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Exact multiplicity of solutions and S-shaped bifurcation curve for a class of semilinear elliptic equations

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Abstract

The set of steady state solutions to a reaction–diffusion equation modeling an autocatalytic chemical reaction is completely determined, when the reactor has spherical geometry, and the spatial dimension is $n = 1$ or 2 for any reaction order, or $n \geqslant 3$ for subcritical reaction order. Bifurcation approach and analysis of linearized problems are used to establish exact multiplicity and precise global bifurcation diagram of positive steady states.

Keywords: Exact multiplicity; Bifurcation; S-shaped curve; Autocatalytic chemical reaction

1. Introduction

The prototype representation for an autocatalytic chemical reaction is

$$A + pB \to (p+1)B, \tag{1.1}$$

and the reaction rate is kab^p, where a and b are the concentrations of the reactant A and the autocatalyst B, and $p \geqslant 1$ is the order of the reaction with respect to the autocatalytic species [13].

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The equations describing the reaction and diffusion of the two reactants A and B in a bounded region are

$$\frac{\partial a}{\partial t} = D_A \Delta a - ab^p, \quad \frac{\partial b}{\partial t} = D_B \Delta b + ab^p, \quad t > 0,\ x \in \Omega, \tag{1.2}$$

where D_A and D_B are the diffusion coefficients of A and B, respectively, and Ω is a bounded reactor in $\mathbf{R}^n$. Here the spatial dimension $1 \leqslant n \leqslant 3$, and the typical geometry of the reactor Ω is spherical ($n = 3$ and $\Omega = B^3$), cylindrical ($n = 2$ and $\Omega = B^2$), and linear ($n = 1$ and $\Omega = (-1, 1)$), where $B^n = \{x \in \mathbf{R}^n\colon\ |x| < 1\}$ is the unit ball. The chemicals A and B can diffuse from a reservoir of constant composition across the boundary $\partial\Omega$ into Ω, thus the boundary conditions of A and B can be taken as

$$a(x,t) = a_0 > 0 \quad \text{and} \quad b(x,t) = b_0 \geqslant 0, \quad x \in \partial\Omega. \tag{1.3}$$

The steady state solutions of (1.2) and (1.3) satisfy

$$\begin{cases} D_A \Delta a - ab^p = 0, \quad D_B \Delta b + ab^p = 0, & x \in \Omega, \\ a(x) = a_0, \quad b(x) = b_0, & x \in \partial\Omega. \end{cases} \tag{1.4}$$

In the following, we shall concentrate on the case when the reactor is the unit ball B^n. By adding the two equations in (1.4), we have $\Delta(D_A a + D_B b) = 0$ in Ω, and $D_A a + D_B b \equiv D_A a_0 + D_B b_0$ on $\partial\Omega$. From the uniqueness of the solution of Laplace equation, $D_A a(x) + D_B b(x) \equiv D_A a_0 + D_B b_0$ in Ω. Thus the system of Eqs. (1.4) can be reduced to a scalar equation:

$$D_A D_B \Delta b + (D_A a_0 + D_B b_0 - D_B b) b^p = 0, \quad x \in B^n, \qquad b(x) = b_0, \quad x \in \partial B^n. \tag{1.5}$$

Let $v(x) = b(x)/(D_A D_B^{-1} a_0 + b_0)$, and let $\lambda = D_A^{-1}(D_A D_B^{-1} a_0 + b_0)^p$. Then $v(x)$ satisfies

$$\Delta v + \lambda(1 - v) v^p = 0, \quad x \in B^n, \qquad v(x) = k, \quad x \in \partial B^n, \tag{1.6}$$

where $k = D_B b_0/(D_A a_0 + D_B b_0) \geqslant 0$. Since $v \geqslant 0$, then $1 \geqslant v(x) \geqslant k$ from the maximum principle (notice that $k \in [0, 1)$). Finally we let $u(x) = v(x) - k$, then $u(x)$ satisfies

$$\begin{cases} \Delta u + \lambda\big[(u+k)^p - (u+k)^{p+1}\big] = 0, & x \in B^n, \\ u(x) = 0, & x \in \partial B^n. \end{cases} \tag{1.7}$$

Here $\lambda > 0$, $k \in [0, 1)$ and $p \geqslant 1$. We are interested in the higher order reactions, so we assume $p > 1$.

Our goal of this paper is to study the existence, multiplicity and exact multiplicity of positive solutions to (1.7). Our main result is for the spherical domain $\Omega = B^n$, which is typical for chemical reactions, and we will assume n to be any positive integer most of time since our results hold for all these cases. From the well-known result of [10], when the domain Ω is the unit ball in $\mathbf{R}^n$, then a positive solution of (1.7) must be radially symmetric and it is decreasing along the radial direction. Thus it satisfies

$$\begin{cases} u'' + \dfrac{n-1}{r} u' + \lambda\big[(u+k)^p - (u+k)^{p+1}\big] = 0, & r \in (0,1), \\ u(r) > 0, \quad u'(r) < 0, \quad r \in (0,1), \qquad u'(0) = u(1) = 0. \end{cases} \tag{1.8}$$

First when $k = 0$, we have an exact multiplicity result for balls of any dimension:

Theorem 1.1. *Consider*

$$\begin{cases} \Delta v(x) + \eta\big(v^p - v^{p+1}\big) = 0, & x \in B^n, \\ v(x) = 0, & x \in \partial B^n. \end{cases} \tag{1.9}$$

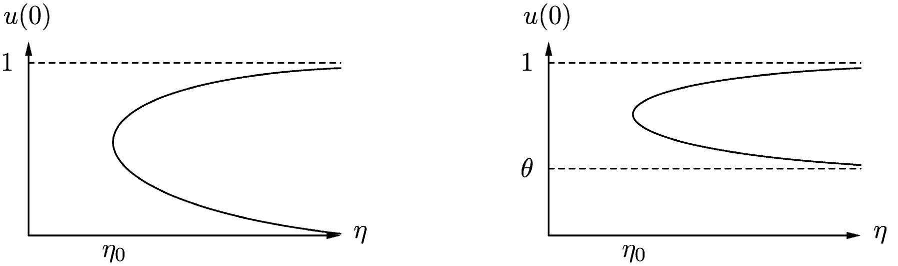

Fig. 1. Bifurcation diagrams for (1.7), $k = 0$. Left: $p \leqslant \frac{n+2}{n-2}$ or $n \leqslant 2$; Right: $p > \frac{n+2}{n-2}$.

Suppose that $p > 1$ and $n \geqslant 1$. Then there exists $\eta_0 > 0$ such that (1.9) *has exactly two positive solutions if $\eta > \eta_0$, exactly one positive solution if $\eta = \eta_0$, and no positive solution if $\eta < \eta_0$, i.e. the bifurcation diagram of* (1.9) *is exactly $\subset$-shaped. Furthermore, all positive solutions of* (1.9) *lie on a single smooth solution curve in the space $\mathbf{R}_+ \times C^2(\bar{\Omega})$, which consists of two branches $v_*(x,\eta) < v^*(x,\eta)$ for $\eta > \eta_0$; the mapping $\eta \mapsto v^*(x,\eta)$ is continuous and increasing,* $\lim_{\eta\to\infty} v^*(0,\eta) = 1$; *the mapping $\eta \mapsto v_*(x,\eta)$ is continuous and decreasing and* $\lim_{\eta\to\infty} v_*(0,\eta) = \theta \geqslant 0$; *$\theta = 0$ if $n \leqslant 2$, or $n \geqslant 3$ and $p \leqslant (n+2)/(n-2)$, and $\theta > 0$ if $n \geqslant 3$ and $p > (n+2)/(n-2)$; for $\eta > \eta_0$, $v^*(x,\eta)$ is stable, and $v_*(x,\eta)$ is unstable with Morse index* 1. (*See Fig.* 1.)

This result is included in [25, Theorem 3], but a detailed proof is omitted in [25]. We will sketch a proof (see Section 4) to this result for the sake of completeness. Our main result is for $k > 0$ but close to 0:

Theorem 1.2. *Suppose that n and p satisfy one of the following*:

$$n = 1 \quad or \quad n = 2 \quad and \quad 1 < p < \infty, \quad or$$
$$n \geqslant 3 \quad and \quad 1 < p \leqslant \frac{n+2}{n-2},$$

then there exists $k_0 > 0$ such that when $k \in (0, k_0)$, the bifurcation diagram of (1.7) *is exactly S-shaped. More precisely, there exist $0 < \lambda_* < \lambda^* < \infty$ such that* (1.7) *has exactly three positive solutions if $\lambda^* > \lambda > \lambda_*$, has exactly one positive solution if $\lambda > \lambda_*$ or $\lambda < \lambda^*$, and has exactly two positive solutions if $\lambda = \lambda_*$ or $\lambda = \lambda^*$. Furthermore, all positive solutions of* (1.7) *lie on a single smooth solution curve in the space $\mathbf{R}_+ \times C^2(\bar{\Omega})$, which consists of three branches*

$$\Gamma_* = \{(\lambda, u_*(x,\lambda)) \colon 0 < \lambda \leqslant \lambda^*\}, \quad \Gamma_m = \{(\lambda, u_m(x,\lambda)) \colon \lambda_* \leqslant \lambda \leqslant \lambda^*\}, \quad and$$
$$\Gamma^* = \{(\lambda, u^*(x,\lambda)) \colon \lambda_* \leqslant \lambda < \infty\};$$

$\lim_{\lambda\to 0^+} u_*(x,\lambda) = 0$, $\lim_{\lambda\to\infty} u^*(0,\lambda) = 1 - k$; *for $\lambda_* < \lambda < \lambda^*$, $u_*(x,\lambda) < u_m(x,\lambda) < u^*(x,\lambda)$; the mappings $\lambda \mapsto u^*(x,\lambda)$ and $\lambda \mapsto u_*(x,\lambda)$ are continuous and increasing*; *$u^*(x,\lambda)$ and $u_*(x,\lambda)$ are stable, and $u_m(x,\lambda)$ is unstable with Morse index* 1. (*See Fig.* 2.)

The model (1.2) and several variants were first formulated by Gray and Scott [11–13], and the specific configuration as in (1.2) and (1.3) was developed in Kay and Scott [17]. The reaction (1.2) in both well-stirred open systems and closes systems have been studied. A feedback mechanism must exist to sustain the reaction in either cases. Here we assume an open system with

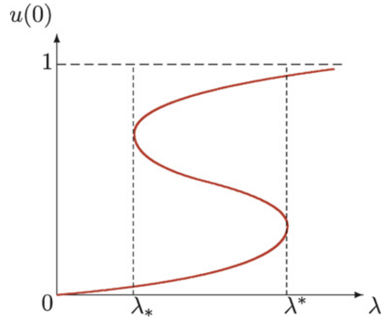

Fig. 2. S-shaped bifurcation diagram for (1.7) with small $k > 0$.

constant concentration outside of the reactor so that reactants are fed into the reactor through the boundary flux. In a closed system (with zero flux), feedback is usually modeled by using additional terms in the reaction equation (see Aris [1]). The closed system of Gray–Scott model has been used in the pioneer work of pattern formation by physicists and chemists, see [24,28], and more recently it has also received more attention by mathematicians, see, for example, [5,36]. However a complete mathematical understanding of the dynamical behavior is still beyond the reach. Notice the order p of the reaction can be one, but mainly the quadratic and cubic reactions were considered in [11–13], and even higher p was recently considered in [15,33].

In this paper we study the open system with Dirichlet boundary condition proposed in [17], and we hope the analysis will shred new light to the understanding of the pattern formation and bifurcation of this basic model of chemical reaction. Numerical results of S-shaped curve for (1.4) and (1.7) were obtained in [17], and here we give theoretical justification of their numerical simulations.

The exact multiplicity of solutions to (1.7) implies the precise bifurcation structure of positive solutions to the system (1.4). The S-shaped bifurcation diagram reveals possible bistability of the reaction: the right turning point λ^* is an ignition point, above which the system jumps suddenly to the larger stable state (with highest reactant consumption), and the left turning point λ_* is an extinction point, below which the system drops to the smaller stable state (with low reactant consumption). The middle steady state represents a threshold state, below which will trigger an extinction, and above which results in conversion of reactant. The S-shaped curve is obtained for small k, which indicates the lower concentration b_0 of the autocatalyst B in the reservoir, and higher concentration b_0 (hence larger k) will result in a unique steady state for all λ (see Section 5 for more detailed discussion).

Bistability is also observed for the same system (1.2) but with the unbounded reactor $\mathbf{R}^n$ (see [15] for numerical and formal results, and [33] for rigorous justification). But the bistability there is only possible when p is supercritical ($n \geqslant 3$ and $p > (n+2)/(n-2)$), while our results here are for subcritical case. We should notice that (1.2) on $\mathbf{R}^n$ can be thought as the limit system of (1.2) on a finite ball with $\lambda \to \infty$. For λ large, (1.4) always has a unique solution with high concentration of autocatalyst and low concentration of reactant, which is consistent with the result in $\mathbf{R}^n$ [15,33] that a reaction wave always can be initiated with subcritical reaction order p. For system (1.4) with supercritical p, the bifurcation diagram is more complicated, and it could have many turning points (see Section 5 for more detailed discussion).

In this paper it is shown that the phenomenon of bistability could be caused by a higher order chemical reaction, but we should be cautious that our results are obtained for a simplified model ignoring the tubular flow and non-isothermal effects. In practical chemical engineering, the chemical kinetics are usually more complex due to the interaction between the reacting substances and the catalysts which usually take place in multiple phase medium, and convection term could also affect the dynamics. Here we ignore these additional effects but concentrate on the bistability caused by this relatively simple kinetics, and we hope our methods can motivate the study of more realistic models in chemical engineering.

In [23,37] Winkin et al. studied the dynamics of non-isothermal tubular reactors with Arrhenius type kinetics. The existence of solutions to evolution equation is established, and in certain cases, the multiplicity of the equilibrium profiles is reported (see [23]). In a related work, Dramé [6] proved a more general existence result, and it was showed that the solution is uniformly bounded with a partial description of the limit set. In the present work, the chemical reactions are assumed to be isothermal following [13] but we concentrate on the multiplicity of equilibrium states caused by the autocatalyst and higher order of the reaction. It would be interesting to combine the effect of temperature and flow as in [23,37] with our setting here.

Exact multiplicity and bifurcation of positive solutions to semilinear elliptic equations have been studied by many people in the last thirty years. A systematic bifurcation approach combining comparison methods has been established by Korman, Li, Ouyang and Shi in the last decade, see, for example, [18,20,21,25,26,32]. The S-shaped bifurcation diagrams for various models have been studied in [2,4,8,9,14,19,22,27,30,31,34], in particular for the perturbed Gelfand's equation arising from combustion theory. More historical remarks on S-shaped curves can be found in [9,30,31]. Our proof of theorems here uses approach developed in [25,26], and also ideas in [9,30]. But we use a different way to prove the positivity of the solution to linearized equation, and we simplify earlier proofs by taking the advantage of the translational perturbation in the problem.

We will recall some preliminaries of bifurcation approach in Section 2. In Section 3, we prove the solution of linearized equation does not change sign under the conditions of our main theorems, and we prove our main theorems in Section 4. Some additional remarks conclude the paper in Section 5.

2. Setup and basics of bifurcation approach

In this section we briefly review the basic setting for bifurcation approach to the set of positive solutions of semilinear elliptic equation

$$\begin{cases} \Delta u + \lambda f(u) = 0, & \text{in } B^n, \\ u = 0, & \text{on } \partial B^n. \end{cases} \tag{2.1}$$

A framework of using the bifurcation method to prove the exact multiplicity of solutions of (2.1) was established in Ouyang and Shi [25,26] (see also [20,21]). Here we briefly recall the approach in [26] without the proof since all proofs can be found in [26]. We summarize some basic facts on (2.1).

Lemma 2.1.

1. *If f is locally Lipschitz continuous in* $[0,\infty)$*, then all positive solutions of* (2.1) *are radially symmetric, and satisfy*

$$\begin{cases} \left(r^{n-1}u'\right)' + \lambda r^{n-1} f(u) = 0, & r \in (0,1), \\ u'(0) = u(1) = 0; \end{cases} \tag{2.2}$$

2. *If u is a positive solution to* (2.1)*, and w is a solution of the linearized problem* (*if it exists*):

$$\begin{cases} \Delta w + \lambda f'(u)w = 0 & \text{in } B^n, \\ w = 0 & \text{on } \partial B^n, \end{cases} \tag{2.3}$$

then w is also radially symmetric and satisfies

$$\begin{cases} \left(r^{n-1}w'\right)' + \lambda r^{n-1} f'(u)w = 0, & r \in (0,1), \\ w'(0) = w(1) = 0; \end{cases} \tag{2.4}$$

3. *For any $d > 0$, there is at most one $\lambda_d > 0$ such that* (2.1) *has a positive solution $u(\cdot)$ with $\lambda = \lambda_d$ and $u(0) = d$. Let $T = \{d > 0\colon$ (2.1) has a positive solution with $u(0) = d\}$, then T is open*; *$\lambda(d) = \lambda_d$ is a well-defined continuous function from T to $\mathbf{R}^+$.*

Because of (3), we call $\mathbf{R}^+ \times \mathbf{R}^+ = \{(\lambda, d)\colon \lambda > 0,\ d > 0\}$ the *phase space*, and $\Sigma = \{(\lambda(d), d)\colon d \in T\}$ the *bifurcation diagram*. A solution (λ, u) of (2.1) or (2.2) is a *degenerate* solution if (2.3) or (2.4) has a nontrivial solution. At a degenerate solution $(\lambda(d), u(d))$, $\lambda'(d) = 0$, and it is referred as a *turning point* of Σ if $\lambda''(0) \neq 0$. We define the *Morse index $M(u)$* of a solution (λ, u) to be the number of negative eigenvalues of the following eigenvalue problem:

$$\begin{cases} \left(r^{n-1}\phi'\right)' + \lambda r^{n-1} f'(u)\phi = -\mu\phi, & r \in (0,1), \\ \phi'(0) = \phi(1) = 0. \end{cases} \tag{2.5}$$

It is well known that the eigenvalues $\mu_1, \mu_2, \ldots$ of (2.5) are all simple, and the eigenfunction ϕ_i corresponding to μ_i has exactly $i - 1$ simple zeros in $(0,1)$ for $i \in \mathbf{N}$. We also call a solution (λ, u) *stable* if $\mu_1(u) > 0$, otherwise it is *unstable*.

For the problem we consider in this paper, we first have

Lemma 2.2. *For the nonlinear function $f(u) = (k+u)^p - (k+u)^{p+1}$, $0 < k < 1$, $T = (0, 1-k)$ for any $n \geqslant 1$, where T is defined in Lemma* 2.1 *part* 3, *and*

$$\lim_{d \to 0^+} \lambda(d) = 0, \qquad \lim_{d \to (1-k)^-} \lambda(d) = \infty. \tag{2.6}$$

Proof. Since $f(0) > 0$, from Theorem 3.2 of [26], $(\lambda, d) = (0,0)$ is a bifurcation point, a curve of positive solution emerges from $(0,0)$, so $(0,\varepsilon) \subset T$ and $\lim_{d\to 0^+} \lambda(d) = 0$. Let $M = \sup\{m > 0\colon (0,m) \subset T\}$. Then from the results in Section 5 of [26], M is a "horizontal asymptote" of $\lambda(d)$. From Proposition 5.8 of [26], either $f(M) = 0$ or $\Delta u + f(u) = 0$ has a radial positive solution $u(x)$ in $\mathbf{R}^n$ so that $\max_{x\in\mathbf{R}^n} u(x) = u(0) = M$. However the latter possibility is ruled out by Lemma 5.1 of [25] since $f(u)$ has no nonnegative zero which is less than M. Hence $f(M) = 0$ and $M = 1-k$. Since we can show (see, for example, [29, Theorem 2.32]) that for every large λ, (1.7) has a positive solution, then $\lim_{d\to(1-k)^-} \lambda(d) = \infty$. □

Notice that the result in Lemma 2.2 does not hold when $k = 0$. The following result is a direct consequence of arguments above and Proposition 6.6 in [26]:

Lemma 2.3. *For the nonlinear function $f(u) = u^p - u^{p+1}$, let T be defined in Lemma* 2.1 *part* 3.

1. *If $n \leqslant 2$, or $n \geqslant 3$ and $p \leqslant (n+2)/(n-2)$, then $T = (0,1)$, and*

$$\lim_{d \to 0^+} \lambda(d) = \infty, \qquad \lim_{d \to 1^-} \lambda(d) = \infty. \tag{2.7}$$

2. *If $n \geqslant 3$ and $p > (n+2)/(n-2)$, then $T = (\theta, 1)$ for some $\theta \in (0,1)$, and*

$$\lim_{d \to \theta^+} \lambda(d) = \infty, \qquad \lim_{d \to 1^-} \lambda(d) = \infty. \tag{2.8}$$

We shall see later that Lemma 2.3 is the main reason that we can only show the exact S-shape for $n \leqslant 2$, or $n \geqslant 3$ and $p \leqslant (n+2)/(n-2)$. In the supercritical case when $n \geqslant 3$ ($p > (n+2)/(n-2)$), there is a family of radial entire solutions of $\Delta u + f(u) = 0$ on $\mathbf{R}^n$ so that $0 < u(0) < \theta$. These solutions are called flame balls [15], and they play important roles in chemical reactions with higher order [15,33].

Now the exact multiplicity of positive solutions to (1.8) is reduced to determine the number of critical point of $\lambda(d)$, or equivalently, the number of degenerate solutions. At a degenerate solution $(\lambda(d_0), u(d_0))$, we have $\lambda'(d_0) = 0$, and $\lambda''(d_0)$ is expressed by

$$\lambda''(d_0) = -\frac{\lambda(d_0)\int_0^1 r^{n-1} f''(u) w^3\, dr}{\int_0^1 r^{n-1} f(u) w\, dr}, \tag{2.9}$$

where w is a nontrivial solution of (2.3). From [25, Lemma 2.3], $\int_0^1 r^{n-1} f(u) w\, dr = (2\lambda)^{-1} w'(1) u'(1) \neq 0$. In the next section, we will show that under the assumptions of our main results, we can show that w does not change sign in $(0,1)$. Without loss of generality, we can assume that $w > 0$ in $(0,1)$.

To consider the Morse indices of the solution, we introduce an auxiliary equation:

$$\begin{cases} \left(r^{n-1} w'\right)' + \lambda r^{n-1} f'(u) w = 0, & r \in (0,1), \\ w'(0) = 0, \qquad w(0) = 1, \end{cases} \tag{2.10}$$

where u is a solution to (2.2). Let $w(\lambda, \cdot)$ be the solution of (2.10), then $w(\lambda, \cdot)$ has the following relation with the Morse index of u (see [26, Lemma 5.2]):

Lemma 2.4. *Suppose that u is a solution of* (2.1)*, and $w(\lambda, \cdot)$ is the solution of* (2.10)*, then $M(u) = k$ if and only if $w(\lambda, \cdot)$ has exactly k zeros in* $(0,1)$.

To conclude this section, we point out a relation between the solution of (1.8) with $k > 0$ and the one with $k = 0$ due to the translational perturbation $u \mapsto u + k$. Suppose that $v(r,d)$ is the solution of

$$\begin{cases} v'' + \dfrac{n-1}{r} v' + \eta\left(v^p - v^{p+1}\right) = 0, & r \in (0,1), \\ v(r) > 0, \quad v'(r) < 0, \quad r \in (0,1), \qquad v'(0) = v(1) = 0, \end{cases} \tag{2.11}$$

which satisfies $v(0) = d$. From Lemma 2.1, $\eta = \eta(d)$ is uniquely determined by d. For any $k \in (0,d)$, there exists a unique $a = a(d,k) \in (0,1)$ such that $v(a,d) = k$.

Lemma 2.5. *Let $\{v(r,d)\colon d \in (0,1)\}$ be the solutions of* (2.11)*, and let $0 < k < 1$. For $d \in (0, 1-k)$, define*

$$u = u(r,d) \equiv v\big(a(d+k,k) r, d+k\big) - k. \tag{2.12}$$

Then $u(r,d)$ is a solution of (1.8) *with $\lambda = \lambda(d) = [a(d+k,k)]^2 \eta(d+k)$. Moreover, suppose that $v(r,d+k)$ has Morse index $M(v(\cdot, d+k))$, then $M(u(\cdot,d)) \leqslant M(v(\cdot,d+k))$.*

Proof. If $v(r)$ is a radial positive solution of

$$\begin{cases} v'' + \dfrac{n-1}{r}v' + \eta f(v) = 0, & r \in (0,1), \\ v'(0) = v(1) = 0, \end{cases} \tag{2.13}$$

then from [10], $v'(r) < 0$ in $(0,1)$. Then for any $k \in (0, v(0))$, there exists a unique $a \in (0,1)$ such that $v(a) = k$, thus $v(r) - k$ with $r \in (0,a)$ solves the equation

$$\begin{cases} v'' + \dfrac{n-1}{r}v' + \eta f(v+k) = 0, & r \in (0,a), \\ v'(0) = v(a) = 0. \end{cases} \tag{2.14}$$

A rescaling of the interval from $(0,a)$ to $(0,1)$ yields a solution u of

$$\begin{cases} u'' + \dfrac{n-1}{r}u' + \lambda f(u+k) = 0, & r \in (0,1), \\ u'(0) = u(1) = 0, \end{cases} \tag{2.15}$$

with $\lambda = a^2\eta$. The main statement in the lemma follows from this argument. From Lemma 2.4, the Morse index of a radial solution u to (2.13) is the number of zeros of the solution to (2.10). Let $\phi(r,d)$ be the solution of (2.10) associated with $u = v(\cdot,d)$ and $\lambda = \eta(d)$, where $v(\cdot,d)$ is a solution of (2.13). Define

$$\psi(r,d) = \phi\big(a(d+k,k)r, d+k\big). \tag{2.16}$$

Then ψ is the solution of (2.10) associated with $u = u(\cdot,d)$ and $\lambda = [a(d+k,k)]^2\eta(d)$, where $u(\cdot,d)$ is the solution of (2.15) as defined in (2.12). In particular, the number of zeros of $\psi(\cdot,d)$ in $(0,1)$ is the number of zeros of $\phi(\cdot,d)$ in $(0, a(d+k,k)) \subset (0,1)$. Thus from Lemma 2.4, $M(u(\cdot,d)) \leqslant M(v(\cdot,d+k))$. □

3. Positivity of solution to linearized equation

In this section, we assume that u is a degenerate solution of (1.8), and w is a nontrivial solution of (2.3). Then we prove that w does not change sign in $(0,1)$ under the assumptions of our main results.

Our first result is proved for the case of $0 < k < 1$, and $n = 1$ or $n = 2$.

Proposition 3.1. *Suppose that* $1 > k > 0$, *$u(r)$ is a degenerate solution of* (1.8), *and $w(x)$ is a nontrivial solution of* (2.3). *If $n = 1$ or $n = 2$, then $w(x)$ does not change sign in B^n.*

Proof. In the following, $f(u) = u^p - u^{p+1}$, and the nonlinearity in (1.8) is $f(u+k)$. From Lemma 2.1, w is also radially symmetric, thus it is a solution of (2.4). We may assume that $w(0) > 0$. Following [9] we use the test function $v(r) = ru'(r) + \beta$, where β is a positive constant to be determined later. By a straightforward calculation: $v' = u' + ru''$, $v'' = 2u'' + ru'''$. Set

$$\begin{aligned} G(r) &= v'' + \frac{n-1}{r}v' + \lambda f'(u+k)v \\ &= (2u'' + ru''') + \frac{n-1}{r}(u' + ru'') + \lambda f'(u+k)(ru' + \beta) \\ &= 2u'' + ru''' + \frac{n-1}{r}u' + \frac{(n-1)r}{r}u'' + \lambda r f'(u+k)u' + \lambda\beta f'(u+k), \end{aligned}$$

note that

$$\frac{d}{dr}\left[u'' + \frac{n-1}{r}u' + \lambda f(u+k)\right] = u''' + \left(\frac{n-1}{r}u'' - \frac{n-1}{r^2}u'\right) + \lambda f'(u+k)u' = 0.$$

Hence

$$\begin{aligned} G(r) &= 2u'' + \frac{2(n-1)}{r}u' + \lambda\beta f'(u+k) \\ &= 2\left[u'' + \frac{n-1}{r}u' + \lambda f(u+k)\right] - 2\lambda f(u+k) + \lambda\beta f'(u+k) \\ &= \lambda\left[\beta f'(u+k) - 2f(u+k)\right] \\ &= \lambda f(u+k)g(r), \end{aligned}$$

where

$$g(r) = \frac{\beta f'(u+k)}{f(u+k)} - 2 = \beta\left[\frac{p}{u+k} + \frac{1}{u+k-1}\right] - 2.$$

We claim

$$\left[r^{n-1}(v'w - vw')\right]' = G(r)r^{n-1}w, \tag{3.1}$$

where $G(r) \equiv \lambda f(u+k)g(r)$. Indeed

$$\begin{aligned} \left[r^{n-1}(v'w - vw')\right]' &= (n-1)r^{n-2}[v'w - vw'] + r^{n-1}\left[(v''w + v'w') - (v'w' + vw'')\right] \\ &= r^{n-2}\left[(n-1)(v'w - vw') + r(v''w - vw'')\right] \\ &= r^{n-2}\left[(n-1)v'w - (n-1)vw' + rv''w - rvw''\right] \\ &= r^{n-2}\left[rw\left(v'' + \frac{n-1}{r}v'\right) - rv\left(w'' + \frac{n-1}{r}w'\right)\right] \\ &= r^{n-2}\left[rw\left(G(r) - \lambda f'(u+k)v\right) - rv\left(-\lambda f'(u+k)w\right)\right] \\ &= r^{n-1}G(r)w. \end{aligned}$$

Clearly $g(r)$ is increasing in r, since $u(r)$ is decreasing in r. Now we suppose $w(r)$ changes sign in $(0,1)$. Let $r_0 \in (0,1)$ be the first root of $w(r) = 0$: $w(r_0) = 0$, and $w(r) > 0$ for $r \in [0, r_0)$. We take $\beta = -r_0 u'(r_0)$. Since

$$v' = -r\lambda f(u+k) + (2-n)u' < 0$$

for any $r \in (0,1]$ when $n = 1$ or 2, we have $v(r) > v(r_0) = 0$ on $[0, r_0)$ and $v(r) < 0$ on $(r_0, 1]$. There are two possibilities:

Case (i): $g(r_0) \leqslant 0$. So we have $g(r) < g(r_0) \leqslant 0$ on $[0, r_0)$ in this case. By integrating (3.1) from 0 to r_0, we obtain

$$\begin{aligned} 0 &> \int_0^{r_0} r^{n-1}\lambda f(u+k)g(r)w(r)\,dr = \int_0^{r_0} r^{n-1}G(r)w(r)\,dr \\ &= \left[r^{n-1}(v'w - vw')\right]_0^{r_0} = r_0^{n-1}\left[v'(r_0)w(r_0) - v(r_0)w'(r_0)\right] = 0, \end{aligned}$$

since $w(r_0) = v(r_0) = 0$. This is a contradiction.

Case (ii): $g(r_0) > 0$. This time we consider the last root r^0 of $w(r) = 0$ before $r = 1$: $r_0 \leqslant r^0 < 1$, $w(r^0) = 0$, $w(r) \neq 0$ on $(r^0, 1)$. We may assume $w(r) > 0$ (otherwise we take $-w(r)$) on $(r^0, 1)$, then $w'(r^0) > 0 > w'(1)$ since $w(r^0) = w(1) = 0$. Now using $g(r) > 0$ and $v(r) \leqslant 0$ on $[r^0, 1]$ and integrating (3.1) from r^0 to 1, we obtain

$$0 < \int_{r^0}^{1} r^{n-1}\lambda f(u+k)g(r)w(r)\,dr = \int_{r^0}^{1} G(r)r^{n-1}w(r)\,dr$$
$$= \left[r^{n-1}(v'w - vw')\right]_{r^0}^{1} = \left(r^0\right)^{n-1} v\left(r^0\right)w'\left(r^0\right) - v(1)w'(1) \leqslant 0,$$

since $v(r^0) < 0$, $v(1) < 0$, $w(1) = w(r^0) = 0$. This is another contradiction, which completes the proof. □

Our second result is for $k = 0$ and $n \geqslant 1$, which has been proved in [25]. But the proof in [25] is for a more general problem thus involving some more technical details, so here we give a direct proof for the sake of completeness.

Proposition 3.2. *Suppose that* $k = 0$, $u(r)$ *is a degenerate solution of* (1.8), *and* $w(r)$ *is a nontrivial solution of* (2.3). *If* $n \geqslant 1$, *then* $w(r)$ *does not change sign in* (0, 1).

Proof. Without loss of generality, we assume that $w(0) > 0$. We use $f(u) = u^p - u^{p+1}$ and $F(u) = \int_0^u f(t)\,dt$. We define

$$H(r) = \frac{1}{2}\left[ru_r^2(r) + (n-2)u_r(r)u(r)\right] + \lambda r F\left(u(r)\right), \tag{3.2}$$

where $F(u) = \int_0^u f(s)\,ds$. The Pohozaev's identity is

$$r_2^{n-1}H(r_2) - r_1^{n-1}H(r_1) = \int_{r_1}^{r_2} \lambda r^{n-1}\left[nF\left(u(r)\right) - \frac{n-2}{2} f\left(u(r)\right)u(r)\right]dr, \tag{3.3}$$

where $0 \leqslant r_1 < r_2 \leqslant 1$.

We also define $K_f(u) = uf'(u)/f(u)$. For $f(u) = u^p - u^{p+1}$, $K_f(u) = p + 1 - (1-u)^{-1}$, which is strictly decreasing for $u \in (0, 1)$, $\lim_{u\to 0^+} K_f(u) = p$ and $\lim_{u\to 1^-} K_f(u) = -\infty$. It is easy to check that

$$\begin{aligned} &f'(u)u - f(u) \geqslant 0, \quad u \in \left(0, (p-1)/p\right), \\ &f'(u)u - f(u) \leqslant 0, \quad u \in \left((p-1)/p, 1\right). \end{aligned} \tag{3.4}$$

We define r_1 to be the unique point in (0, 1) such that $u(r_1) = (p-1)/p$ if $u(0) > (p-1)/p$, and $r_1 = 0$ if $u(0) \leqslant (p-1)/p$. In the following L is defined as $Lv = (r^{n-1}v_r)_r + \lambda r^{n-1}f'(u)v$. We use a comparison function $v(r) = ru_r(r) + \mu u(r)$, where $\mu > 0$ is a constant to be specified later. Then

$$Lv(r) = \lambda r^{n-1}\left\{\mu\left[f'(u)u - f(u)\right] - 2f(u)\right\} = \lambda r^{n-1} g\left(u(r)\right), \tag{3.5}$$

where $g(u) = \mu[f'(u)u - f(u)] - 2f(u)$. Define

$$h(r) = -\frac{ru'(r)}{u(r)} \quad \text{in } (0,1), \tag{3.6}$$

$$\mu(r) = \frac{2f(u(r))}{f'(u(r))u(r) - f(u(r))} \quad \text{in } (r_1, 1). \tag{3.7}$$

Then

$$\begin{aligned} h'(r) &= \frac{(n-2)uu_r + ru_r^2 + \lambda fru}{u^2} = \frac{2H(r) - 2\lambda rF(u(r)) + \lambda rf(u(r))u(r)}{u^2(r)} \\ &= \frac{2H(r)}{u^2(r)} + \lambda r\frac{f(u(r))u(r) - 2F(u(r))}{u^2(r)}. \end{aligned} \tag{3.8}$$

Here in the second equality, we use Pohozaev's identity (3.3). We claim that $H(r) > 0$ for all $r \in (0,1)$. Indeed $H(0) = 0$ and $H(1) = (1/2)[u'(1)]^2 > 0$, and $[r^{n-1}H(r)]' = \lambda r^{n-1}G(u(r))$, where $G(u) = nF(u) - (n-2)/2uf(u)$. Since

$$G'(u) = \frac{n+2}{2}f(u) - \frac{n-2}{2}uf'(u) = \frac{n-2}{2}f(u)\left[\frac{n+2}{n-2} - K_f(u)\right], \tag{3.9}$$

and $K_f(u)$ is strictly decreasing in $(0,1)$, then $[r^{n-1}H(r)]'$ changes sign at most once. From $H(1) > 0$ we conclude that $H(r) > 0$ for all $r \in (0,1)$. From (3.4), $uf(u) - 2F(u) > 0$ for $u \in (0, (p-1)/p]$, hence $h'(r) > 0$ for $r \in (r_1, 1)$. On the other hand, since $\mu(r) = 2/(K_f(u(r)) - 1)$, then $\mu'(r) < 0$ for $r \in (r_1, 1)$. In summary $h(r)$ and $\mu(r)$ are both positive functions in $(r_1, 1)$ such that $h'(r) > 0$ and $\mu'(r) < 0$, and $\lim_{r\to r_1^+}\mu(r) = \lim_{r\to 1^-}h(r) = \infty$. Hence there exists a unique $r_2 \in (r_1, 1)$ such that $h(r_2) = \mu(r_2) \equiv \mu_0$. With this choice of $\mu = \mu_0$, we have

$$\begin{aligned} &v(r) \geqslant 0, \quad r \in [r_1, r_2], \qquad v(r) \leqslant 0, \quad r \in [r_2, 1], \\ &Lv(r) \leqslant 0, \quad r \in [r_1, r_2], \qquad Lv(r) \geqslant 0, \quad r \in [r_2, 1]. \end{aligned} \tag{3.10}$$

From a well-known Sturm comparison lemma (see [26, Lemma 4.1]), w has at most one zero in $[r_1, 1)$.

For $r \in [0, r_1]$, we use test function $v(r) = u(r) > 0$, and $Lv(r) = \lambda r^{n-1}[u(r)f'(u(r)) - f(u(r))] \leqslant 0$. Again by Sturm comparison lemma, w has no zero in $[0, r_1]$. Therefore w has at most one zero in $[0,1)$. Suppose that w has exactly one zero, which we denote by r_3, in $[0,1)$. From Sturm comparison lemma, $r_3 \in [r_1, r_2)$. Since we assume $w(0) > 0$, then $w(r) > 0$ in $[0, r_3)$. With $v(r) = ru_r(r) + \mu_0 u(r)$, we evaluate the integral identity

$$r^{n-1}(wv' - w'v)\Big|_0^{r_3} = \int_0^{r_3}(wLv - vLw)\,dr = \int_0^{r_3}\lambda r^{n-1}w(r)g\big(u(r)\big)\,dr. \tag{3.11}$$

Observe that $g(u(r)) < 0$ for $r \in [0, r_2)$, and $w(r) > 0$ in $(0, r_3)$, then $\int_0^{r_3}\lambda r^{n-1}w(r) \times g(u(r))\,dr < 0$. However $r^{n-1}(wv' - w'v)|_0^{r_3} = -r_3^{n-1}w'(r_3)v(r_3) > 0$, which is a contradiction. This rules out the possibility of w having exactly one zero in $[0,1)$. Therefore $w(r) > 0$ for all $r \in [0,1)$. □

The proof of Proposition 3.2 also clearly implies the following observation, which will be useful for proving the positivity of w when $k > 0$ in (1.8).

Corollary 3.3. *Suppose that $k = 0$, $u(r)$ is a solution of* (1.8)*, and w is the corresponding solution of* (2.10)*. Then w has at most one zero in* $[0,1]$*, and hence the Morse index of u is either* 0 *or* 1.

Now we are ready to prove the positivity of w for higher dimensions but subcritical p:

Lemma 3.4. *Suppose that* $1 > k > 0$, *and* $n \leqslant 2$ *or* $n \geqslant 3$ *and* $p \leqslant (n+2)/(n-2)$.

1. *If* $u(r)$ *is a degenerate solution of* (1.8), *and* $w(r)$ *is a nontrivial solution of* (2.3), *then* $w(r)$ *does not change sign in* $(0,1)$.
2. *If* $u(r)$ *a solution of* (1.8), *and* w *is the corresponding solution of* (2.10), *then* w *has at most one zero in* $[0,1]$, *and the Morse index of* u *is either* 0 *or* 1.

Proof. Suppose that $u(r,d)$ is a degenerate solution of (1.8) so that $u(0,d) = d \in (0, 1-k)$, then from Lemma 2.5, $u(r,d) = v(a(d+k,k)r, d+k) - k$, where $v(\cdot, d+k)$ is the solution of (1.8) with $k=0$. Here we use that fact that $T = (0,1)$ for (1.8) with $k=0$ and $n \leqslant 2$ or $n \geqslant 3$ and $p \leqslant (n+2)/(n-2)$, which is stated in Lemma 2.3. From Proposition 3.2, the Morse index of $v(\cdot, d+k)$ is either 0 or 1, thus from Lemma 2.5, $M(u(\cdot,d)) \leqslant M(v(\cdot, d+k)) \leqslant 1$. If $M(u(\cdot,d)) = 1$ and $u(\cdot,d)$ is degenerate, then $\psi(\cdot,d)$ has at least one zero in $(0,1)$ and another zero at $r=1$, where $\psi(\cdot,d)$ is the solution of (2.10) associated with $u(\cdot,d)$. But $a(d+k,k) < 1$, so the solution $\phi(\cdot, d+k)$ of (2.10) associated with $v(\cdot, d+k)$ has at least two zeros in $(0,1)$, which implies that $M(v(\cdot, d+k)) \geqslant 2$, that is a contradiction. Hence $M(u(\cdot,d)) = 0$, and $w(r)$ does not change sign in $(0,1)$. The second part follows from above argument and Corollary 3.3. □

We notice that Lemma 3.4 covers the result in Proposition 3.1, but the proof of Proposition 3.1 is direct and it does not rely on the translation $u \mapsto u+k$, thus we still include it here for interested readers.

4. Proof of main theorems

Proof of Theorem 1.1. From Theorem 2.32 of [29], there exists $\eta_1 > 0$ such that (1.9) has at least one positive solution if $\eta > \eta_1$. Set

$$\eta_0 = \inf\{\eta > 0\colon\ (1.9) \text{ has at least a positive solution for such } \eta\}.$$

We claim that $\eta_0 > 0$. Since $f(u)/u = u^{p-1}(1-u)$ is continuous for $u \in [0,1]$, so there exists $k > 0$ such that $f(u) \leqslant mu$ for $u(r) \in [0,1]$. If $v(x)$ is a positive solution of (1.9) for $\eta > 0$, multiplying (1.9) by v, and integrating over Ω, we get $\int_\Omega v[\Delta v + \eta f(v)]\,dx = 0$. Integrate by parts, we have

$$\int_\Omega \left|\nabla v(x)\right|^2 dx = \eta \int_\Omega f\big(v(x)\big)v(x)\,dx \leqslant \eta m \int_\Omega v^2(x)\,dx. \tag{4.1}$$

One the other hand, let λ_1 be the principal eigenvalue of $-\Delta$ in $H_0^1(B^n)$. Then

$$\int_\Omega |\nabla v|^2\,dx \geqslant \lambda_1 \int_\Omega v^2\,dx. \tag{4.2}$$

Combining (4.1) and (4.2), we obtain $\lambda_1 \int_\Omega v^2\,dx \leqslant \eta m \int_\Omega v^2\,dx$, and hence $\eta_0 \geqslant \lambda_1/m > 0$.

Next we can show that (1.9) has a solution when $\eta = \eta_0$ from standard elliptic equation regularity theory. We may choose one of them and again denote by $v_0(x)$. This positive solution $v_0(x)$

of (1.9) at $\eta = \eta_0$ must be a degenerate solution. If not, from implicit function theorem, there would be a positive solution of (1.9) for $\eta < \eta_0$, and this contradicts the definition of η_0. From Proposition 3.2, the solution w of linearized problem (2.3) does not change sign in B^n, and we choose w to be positive in B^n. Thus $\int_{B^n} f(v_0) w\, dx > 0$ since $f(u) > 0$ for $0 \leqslant u \leqslant v_0(0)$ and $w > 0$. By a bifurcation theorem of Crandall–Rabinowitz [3], all solutions of (1.9) near (η_0, v_0) have the form $(\eta_0 + \tau(s), v_0 + sw + z(s))$, with $\tau(0) = \tau'(0) = 0$, $z(0) = z'(0) = 0$, and $\tau''(0) > 0$ from Theorem 2.2 of [25]. So the solution curve "turns right" at (η_0, v_0). We may denote the upper and lower branches by $v^*(\cdot, \eta)$ (with $s > 0$) and $v_*(\cdot, \eta)$ (with $s < 0$), respectively, for $\eta > \eta_0$.

We claim there is no any other turning point on either upper branch or lower branch. Suppose there is, for example, a turning point on the upper branch. Let $(\eta^*, v^*(\cdot, \eta^*))$ be the first degenerate solution on the upper branch when we continue the solution curve rightward in η from η_0. Then Proposition 3.2 shows that the corresponding w does not change sign, and Theorem 2.2 of [25] is applicable, then $\eta''(d) < 0$, but that is impossible, since we continue the solution curve from left to right, there always exists solution for η near η^* and $\eta < \eta^*$. So (η_0, v_0) is the unique degenerate solution on this component of solution curve. Thus both upper and lower branches can be continued for $\eta > \eta_0$ without turning points.

By using the same proof in [25, pp. 143–145], we can show that the solutions on the upper branch are strictly increasing in η, and $\lim_{\eta\to\infty} v^*(x, \eta) = 1$. Similarly we can show that the solutions on the lower branch are strictly decreasing with respect to η (see [20]), and $\lim_{\eta\to\infty} v_*(x, \eta) = \theta \geqslant 0$. From Lemma 2.3, $\theta = 0$ if $n \leqslant 2$, or $n \geqslant 3$ and $p \leqslant (n+2)/(n-2)$, and $\theta > 0$ if $n \geqslant 3$ and $p > (n+2)/(n-2)$. The fact that $\lim_{\eta\to\infty} v^*(x, \eta) = 1$ also rules out the possibility of the bifurcation diagram having more than one component, from Lemma 2.1. The stability of the solution comes directly from Corollary 5.6 in [26] and Theorem 3.12 of [25], and here we omit the details. □

Finally we prove Theorem 1.2. Before we go into the technical details, we sketch the main ideas, which are close to those in [9] and [30,31]. Since the equation with small $k > 0$ is a perturbation of the one with $k = 0$, one can use a result of Dancer [4] to conclude that any compact solution branch remains basically same. Hence the perturbation of the middle section of bifurcation curve of (1.9) is still a bounded $\subset$-shaped curve. The sections of curve near $d = 0$ and $d = 1$ are unbounded for (1.9), thus the perturbation of these sections may not have same profile. For the upper section with $d \to 1$, we notice that the perturbation from (1.9) to (1.7) is a translation $u \mapsto u + k$, hence we can use Lemma 2.5 to conclude that there is no degenerate solution on that section. For the lower section, we know that $(\lambda, u) = (0, 0)$ is a bifurcation point, and the bifurcation curve $\lambda(d)$ emerges from $(0, 0)$. To connect to the middle section which tends to a large λ, the lower branch emerging from $(0, 0)$ must turn back at some $\lambda^* > 0$. We need to show that there is only one turning point in that section. For that part, the bifurcation approach developed in [25,26] can be employed, provided that w does not change sign at any degenerate solution (which has been proved in Section 3).

Proof of Theorem 1.2. From Lemmas 2.1 and 2.2, the bifurcation curve of (1.8) can be represented by $\Gamma = \{(\lambda(d), u(\cdot, d))\colon d \in (0, 1-k)\}$. Recall that $(\eta_0, v_0(\cdot))$ is the degenerate solution of (1.9), $\{(\eta(d), v(\cdot, d) = v^*(\cdot, \eta(d))\colon 1 > d > v_0(0),\ \eta(d) > \eta_0\}$ and $\{(\eta(d), v(\cdot, d) = v_*(\cdot, \eta(d))\colon v_0(0) > d > 0,\ \eta(d) > \eta_0\}$ are the upper and lower branches of bifurcation curve of (1.9), respectively.

From Lemma 2.5, for any $d \in (0, 1-k)$, $u(\cdot, d)$ can be represented by

$$u(r,d) = v\big(a(d+k,k)r, d+k\big) - k, \tag{4.3}$$

and $M(u(\cdot,d)) \leqslant M(v(\cdot,d+k))$. Since each solution $v(\cdot,d)$ on the upper branch is stable with Morse index 0, then immediately, for any $d \in (v_0(0)-k, 1-k)$, $u(\cdot,d)$ is also stable with Morse index 0. At $d = v_0(0)-k$, $u(\cdot,d)$ is also stable since $a(d+k,k) < 1$. Thus there is no degenerate solution for (1.8) and $\lambda'(d) > 0$ when $d \in [v_0(0)-k, 1-k)$. Hence we need to show that $\lambda(d)$ has exactly two critical points for $d \in (0, v_0(0)-k)$.

It is easy to check that for $f(u) = u^p - u^{p+1}$, $f''(u) > 0$ for $u \in (0, (p+1)/(p-1))$ and $f''(u) < 0$ for $u \in ((p+1)/(p-1), 1)$. Fix a small $\varepsilon > 0$ such that $\varepsilon < \min\{(p+1)/4(p-1), (1-v_0(0))/4\}$. Let $d_1 = (p+1)/(p-1) - \varepsilon$ and $d_2 = v_0(0) + \varepsilon$. We claim that on the portion of Γ with $d_1 \leqslant d \leqslant d_2$, there is exactly one degenerate solution if $k > 0$ is small enough. To prove the claim, we apply Theorems 2.1 and 2.3 in [30]. Since when $k = 0$, the portion of solution curve to (1.9) for $d_1 \leqslant d \leqslant d_2$ is $\subset$-shaped with a unique degenerate point $(\eta(d_0), v_0)$ with $d_0 \equiv v_0(0)$, and $\eta''(d_0) > 0$. Then the conditions of Theorems 2.1 and 2.3 in [30] are satisfied, for small k near 0, (1.8) has a unique degenerate solution $(\lambda(d_*), u(\cdot, d_*))$ with d_* a perturbation of d_0. Since the portion of solution curve for $d_1 \leqslant d \leqslant d_2$ is compact, then $(\lambda(d_*), u(\cdot,d_*))$ is the only degenerate solution on that portion and $\lambda''(d_*) > 0$ if k is properly chosen. So the claim is proved.

Finally we prove that on the portion of Γ with $0 \leqslant d \leqslant d_1$, there is exactly one degenerate solution. Since the portion $d_1 \leqslant d \leqslant d_2$ is $\subset$-shaped, then $\lambda'(d_1) < 0$. On the other hand, at $(\lambda, u) = (0,0)$, from implicit function theorem, we obtain $\lambda'(d) > 0$ for $d \in (0, \delta)$. Thus there is at least one $d_3 \in (0, d_1)$ such that $\lambda'(d) = 0$. We show that at any critical point $d^* \in (0, d_1)$ of $\lambda(d)$, we must have $\lambda''(d^*) < 0$. In fact $f''(u(r,d^*)) > 0$ for $r \in [0,1)$ if $d^* < d_1 < (p+1)/(p-1)$. From Corollary 3.4, the solution w of (2.3) can be chosen as positive. Then from (2.9), we obtain $\lambda''(d^*) < 0$, and this completes the proof of exact S-shaped.

The order relation $u_*(x,\lambda) < u_m(x,\lambda) < u^*(x,\lambda)$ can be proved by showing that u_* is the minimal solution, and u^* is the maximal solution, by using comparison methods. That is rather standard, so we omit the details. u_* and u^* are increasing in λ by, for example, Lemma 5.7 of [26]. From Lemma 3.4, all solutions have Morse index 0 or 1, then the statement on the stability follows from Corollary 5.6 of [26]. □

5. Concluding remarks

1. For (1.7), the bifurcation diagram is not always S-shaped. Indeed one can easily show that when $(p-1)^2/(p+1)^2 \leqslant k < 1$, (1.7) has a unique positive solution for each $\lambda > 0$ even for the general bounded smooth domain Ω, since $f(u+k)$ is concave for $0 \leqslant u \leqslant 1-k$. When $n = 1$, it was shown by Wang and Lee [35] that for $(p-1)^2/[(p+1)(p+2)] < k < 1$, the bifurcation curve is monotone. We conjecture that there exists $k_0 \in (0, (p-1)^2/(p+1)^2)$ such that the bifurcation diagram is S-shaped if $0 < k < k_0$, and it is monotone if $k_0 < k < 1$ (see Fig. 3). Similar evolution of bifurcation diagrams have been observed for perturbed Gelfand's equation (see [4,9]), and related discussion can be found in [30,31].

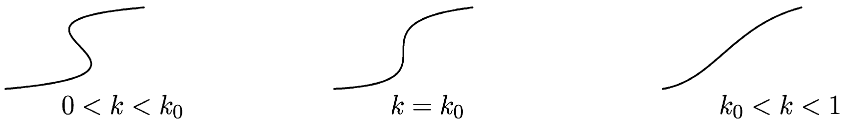

Fig. 3. Possible evolution of bifurcation diagrams.

2. Our result in this paper is optimal for p. Indeed for $n \geqslant 3$ and $p > (n+2)/(n-2)$, (1.8) could have more than three positive solutions. Using a change of variables: $u = \varepsilon v$, $\mu = \lambda \varepsilon^{p-1}$, $K = k/\varepsilon$, (1.7) becomes

$$\Delta v + \mu\big[(v+K)^p - \varepsilon(v+K)^{p+1}\big] = 0, \quad x \in B^n, \qquad v(x) = 0, \quad x \in \partial B^n, \tag{5.1}$$

which is a perturbation of the classical equation considered by Joseph and Lundgren [16]:

$$\Delta v + \mu(v+K)^p = 0, \quad x \in B^n, \qquad v(x) = 0, \quad x \in \partial B^n. \tag{5.2}$$

For fixed $K > 0$, it is known [16] that when $3 \leqslant n \leqslant 10$ and $(n+2)/(n-2) \leqslant p < \infty$, or $n \geqslant 11$ and $(n+2)/(n-2) \leqslant p < (n - 2\sqrt{n-1})/(n - 4 - 2\sqrt{n-1})$, then the bifurcation diagram of (5.2) has infinitely many turning points. Since the perturbation of a compact portion of bifurcation diagram still has all the turning points, thus for any integer $N > 0$, there exists small $\varepsilon_0 > 0$, such that the bifurcation diagram of (5.1) can have at least N turning points. A similar discussion for the higher dimension perturbed Gelfand's equation can be found in [7].

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