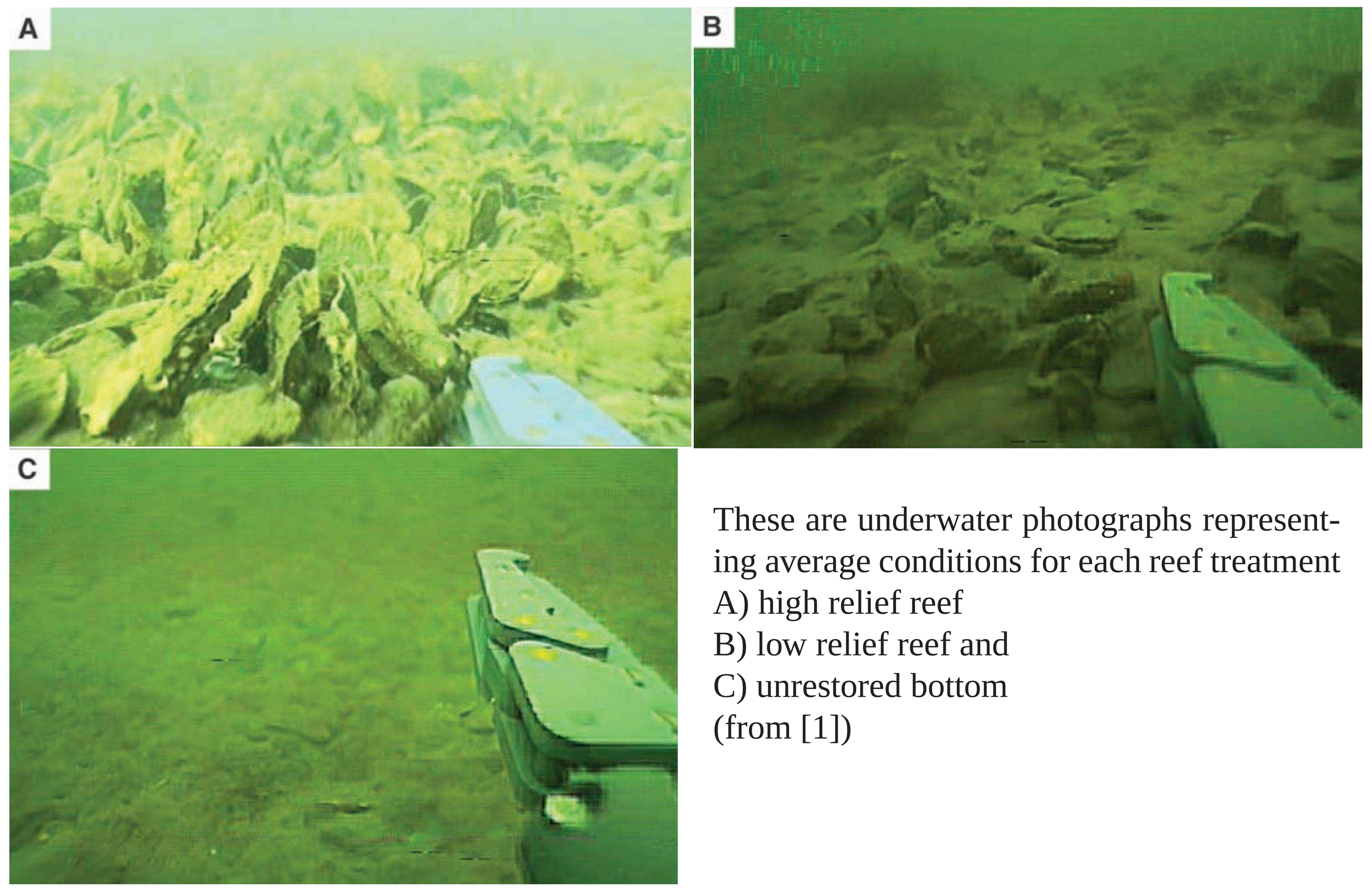
Abstract

To determine the impact of reef height on the restoration of the native oyster population in Chesapeake bay, we propose a mathematical model of live oyster, dead oyster, and sediment volumes. The model is a system of three ordinary differential equations. Interaction between oyster volume and sedimentation can produce alternative stable states. The existence of alternative stable states with positive feedbacks means that native oyster restoration from a degraded state requires artificial reefs whose initial height determines long term viability.

Restoration and Ecological Question

In colonial times, Chesapeake Bay oysters were so plentiful that reefs posed grounding hazards to ships. In 1880's the yearly oyster harvest was 20 to 25 millions bushels. Today, harvests are typically around $200,000$ bushels and the oyster population is reduced to approximately 1% of historic values. Restoration efforts have been extensive but largely ineffectual. However, one effort within the Great Wicomico River has experienced unprecedented success [1,2]. The determining factor has been the initial height of artificial reefs. It seems so called “high relief reefs” are able to persist and thrive where “low relief reefs” disappear in three to five years. Our model will determine what set of parameters produce alternative stable states for the Chesapeake Bay oyster reef, and above what reef height the oyster population will persist [3].



These are underwater photographs representing average conditions for each reef treatment
A) high relief reef
B) low relief reef and
C) unrestored bottom
(from [1])

Model

$$\frac{dO}{dt} = rOf\left(1-\frac{O}{k}\right) - \mu f O - \epsilon(1-f)O, \quad (1)$$

$$\frac{dB}{dt} = rf\frac{O^2}{k} + \mu f O - \gamma B + \epsilon(1-f)O, \quad (2)$$

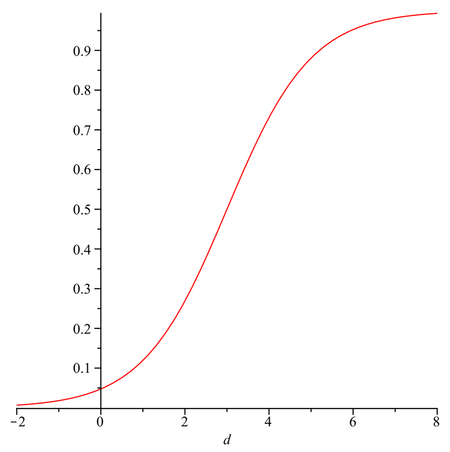
$$\frac{dS}{dt} = Cge^{-\frac{FO}{Cg}} - \beta S. \quad (3)$$

Variable Name	*Variable meaning*	*Dimension*	*Variable defining domain*
t	time	years	$t \geq 0$
$O(t)$	live oysters	m^3	$O \geq 0$
$B(t)$	dead oysters	m^3	$B \geq 0$
$S(t)$	sediment	m^3	$S \geq 0$

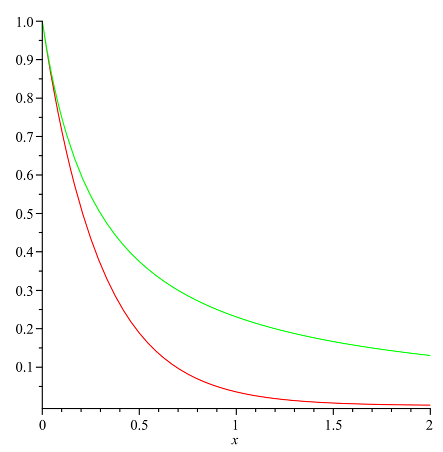
Table 1: Table of variables in the equations.

Composite Functions

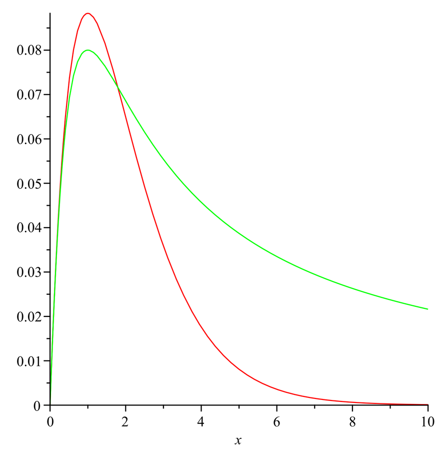
The symbol $f = f(d)$ where $d = \lambda(O+B) - S - \frac{\lambda O}{2}$; f is a sigmoid function bound by 0 and 1 and it represents the proportion of oysters submerged in sediment; k represents the carrying of an oyster reef; μ and γ are the natural death rate and oyster shell degradation rate, ϵ is the death rate of submerged oyster and β is the sediment erosion rate; C is the maximum sediment deposition rate; $g = g(O+B)$ represents the proportion of sediment which settles on the reef; $F = F(Cg)$ is the rate at which oysters filter the sediment.



$f(d) = \frac{1}{1+e^{-hd}}$



$g(x) = e^{-\eta x}$



$F(x) = Mxe^{-\theta x}$

Parameter	Meaning	Dimension	(Approximate) Value
r	birth minus death by competition rate	$(year)^{-1}$	1
k	oyster capacity	m^3	0.11
μ	natural death rate	$(year)^{-1}$	0.4
ϵ	death rate of submerged oyster	$(year)^{-1}$	0.05
λ	packing density	none	1
h	scaling factor	m^{-3}	10
γ	oyster shell degradation rate	$(year)^{-1}$	0.7
M	maximum sediment filtration	$(year)^{-1}$	$35e$
C	maximum sediment deposition rate	$m^3(year)^{-1}$	0.02
θ	sediment amount where the filtration is maximum	$year \cdot m^{-3}$	50
η	decay rate of sediment deposition on the reef height	m^{-3}	3.33
β	sediment erosion rate	m^{-3}	0.01

Table 2: Table of parameters in the equations.

Analysis

Equilibrium points are solved when equations (1)-(3) are set equal to zero. A trivial solution exists with $O = B = 0$ and $S = C/\beta$. The Jacobian matrix $J(0,0,C/\beta)$ is diagonal so the three diagonal entries are eigenvalues:

$$J(0,0,C/\beta) = \begin{pmatrix} f(-C/\beta)(r-\mu+\epsilon)-\epsilon & 0 & 0 \\ f(-C/\beta)(\mu-\epsilon)+\epsilon & -\gamma & 0 \\ Cg'(0) & Cg'(0) & -\beta \end{pmatrix}.$$

With reasonable choice of parameters, there exists a unique $C_* > 0$ such that $f(-C/\beta)(r-\mu+\epsilon)-\epsilon > 0$ for $C > C_*$, and $f(-C/\beta)(r-\mu+\epsilon)-\epsilon < 0$ for $C < C_*$. Hence C_* is a bifurcation point where a branch of positive equilibrium points bifurcate from the line of trivial solutions. We have the following result regarding the stability of the trivial state and existence of equilibrium points:

Theorem: The equilibrium $(0,0,C/\beta)$ is stable when $C > C_*$, and it is unstable for when $0 < C < C_*$. Moreover when $0 < C < C_*$, the system has at least one positive equilibrium point.

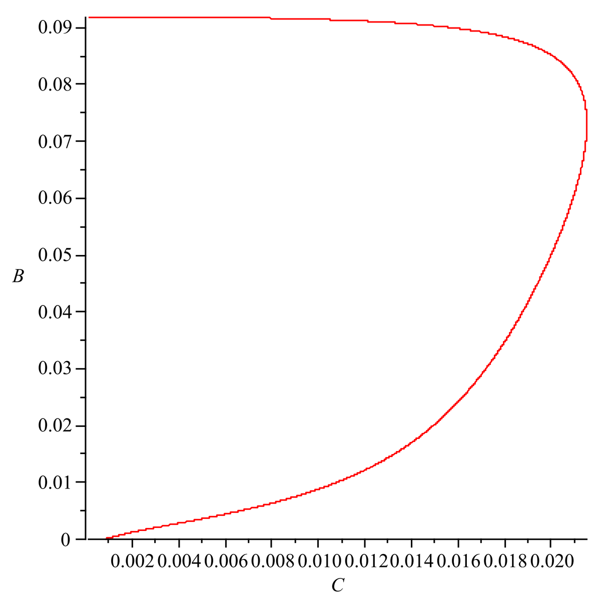
The critical value $C = C_*$ is a bifurcation point where a branch of nontrivial equilibrium points emanates from the line of trivial equilibria $(C,O,B,S) = (C,0,0,C/\beta)$. The bifurcation is called subcritical if the branch of bifurcating positive equilibria bends to the left of $C = C_*$, otherwise it is supercritical.

We find that the direction of the branch of bifurcating positive equilibria is determined by

$$I = \frac{\lambda(r-\mu+\epsilon)k(0)}{2r} - \frac{f(-C_*/\beta)}{f'(-C_*/\beta)}.$$

If $I < 0$, then the bifurcation is subcritical and a unique equilibrium exists for $0 < C < C_*$, and there is no equilibrium for $C > C_*$; on the other hand, if $I > 0$, then the bifurcation is supercritical, then there is a range of values of $C > C_*$ for which the system has two equilibria.

With practical parameter values given in Table 2, the bifurcation at $(C,O,B,S) = (C_*,0,0,C/\beta)$ is supercrtical, hence the bistability holds. (see bifurcation diagram below; the horizontal axis is parameter C, and the vertical axis is the value of B.



Conclusion

The initial value B_0 of $B(t)$ represents the height of built oyster reef. With larger B_0, the oyster population sustains, while extinction occurs with smaller B_0. Hence we have provided scientific support that higher reef is better!

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References

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[3] Jordan-Cooley, William; Lipcius, Romuald N.; Shaw, Leah; Shen, Jian; Shi, Junping, Bistability in Oyster Population and Sediment Volume Model. Preprint, (2010).