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Hair-triggered instability of radial steady states, spread and extinction in semilinear heat equations

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Abstract

We first study the initial value problem for a general semilinear heat equation. We prove that every bounded nonconstant radial steady state is unstable if the spatial dimension is low ($n \leqslant 10$) or if the steady state is flat enough at infinity: the solution of the heat equation either becomes unbounded as t approaches the lifespan, or eventually stays above or below another bounded radial steady state, depending on if the initial value is above or below the first steady state; moreover, the second steady state must be a constant if $n \leqslant 10$.

Using this instability result, we then prove that every nonconstant radial steady state of the generalized Fisher equation is a hair-trigger for two kinds of dynamical behavior: extinction and spreading. We also prove more criteria on initial values for these types of behavior. Similar results for a reaction–diffusion system modeling an isothermal autocatalytic chemical reaction are also obtained.

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Keywords: Semilinear heat equation; Instability; Autocatalytic chemical reaction

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1. Introduction

In 1992, Gui, Ni and Wang [9] proved that every positive radial steady state solution of the Cauchy problem

$$\begin{cases} \dfrac{\partial v}{\partial t} = \Delta v + v^p, & t > 0,\ x \in \mathbf{R}^n, \\ v(x,0) = v_0(x) \geqslant 0, & x \in \mathbf{R}^n, \end{cases} \tag{1.1}$$

is unstable in any reasonable sense if $p < p_c$; and is stable (even "weakly asymptotically stable") in a scale of weighted L^∞ norms if $p \geqslant p_c$, where

$$p_c = \begin{cases} \frac{(n-2)^2-4n+4\sqrt{n^2-(n-2)^2}}{(n-2)(n-10)} & \text{when } n \geqslant 11, \\ \infty & \text{when } 3 \leqslant n \leqslant 10. \end{cases} \tag{1.2}$$

See also [10]. Specifically, in the case of $p < p_c$, $v(x,t) \to 0$ as $t \to \infty$ if $0 \leqslant v_0(x) \leqslant (\not\equiv)\, \tilde{v}(x)$, and v blows up in finite time if $v_0(x) \geqslant (\not\equiv)\, \tilde{v}(x)$, where $\tilde{v}(x)$ is a positive radial steady state; in the case $p \geqslant p_c$, v stays close and even converges to a radial steady state as $t \to \infty$ if v_0 is close to the steady state enough (closeness measured in weighted L^∞ norms).

In this paper, we first generalize the instability part of the above result to the following general semilinear heat equation:

$$\begin{cases} \dfrac{\partial v}{\partial t} = \Delta v + f(v), & t > 0,\ x \in \mathbf{R}^n, \\ v(x,0) = v_0(x), & x \in \mathbf{R}^n, \end{cases} \tag{1.3}$$

where $f \in C^1(\mathbf{R})$. We prove that for any bounded nonconstant radial steady state $\tilde{v}$ of (1.3), if $n \leqslant 10$ or if $\tilde{v}$ is flat enough at infinity, i.e., there exists β such that

$$\beta > \frac{n-2-2\sqrt{n-1}}{2} \quad \text{and} \quad \left|\tilde{v}'(r)\right| = O\left(r^{-\beta}\right) \quad \text{as } r \to \infty, \tag{1.4}$$

then $\tilde{v}$ is unstable in the following hair-trigger sense: if $v_0(x) \leqslant (\not\equiv)\, \tilde{v}(x)$, then the solution $v(x,t)$ of (1.3) either becomes unbounded in finite or infinite time, or $\limsup_{t\to\infty} v(x,t) \leqslant$ another bounded radial steady state $\tilde{\tilde{v}}$ which is strictly below $\tilde{v}$ (similar result holds if $v_0(x) \geqslant (\not\equiv)\, \tilde{v}(x)$); moreover if $n \leqslant 10$, $\tilde{\tilde{v}}$ must be a constant. See Theorem 2.2. By the result of [9] on (1.1), the cut-off dimension $n = 11$ and the cut-off decay rate (1.4) are the best possible.

Cabré and Capella [2] proved that if $n \leqslant 10$, every bounded radial nonconstant steady state $\tilde{v}$ is unstable in the sense that there exists $\eta \in C_0^\infty(\mathbf{R}^n)$ such that

$$\int_{\mathbf{R}^n} \left(|\nabla \eta|^2 - f'(\tilde{v})\eta^2\right) dx < 0. \tag{1.5}$$

If we were in the case of bounded domain, this would imply the linearized instability of $\tilde{v}$. But we are working in the whole space $\mathbf{R}^n$. By Ghoussoub and Gui [7], (1.5) is equivalent to that every solution of the linearized equation

$$\Delta \phi + f'(\tilde{v})\phi = 0, \quad x \in \mathbf{R}^n, \tag{1.6}$$

must change sign. We will prove that this, in turn, implies that $\tilde{v}$ intersects all nearby radial steady states with the first intersection point remaining bounded. We use this fact to construct upper and lower steady states, staying below and above $\tilde{v}$, respectively, and being arbitrarily close to $\tilde{v}$. We then use comparison arguments to show the hair-triggered instability of $\tilde{v}$. Our contribution (Theorem 2.2) is to make the instability result a truly dynamical and global result.

If (1.3) has two constant steady states, say, 0 and 1 with 0 being unstable and 1 being stable with respect to the ODE dynamics, then there exists a competition between diffusion and reaction: the reaction term pulls the solution towards 1, while the diffusion demolishes bumps and possibly pushes the solution down to 0 if the solution is "held" zero at infinity all the time. Thus it is possible that 0 is stable with respect to the dynamics of (1.3) (e.g., in a weighted L^∞ norm, as proved for (1.1) in [10] if $p > (n+2)/n$, i.e., if $f(v)$ is degenerate enough at 0). We say the solution spreads if $v(x,t) \to 1$ as $t \to \infty$; the solution is extinguished if $v(x,t) \to 0$ as $t \to \infty$. If both spread and extinction occur, we have the situation of bistability. In this case, we would like to know the size of the domains of attraction for 0 and 1; and in light of our instability result, we tend to believe that the nonconstant steady states, if any, should be at the separatrices of these domains of attraction.

To verify this "mental picture," we shall look at an important example: the generalized Fisher's equation

$$\begin{cases} \dfrac{\partial v}{\partial t} = \Delta v + (1-v)v^p, & t > 0,\ x \in \mathbf{R}^n, \\ v(x,0) = v_0(x) \geqslant 0, & x \in \mathbf{R}^n. \end{cases} \tag{1.7}$$

The results of Ouyang and Shi [21,22] imply that there exists a continuum of radial steady states v_d, $0 < d \leqslant d_1 \in (0,1)$ if and only if $n \geqslant 3$ and $p > (n+2)/(n-2)$ (see our Proposition 3.1). We prove that in the case $(n+2)/(n-2) < p < p_c$, if the initial value v_0 stays below some v_d, then the solution $v(x,t)$ of (1.7) becomes extinct, i.e., decays to zero uniformly for all x as $t \to \infty$, while if v_0 stays above some v_d, then $v(x,t)$ spreads, i.e., converges to 1 uniformly for bounded x as $t \to \infty$. Thus each v_d is a hair-trigger (threshold) for extinction and spreading behavior of (1.7). These are also true if $p \geqslant p_c$ and $d = d_1$ (see Theorem 3.3). More conditions on the initial value v_0 for extinction and spread of $v(x,t)$ are supplied by using comparison arguments, including comparing (1.7) with (1.1) and then using the results of [10,26]. We mention that our methods can also be applied to the case $f(u) = e^{-1/u}(1-u)$ instead of $f(u) = u^p(1-u)$, the Arrhenius combustion nonlinearity, see the remark at the end of Section 5.

Our results on the generalized Fisher equation will be used to study the spread and extinction of solutions to the Gray and Scott [8] isothermal autocatalytic chemical reaction system (in dimensionless form)

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - uv^p, \quad \dfrac{\partial v}{\partial t} = D\Delta v + uv^p, \quad t > 0,\ x \in \mathbf{R}^n, \\ u(x,0) = u_0(x) \geqslant 0, \quad v(x,0) = v_0(x) \geqslant 0, \quad x \in \mathbf{R}^n, \\ \lim\limits_{|x|\to\infty} u(x,t) = 1 \quad \text{and} \quad \lim\limits_{|x|\to\infty} v(x,t) = 0, \end{cases} \tag{1.8}$$

where u and v are the concentrations of the reactant and the autocatalyst, respectively, and $p \geqslant 1$ is the order of the reaction with respect to the autocatalytic species [13]. To gain mathematical insights we assume only that the spatial dimension $n \geqslant 1$ and the kinetic order $p \geqslant 1$—we allow $n \geqslant 4$.

Because of “the analogy between thermal feedback in exothermic combustion and chemical feedback in isothermal autocatalytic systems” ([13] and references therein), we say the “flame” is *extinguished* in the long run if

$$\lim_{t\to\infty} u(x,t) = 1 \quad \text{and} \quad \lim_{t\to\infty} v(x,t) = 0;$$

and we say the “flame” *spreads* if

$$\lim_{t\to\infty} u(x,t) = 0 \quad \text{and} \quad \liminf_{t\to\infty} v(x,t) > 0.$$

We shall prove that spread and extinction of the flame can be triggered by steady states of (1.8), at least when $D = 1$. From now on, we assume $D = 1$ in (1.8). The steady state solutions of (1.8) satisfy

$$\begin{cases} \Delta u - uv^p = 0, \quad \Delta v + uv^p = 0, \quad x \in \mathbf{R}^n, \\ \lim_{|x|\to\infty} u(x) = 1 \quad \text{and} \quad \lim_{|x|\to\infty} v(x) = 0. \end{cases} \tag{1.9}$$

As noticed in [13], by adding the two equations in (1.9), one obtains

$$\Delta(u+v) = 0, \quad x \in \mathbf{R}^n, \tag{1.10}$$

and $u + v$ is bounded since $\lim_{|x|\to\infty}(u+v)(x) = 1$. Thus $u + v \equiv 1$ from Liouville’s theorem. Hence (1.9) can be reduced to a scalar equation

$$\Delta v + (1-v)v^p = 0, \quad x \in \mathbf{R}^n, \quad \lim_{|x|\to\infty} v(x) = 0, \tag{1.11}$$

which is the steady state equation of the generalized Fisher equation (1.7). Thus steady state solutions of (1.8) can be constructed from the solutions of (1.11).

By using our results on the generalized Fisher equation (1.7), we prove that when $3 \leqslant n \leqslant 10$ and $p > (n+2)/(n-2)$ or when $n \geqslant 11$ and $(n+2)/(n-2) < p < p_c$ (p_c defined by (1.2)), each radial steady state of (1.8) is a hair-trigger for extinction and spread (see Theorem 4.1 and Corollary 4.2). We also provide some more criteria (some of which rather concrete and explicit) for the spread and extinction of the “flame”—see Theorem 4.3.

Radially symmetric steady states of (1.8) are studied in [13] via formal asymptotic and numerical analysis when $D = 1$, $n = 3$ and $p > 5$. It is claimed, but neither proved nor analyzed formally, that these steady states, known as “flame balls,” are unstable and may indicate the minimal size for the initiation of a traveling wave solution. Merkin and Needham [20] studied formally the wave front propagation, assuming equal diffusion coefficients $D = 1$, and $u(x,0) = 1$, $v(x,0) = b_0 g(x)$ with b_0 being a positive constant, $g(x)$ being a nonnegative function with a compact support. They focused on the roles of p and b_0 played on the spread and extinction. The results on this system (1.8) with boundary conditions (on bounded or unbounded spatial domains) other than the one in (1.8) were well surveyed in the recent paper [17].

The spread and extinction of the single equation (1.3) have long been intensively studied under the assumptions $f(0) = 0 = f(1)$ and f being of either the generalized Fisher type (e.g., $f(u) = u^p(1-u)$), or bistable type (e.g., $f(u) = u(1-u)(u-a)$, $a \in (0,1)$), or ignition type (e.g., $f(u) = 0$ for $u \in (0,a)$; $f(u) > 0$ for $u \in (a,1)$). Pioneering works include [1,6,14]. For the special case (1.7), Theorem 3.1 of Aronson and Weinberger [1] implies that if $1 \leqslant p \leqslant (n+2)/n$, then the extinction does not occur and the spread always occurs: as long as the initial value $v_0 \geqslant (\not\equiv)\, 0$, $v(x,t) \to 1$ as $t \to \infty$; their Theorem 3.2 implies that if $p > (n+2)/n$, then extinction occurs for small initial value (e.g., $v_0(x) \leqslant$ small constant multiple of the Gaussian $\exp(-|x|^2)$—this can be relaxed: the Gaussian can be replaced by $(1+|x|)^{-2/(p-1)}$ [18]). Wu and Xing [27] proved that in the 1-D case ($n=1$) if v_0 is bounded below from 0 at $-\infty$, and decays exponentially at $+\infty$, then $v(x,t)$ converges to 1 in the form of a traveling wave with the lowest speed, provided $p > 1$. For the ignition and bistable types, Zlatoš [28] deals with the 1-D case again and studies the effect of the size of the support of the initial value v_0, generalizing and sharpening the old result of Kanel' [14].

2. Instability of radial steady states

In the following we always assume that $x \in \mathbf{R}^n$ and $r = |x|$, and sometimes we use $v(r)$ to denote $v(|x|)$, a radially symmetric function in $\mathbf{R}^n$. Of concern in this section is

$$\begin{cases} \dfrac{\partial v}{\partial t} = \Delta v + f(v), & t > 0,\ x \in \mathbf{R}^n, \\ v(x,0) = v_0(x) \geqslant 0, & x \in \mathbf{R}^n, \end{cases} \tag{2.1}$$

where $f \in C^1(\mathbf{R})$ and v_0 is bounded and continuous in $\mathbf{R}^n$. We shall prove the instability of nonconstant bounded radial steady states of (2.1). The steady state equation of (2.1) is

$$\Delta v + f(v) = 0, \quad x \in \mathbf{R}^n. \tag{2.2}$$

Proposition 2.1. *Let $\tilde{v}$ be a bounded* (*not necessarily radial*) *solution of* (2.2).

(1) $$\int_{\mathbf{R}^n} \big(|\nabla \eta|^2 - f'(\tilde{v})\eta^2\big)\, dx \geqslant 0, \quad \textit{for any } \eta \in C_0^\infty\big(\mathbf{R}^n\big), \tag{2.3}$$

if and only if the linearized equation

$$\Delta \phi + f'(\tilde{v})\phi = 0, \quad x \in \mathbf{R}^n, \tag{2.4}$$

has a solution $\phi(x) > 0$ in $\mathbf{R}^n$.

(2) *In the case that $\tilde{v}$ is also radial and nonconstant, then* (2.3) *does not hold if either $1 \leqslant n \leqslant 8$ or $n = 9, 10$ and f satisfies one more condition* (*not needed for $n \leqslant 8$*): *whenever there exists a critical point c of $f(v)$, there exists $q > 0$ such that*

$$\lim_{v \to c} \frac{|f'(v)|}{|v-c|^q} \in (0,\infty). \tag{2.5}$$

(3) *In the case that $\tilde{v}$ is also radial and nonconstant, then* (2.3) *does not hold if there exists β such that*

$$\beta > \frac{n-2-2\sqrt{n-1}}{2} \quad \textit{and} \quad \big|\tilde{v}'(r)\big| = O\big(r^{-\beta}\big) \quad \textit{as } r \to \infty. \tag{2.6}$$

Part (1) of Proposition 2.1 is just [7, Proposition 2.3], and part (2) is [2, Theorem 1]. Part (3) is not stated in [2] but follows from the proof of Theorem 1 there: if we choose $\alpha = \sqrt{n-1}$ in [2, (8)], then (2.6) ensures (9); but then (10) is violated. In [2], if (2.3) holds, $\tilde{v}$ is said to be stable; otherwise, it is said to be unstable. But as pointed put in Section 1, it is not immediately clear what this notion of stability bears on the dynamics of the corresponding parabolic equation (2.1). Using Proposition 2.1, we prove the dynamical instability of $\tilde{v}$:

Theorem 2.2. *Suppose that $f \in C^1(\mathbf{R})$. Let $\tilde{v}$ be a bounded nonconstant radial solution of* (2.2) *in $\mathbf{R}^n$, and let one of the assumptions in parts* (2) *and* (3) *of Proposition* 2.1 *hold. Then $\tilde{v}$ is unstable with respect to the dynamics of* (2.1) *in the following sense*: *given any bounded and continuous initial value $v_0(x)$,*

(1) *If $v_0(x) \leqslant (\not\equiv)\tilde{v}(r)$, then the solution $v(x,t)$ of* (2.1) *either becomes unbounded as t increases* (*i.e., $\|v(x,t)\|_{L^\infty(\mathbf{R}^n)} \to \infty$ as $t \to t_{\max}$, the life span of v*), *or* $\limsup_{t\to\infty} v(x,t) \leqslant \tilde{\tilde{v}}(r)$, *a radial bounded solution of* (2.2), *uniformly for bounded x, with $\tilde{\tilde{v}}(r) < \tilde{v}(r)$.*
(2) *If $v_0(x) \geqslant (\not\equiv)\tilde{v}(r)$, then the solution $v(x,t)$ of* (2.1) *either becomes unbounded as t increases, or* $\liminf_{t\to\infty} v(x,t) \geqslant \tilde{\tilde{v}}(r)$, *a radial bounded solution of* (2.2), *uniformly for bounded x, with $\tilde{\tilde{v}}(r) > \tilde{v}(r)$.*

In both cases ($v_0(x) \geqslant \tilde{v}(r)$ or $\leqslant \tilde{v}(r)$), if $n \leqslant 8$ or if $n = 9, 10$ and (2.5) *holds, then $\tilde{\tilde{v}}(r)$ is actually a constant solution.*

Remark 2.3. A result in [9] implies that the dimension $n = 11$ and the decay rate (2.6) for stability are critical. It was proved that when $n \geqslant 11$ and $p \geqslant p_c$, each radial steady state $w_\alpha(r)$ of (1.1) is "weakly asymptotically stable." In particular, the conclusion of the above theorem does not hold. Observe that for $p > (n+2)/(n-2)$, $w'_\alpha(r) = O(r^{-1-(2/(p-1))})$ at $r = \infty$ and

$$1 + \frac{2}{p-1} > \frac{n-2-2\sqrt{n-1}}{2} \quad \Leftrightarrow \quad 1 < p < p_c. \tag{2.7}$$

Proof of Theorem 2.2. Suppose that the initial value $v_0(x) \leqslant (\not\equiv)\,\tilde{v}(r)$ in $\mathbf{R}^n$, and $v(x,t)$ remains bounded as $t \to t_{\max}$, the lifespan of v (and hence $t_{\max} = \infty$). By the strong maximum principle, $v(x,t)$ and $\tilde{v}(x)$ separate from each other right after $t = 0$. So, without loss of generality, assume $v_0(x) < \tilde{v}(x)$ (or we take the initial value to be $v(x,1)$). Suppose we can build a radial bounded continuous weak upper solution $\bar{v}(r)$ of (2.2), staying above $v_0(x)$ and below $\tilde{v}(r)$. Denote by $\bar{v}(x,t)$ the solution of (2.1) with initial value $\bar{v}(r)$. Then by the comparison principle, $v(x,t) < \bar{v}(x,t) < \tilde{v}(r)$ for $x \in \mathbf{R}^n$ and $t > 0$; moreover, $\bar{v}(x,t)$ is radial in x and decreasing as t increases (see [10, Proposition 2.2]). Since $v(x,t)$ is assumed to be bounded on $\mathbf{R}^n \times [0,\infty)$, so is $\bar{v}(x,t)$. Thus $\bar{v}(x,t) \to$ some radial bounded steady state $\tilde{\tilde{v}}(r)$ of (2.1) as $t \to \infty$ uniformly for bounded x, and we have that $\limsup_{t\to\infty} v(x,t) \leqslant \tilde{\tilde{v}}(r)$ uniformly for bounded x. Notice that $\tilde{\tilde{v}}(r) < \tilde{v}(r)$.

How do we construct such an upper solution $\bar{v}(r)$? The basic idea is to glue another radial solution $v(r,\alpha)$ of (2.2) with $\tilde{v}(r)$ ($v(0,\alpha) = \alpha < \tilde{v}(0)$). To be able to do so, we of course require that $v(r,\alpha)$ intersects $\tilde{v}(r)$. Let the first intersection be $z(\alpha)$. $\bar{v}(r)$ is defined to be $v(r,\alpha)$ for $0 \leqslant r \leqslant z(\alpha)$, and $\tilde{v}(r)$ for $r \geqslant z(\alpha)$. Then it is a bounded continuous weak upper solution of (2.2) (see [10, Proposition 2.1]). Note that by the continuous dependence of solutions of ODE on the initial value, $\bar{v}(r)$ converges to $\tilde{v}(r)$ as α converges to $\tilde{v}(0)$ uniformly for bounded r. But to ensure that $\bar{v}(r)$ stays above $v_0(x)$, we need that $z(\alpha)$ remains bounded as α converges to $\tilde{v}(0)$.

To prove the existence and boundedness of $z(\alpha)$, we first write down the ODE satisfied by radial solutions of (2.2):

$$\begin{cases} v'' + \dfrac{n-1}{r}v' + f(v) = 0, & r \in (0,\infty), \\ v(0) = \alpha > 0, & v'(0) = 0. \end{cases} \tag{2.8}$$

The existence and uniqueness of solution to (2.8) is well known, and the unique solution is denoted by $v(r,\alpha)$. Let $\alpha_0 = \tilde{v}(0)$. Define

$$w(r) = \left.\frac{\partial v(r,\alpha)}{\partial \alpha}\right|_{\alpha=\alpha_0}. \tag{2.9}$$

Then $w(r)$ satisfies the linear differential equation

$$\begin{cases} w'' + \dfrac{n-1}{r}\, w' + f'\big(\tilde{v}(r)\big)w = 0, & r \in (0,\infty), \\ w(0) = 1, & w'(0) = 0. \end{cases} \tag{2.10}$$

It is easy to show that if $w(r)$ changes sign, then for α close to α_0, $z(\alpha)$ exists and converges to the first zero of w as $\alpha \to \alpha_0$. If one of the assumptions in parts (2) and (3) of Proposition 2.1 holds, then by part (1) of Proposition 2.1, w must change sign. This completes the proof of part (1). The proof of part (2) is just the same.

Finally, if $n \leqslant 10$ (we assume (2.5) if $n = 9, 10$) and if $\tilde{\tilde{v}}(r)$ is nonconstant, then $\tilde{\tilde{v}}(r)$ is also unstable. Hence by what has been proved, we should not have that $\bar{v}(x,t) \to \tilde{\tilde{v}}(r)$ as $t \to \infty$ uniformly for bounded x. Thus $\tilde{\tilde{v}}(r)$ is a constant in this case. □

3. Generalized Fisher's equation

In this section, we first study the instability of positive radial steady states of the generalized Fisher equation

$$\begin{cases} \dfrac{\partial v}{\partial t} = \Delta v + (1-v)v^p, & t > 0,\ x \in \mathbf{R}^n, \\ v(x,0) = v_0(x) \geqslant 0, & x \in \mathbf{R}^n. \end{cases} \tag{3.1}$$

These steady states satisfy the ODE problem

$$\begin{cases} v'' + \dfrac{n-1}{r}v' + v^p - v^{p+1} = 0, & r \in (0,\infty), \\ v(0) = d > 0, \quad v'(0) = 0, & v'(r) < 0. \end{cases} \tag{3.2}$$

It is well known that (3.2) has a unique C^2 solution $v(r,d)$ for $r \in [0, r_d)$ with some $r_d > 0$ if $d \in (0,1)$, and $v(r,d) > 0$, $v_r(r,d) < 0$ for $r \in (0, r_d)$. Obviously, $v(r,d) \equiv 1$ if $d = 1$; and $v(r,d)$ is unbounded if $d > 1$. These are not interesting solutions and we shall not deal with them.

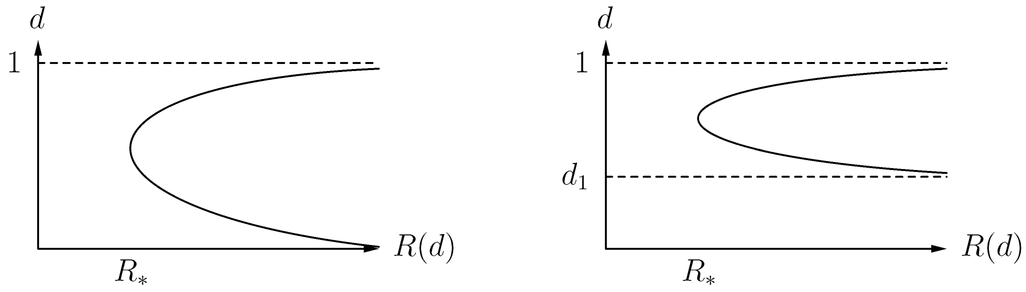

Fig. 1. Bifurcation diagrams for (3.2). Left: $p \leqslant \frac{n+2}{n-2}$; Right: $p > \frac{n+2}{n-2}$.

We define

$$R(d) = \sup\{r_0 > 0\colon\ v(r,d) > 0,\ v_r(r,d) < 0 \text{ for } r \in (0, r_0)\} \quad \text{and} \tag{3.3}$$

$$N = \{d \in (0,1)\colon\ R(d) < \infty\},$$

$$G = \{d \in (0,1)\colon\ R(d) = \infty\}. \tag{3.4}$$

A solution with $d \in N$ satisfies $v(R(d),d) = 0$ and $v_r(R(d),d) < 0$, and it is called a *crossing solution*. If $d \in G$, then $v(r,d) > 0$ and $v_r(r,d) < 0$ for all $r \in (0,\infty)$ and $\lim_{r\to\infty} v(r,d) = 0$, and it is called a *decaying solution*. The structure of the N and G for (3.2) is obtained in [22, Theorem 6.19] (see also [15,21,25]).

Proposition 3.1. *Assume that* $n \geqslant 1$.

(1) $N \cup G = (0,1)$.
(2) *If* $1 < p \leqslant \frac{n+2}{n-2}$ $(= \infty$ *if* $n = 1,2)$, *then* $N = (0,1)$ *and* $G = \emptyset$, $R(d)$ *is a smooth function with a unique local minimum point* $R(d_*) = R_*$, *and* $\lim_{d\to 0^+} R(d) = \lim_{d\to 1^-} R(d) = \infty$ (*see Fig.* 1).
(3) *If* $n \geqslant 3$ *and* $p > \frac{n+2}{n-2}$, *then there exists* $d_1 \in (0,1)$ *such that* $N = (d_1, 1)$, *and* $G = (0, d_1]$, *and* $R(d)$ *is a smooth function with a unique local minimum point* $R(d_*) = R_*$, *and* $\lim_{d\to d_1^+} R(d) = \lim_{d\to 1^-} R(d) = \infty$ (*see Fig.* 1).

Decaying solutions can be further characterized by their limiting behavior at infinity. It can be shown that for each $d \in G$, $[r^{n-2}v(r)]' \geqslant 0$ for all $r > 0$, thus $\lim_{r\to\infty} r^{n-2}v(r,d) = c(d) \in (0,\infty]$ exists (see, for example, [25, Lemma 2.1]). If $c(d) < \infty$, then we call $v(r,d)$ a *fast decaying solution*, and if $c(d) = \infty$, then we call $v(r,d)$ a *slow decaying solution*. The structure and properties of the decaying solutions are summarized in the following proposition.

Proposition 3.2. *Assume that* $n \geqslant 3$ *and* $p > \frac{n+2}{n-2}$.

(1) (3.2) *has a unique fast decaying solution* $v(r, d_1)$.
(2) (3.2) *has a family of slow decaying solutions* $v(r,d)$, $d \in (0, d_1)$, *and for each slow decaying solution,* $\lim_{r\to\infty} r^{2/(p-1)} v(r,d) = L$, *where*

$$L = \left[\frac{2(n-2)}{(p-1)^2}\left(p - \frac{n}{n-2}\right)\right]^{1/(p-1)}. \tag{3.5}$$

(3) *For* $d \in (0, d_1)$, $v(r, d_1) < v(r, d)$ *for* $r > r_d^*$ *where* $r_d^* > 0$ *is a constant.*
(4) *If* $\frac{n+2}{n-2} < p < p_c$, *where* p_c *is defined in* (1.2), *then for any* $d_a, d_b \in (0, d_1)$, $v(r, d_a)$ *and* $v(r, d_b)$ *intersect each other infinitely many times in* $(0, \infty)$.

The classification of the decaying solutions is proved in [25, Theorem 2]; the limit in part (2) is shown in [16, Theorem 1]; part (3) is obvious since the fast decaying solution decays faster than the slow decaying solution; and the result of part (4) for the equation $\Delta u + u^p = 0$ is proved in [26, Proposition 3.5], and the proof can also be carried over for this case. We also notice that a fast decay solution always satisfies (2.6) regardless of spatial dimension n, thus unstable in the sense of Theorem 2.2.

Theorem 3.3. *Suppose that* $n \geqslant 3$ *and* $p > \frac{n+2}{n-2}$. *Let* $\{v_d = v(r, d)\}$ *be the radially symmetric steady state solutions of* (3.1) *where* $d \in (0, d_1]$ *for* $d_1 \in (0, 1)$ *defined in Proposition* 3.1, *and let* $v(x, t; v_0)$ *be the solution of* (3.1) *with initial value* v_0. *Assume that the initial value* v_0 *in* (3.1) *is bounded, nonnegative and continuous in* $\mathbf{R}^n$. *If* $\frac{n+2}{n-2} < p < p_c$, *then we have the "hair-trigger" effect*: *if* $v_0(x) \leqslant (\not\equiv)\, v_d(x)$ *for some* $d \in (0, d_1]$, *then* $v(x, t; v_0) \to 0$ *as* $t \to \infty$ *uniformly for* $x \in \mathbf{R}^n$; *and if* $v_0(x) \geqslant (\not\equiv)\, v_d(x)$ *for some* $d \in (0, d_1]$, *then* $v(x, t; v_0) \to 1$ *as* $t \to \infty$ *uniformly for* x *in any bounded subset of* $\mathbf{R}^n$. *These are also true if* $n \geqslant 11$, $p \geqslant p_c$ *and* $d = d_1$ (*i.e*, v_d *is the fast decaying steady state*).

Proof. Observe first that by the comparison principle, we have

$$0 \leqslant v(x, t; v_0) \leqslant v\big(x, t; 1 + \|v_0\|_{L^\infty(\mathbf{R}^n)}\big), \quad x \in \mathbf{R}^n,\ t > 0, \tag{3.6}$$

the latter actually being a solution of the ODE $v' = (1 - v)v^p$ and hence bounded. Thus $v(x, t; v_0)$ is bounded on $\mathbf{R}^n \times [0, \infty)$. Suppose that $v_0 \leqslant (\not\equiv)\, v_d$. By Proposition 3.2, $v_d(r) = O(r^{-2/(p-1)})$ if $d \in (0, d_1)$, and $v_d(r) = O(r^{2-n})$ if $d = d_1$. It is not hard to show that $v_d'(r) = O(r^{-2/(p-1)-1})$ if $d \in (0, d_1)$, and $v_d'(r) = O(r^{1-n})$ if $d = d_1$. Thus under the assumptions on p and d in the statement of part (1) of the present theorem, at least one of the assumptions in parts (2) and (3) of Proposition 2.1 holds. By Theorem 2.2, $\limsup_{t\to\infty} v(x, t; v_0) \leqslant$ some bounded radial steady state, i.e., some $v_{\tilde d}(r)$, which is $< v_d(r)$, and the limsup is uniform for bounded x. If $\tilde d > 0$, then by parts (3) and (4) of Proposition 3.2, $v_{\tilde d}(r)$ and $v_d(r)$ intersect, that is a contradiction. Thus $\tilde d = 0$. Now we have $\lim_{t\to\infty} v(x, t) = 0$ uniformly for bounded x. The limit is also uniform for all $x \in \mathbf{R}^n$ because $v(x, t) \leqslant v_d(r)$, and $v_d(r) \to 0$ as $r \to \infty$.

Now suppose that $v_0 \geqslant (\not\equiv)\, v_d$. By Theorem 2.2 again, $\liminf_{t\to\infty} v(x, t; v_0) \geqslant v_{\tilde d}(r)$ uniformly for bounded x and for some $\tilde d \in (0, 1]$, and $v_{\tilde d}(r) > v_d(r)$. If $d = d_1$, this is impossible by Proposition 3.1 unless $\tilde d = 1$. If $d \in (0, d_1)$, by Proposition 3.2 parts (3) and (4), $\tilde d \notin (0, d_1]$. Again by Proposition 3.1, $\tilde d \notin (d_1, 1)$. Thus $\tilde d = 1$, $\liminf_{t\to\infty} v(x, t; v_0) \geqslant 1$ uniformly for bounded x. From (3.6), $\limsup_{t\to\infty} v(x, t; v_0) \leqslant 1$ uniformly for bounded x. □

Next, we supply more criteria on v_0 for the spread and extinction of $v(x, t; v_0)$. In proving extinction results, we need to compare the generalized Fisher equation (3.1) with the Fujita equation

$$\begin{cases} \dfrac{\partial w}{\partial t} = \Delta w + w^p, & t > 0,\ x \in \mathbf{R}^n, \\ w(x, 0) = w_0(x) \geqslant 0, & x \in \mathbf{R}^n, \end{cases} \tag{3.7}$$

because by the comparison principle, $v(x,t;v_0) \leqslant w(x,t;w_0)$ if $v_0 \leqslant w_0$. The following facts concerning radial steady states of (3.7) are well known (see, e.g., [26, Proposition 3.4]):

(1) For $n \geqslant 3$ and $p > \frac{n}{n-2}$, $W_s(x) = Lr^{-2/(p-1)}$, $r = |x|$, is a singular steady state of (3.7). Here L is defined in (3.5).
(2) If $1 < p < \frac{n+2}{n-2}$ ($< \infty$ if $n = 1, 2$), classical steady states that are positive on $\mathbf{R}^n$ do not exist.
(3) When $n \geqslant 3$ and $p \geqslant \frac{n+2}{n-2}$, all the classical positive radial solutions are given by the family $\{w_\alpha\}_{\alpha>0}$ satisfying $w_\alpha(0) = \alpha$,

$$w_\alpha(r) = \begin{cases} \alpha w_1(\alpha^{(p-1)/2} r), & p > \frac{n+2}{n-2}, \\ \alpha\left(\frac{n(n-2)}{n(n-2)+\alpha^{4/(n-2)} r^2}\right)^{(n-2)/2}, & p = \frac{n+2}{n-2}. \end{cases} \tag{3.8}$$

(4) w_α mentioned above is strictly decreasing in r, $\lim_{r\to\infty} r^{2/(p-1)} w_\alpha(r) = L$, except when $p = \frac{n+2}{n-2}$.

Theorem 3.4. *Suppose that $n \geqslant 3$. Assume that the initial value v_0 in* (3.1) *is bounded, nonnegative and continuous in $\mathbf{R}^n$.*

(1) *$v(x,t) \to 0$ uniformly for $x \in \mathbf{R}^n$ as $t \to \infty$, if one of the following conditions on the initial value v_0 holds*:
   (a) *When $n \geqslant 3$ and $p > \frac{n}{n-2}$, $v_0(x) \leqslant \lambda W_s(x) = \lambda L r^{-2/(p-1)}$, or when $p \geqslant \frac{n+2}{n-2}$, $v_0(x) \leqslant \lambda w_\alpha(x)$ for some constant $\lambda \in (0,1)$ and $\alpha > 0$. In this case, we have the decay rate of v,*

$$v(x,t) \leqslant \left[\left(\lambda^{1-p} - 1\right)(p-1)\right]^{-1/(p-1)} t^{-1/(p-1)}, \quad x \in \mathbf{R}^n,\ t > 0; \tag{3.9}$$

   if $\frac{n}{n-2} < p < p_c$, then λ can be taken to be 1, *but then we only know for some constant $C > 0$, we have*

$$v(x,t) \leqslant C t^{-1/(p-1)}, \quad x \in \mathbf{R}^n,\ t > 0. \tag{3.10}$$

   This is also true if we merely require $n \geqslant 1$ and $p > (n+2)/n$, but $v_0(x) \leqslant \lambda(1+r)^{-2/(p-1)}$ with λ small enough.
   (b) *When $n \geqslant 11$ and $p \geqslant p_c$, $v_0(x) \leqslant W_s(x)$ for $x \in \mathbf{R}^n$ and $v_0(x) \leqslant W_s(x) - a|x|^{-(m+\mu)}$ for large $x \in \mathbf{R}^n$, and for some constants $a > 0$, $\mu \in (0, \mu_1)$, where $m = 2/(p-1)$, and μ_1 is the smaller* (*positive*) *root of the characteristic equation*

$$\mu^2 - (n-2-2m)\mu + 2(n-2-m) = 0. \tag{3.11}$$

(2) *$v(x,t;v_0) \to 1$ uniformly for x in any bounded subset of $\mathbf{R}^n$ as $t \to \infty$ if*
   (c) *When $p > 1$,*

$$v_0(x) \geqslant \begin{cases} v(|x|, d), & |x| < R(d), \\ 0, & |x| \geqslant R(d), \end{cases} \tag{3.12}$$

   for some $d \in (d_1, 1)$ (d_1 is understood to be 0 *if $p \leqslant \frac{n+2}{n-2}$), where $v(r,d)$ is the solution of* (3.2).

Remark 3.5. The very last part of (a) was shown in [20] for special initial values via formal arguments. Result (c) was attributed to Fife [5] in [13]. We could not find the proof there or anywhere else, though the idea of proof was certainly contained in [5].

Proof of Theorem 3.4. (1) To prove cases (a) and (b), take the initial values $w_0(x)$ of (3.7) to be the same as $v_0(x)$ of (3.1). Then $0 \leqslant v(x,t;v_0) \leqslant w(x,t;v_0)$. Now the conclusion follows from [26, Theorem 4.1 and Corollary 4.2], and [10, Theorem 3] if (a) holds; [10, Theorem 4(ii)] if (b) holds. To show (3.10) when $p > (n+2)/n$, we use a result of [18] for w.

(2) To prove the case when (c) holds, define

$$\widehat{v_d}(r) = \begin{cases} v(r,d), & r < R(d), \\ 0, & r \geqslant R(d), \end{cases} \tag{3.13}$$

where $d \in (d_1, 1)$ and $v(r,d)$ is a crossing solution of (3.2). Then $\widehat{v_d}(r)$ is a continuous weak lower steady state of (3.1). Let the solution of (3.1) with initial value $\widehat{v_d}(r)$ be $\widehat{v_d}(x,t)$. It is radial in x, increasing in t (by [10, Proposition 2.2]) and is bounded above by $v = 1$. Thus $\lim_{t\to\infty} \widehat{v_d}(x,t)$, which is denoted by $\hat{v}(r)$, exists and is a positive radial steady state of (3.1). From Proposition 3.2, $\hat{v}(r) \equiv 1$ because $\hat{v}(0) > d_1$ and $\hat{v}(r) > 0$ in $\mathbf{R}^n$. From this it follows that $v(x,t;v_0) \to 1$ as $t \to \infty$ uniformly for any bounded subset of $\mathbf{R}^n$ if (c) holds. □

Most spread/extinction criteria in Theorems 3.3 and 3.4 are for the case $p < p_c$. We conjecture that when $n \geqslant 11$ and $p \geqslant p_c$, the results in Theorems 3.3 may not always hold. However we still obtain the following criterion when the initial value is more restrictive:

Theorem 3.6. *Suppose that $n \geqslant 11$. Assume that the initial value v_0 in* (3.1) *is bounded, nonnegative and continuous in $\mathbf{R}^n$.*

(1) $v(x,t) \to 0$ *uniformly for $x \in \mathbf{R}^n$ as $t \to \infty$, if $p \geqslant p_c$, $v_0(x) \leqslant \lambda v_d(x)$ for some constant* $\lambda \in (0,1)$ *and* $0 < d < \min\{(p-1)/p, d_1\}$.
(2) $v(x,t;v_0) \to 1$ *uniformly for x in any bounded subset of $\mathbf{R}^n$ as $t \to \infty$, if $p \geqslant p_c$, $v_0(x) \geqslant \lambda v_d(x)$ for some constant $\lambda > 1$ and* $0 < d < \min\{(p-1)/p, d_1\}$.

Proof. (1) We first check that $\lambda v_d(r)$, $\lambda \in (0,1)$, is an upper steady state of (3.1) if $d \leqslant \min\{(p-1)/p, d_1\}$:

$$\begin{aligned} \Delta(\lambda v_d) + (1-\lambda v_d)(\lambda v_d)^p &= -\lambda(1-v_d)v_d^p + (1-\lambda v_d)(\lambda v_d)^p \\ &= \lambda v_d^p\left(1-\lambda^p\right)\left[v_d - \frac{1-\lambda^{p-1}}{1-\lambda^p}\right] \\ &\leqslant \lambda v_d^p\left(1-\lambda^p\right)\left[d - g(\lambda)\right], \end{aligned} \tag{3.14}$$

where $g(\lambda) = \frac{1-\lambda^{p-1}}{1-\lambda^p}$. The function $g(\lambda)$ is decreasing on $(0,\infty)$ and $g(1) = (p-1)/p$. Thus if $d \leqslant (p-1)/p$, λv_d is an upper steady state of (3.1) for any $\lambda \in (0,1)$. Now that the solution $\tilde{v}_{\lambda,d}(x,t)$ of (3.1) initiated at $\lambda v_d(r)$ is decreasing in t, radial in x (see [10, Proposition 2.2]), and so it converges to a radial steady state $v_{\lambda,d}(r)$ as $t \to \infty$. Since $v_{\lambda,d}(r) \leqslant \lambda v_d(r)$, $\lim_{r\to\infty} r^{2/(p-1)} v_{\lambda,d}(r) \leqslant \lambda L < L$. From Proposition 3.2, this implies $v_{\lambda,d}(r) \equiv 0$.

(2) We fix a $d < \min\{(p-1)/p, d_1\}$. Then for $\lambda > 1$ but close to 1, $g(\lambda) > d$. So for such λ,

$$\begin{aligned}\Delta(\lambda v_d) + (1-\lambda v_d)(\lambda v_d)^p &= \lambda v_d^p\bigl(1-\lambda^p\bigr)\left[v_d - \frac{1-\lambda^{p-1}}{1-\lambda^p}\right] \\ &\geqslant \lambda v_d^p\bigl(1-\lambda^p\bigr)\bigl[d - g(\lambda)\bigr] \geqslant 0,\end{aligned} \tag{3.15}$$

which means λv_d is a lower steady state of (3.1). Let $\widehat{v_{\lambda,d}}(x,t)$ be the solution of (3.1) initiated at $\widehat{v_{\lambda,d}}(x,0) = \lambda v_d(x)$. Then $\widehat{v_{\lambda,d}}(x,t)$ is radial in x, increasing in t and bounded by $v = 1$ for $t \geqslant 0$. Thus $\lim_{t\to\infty} \widehat{v_{\lambda,d}}(x,t)$, which is denoted by $\widehat{v_{\lambda,d}}(r)$, exists and is a positive radial steady state of (3.1). Since $\widehat{v_{\lambda,d}}(r) \geqslant \lambda v_d(r)$, $\lim_{r\to\infty} r^{2/(p-1)}\widehat{v_{\lambda,d}}(r) \geqslant \lambda L > L$. From Proposition 3.2, this implies $\widehat{v_{\lambda,d}}(r) \equiv 1$, and the theorem is proved when $\lambda > 1$ and is close to 1. For arbitrary $\lambda > 1$, the same conclusion holds because the larger λ is, the larger $\widehat{v_{\lambda,d}}(x,t)$ is. □

4. Isothermal autocatalytic chemical reaction system

In this section, we consider system (1.8) with equal diffusion coefficients ($D = 1$):

$$\begin{cases} u_t = \Delta u - uv^p, \quad v_t = \Delta v + uv^p, \quad t > 0,\ x \in \mathbf{R}^n, \\ u(x,0) = u_0(x) \geqslant 0, \quad v(x,0) = v_0(x) \geqslant 0, \quad x \in \mathbf{R}^n, \\ \lim_{|x|\to\infty} u(x,t) = 1 \quad \text{and} \quad \lim_{|x|\to\infty} v(x,t) = 0. \end{cases} \tag{4.1}$$

We assume that u_0 and v_0 are nonnegative, bounded and continuous on $\mathbf{R}^n$. Because of the boundary condition at $x = \infty$, we also require that

$$\lim_{|x|\to\infty} u_0(x) = 1 \quad \text{and} \quad \lim_{|x|\to\infty} v_0(x) = 0. \tag{4.2}$$

Indeed, these assumptions on the initial values are satisfied in the physical situation (see [13, Section II]). Then (4.1) is well posed: by converting the u and v equations in (4.1) into integral equations (via "variation of constants formula"), and by the standard Picard-like argument, one can show that (4.1) has one and only one classical solution $(u(x,t), v(x,t))$, existing for $x \in \mathbf{R}^n$ and $t \in [0,T)$, where T is the "life-span" of the solution. Here $T = \infty$ because $u + v$ satisfies the standard linear heat equation, $u(x,t) \geqslant 0$ and $v(x,t) \geqslant 0$, and hence $u(x,t)$ and $v(x,t)$ are bounded.

When $n \geqslant 3$, and $p > \frac{n+2}{n-2}$, (4.1) has a family of nonconstant radially symmetric steady state solutions

$$E = \bigl\{\bigl(u_d\bigl(|x|\bigr), v_d\bigl(|x|\bigr)\bigr)\colon d \in (0,d_1],\ v_d \text{ solves (3.2) and } u_d = 1 - v_d\bigr\}. \tag{4.3}$$

Our first result in this section says that any steady state $(u_d(r), v_d(r))$ is unstable in any reasonable sense:

Theorem 4.1. *Suppose that $n \geqslant 3$, and either $\frac{n+2}{n-2} < p < p_c$ and $d \in (0,d_1]$, or $p \geqslant p_c$ and $d = d_1$. Let the initial value u_0 and v_0 satisfy* (4.2).

(1) *If* $v_0(x) \geqslant v_d(r)$ *and* $u_0(x) + v_0(x) \geqslant 1$*, but not simultaneously* $\equiv v_d(r)$ *and* 1*, respectively, then*

$$\lim_{t\to\infty} u(x,t) = 0, \qquad \lim_{t\to\infty} v(x,t) = 1, \tag{4.4}$$

uniformly for x *in any bounded subset of* $\mathbf{R}^n$ *as* $t \to \infty$.

(2) *If* $v_0(x) \leqslant v_d(r)$ *and* $u_0(x) + v_0(x) \leqslant 1$*, but not simultaneously* $\equiv v_d(r)$ *and* 1*, respectively, then*

$$\lim_{t\to\infty} u(x,t) = 1, \qquad \lim_{t\to\infty} v(x,t) = 0, \tag{4.5}$$

uniformly for x *in* $\mathbf{R}^n$ *as* $t \to \infty$.

Proof. (1) Let $h(x,t) = u(x,t) + v(x,t)$. Then $h(x,t)$ satisfies

$$\begin{cases} \dfrac{\partial h}{\partial t} = \Delta h, & t > 0,\ x \in \mathbf{R}^n, \\ h(x,0) = u_0(x) + v_0(x) \geqslant 0, & x \in \mathbf{R}^n, \end{cases} \tag{4.6}$$

and

$$h(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbf{R}^n} \exp\left(-\frac{|x-y|^2}{4t}\right)\big[u_0(y) + v_0(y)\big]\,dy. \tag{4.7}$$

By the assumptions on u_0 and v_0, we have $h(x,t) \geqslant 1$ for $x \in \mathbf{R}^n$, $t > 0$. Thus

$$v_t = \Delta v + uv^p = \Delta v + (h - v)v^p \geqslant \Delta v + (1 - v)v^p, \quad t > 0,\ x \in \mathbf{R}^n, \tag{4.8}$$

and hence, $v(x,t)$ is an upper solution of (3.1). Since v_d is a steady state of (3.1), by the comparison principle and the strong maximum principle, $v(x,t) > v_d(x)$, for $x \in \mathbf{R}^n$ and $t > 0$. Denote by $\underline{v}(x,t)$ the solution of (3.1) with initial value $\underline{v}(x,0) = v(x,1)$. From Theorem 3.3, it follows that $\underline{v}(x,t) \to 1$ as $t \to \infty$ uniformly for bounded x. By the comparison principle, $v(x,t+1) \geqslant \underline{v}(x,t)$. So

$$\liminf_{t\to\infty} v(x,t) \geqslant 1, \quad \text{uniformly for bounded } x \in \mathbf{R}^n. \tag{4.9}$$

On the other hand,

$$\limsup_{t\to\infty} v(x,t) \leqslant \lim_{t\to\infty} h(x,t) \equiv 1, \quad \text{uniformly for } x \in \mathbf{R}^n, \tag{4.10}$$

with the last equality easily proved by using (4.2) and (4.7). Finally, $u(x,t) = h(x,t) - v(x,t) \to 0$ as $t \to \infty$. This completes the proof of (i).

(2) In this case, $h(x,t) \leqslant 1$ for $x \in \mathbf{R}^n$, $t > 0$, and $v(x,t)$ is a lower solution of (3.1). Let $\bar{v}(x,t)$ be the solution of (3.1) with initial value $\bar{v}(x,0) = v(x,1)(< v_d(x))$. From Theorem 3.3, it follows that $\underline{v}(x,t) \to 0$ as $t \to \infty$ uniformly for $x \in \mathbf{R}^n$, so does $v(x,t)$. On the other hand, $h(x,t) \to 1$ uniformly in $\mathbf{R}^n$ as $t \to \infty$. Thus $u(x,t) = h(x,t) - v(x,t) \to 1$ as $t \to \infty$ uniformly for $x \in \mathbf{R}^n$. □

Corollary 4.2. *Suppose that $n \geqslant 3$, and either $\frac{n+2}{n-2} < p < p_c$ and $d \in (0, d_1]$, or $p \geqslant p_c$ and $d = d_1$. Let the initial value u_0 and v_0 satisfy* (4.2).

(1) *If $u_0(x) \geqslant u_d(r)$ and $v_0(x) \geqslant v_d(r)$, but not simultaneously $\equiv u_d(r)$ and $v_d(r)$, respectively, then*

$$\lim_{t\to\infty} u(x,t) = 0, \qquad \lim_{t\to\infty} v(x,t) = 1, \tag{4.11}$$

uniformly for x in any bounded subset of $\mathbf{R}^n$ as $t \to \infty$.

(2) *If $u_0(x) \leqslant u_d(r)$ and $v_0(x) \leqslant v_d(r)$, but not simultaneously $\equiv u_d(r)$ and $v_d(r)$, respectively, then*

$$\lim_{t\to\infty} u(x,t) = 1, \qquad \lim_{t\to\infty} v(x,t) = 0, \tag{4.12}$$

uniformly for x in $\mathbf{R}^n$ as $t \to \infty$.

Other extinction/spread results can be obtained as a consequence of Theorem 3.4 as follows:

Theorem 4.3. *Let the initial value u_0 and v_0 of* (4.1) *satisfy* (4.2).

(1) *$v(x,t) \to 0$ and $u(x,t) \to 1$, uniformly for $x \in \mathbf{R}^n$ as $t \to \infty$, if $u_0(x) + v_0(x) \leqslant 1$ for $x \in \mathbf{R}^n$, and one of* (a)–(b) *in Theorem* 3.4 *or part* (1) *in Theorem* 3.6 *holds. The t-decay rate of v in* (3.9) *and* (3.10) *still holds if* (a) *of Theorem* 3.4 *holds.*
(2) *$u(x,t) \to 0$ and $v(x,t) \to 1$ uniformly for bounded x as $t \to \infty$, if $u_0(x) + v_0(x) \geqslant 1$ for $x \in \mathbf{R}^n$, and either* (c) *in Theorem* 3.4 *or part* (2) *in Theorem* 3.6 *holds.*
(3) *$u(x,t) \to 0$ and $v(x,t) \to 1$ uniformly for bounded x as $t \to \infty$, if $v_0(x) \not\equiv 0$, $u_0(x) + v_0(x) \geqslant 1$ for $x \in \mathbf{R}^n$, and $1 \leqslant p \leqslant (n+2)/n$.*

(The last part was shown in [20] for special initial values as mentioned in Section 1, by using formal arguments.)

Proof. (1) We use the notation in the proof of Theorem 4.1. Since $u_0(x) + v_0(x) \leqslant 1$, $h(x,t) \leqslant 1$ for $x \in \mathbf{R}^n$, $t \geqslant 0$. Thus $v(x,t)$ is a lower solution of (3.1). Let the solution of (3.1) with initial value $v_0(x)$ be $\tilde{v}(x,t)$. Then by the comparison principle, $0 \leqslant v(x,t) \leqslant \tilde{v}(x,t)$, $x \in \mathbf{R}^n$, $t \geqslant 0$. It follows from Theorem 3.4/3.6 that $\lim_{t\to\infty} \tilde{v}(x,t) = 0$, and hence $\lim_{t\to\infty} v(x,t) = 0$ uniformly for $x \in \mathbf{R}^n$. Since $\lim_{t\to\infty} h(x,t) = 1$ uniformly for $x \in \mathbf{R}^n$, we have $u(x,t) = h(x,t) - v(x,t) \to 1$ uniformly for $x \in \mathbf{R}^n$ as $t \to \infty$.

(2) and (3) In these two cases, $h \geqslant 1$ for $x \in \mathbf{R}^n$, $t \geqslant 0$. Then $v(x,t)$ is an upper solution of (3.1). Denote by $\hat{v}(x,t)$ the solution of (3.4) with $\hat{v}(x,0) = v_0(x)$. Then $\hat{v}(x,t) \leqslant v(x,t) \leqslant h(x,t)$, $x \in \mathbf{R}^n$, $t \geqslant 0$. We conclude that $\lim_{t\to\infty} \hat{v}(x,t) = 1$ uniformly for bounded x, by using (c) in Theorem 3.4 or part (2) in Theorem 3.6 for (2), or by using [1, Theorem 3.1] for (3). Since $\lim_{t\to\infty} h(x,t) = 1$, then $\lim_{t\to\infty} u(x,t) = 0$ uniformly for bounded x. □

5. Concluding remarks

Bistability has been observed in many physical, chemical and biological systems, and it can be characterized as the existence of two stable steady states and the partition of phase space into separate basins of attraction of these stable states. Our results here rigorously establish the bistable dynamics for some fundamental reaction–diffusion equations and systems arising from chemistry and biology. Moreover we are able to partially characterize the attracting sets of two stable states (extinction/spread). When the system has nonconstant radial steady states, by our instability result, it is very likely that they are located on the boundary of the domains of attraction of both stable steady states, triggering spread and extinction, especially in low (but $n \geqslant 3$) spatial dimensions or when the steady state is too flat at infinity.

The bistability phenomenon has been found in (1.1) earlier where one may think of the value ∞ as a (stable) constant steady state. The characterization of the separatrix between the two attracting basins is a rather delicate question. For Fujita equation (1.1), it has been shown positive initial values on the separatrix must have certain decaying rates [18,26]. We also note that $p < p_c$ is necessary for some instability results. When $p \geqslant p_c$, many other phenomena have been discovered for Fujita equation such as weakly asymptotically stable steady states [9,10], nonconverging orbits [23,24], and faster blowup [12,19].

For the generalized Fisher equation (1.7) in the case of $p \geqslant p_c$, even though we have Theorem 3.6, which indicates that each v_d (d small) is unstable in L^∞ norm, it is still possible that v_d is stable (even weakly asymptotically stable) in a weighted L^∞ norm as in the Fujita case. The proof of this would require that these radial steady states do not intersect each other (i.e., v_d is monotone in d). This is a delicate issue: v_{d_1} intersects all v_d, $d \in (0, d_1)$, and thus v_d for $d \approx d_1$ intersects some other v_d's. We believe that for $d <, \approx d_1$, v_d is unstable in the hair-trigger sense (in the sense of Theorem 3.3), while for small d, v_d is stable in a weighted norm.

We mention that it is unclear that if all nontrivial nonnegative steady states of generalized Fisher equation (1.7) with the zero boundary condition at infinity are radially symmetric. For the subcritical case ($p < (n+2)/(n-2)$), it is known that every nonnegative steady states must be radially symmetric, thus constant (see [3] and references therein).

We remark that some results for generalized Fisher equation (1.7) also hold for more general linearities which satisfy $f(0) = f(1) = 0$, $f(u) > 0$ in $(0, 1)$, and $\lim_{u\to 0^+} K_f(u) > (n+2)/(n-2)$, where $K_f(u) = uf'(u)/f(u)$ (see [22] for explanation of K_f). Another significant example is $f(u) = e^{-1/u}(1-u)$, the Arrhenius combustion case. One can check that the conditions of [22, Theorem 6.19] are also satisfied by this nonlinearity, hence Proposition 3.1 also holds. The decaying rate for radial solutions is unknown, but from our Theorem 2.2, when $3 \leqslant n \leqslant 8$ any nonconstant radial steady state is unstable, and we have the hair trigger effect for that steady state just as Theorem 3.3 at least when $3 \leqslant n \leqslant 8$.

The assumption $D = 1$ in (1.8) is not unreasonable from the physical point of view: u and v are molecules that do not differ too much in sizes (and hence their diffusion rates should be close, if not equal). On the other hand, mathematically, this assumption allows one to add the two equations in (1.8) to obtain the heat equation for $u + v$. If the initial values of u and v satisfy the boundary conditions in (1.8), then $u(x,t) + v(x,t)$ converges to 1 as $t \to \infty$. Thus the dynamics of (1.8) is closely related to the generalized Fisher equation (1.7). If D is not equal to 1, the global existence of solutions of (1.8) is known [11], but it is hard to obtain the kind of instability results presented here; in this case, one may have to be content with a linearized (and hence local) instability result in a weighted space. Also (1.8) is not a monotone dynamical system for which comparison methods can be directly applied to the whole system, especially when $D \neq 1$. Bistability has also been observed in other nonmonotone biological system such as predator–prey ecological systems [4].

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