Global bifurcations of concave semipositone problems

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ABSTRACT. We study semilinear elliptic equations on general bounded domains with concave *semipositone* nonlinearities. We prove the existence of the maximal solutions, and describe the global bifurcation diagrams. When a parameter is small, we obtain the exact global bifurcation diagram. We also discuss the related symmetry breaking bifurcation when the domains have certain symmetries.

1 INTRODUCTION

We study the boundary value problem:

$$\begin{cases} \Delta u + \lambda f(u) = 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where λ is a positive parameter, and Ω is a smooth bounded region in $\mathbf{R}^n$ for $n \geq 1$. We assume that the nonlinear function f in this paper satisfies

(f1) $f \in C^2[0, \infty)$, $f(0) < 0$, $f(u)(u - b) > 0$ for $u \in (0, M) \backslash \{b\}$ for some $b > 0$, where either $M = \infty$ or $M < \infty$, $f(M) = 0$ and $f'(M) < 0$;

(f2) $f''(u) < 0$ for $u \geq 0$;

(f3) $\int_0^M f(u)du > 0$;

(f4) If $M = \infty$, then $\lim_{u\to\infty} \dfrac{f(u)}{u} = 0$.

The semilinear problem (1.1) arises in population biology, where $g(u) = f(u) - f(0) > 0$ is the growth rate, and $\varepsilon = -f(0)$ is the harvesting effort. An example for the case of $M < \infty$ in (f1) is the logistic growth $g(u) = au - bu^2$ for some $a, b > 0$, and examples for the case of $M = \infty$ are $g(u) = au/(1 + bu)$ for $a, b > 0$ and $g(u) = 1 - e^{-au}$ for $a > 0$, which are sublinear functions modeling the saturating effect. With the nonlinearity $f(u)$ satisfying $f(0) < 0$, we call it a semipositone problem (see [4]) to compare with the positone case of $f(0) \geq 0$.

Semilinear semipositone problems have been studied for more than a decade, see [4] for a survey of the results. The complete bifurcation diagrams of the equation (1.1) with f satisfying conditions similar to (f1)-(f4) for Ω being a ball have been obtained in [1], [2], [5], and [14]. In this paper, we study the bifurcation diagram with Ω being a general bounded smooth domain in $\mathbf{R}^n$. Here we are mainly concerned with bifurcation diagrams with λ as the bifurcation parameter. (Note that λ can be interpreted as the reciprocal of the diffusion coefficient.) In a separate paper, the authors and Oruganti [12] study bifurcation diagrams with the harvesting effort as the bifurcation parameter. Also in [12] the harvesting effort is allowed to be nonhomogeneous.

To state our results, we recall a few commonly used definitions. Suppose that (λ, u) is a solution of (1.1). (λ, u) is a *stable* solution if the principal eigenvalue $\mu_1(u)$ of

$$\begin{cases} \Delta\psi + \lambda f'(u)\psi = -\mu\psi & \text{in} \quad \Omega, \\ \psi = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.2}$$

is positive, otherwise it is *unstable*. When $\mu_i(u) = 0$ for some $i \geq 1$, (λ, u) is a *degenerate solution*. The solutions of (1.1) are also the solutions of operator equation $F(\lambda, u) = 0$ where

$$F(\lambda, u) = \Delta u + \lambda f(u), \quad \lambda \in \mathbf{R}, \quad u \in X, \tag{1.3}$$

and $X = \{u \in C^{2,\alpha}(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\}$. Thus (λ, u) is degenerate solution if the linearized operator $F_u(\lambda, u) = \Delta + \lambda f'(u)$ is not an invertible operator. We call $v(x)$ a *maximal solution* of (1.1) if for any solution $u(x)$ of (1.1), we have $v(x) \geq u(x)$ for all $x \in \overline{\Omega}$.

Our first result proves that (1.1) has a unique stable nonnegative solution, which is also the maximal solution among all solutions:

THEOREM 1.1. *Suppose that f satisfies (f1)-(f4). Then there exists $\lambda_* > 0$ such that*

(i) *(1.1) has at least one positive solution u_λ for $\lambda > \lambda_*$, and has no nonnegative solution for $0 < \lambda < \lambda_*$;*

(ii) *For $\lambda > \lambda_*$, (1.1) has a maximal positive solution $\overline{u}_\lambda$, and $\overline{u}_\lambda$ is increasing with respect to λ;*

(iii) *At $\lambda = \lambda_*$, (1.1) has a maximal non-negative solution $\overline{u}_{\lambda_*}$;*

(iv) *The principal eigenvalue* $\mu_1(\overline{u}_\lambda) \geq 0$ *when* $\lambda > \lambda_*$ *and* $\mu_1(\overline{u}_{\lambda_*}) = 0$. *If in addition* Ω *is star-shaped, then* $\mu_1(\overline{u}_\lambda) > 0$ *and* $\overline{u}_\lambda$ *is a stable solution for* $\lambda > \lambda_*$;

(v) *If* Ω *is star-shaped, then for* $\lambda > \lambda_*$, $\overline{u}_\lambda(x) > 0$ *for any* $x \in \Omega$, *and* $\dfrac{\partial \overline{u}_\lambda(x)}{\partial \nu} < 0$ *for any* $x \in \partial\Omega$;

(vi) *If for some* $\lambda \geq \lambda_*$, *(1.1) has another nonnegative solution* w_λ, *then* $\mu_1(w_\lambda) < 0$.

The existence of a positive solution for f satisfying (f1), (f3) and (f4) and large λ was proved in [3] for the case of $M = \infty$, and in [7] for the case of $M < \infty$. Here with the extra condition (f2), we show that the branch of maximal solutions can be extended to a critical λ_*, which is the minimum λ for which one can have a nonnegative solution. Since $f(0) < 0$, it is not known whether $\overline{u}_{\lambda_*}$ satisfies (v) in Theorem 1.1. But we will show that in the case $\varepsilon = -f(0)$ is small, $\overline{u}_{\lambda_*}$ will satisfy (v).

In Section 4, we study the behavior of the bifurcation diagrams in (λ, u) space when $\varepsilon = -f(0)$ is small, for both positive and sign-changing solutions. Here the definition of $f(u)$ is extended to $\mathbf{R}$, and we assume that f satisfies

(f1') $f \in C^2(\mathbf{R})$, $f(0) < 0$, $f(u)(u-b) > 0$ for $u \in (-\infty, M)\backslash\{b\}$ for some $b > 0$, where either $M = \infty$ or $M < \infty$, $f(M) = 0$ and $f'(M) < 0$;

(f2') $f''(u) < 0$ for $u \in \mathbf{R}$;

and (f3), (f4) are as defined before. In this part we continue some discussions started in Shi [16]. To state the results, we rewrite the equation (1.1) in the form:

$$\begin{cases} \Delta u + \lambda[g(u) - \varepsilon] = 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.4}$$

where $g(u) = f(u) - f(0)$, and $\varepsilon = -f(0)$. We denote by λ_k the k-th eigenvalue of

$$\begin{cases} \Delta \phi + \lambda \phi = 0 & \text{in} \quad \Omega, \\ \phi = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.5}$$

It is well-known that λ_1 is simple, and its eigenfunction does not change sign. We define $\lambda_k^0 = \lambda_k / g'(0)$.

THEOREM 1.2. *Suppose that* $f(u) = g(u) - \varepsilon$ *satisfies (f1'), (f2'), (f3) and (f4). We assume that* λ_2 *is a simple eigenvalue of (1.5) and*

$$\int_\Omega \phi_2(x)dx \cdot \int_\Omega \phi_2^3(x)dx > 0, \tag{1.6}$$

where ϕ_2 *is an eigenfunction corresponding to* λ_2. *Let* $\Sigma = \{(\lambda, u) \in \mathbf{R} \times X : (\lambda, u)$ *solves* $(1.4)\}$, *and* $T(a,b,c) = \{(\lambda, u) : a < \lambda < b, ||u||_X < c\}$. *Then for any small* $\delta_1, \delta_2 > 0$, *there exists* $\varepsilon_1 = \varepsilon_1(\delta_1, \delta_2, g)$ *such that for any* $\varepsilon \in (0, \varepsilon_1)$,

$$\Sigma_0 \equiv \Sigma \bigcap T(\lambda_1^0 - \delta_1, \lambda_2^0 + \delta_1, \delta_2) = \bigcup_{i=1}^{3} \Sigma_i,$$

where Σ_i *is a connected component of* Σ_0*,* $(i = 1, 2, 3)$*. Moreover,*

(i) *Each* Σ_i $(i = 1, 2, 3)$ *is a smooth curve in* $\mathbf{R} \times X$*;*

(ii) Σ_1 *is exactly* $\supset$*-shaped, there is a unique degenerate solution on* Σ_1*, and each solution on* Σ_1 *is negative;*

(iii) Σ_3 *is exactly* $\subset$*-shaped, there is a unique degenerate solution on* Σ_3*, and each solution on* Σ_3 *is sign-changing;*

(iv) Σ_2 *is exactly S-shaped, and there are exactly two degenerate solutions on* Σ_2*;* Σ_2 *can be parameterized as* $(\lambda(s), u(s))$*,* $s \in (s_1, s_4)$*, such that for* $s \in (s_1, s_2)$*,* $u(s)$ *is positive, and for* $s \in (s_3, s_4)$*,* $u(s)$ *is sign-changing, where* $s_1 < s_2 < s_3 < s_4$*; The portion of* Σ_2 *with* $s \in (s_1, s_2)$ *contains the degenerate solution on the left, and the portion of* Σ_2 *with* $s \in (s_3, s_4)$ *contains the degenerate solution on the right (see Fig. 1);*

(v) *There exist* $\delta_i = \delta_i(\varepsilon) > 0$*,* $(i = 3, 4, 5, 6)$ *such that*

$$proj\Sigma_1 = [\lambda_1^0 - \delta_1, \lambda_1^0 - \delta_3],\ proj\Sigma_2 = [\lambda_1^0 + \delta_4, \lambda_2^0 - \delta_5],\ proj\Sigma_3 = [\lambda_2^0 + \delta_6, \lambda_2^0 + \delta_1],$$

where $proj\Sigma_i$ *is the projection of* Σ_i *into* $\mathbf{R} = (\lambda)$*.*

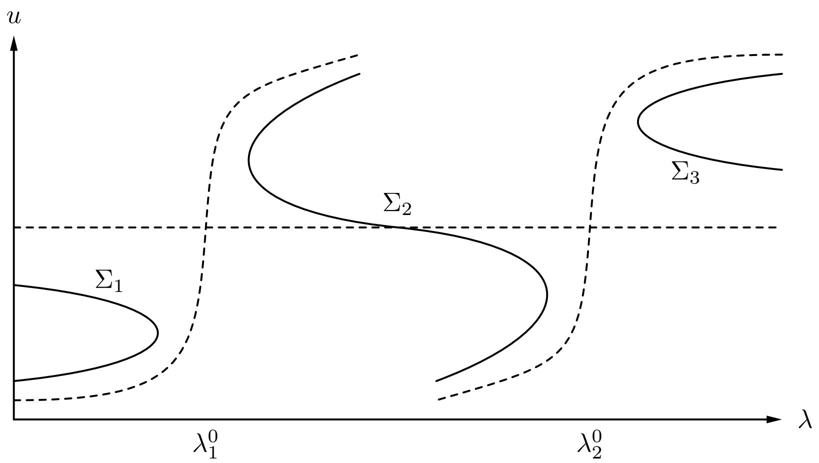

Fig. 1: Precise bifurcation diagram when (1.6) holds for $||u||$ small
solid curve: small $\varepsilon > 0$, dashed curve: $\varepsilon = 0$

The diagrams in Fig. 1 are well-known as the imperfect bifurcation under small perturbation. Here we emphasize on a rigorous proof of the exactness of the shape of the bifurcation diagrams. If (1.6) is changed to

$$\int_\Omega \phi_2(x)dx \cdot \int_\Omega \phi_2^3(x)dx < 0, \tag{1.7}$$

then the diagram in Fig. 1 becomes the one in Fig. 2. More explanation is given in Section 4. For all connected bounded smooth domains except for a zero measure set, either (1.6) or (1.7) holds, so the diagrams in Fig. 1 and Fig. 2 are generic. But we should comment that for domains with symmetry, the second eigenvalue may not be simple (example: balls), and even when it is simple, (1.6) or (1.7) may not be true (example: rectangles). We will briefly discuss the structure of the bifurcation diagrams for these symmetric domains in Section 5.

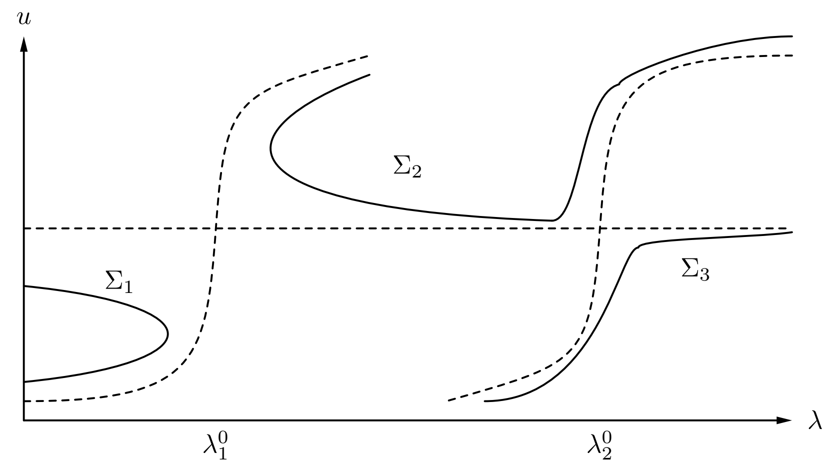

Fig. 2: Precise bifurcation diagram when (1.7) holds for $||u||$ small
solid curve: small $\varepsilon > 0$, dashed curve: $\varepsilon = 0$

We remark that Theorem 1.2 not only shows an interesting bifurcation diagram, but also implies the following fact:

COROLLARY 1.3. *Suppose that the domain Ω has a simple second eigenvalue λ_2, and satisfies (1.6) or (1.7). Then when $\varepsilon > 0$ is sufficiently small, (1.4) has a nonnegative solution (λ, u) satisfying either (i) there exists $x_1 \in \Omega$ such that $u(x_1) = 0$; or (ii) there exists $x_2 \in \partial\Omega$ such that $\partial u(x_2)/\partial\nu = 0$.*

In the description of Theorem 1.2, such a solution is on the middle part of the S-shaped curve when $s \in [s_2, s_3]$. We do not know if such solution is unique. A nontrivial nonnegative solution of (1.4) is always positive if $f(0) \geq 0$ and thus satisfies the strong maximum principle and the Hopf boundary lemma, *i.e.* $u(x) > 0$ for $x \in \Omega$ and $\partial u(x)/\partial\nu < 0$ for $x \in \partial\Omega$. For the case of $f(0) < 0$, it is easy to show the existence of a solution with zero derivative at the boundary points when $n = 1$ by the quadrature method. When Ω is a ball in $\mathbf{R}^n$, the existence of such a solution (zero gradient on the boundary) is also well-known (see [20], [14] and [11].) Corollary 1.3 confirms the existence of such a solution for a large class of domains.

In Section 2, we study the bifurcation diagram of the positone concave equation, which will be used in later proofs. In Section 3, we prove Theorem 1.1, and in Section 4, we prove Theorem 1.2. We discuss the bifurcation diagrams for symmetric domains in Section 5.

2 BIFURCATION DIAGRAM OF LOGISTIC TYPE EQUATIONS

For f satisfying (f1), (f2) and (f4), the existence of a non-negative solution for (1.1) for λ large was established in [3], for the case $M = \infty$, and in [7] for the case of $M < \infty$:

LEMMA 2.1. *Assume that f satisfies (f1), (f2) and (f4). Then there exists $\lambda_a > 0$ such that if $\lambda > \lambda_a$ then (1.1) has a non-negative solution u_λ.*

Next we recall the well-known bifurcation diagram when $\varepsilon = -f(0) = 0$. First we quote Lemma 3 in [19], which will be repeatedly used in our proofs:

LEMMA 2.2. *Suppose that $f : \Omega \times \mathbf{R}^+ \to \mathbf{R}$ is a continuous function such that $\dfrac{f(x,s)}{s}$ is strictly decreasing for $s > 0$ at each $x \in \Omega$. Let $w, v \in C(\overline{\Omega}) \cap C^2(\Omega)$ satisfy*
(a) $\Delta w + f(x,w) \leq 0 \leq \Delta v + f(x,v)$ *in* Ω,
(b) $w, v > 0$ *in* Ω *and* $w \geq v$ *on* $\partial\Omega$,
(c) $\Delta v \in L^1(\Omega)$.
Then $w \geq v$ *in* $\overline{\Omega}$.

By using Lemma 2.2 and bifurcation theory, we prove the following result: (recall that $\lambda_1^0 = \lambda_1/g'(0)$)

THEOREM 2.3. *Assume that $g \in C^1[0,\infty)$ satisfies*

$$g(0) = 0, \;\; g'(0) > 0, \;\; \frac{d}{du}\left(\frac{g(u)}{u}\right) < 0 \text{ for all } u > 0, \tag{2.1}$$

and, either $g(u) > 0$ for all $u > 0$ and $\lim\limits_{u\to\infty} \dfrac{g(u)}{u} = 0$, or $g(M) = 0$ for some $M > 0$. Then

$$\begin{cases} \Delta v + \lambda g(v) = 0 & \text{in} \quad \Omega, \\ v = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.2}$$

has no positive solution if $\lambda \leq \lambda_1^0$, and has exactly one positive solution v_λ if $\lambda > \lambda_1^0$. Moreover, all v_λ's lie on a smooth curve, v_λ is stable and v_λ is increasing with respect to λ.

Proof. Suppose that (λ, v) is a positive solution of (2.2), and (λ_1, ϕ_1) is the principal eigen-pair of (1.5). Then from (2.2) and (1.5), we have

$$\begin{aligned} &\left(\lambda - \frac{\lambda_1}{g'(0)}\right) g'(0) \int_\Omega u(x)\phi_1(x)dx \\ &+\lambda \int_\Omega \left[\frac{g(u(x))}{u(x)} - g'(0)\right] u(x)\phi_1(x)dx = 0. \end{aligned} \tag{2.3}$$

Since $g(u)/u$ is decreasing, the second term in the equality is negative. This implies (2.2) has no positive solution if $\lambda \leq \lambda_1^0$.

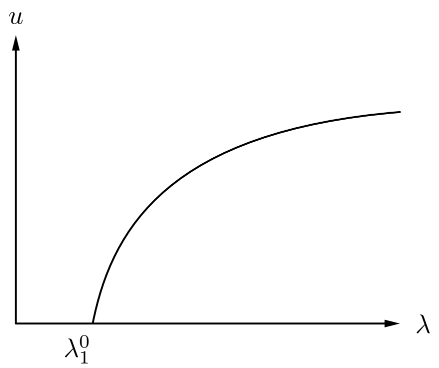

Fig. 3: Precise bifurcation diagram for the Logistic type equation

Next we apply the bifurcation from simple eigenvalue result by Crandall and Rabinowitz [8]: $(\lambda, u) = (\lambda_1^0, 0)$ is a bifurcation point; near $(\lambda_1^0, 0)$, the solutions of (2.2) are on two branches $\Sigma_0 = \{(\lambda, 0)\}$ and $\Sigma_1 = \{(\lambda(s), v(s)) : |s| \leq \delta\}$, where $\lambda(0) = 0$, $v(s) = s\phi_1 + O(s^2)$. We assume that $\phi_1 > 0$, then $v(s)$ is a positive solution when $s \in (0, \delta)$. Moreover, from (2.3), positive solutions only exist for $\lambda > \lambda_1^0$. Therefore there exists $\varepsilon > 0$ such that for $\lambda \in (\lambda_1^0, \lambda_1^0 + \varepsilon)$, (2.2) has a positive solution v_λ. We prove that any positive solution (λ, v) of (2.2) is stable. Let (μ_1, ψ_1) be the principal eigen-pair of (1.2) for $g(u)$ and (λ, v). Then by (2.2) and (1.2), we have

$$\begin{aligned} -\mu_1 \int_\Omega \psi_1 v dx &= \int_\Omega (\Delta v \cdot \psi_1 - \Delta\psi_1 \cdot v) dx \\ &= \lambda \int_\Omega [g'(v)v - g(v)]\psi_1 dx. \end{aligned} \tag{2.4}$$

Since $g(v)/v$ is decreasing, then $g'(v)v - g(v) < 0$ for $v > 0$. Thus $\mu_1 > 0$. In particular, any positive solution (λ, v) is non-degenerate. Therefore, at any positive solution (λ^*, v^*), we can apply the implicit function theorem to $F(\lambda, u) = 0$, and all the solutions of $F(\lambda, u) = 0$ near (λ^*, v^*) are on a curve $(\lambda, v(\lambda))$ with $|\lambda - \lambda^*| \leq \varepsilon$ for some small $\varepsilon > 0$. Hence the portion of Σ_1 with $s > 0$ can be extended to a maximal set

$$\Sigma_1 = \{(\lambda, v_\lambda) : \lambda \in (\lambda_1^0, \lambda_M)\}, \tag{2.5}$$

where λ_M is the supreme of all $\lambda > \lambda_1^0$ such that v_λ exists. We claim that $\lambda_M = \infty$. Suppose not, then $\lambda_M < \infty$, and there are two possibilities: (a) $\lim\limits_{\lambda \to \lambda_M^-} ||v_\lambda|| = \infty$, or (b) $\lim\limits_{\lambda \to \lambda_M^-} v_\lambda = 0$, otherwise we can extend Σ_1 further beyond λ_M. The case (a) is impossible since if either $g(M) = 0$ for some $M > 0$ or $\lim\limits_{u \to \infty} \dfrac{g(u)}{u} = 0$, then the solution curve cannot blow up at finite λ_M (see details in [17] Theorem 1.3.) The case (b) is not possible either, since if so, $\lambda = \lambda_M$ must be a point where a bifurcation from the trivial solutions $v = 0$ occurs. That is $\lambda_M g'(0)$ must be an eigenvalue λ_i of (1.5) with $i \geq 2$, and the eigenfunction ϕ_i is not of one sign. But the positive solution v_λ satisfies $v_\lambda/||v_\lambda|| \to \phi_i$ as $\lambda \to \lambda_M^-$, which is a contradiction. Thus $\lambda_M = \infty$.

We prove v_λ is increasing with respect to λ. Since v_λ is differentiable with respect to λ (as a consequence of the implicit function theorem), $\frac{dv_\lambda}{d\lambda}$ satisfies $\Delta\frac{dv_\lambda}{d\lambda} + \lambda g'(v_\lambda)\frac{dv_\lambda}{d\lambda} = -g(v_\lambda) \leq 0$, and v_λ is stable, hence $\mu_1(v_\lambda) > 0$. Then by a standard variant of the maximum principle (see for example Lemma 2.16 in [10]), $\frac{dv_\lambda}{d\lambda} \geq 0$. Finally, by Lemma 2.2, (2.2) has at most one positive solution for any possible $\lambda > 0$, which completes the proof. ■

REMARK. If g satisfies $\lim_{u\to\infty} \frac{g(u)}{u} = k > 0$, the proof of Theorem 2.3 still works except that the solution curve (λ, v_λ) exists only for $\lambda \in (\lambda_1^0, \lambda_1^\infty)$, where $\lambda_1^\infty = \lambda_1/k$. $(\lambda_1^\infty, \infty)$ is a point where a bifurcation from infinity occurs. (see [14] or [17] for details.)

3 THE BRANCH OF MAXIMAL SOLUTIONS

Proof of Theorem 1.1 Let v_λ be the unique positive solution of (2.2) for $\lambda > \lambda_1^0$. First we prove that if (1.1) has a solution (λ, u), then $\lambda > \lambda_1^0$ and $v_\lambda \geq u$. Indeed, if (λ, u) is a positive solution of (1.1), then similar to (2.3), we have (using $f(u) = g(u) - \varepsilon$)

$$\begin{aligned}&\left(\lambda - \frac{\lambda_1}{g'(0)}\right) g'(0) \int_\Omega u(x)\phi_1(x)dx - \lambda\varepsilon \int_\Omega \phi_1(x)dx \\ &+\lambda \int_\Omega \left[\frac{g(u(x))}{u(x)} - g'(0)\right] u(x)\phi_1(x)dx = 0.\end{aligned} \tag{3.1}$$

So $\lambda > \lambda_1^0$. On the other hand,

$$\Delta v_\lambda + \lambda g(v_\lambda) = 0 < \lambda\varepsilon = \Delta u + \lambda g(u),$$

and $u = v_\lambda = 0$ on the boundary. By Lemma 2.2, $v_\lambda(x) \geq u(x)$ for $x \in \Omega$.

By Lemma 2.1, (1.1) has a non-negative solution u_λ for $\lambda > \lambda_a$. From the last paragraph, $v_\lambda \geq u_\lambda$. For fixed λ we define a sequence $\{u_n(\lambda)\}$: $u_0(\lambda) = v_\lambda$, and for $n \geq 1$

$$\begin{cases} -\Delta u_n + K u_n = \lambda f(u_{n-1}) + K u_{n-1} & \text{in} \quad \Omega, \\ u_n = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.2}$$

where $K > 0$ is a constant such that $|\lambda f'(u)| \leq K$ for all $u \in [0, M]$ (from (f2)). Then it is standard to show that $v_\lambda \equiv u_0(\lambda) \geq u_1(\lambda) \geq \cdots \geq u_n(\lambda) \geq u_{n+1}(\lambda) \geq \cdots$. On the other hand, we claim that $u_n(\lambda) \geq u_\lambda$ for $n \geq 0$. It is true when $n = 0$. Suppose it is true for $n = k$, then when $n = k + 1$,

$$\Delta(u_{k+1} - u_\lambda) - K(u_{k+1} - u_\lambda) = f_1(u_\lambda) - f_1(u_k) \leq 0,$$

where $f_1(u) = \lambda f(u) + Ku$, and $f_1'(u) > 0$. And $u_{k+1} - u_\lambda = 0$ on the boundary, thus $u_{k+1} - u_\lambda \geq 0$ by the maximum principle. Define $\overline{u}_\lambda(x) = \lim_{n\to\infty} u_n(\lambda)(x)$. It is standard to verify that $\overline{u}_\lambda \in C^{2,\alpha}(\overline{\Omega})$ and $\overline{u}_\lambda$ is a non-negative solution of (1.1) such that $\overline{u}_\lambda \geq u_\lambda$. Since we can use any solution u in the place of u_λ, $\overline{u}_\lambda$ is the maximal solution of (1.1).

Define

$$H = \{\lambda > 0 : \text{(1.1) has at least one non-negative solution}\},$$

and $\lambda_* = \inf\{\lambda \in H\}$. Then $\lambda_* \geq \lambda_1^0 > 0$. We claim that $H \supset (\lambda_*, \infty)$. Suppose that $\lambda_b \in H$, then for $\lambda_c > \lambda_b$, $\overline{u}_{\lambda_b} \leq v_{\lambda_b} \leq v_{\lambda_c}$. Thus the iteration sequence with $u_0(\lambda_c) = v_{\lambda_c}$ and defined as in (3.2) has $\overline{u}_{\lambda_b}$ as lower bound similar to the proof in the last paragraph. Therefore the maximal solution $\overline{u}_{\lambda_c}$ for $\lambda = \lambda_c$ also exists, and

$$H \supset \cup_{\lambda \in H}(\lambda, \infty) = (\lambda_*, \infty).$$

Moreover, we have also proved that $\overline{u}_\lambda$ is increasing with respect to λ.

Next we prove that $\lambda_* > \lambda_1^0$. We have established that $\lambda_* \geq \lambda_1^0$. In fact, if $\lambda_* = \lambda_1^0$, then $0 \leq \lim\limits_{\lambda \to \lambda_*^+} \max\limits_{x \in \overline{\Omega}} \overline{u}_\lambda(x) \leq \lim\limits_{\lambda \to \lambda_*^+} \max\limits_{x \in \overline{\Omega}} v_\lambda(x) = 0$. But it is well-known that each solution u of (1.1) satisfies $\max_{x \in \overline{\Omega}} u(x) > b$ (from the maximum principle). So we reach a contradiction. Thus $\lambda_* > \lambda_1^0$.

We now prove that $\overline{u}_\lambda$ is stable for $\lambda > \lambda_*$ if Ω is a star-shaped domain. Since $\overline{u}_\lambda$ is obtained via the iteration from a supersolution, then from [15] pg. 992, the principal eigenvalue $\mu_1(\overline{u}_\lambda) \geq 0$. We exclude the possibility of $\mu_1(\overline{u}_\lambda) = 0$. Suppose there exists $\lambda_d > \lambda_*$ such that $\mu_1(\overline{u}_{\lambda_d}) = 0$. Then the principal eigenfunction $\psi_1 > 0$ satisfies

$$\begin{cases} \Delta\psi + \lambda_d f'(\overline{u}_{\lambda_d})\psi = 0 & \text{in} \quad \Omega, \\ \psi = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{3.3}$$

At $(\lambda_d, \overline{u}_{\lambda_d})$, we apply a bifurcation theorem by Crandall and Rabinowitz [9]. Since 0 is the principal eigenvalue of (1.2), so it must be a simple eigenvalue. By Lemma 2.3 in [13], we have

$$\int\limits_\Omega f(\overline{u}_{\lambda_d})\psi_1 dx = \frac{1}{2\lambda} \int\limits_{\partial\Omega} |\nabla \overline{u}_{\lambda_d}| \cdot |\nabla\psi_1|(x \cdot \nu) ds, \tag{3.4}$$

where ν is the outer unit normal vector. If Ω is star-shaped, then $\int_\Omega f(\overline{u}_{\lambda_d})\psi_1 dx > 0$ since $\dfrac{\partial \overline{u}_{\lambda_d}}{\partial n}(x) < 0$ and $\dfrac{\partial \psi_1}{\partial n}(x) < 0$ for $x \in \partial\Omega$. Hence by the result of [9], the solutions near $(\lambda_d, \overline{u}_{\lambda_d})$ form a curve $\{(\lambda(s), u(s)) : |s| \leq \delta\}$ such that $\lambda(0) = \lambda_d$, $\lambda'(0) = 0$, $u(s) = \overline{u}_{\lambda_d} + s\psi_1 + o(|s|)$. Moreover, (see [16] pg. 506)

$$\lambda''(0) = \frac{-\lambda_d \int_\Omega f''(\overline{u}_{\lambda_d})\psi_1^3 dx}{\int_\Omega f(\overline{u}_{\lambda_d})\psi_1 dx}. \tag{3.5}$$

$\lambda''(0) > 0$ since $f''(u) \leq 0$, and the solution curve is $\subset$-shaped near $(\lambda_d, \overline{u}_{\lambda_d})$. Therefore for $\lambda \in (\lambda_d - \varepsilon, \lambda_d)$, $||\overline{u}_\lambda - \overline{u}_{\lambda_d}|| \geq \delta > 0$. But on the other hand, $\overline{u}_\lambda$ is continuous with respect to λ since $\overline{u}_\lambda$ is obtained from a family of continuous supersolution v_λ. This is a contradiction. So $\mu_1(\overline{u}_\lambda) > 0$ for $\lambda > \lambda_*$.

Now we are able to prove that if Ω is star-shaped, then for $\lambda > \lambda_*$, $\overline{u}_\lambda(x) > 0$ for any $x \in \Omega$, and $\dfrac{\partial \overline{u}_\lambda(x)}{\partial \nu} < 0$ for any $x \in \partial\Omega$. Indeed since $\mu_1(\overline{u}_\lambda) > 0$, we obtain this result by a variant of the maximum principle (see Theorem 2.1 in [12], or [6].)

Finally we use an idea of [10] to prove that all other nonnegative solutions of (1.1) must be unstable. Suppose that for some $\lambda \geq \lambda_*$, (1.1) has another nonnegative solution w_λ such that $\mu_1(w_\lambda) \geq 0$. Since f is concave, for any $\tau \in [0,1]$,

$$\Delta(\tau w_\lambda + (1-\tau)\overline{u}_\lambda) + \lambda f(\tau w_\lambda + (1-\tau)\overline{u}_\lambda) \geq 0. \tag{3.6}$$

But since at $\tau = 0$, (3.6) is an identity, the derivative of the left hand side of (3.6) with respect to τ at $\tau = 0$ is non-negative, that is

$$\Delta(w_\lambda - \overline{u}_\lambda) + \lambda f'(\overline{u}_\lambda)(w_\lambda - \overline{u}_\lambda) \geq 0. \tag{3.7}$$

Similarly, at $\tau = 1$, we get

$$\Delta(w_\lambda - \overline{u}_\lambda) + \lambda f'(w_\lambda)(w_\lambda - \overline{u}_\lambda) \leq 0. \tag{3.8}$$

From Lemma 2.26 of [10], if $\mu_1(\overline{u}_\lambda) > 0$ and $\mu_1(w_\lambda) > 0$, then $w_\lambda - \overline{u}_\lambda \leq 0$ and $w_\lambda - \overline{u}_\lambda \geq 0$, thus $w_\lambda \equiv \overline{u}_\lambda$. If one of $\mu_1(\overline{u}_\lambda)$ and $\mu_1(w_\lambda)$ is zero, say $\mu_1(\overline{u}_\lambda) = 0$, then (3.7) is an identity, and the second derivative of the left hand side of (3.6) with respect to τ at $\tau = 0$ must be non-negative, that is

$$\lambda f''(\overline{u}_\lambda)(w_\lambda - \overline{u}_\lambda)^2 \geq 0.$$

But $\lambda > 0$ and $f'' < 0$, so $w_\lambda \equiv \overline{u}_\lambda$. Hence for any nonnegative solution w_λ of (1.1) other than $\overline{u}_\lambda$, $\mu_1(w_\lambda) < 0$.

At $\lambda = \lambda_*$, $\overline{u}_{\lambda_*}$ still exists as the iteration of v_{λ_*} and $\overline{u}_{\lambda_*}$ is non-negative since $\overline{u}_{\lambda_*} = \lim_{\lambda \to \lambda_*^+} \overline{u}_\lambda$. Now $\mu_1(\overline{u}_{\lambda_*}) \geq 0$ and if it is positive, it contradicts the definition of λ_*. Hence $\mu_1(\overline{u}_{\lambda_*}) = 0$. ∎

4 IMPERFECT BIFURCATION NEAR $U = 0$

In this section, we prove Theorem 1.2. We assume $\int_\Omega \phi_2^3(x)dx > 0$, and the case when $\int_\Omega \phi_2^3(x)dx < 0$ can be shown similarly. First we show that the bifurcation diagram when $\varepsilon = 0$ (equation (2.2)) is as described by the dashed curves in Fig. 1. From our assumptions, λ_1 and λ_2 are both simple eigenvalues. Thus we can apply the result of [8] to conclude that near $(\lambda_i^0, 0)$, $(i = 1,2)$, the solutions of (2.2) are on two branches $\Sigma^0 = \{(\lambda, 0)\}$ and $\Sigma^i = \{(\lambda_i(s), v_i(s)) : |s| \leq \delta\}$, where $\lambda_i(0) = \lambda_i^0$, $v_i(s) = s\phi_i + o(s^2)$, and $\lambda_i'(0)$ satisfies the expression (see [16] page 507)

$$\lambda_i'(0) = -\frac{\int_\Omega g''(0)\phi_i^3(x)dx}{2\int_\Omega g'(0)\phi_i^2(x)dx} > 0. \tag{4.1}$$

From the positivity of ϕ_1 and $\int_\Omega \phi_2^3(x)dx > 0$, $\lambda_i'(0) > 0$, thus both bifurcation are transcritical, and can be drawn as the dashed curves in Fig.1.

Next we recall the discussion of (1.4) in [16] Theorem 2.5 and subsection 6.1. In fact, we show that, when $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ for some small $\varepsilon_0 > 0$, the local bifurcation picture near $(\lambda_1^0, 0)$ is as in the following diagrams:

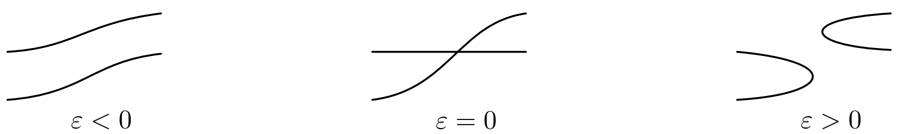

Fig. 4: Bifurcation of solution curves in a transcritical bifurcation

The rigorous proof of the diagrams relies on an abstract theorem (Theorem 2.5 in [16]), in which the degenerate solutions of (1.4) are tracked when ε varies. Indeed, the degenerate solutions of (1.4) form a curve $\{(\varepsilon(s), \lambda(s), u(s), w(s)) : |s| \leq \delta\}$, $\lambda'(0) = 1$, $\varepsilon'(0) = 0$ and

$$\varepsilon''(0) = \frac{[g'(0)]^2 \left(\int_\Omega \phi_1^2(x)dx\right)^2}{2(\lambda_1^0)^2(\int_\Omega \phi_1^3(x)dx) \cdot (\int_\Omega \phi_1(x)dx)} > 0. \tag{4.2}$$

Hence there are two degenerate solutions near $(\lambda_1^0, 0)$ when $\varepsilon > 0$, and no degenerate solution when $\varepsilon < 0$. These arguments are still valid near $(\lambda_2^0, 0)$ as long as λ_2 is simple and (1.6) holds. Therefore we obtain the parts of the bifurcation diagram in Fig. 1 when λ is near λ_1^0 and λ_2^0. Moreover, since the solutions near $(\lambda_1^0, 0)$ are all in a form $(\lambda, t\phi_1 + o(|t|))$, then the solutions on $\supset$-branch are all negative, and the solutions on $\subset$-branch are all positive. Similarly since the solutions near $(\lambda_2^0, 0)$ are all in a form $(\lambda, t\phi_2 + o(|t|))$, then the solutions on both $\supset$-branch and $\subset$-branch are sign-changing solutions.

To be more precise, we select $\widetilde{\delta_1} > 0$ and $\varepsilon_2 > 0$ such that when $\varepsilon \in (0, \varepsilon_2)$, (1.4) has exactly two degenerate solutions in each of the cube $C_i = \{(\lambda, u) : |\lambda - \lambda_i^0| \leq \widetilde{\delta_1}, ||u||_X \leq \widetilde{\delta_1}\}$, $(i = 1, 2)$, from Theorem 2.5 in [16]. Then the portion of the bifurcation diagram in each cube is exactly same as the third diagram in Fig. 4. Note that there is a gap on the λ-axis between the projections of two connected components. This follows from the fact that for the curve of degenerate solutions $\{(\varepsilon(s), \lambda(s), u(s), w(s)\}$, $\lambda(0) = \lambda_i^0$ and $\lambda'(0) = 1$, thus $\lambda(s) < \lambda_i^0$ for $s < 0$ and $\lambda(s) > \lambda_i^0$ for $s > 0$.

Let $\widetilde{\delta_2} = \widetilde{\delta_1}/2$. For $\lambda \in [\lambda_1^0 + \widetilde{\delta_2}, \lambda_2^0 - \widetilde{\delta_2}]$, the trivial solution $(\lambda, 0)$ for (1.4) with $\varepsilon = 0$ is nondegenerate. We choose $\widetilde{\delta_3} > 0$ such that the solutions on the line $(\lambda, 0)$ are the only solutions of (1.4) when $\varepsilon = 0$ in the cube $\{(\lambda, u) : \lambda_1^0 + \widetilde{\delta_2} < \lambda < \lambda_2^0 - \widetilde{\delta_2}, ||u||_X \leq \widetilde{\delta_3}\}$. Thus by the implicit function theorem, there exists $\varepsilon_3 > 0$ such that when $\varepsilon \in (0, \varepsilon_3)$, for each $\lambda \in [\lambda_1^0 + \widetilde{\delta_2}, \lambda_2^0 - \widetilde{\delta_2}]$, (1.4) has exactly one solution $(\lambda, u(\lambda))$ such that $||u(\lambda)||_X \leq \widetilde{\delta_3}$, and they are all nondegenerate. From the nondegeneracy of the solutions, we can see that the curve $(\lambda, u(\lambda))$ joins the lower branch of $\subset$-branch in C_1. We can use the Morse index of the solutions to conclude that $(\lambda, u(\lambda))$ joins the lower branch but not the upper branch, since the solutions on the upper branch have Morse index 0 and the ones on the lower branch have Morse index 1. All solutions on $(\lambda, u(\lambda))$ have Morse index 1 since they are perturbations of $(\lambda, 0)$ when $\varepsilon = 0$, which have Morse index 1. Similarly, $(\lambda, u(\lambda))$ joins the $\supset$-branch in C_2, and here the terms "lower" and "upper" branches are not appropriate as the solutions are not ordered. But $(\lambda, u(\lambda))$ will connect with the branch with Morse index 1 and smaller X-norm. Therefore the $\supset$-branch in C_2 and the $\subset$-branch in C_1 are connected, and it forms a S-shaped curve in a cube near $(\lambda, 0)$. Let $\varepsilon_1 = \min(\varepsilon_2, \varepsilon_3)$, $\delta_2 = \min(\widetilde{\delta_1}, \widetilde{\delta_3})$, and $\delta_1 = \widetilde{\delta_1}$. Then we obtain the results claimed in Theorem 1.2.

Since any solution on the $\subset$-branch in C_1 is positive, and any solution on the $\supset$-branch in C_2 is sign-changing, then there exists $\lambda \in (\lambda_1^0 + \widetilde{\delta_2}, \lambda_2^0 - \widetilde{\delta_2})$ such that the solution $(\lambda, u(\lambda, \cdot))$ satisfies $u(\lambda, x) \geq 0$ for all $x \in \Omega$ but either there exists $x_0 \in \Omega$ such that $u(x_0) = 0$ or there exists $x_1 \in \partial\Omega$ such that $\partial u(x_1)/\partial\nu = 0$.

Finally if we replace (1.6) by (1.7), and assume that

$$\int_\Omega \phi_2^3(x) > 0 \quad \text{and} \quad \int_\Omega \phi_2(x)dx < 0,$$

then from (4.2) and (4.1), we will obtain the first diagram in Fig. 4 for $\varepsilon > 0$, and using the exact arguments above, we will obtain Fig. 2.

5 SYMMETRY BREAKING BIFURCATIONS

For the domains with certain symmetry, it is often that ϕ_2 is an odd function with respect to the symmetry, thus $\int_\Omega \phi_2(x)dx = \int_\Omega \phi_2^3(x)dx = 0$ and Theorem 1.2 is not applicable. Here we describe the bifurcation diagrams for two typical symmetric domains: a simple rectangle and a ball in $\mathbf{R}^n$.

When $\Omega = R \equiv \Pi_{i=1}^n(0, a_i)$, and a_i/a_j is irrational when $i \neq j$, Ω is called a simple rectangle. Without loss of generality, we assume that $a_1 > a_2 > \cdots > a_n$. Then the second eigenvalue and corresponding eigenfunction of R are

$$\lambda_2 = \frac{4\pi^2}{a_1^2} + \sum_{i=2}^n \frac{\pi^2}{a_i^2}, \;\; \phi_2 = \sin\left(\frac{2\pi x_1}{a_1}\right)\prod_{i=2}^n \sin\left(\frac{2\pi x_i}{a_i}\right). \tag{5.1}$$

Thus $\int_R \phi_2(x)dx = 0$. Since λ_2 is a simple eigenvalue, then we can apply the result of [9] to obtain a curve of nontrivial solutions of (2.2) $\Sigma^2 = \{(\lambda_2(s), u_2(s)) : |s| \leq \delta\}$. But from (4.1), $\lambda_2'(0) = 0$ since $\int_\Omega \phi_2^3(x)dx = 0$. In fact, the bifurcation at λ_2 cannot be transcritical and the branch Σ^2 must be entirely on the side of $\lambda < \lambda_2^0$ or $\lambda > \lambda_2^0$, because when $(\lambda, u(x_1, x'))$ is a solution of (2.2), so is $(\lambda, u(a_1 - x_1, x'))$. Here $x' = (x_2, x_3, \cdots, x_n)$. Thus a pitchfork bifurcation occurs at λ_2. It is not possible to determine whether the bifurcation is subcritical or supercritical with only conditions (f1)-(f4). But if we assume that $g \in C^3[0, m)$ for some $m > 0$ and $g'''(0) \geq 0$, then from the arguments in page 525 of [16], we can show that $\lambda_2''(0) > 0$, hence the bifurcation is supercritical.

Now we consider the perturbed problem (1.4). We can show that when $\varepsilon > 0$ is small, there is a unique degenerate solution $(\lambda(\varepsilon), u(\varepsilon))$ near $(\lambda_2, 0)$. Moreover $u(\varepsilon)$ is positive in R but $\partial u(\varepsilon)/\partial n = 0$ when $x_1 = 0$ or $x = a_1$. Thus a pitchfork bifurcation occurs at $(\lambda(\varepsilon), u(\varepsilon))$. If we assume $g'''(0) > 0$ as in the last paragraph, then for $\lambda < \lambda(\varepsilon)$, (1.4) has exactly one solution in a neighborhood of $(\lambda(\varepsilon), u(\varepsilon))$, which is positive and symmetric with respect to $x_1 = a_1/2$. And for $\lambda > \lambda(\varepsilon)$, (1.4) has exactly three solutions in a neighborhood of $(\lambda(\varepsilon), u(\varepsilon))$, and none of them are positive. In fact, among these three solutions, one is still symmetric with three nodal domains, and the other two are not symmetric with two nodal domains. Thus $(\lambda(\varepsilon), u(\varepsilon))$ is where the symmetry breaking bifurcation occurs. The detail proof of this bifurcation diagram will appear in [18] under a more general framework.

The bifurcation diagram for $\Omega = B^n$, the unit ball in $\mathbf{R}^n$, is similar to that of R. In this case, λ_2 is an eigenvalue with multiplicity n. But when $\varepsilon > 0$ is small, there is still a unique degenerate solution $(\lambda(\varepsilon), u(\varepsilon))$ near $(\lambda_2^0, 0)$, and it is where the symmetry breaking bifurcation occurs. The symmetry breaking bifurcation in this particular case (f satisfying (f1)-(f4)) has been studied by Smoller and Wasserman [20], Korman [11] and others (references can be found in [11]). In this case instead of a curve of nonsymmetric solutions, an n-parameter family of non-radial solutions bifurcates out, while a sign-changing radial solution still exists for $\lambda > \lambda(\varepsilon)$. Here we show that the symmetry breaking bifurcation can result from a perturbation of the bifurcation at the second eigenvalue. On the other hand, we also show that the bifurcation of non-radial solutions should be supercritical if $g'''(0) > 0$, since when $g'''(0) > 0$ and $\varepsilon = 0$, the bifurcation of non-trivial solutions from the trivial solutions is supercritical.

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