Multi-Parameter Bifurcation and Applications

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1 Introduction

In this article we discuss some new abstract bifurcation theorems and their applications to the global bifurcation problem of a semilinear boundary value problem:

$$\Delta u + \lambda f(\varepsilon, u) = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{1}$$

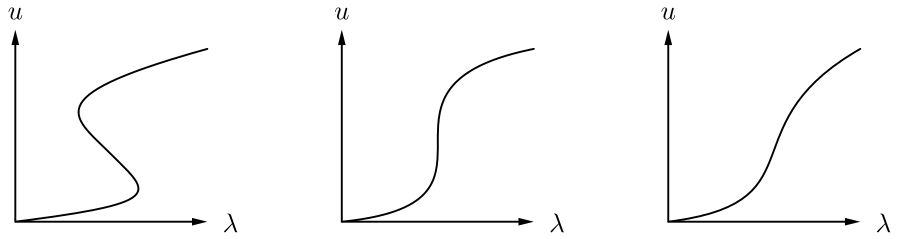
where $\lambda, \varepsilon > 0$, f is a nonlinear function, and Ω is a smooth bounded domain in $\mathbf{R}^n$. The global bifurcation diagrams and exact multiplicity of equation (1) have been studied in many recent work. In particular, a systematic approach for the positive radially symmetric solutions of (1) when Ω is a ball was presented in Ouyang and Shi [OS1], [OS2] based on previous results by Korman, Li and Ouyang [KLO1], [KLO2]. The techniques we developed in this article are extensions of the approach in these work, but emphasizing on different aspects of the bifurcation problems.

First in many bifurcation problems in application, "imperfect bifurcations" occur due to the small perturbations or noises in the physical system. For example, in a pitch-fork bifurcation, an imperfect bifurcation diagram as in Fig. 1b appears often in the literature of bifurcation theory:

Figure 1: (a) Pitchfork bifurcation; (b) Imperfect pitchfork bifurcation

Such diagrams can be rigorously derived by using Lyapunov-Schmidt reduction method, see for example [GS]. We will use bifurcation theorems by Crandall and Rabinowitz [CR1], [CR2] to give a more direct approach. On the other hand, there is a considerable interest in the evolution of global bifurcation diagrams in (λ, u) space when another parameter ε changes. For example, it has been conjectured that for $f(\varepsilon, u) = e^{u/(1+\varepsilon u)}$ (this equation arises from combustion theory, and it is usually called perturbed Gelfand's equation), the bifurcation diagrams of (1) are as in Fig. 2. We will also show that our abstract results can be applied to these evolution problems of bifurcation diagrams.

In Section 2, we introduce some abstract multi-parameter bifurcation results. We apply the abstract imperfect bifurcation results to an equation from natural resource management



(a) $0 < \varepsilon < \varepsilon_0$ (b) $\varepsilon = \varepsilon_0$ (c) $\varepsilon > \varepsilon_0$

Figure 2: Evolution of bifurcation diagrams

in Section 3, and in Section 4, we give some applications of fold and cusp bifurcations. Some open problems and conjectures are mentioned at the end of Section 4. For a linear operator L, we use $N(L)$ to denote the null space of L, and $R(L)$ to denote the range of L.

2 Abstract Theory

In this section, we recall some abstract bifurcation results from [S]. We consider an equation

$$F(\varepsilon, \lambda, u) = 0, \tag{2}$$

where $F : M \equiv \mathbf{R} \times \mathbf{R} \times X \to Y$ is a nonlinear differentiable map and X, Y are Banach spaces. We call $(\varepsilon_0, \lambda_0, u_0)$ a *degenerate* solution of (2) if $F_u(\varepsilon_0, \lambda_0, u_0)$ is not invertible.

First we consider the case when $F(\varepsilon_0, \lambda, u_0) \equiv 0$ for all $\lambda \in \mathbf{R}$, a bifurcation from the trivial solution $u = u_0$ occurs if $F_u(\varepsilon_0, \lambda_0, u_0)$ is not invertible. In [CR1], Crandall and Rabinowitz proved:

Theorem 2.1. ([CR1] Theorem 1.7) *Let $F : M \to Y$ be continuously differentiable. Suppose that $F(\varepsilon_0, \lambda, u_0) = 0$ for $\lambda \in \mathbf{R}$, the partial derivative $F_{\lambda u}$ exists and is continuous. At $(\varepsilon_0, \lambda_0, u_0) \in M$, F satisfies*

(F1) *dim $N(F_u(\varepsilon_0, \lambda_0, u_0))$=codim $R(F_u(\varepsilon_0, \lambda_0, u_0)) = 1$, and $N(F_u(\varepsilon_0, \lambda_0, u_0)) = span\{w_0\}$,*

and

(F2) $F_{\lambda u}(\varepsilon_0, \lambda_0, u_0)[w_0] \notin R(F_u(\varepsilon_0, \lambda_0, u_0))$.

Then for fixed $\varepsilon = \varepsilon_0$, the solutions of (2) near (λ_0, u_0) consists precisely of the curves $u = u_0$ and $(\lambda(s), u(s))$, $s \in I = (-\delta, \delta)$, where $(\lambda(s), u(s))$ are C^1 functions such that $\lambda(0) = \lambda_0$, $u(0) = u_0$, $u'(0) = w_0$. Moreover, if F is C^2 in u, then

$$\lambda'(0) = -\frac{\langle l, F_{uu}(\varepsilon_0, \lambda_0, u_0)[w_0, w_0]\rangle}{2\langle l, F_{\lambda u}(\varepsilon_0, \lambda_0, u_0)[w_0]\rangle}, \tag{3}$$

where $l \in Y^*$ satisfying $N(l) = R(F_u(\varepsilon_0, \lambda_0, u_0))$, and Y^* is the dual space of Y.

In Theorem 2.1, if

(F3) $F_{uu}(\varepsilon_0, \lambda_0, u_0)[w_0, w_0] \notin R(F_u(\varepsilon_0, \lambda_0, u_0))$

Figure 3: (a) Transcritical bifurcation; (b) Pitchfork bifurcation

is satisfied, then $\lambda'(0) \neq 0$, and a *transcritical* bifurcation occurs at (λ_0, u_0); and if $\lambda'(0) = 0$ and $\lambda''(0) \neq 0$, then a *pitchfork* bifurcation occurs at (λ_0, u_0).

By studying the degenerate solutions of (2), we proved the following result of imperfect bifurcation:

Theorem 2.2. ([S] Theorem 2.5) *Assume the conditions in Theorem 2.1 are satisfied. In addition we assume that (F3) and*

(F4) $F_\varepsilon(\varepsilon_0, \lambda_0, u_0) \notin R(F_u(\varepsilon_0, \lambda_0, u_0))$

are satisfied. Then there exists $\rho_1, \delta_1, \delta_2 > 0$*, such that for* $N = \{(\lambda, u) \in \mathbf{R} \times X : |\lambda - \lambda_0| \leq \delta_1, ||u|| \leq \delta_2\}$,
(A) *for* $\varepsilon \in (\varepsilon_0 - \rho_1, \varepsilon_0)$*, (or* $\varepsilon \in (\varepsilon_0, \varepsilon_0 + \rho_1)$*),*

$$F^{-1}(0) \bigcap N = \Sigma_\varepsilon^1 \bigcup \Sigma_\varepsilon^2,\ \Sigma_\varepsilon^i = \{(\lambda, \overline{u_i}(\lambda)) : \lambda \in [\lambda_0 - \delta_1, \lambda_0 + \delta_1]\},\ i = 1, 2, \tag{4}$$

and $\overline{u_i}'(\lambda) > 0$ *for* $\lambda \in [\lambda_0 - \delta_1, \lambda_0 + \delta_1]$, $i = 1, 2$*;*
(B) *for* $\varepsilon = \varepsilon_0$,

$$F^{-1}(0) \bigcap N = \{(\lambda, 0) : \lambda \in [\lambda_0 - \delta_1, \lambda_0 + \delta_1]\} \bigcup \Sigma_0,\ \Sigma_0 = \{(\overline{\lambda}(t), \overline{u}(t)) : t \in [-\eta, \eta]\}, \tag{5}$$

and $\overline{u}'(t) > 0$ *for* $t \in [-\eta, \eta]$*;*
(C) *for* $\varepsilon \in (\varepsilon_0, \varepsilon_0 + \rho_1)$*, (or* $\varepsilon \in (\varepsilon_0 - \rho_1, \varepsilon_0)$*),*

$$F^{-1}(0) \bigcap N = \Sigma_\varepsilon^+ \bigcup \Sigma_\varepsilon^-,\ \Sigma_\varepsilon^\pm = \{(\overline{\lambda}_\pm(t), \overline{u}_\pm(t)) : t \in [-\eta, \eta]\}, \tag{6}$$

$\overline{\lambda}_+(\pm\eta) = \lambda_0 + \delta_1$, $\overline{\lambda}_-(\pm\eta) = \lambda_0 - \delta_1$, $\overline{\lambda}'_\pm(0) = 0$, $\overline{\lambda}''_+(0) < 0$, $\overline{\lambda}''_-(0) > 0$, *and there are exactly one turning point on each component* $\Sigma_\varepsilon^\pm$*. (See Fig. 4.)*

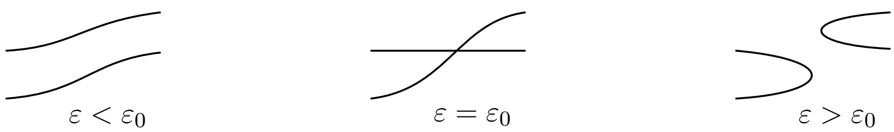


Figure 4: Imperfect bifurcation in a transcritical bifurcation

Secondly, we consider the case when the degenerate solution is a turning point on the bifurcation diagram. The following result was proved by Crandall and Rabinowitz [CR2]:

Theorem 2.3. ([CR2] Theorem 3.2) *Let* $F : M \to Y$ *be continuously differentiable. At* $(\varepsilon_0, \lambda_0, u_0) \in M$*,* $F(\varepsilon_0, \lambda_0, u_0) = 0$*, F satisfies (F1) and*

(F5) $F_\lambda(\varepsilon_0, \lambda_0, u_0) \notin R(F_u(\varepsilon_0, \lambda_0, u_0))$.

Then for fixed $\varepsilon = \varepsilon_0$, the solutions of (2) *near* (λ_0, u_0) *form a* C^1 *curve* $(\lambda(s), u(s))$, $\lambda(0) = \lambda_0$, $u(0) = u_0$, $\lambda'(0) = 0$ *and* $u'(0) = w_0$. *Moreover, if* F *is* C^2 *in* u, *then*

$$\lambda''(0) = -\frac{\langle l, F_{uu}(\varepsilon_0, \lambda_0, u_0)[w_0, w_0]\rangle}{\langle l, F_\lambda(\varepsilon_0, \lambda_0, u_0)\rangle}, \tag{7}$$

where $l \in Y^*$ *satisfying* $N(l) = R(F_u(\varepsilon_0, \lambda_0, u_0))$, *and* Y^* *is the dual space of* Y.

In Theorem 2.3, if (F3) is also satisfied, then $\lambda''(0) \neq 0$, and a *saddle-node* bifurcation occurs at (λ_0, u_0). In singularity theory, a degenerate solution with $\lambda''(0) \neq 0$ is called a *fold*, and one with $\lambda''(0) = 0$ but $\lambda'''(0) \neq 0$ is a *cusp*. (For applications of singularity theory in partial differential equations, see [CT].) In [S], we proved

Theorem 2.4. ([S] Theorem 2.3)

1. *If* F *satisfies (F1), (F3) and (F5) at* $(\lambda_0, \varepsilon_0, u_0)$, *then the saddle-node degenerate solution persists for* ε *near* ε_0.
2. *If* F *satisfies (F1), (F5),*

(F3') $F_{uu}(\varepsilon_0, \lambda_0, u_0)[w_0, w_0] \in R(F_u(\varepsilon_0, \lambda_0, u_0))$,

and

(F6) $D_\varepsilon \begin{pmatrix} F(\cdot) \\ F_u(\cdot)[w_0] \end{pmatrix} \notin R\left(D_{(\lambda,u,w)} \begin{pmatrix} F(\cdot) \\ F_u(\cdot)[w_0] \end{pmatrix}\right)$,

where $\cdot = (\varepsilon_0, \lambda_0, u_0)$. *In addition we assume that* $F \in C^3(M)$ *and*

(F7) $F_{uuu}(\varepsilon_0, \lambda_0, u_0)[w_0, w_0, w_0] + 3F_{uu}(\varepsilon_0, \lambda_0, u_0)[\theta, w_0] \notin R(F_u(\varepsilon_0, \lambda_0, u_0))$, *where* θ *is the solution of* $F_u(\cdot)[\theta] + F_{uu}(\cdot)[w_0, w_0] = 0$.

Then there exists $\rho_1 > 0$ *such that for* $\varepsilon \in (\varepsilon_0 - \rho_1, \varepsilon_0 + \rho_1)$, *all the solutions of* (2) *near* (λ_0, u_0) *are on a curve* $\Sigma_\varepsilon = (\overline{\lambda}(t), \overline{u}(t))$, *where* $t \in I = (-\eta, \eta)$ *for* $\eta = \eta(\varepsilon)$. *Moreover,*
(A) *for* $\varepsilon \in (\varepsilon_0 - \rho_1, \varepsilon_0)$, *(or* $\varepsilon \in (\varepsilon_0, \varepsilon_0 + \rho_1)$*)*, $\overline{\lambda}'(t) > 0$ *for* $t \in I$;
(B) *for* $\varepsilon = \varepsilon_0$, $\overline{\lambda}(0) = \lambda_0$, $\overline{\lambda}'(0) = \overline{\lambda}''(0) = 0$, $\overline{\lambda}'''(0) > (<)0$ *and* $\overline{\lambda}'(t) > (<)0$ *for* $t \in I\backslash\{0\}$;
(C) *for* $\varepsilon \in (\varepsilon_0, \varepsilon_0 + \rho_1)$, *(or* $\varepsilon \in (\varepsilon_0 - \rho_1, \varepsilon_0)$*)*, *there exists* $t_1, t_2 \in I$ *such that* $\overline{\lambda}'(t_1) = \overline{\lambda}'(t_2) = 0$, $\overline{\lambda}''(t_1) < (>)0$, $\overline{\lambda}''(t_2) > (<)0$, $\overline{\lambda}'(t) > (<)0$ *in* $(-\eta, t_1) \bigcup (t_2, \eta)$ *and* $\overline{\lambda}'(t) < (>)0$ *in* (t_1, t_2). *(See Figure 5.)*

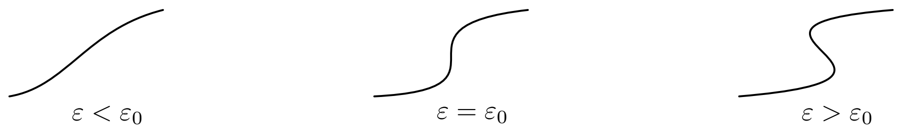


Figure 5: Cusp Bifurcation

The proof of Theorems 2.2 and 2.4 are based on implicit function theorem and bifurcation theorems in [CR1, CR2] (which are also based on implicit function theorems). For details of proof, we refer to [S].

3 Application to a Harvesting Problem

As applications of the abstract results established in Section 2, we consider a diffusive logistic equation with harvesting, which arises from population biology:

$$\frac{\partial u}{\partial t} = D\Delta u + au\left(1 - \frac{u}{N}\right) - c \cdot h(x), \qquad (t,x) \in (0,T) \times \Omega, \tag{8}$$

with the initial and boundary conditions:

$$\begin{aligned} &u(t,x) = 0 \qquad (t,x) \in (0,T) \times \partial\Omega, \\ &u(0,x) = u_0(x) \geq 0, \qquad x \in \Omega, \end{aligned} \tag{9}$$

where $D > 0$ is the diffusion coefficient, $a > 0$ is the linear reproduction rate and $N > 0$ is the carrying capacity of the environment. (8) and (9) arise from the studies of population biology of one species. Here $u(x,t)$ is the concentration of a substance or the density of a population. We assume that (a) the individuals of the species disperse randomly in a bounded habitat Ω; (b) the reproduction of the species follows the logistic growth; (c) the boundary of the environment $\partial\Omega$ is hostile to the species; (d) the habitat Ω is homogeneous, (*i.e.* the diffusion and the growth do not depend on x); and (e) the population is harvested at a constant rate $c \cdot h(x)$, where $c > 0$ is a parameter which represents the level of harvesting, and $h(x)$ represents the natural heterogeneous fishing rate with $h(x) > 0$ for $x \in \Omega$, and $||h||_\infty = 1$. Such harvesting pattern arises naturally from the fishery management problems, where $c \cdot h(x)$ is related to the fishing quota imposed by fishing regulating authorities. The equation (8) with (9) are generalization of well-known ordinary differential equation logistic model with constant yield harvesting $u' = au(1 - u/N) - c$ (see [Cl], [BC].)

We study the steady state solutions of (8) and (9). In fact, we consider a more general nonlinearity. We consider a semilinear elliptic equation:

$$\Delta u + \lambda f(u) - ch(x) = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{10}$$

where $\lambda, c > 0$, $h(x)$ is defined as in last paragraph, and $f(u)$ satisfies

(f1) $f \in C^2(\mathbf{R})$, $f(0) = 0$, $f'(0) > 0$, $f(u) > 0$ for $u \in (0, M)$, where either $M = \infty$ or $M < \infty$, $f(M) = 0$ and $f'(M) < 0$;

(f2) $f''(u) < 0$ for $u \in \mathbf{R}$;

(f3) If $M = \infty$, then $\displaystyle\lim_{u \to \infty} \frac{f(u)}{u} = 0$.

An example for the case of $M < \infty$ in (f1) is the logistic growth $f(u) = au - bu^2$ for some $a, b > 0$, and examples for the case of $M = \infty$ are $g(u) = au/(1 + bu)$ for $a, b > 0$ and $g(u) = 1 - e^{-au}$ for $a > 0$, which are sublinear functions modeling the saturating effect. We denote by λ_k the k-th eigenvalue of

$$\Delta\phi + \lambda\phi = 0 \quad \text{in } \Omega, \quad \phi = 0 \quad \text{on } \partial\Omega. \tag{11}$$

It is well-known that λ_1 is simple, and its eigenfunction does not change sign. We define $\lambda_k^0 = \lambda_k / f'(0)$. It is easy to show that when $\lambda \leq \lambda_1^0$, then (10) has no non-negative solutions if $c \geq 0$. General existence results via comparison and bifurcation methods were obtained in [OSS], and we also proved an exact multiplicity result which can be obtained by using the abstract bifurcation in Section 2:

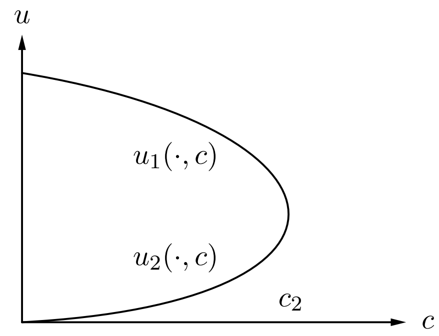


Figure 6: Precise Bifurcation Diagram for Diffusive Logistic Equation with Harvesting

Theorem 3.1. ([OSS] Theorem 4.1) *Suppose that f satisfies (f1)-(f3). There exists $c_2, \delta_2 > 0$ such that when $\lambda \in (\lambda_*, \lambda_* + \delta_2)$,*

1. (10) *has exactly two positive solutions $u_1(\cdot, c)$ and $u_2(\cdot, c)$ for $c \in [0, c_2)$, exactly one positive solution $u_1(\cdot, c)$ for $c = c_2$, and no positive solution for $c > c_2$;*
2. *$u_1(\cdot, c)$ is stable and $u_2(\cdot, c)$ is unstable with Morse index is 1 for $c \in [0, c_2)$, $u_1(\cdot, c_2)$ is degenerate with and neutrally stable;*
3. *All solutions lie on a smooth curve Σ. On (c, u) space, Σ starts from $(0, 0)$, continues to the right, reaches the unique turning point at $c = c_2$ where it turns back, then continues to the left without any turnings until it reaches $(0, v_a)$, where v_a is the unique positive solution of* (10) *with $c = 0$. (see Fig. 6).*

In Theorem 3.1, we use the harvesting rate c as the bifurcation parameter. Note that in this case, the bifurcation diagram is exactly same as that of ODE: $u' = au(1 - u/N) - c$, but the dynamics in function space is much more complicated. Next we use the recipcle of the diffusion coefficient as the bifurcation parameter, and we obtain a local exact multiplicity result. Here we consider

$$\Delta u + \lambda[f(u) - \varepsilon] = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{12}$$

where $\varepsilon, \lambda > 0$, and $f(u)$ satisfies (f1)-(f3).

Theorem 3.2. ([SS] Theorem 1.2) *Suppose that $f(u)$ satisfies (f1)-(f3). We assume that λ_2 is a simple eigenvalue of* (11) *and*

$$\int_\Omega \phi_2(x)dx \cdot \int_\Omega \phi_2^3(x)dx > 0, \tag{13}$$

where ϕ_2 is an eigenfunction corresponding to λ_2. Let $\Sigma = \{(\lambda, u) \in \mathbf{R} \times X : (\lambda, u) \text{ solves } (12)\}$, and $T(a, b, c) = \{(\lambda, u) : a < \lambda < b, ||u||_X < c\}$. Then for any small $\delta_1, \delta_2 > 0$, there exists $\varepsilon_1 = \varepsilon_1(\delta_1, \delta_2, g) > 0$ such that for any $\varepsilon \in (0, \varepsilon_1)$,

$$\Sigma_0 \equiv \Sigma \bigcap T(\lambda_1^0 - \delta_1, \lambda_2^0 + \delta_1, \delta_2) = \bigcup_{i=1}^{3} \Sigma_i,$$

where Σ_i is a connected component of Σ_0, ($i = 1, 2, 3$). Moreover,

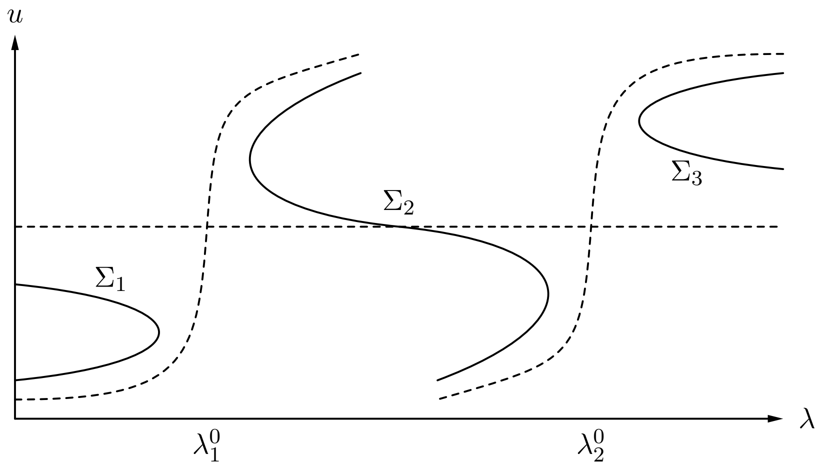


Figure 7: Bifurcation diagram in Theorem 3.2; solid curve: small $\varepsilon > 0$, dashed curve: $\varepsilon = 0$

(A) Each Σ_i $(i = 1, 2, 3)$ is a smooth curve in $\mathbf{R} \times X$;

(B) Σ_1 is exactly $\supset$-shaped, there is a unique degenerate solution on Σ_1, and each solution on Σ_1 is negative;

(C) Σ_3 is exactly $\subset$-shaped, there is a unique degenerate solution on Σ_3, and each solution on Σ_3 is sign-changing;

(D) Σ_2 is exactly S-shaped, and there are exactly two degenerate solutions on Σ_2; Σ_2 can be parameterized as $(\lambda(s), u(s))$, $s \in (s_1, s_4)$, such that for $s \in (s_1, s_2)$, $u(s)$ is positive, and for $s \in (s_3, s_4)$, $u(s)$ is sign-changing, where $s_1 < s_2 < s_3 < s_4$; The portion of Σ_2 with $s \in (s_1, s_2)$ contains the degenerate solution on the left, and the portion of Σ_2 with $s \in (s_3, s_4)$ contains the degenerate solution on the right (see Fig. 7);

(E) There exist $\delta_i = \delta_i(\varepsilon) > 0$, $(i = 3, 4, 5, 6)$ such that

$$proj\Sigma_1 = [\lambda_1^0 - \delta_1, \lambda_1^0 - \delta_3],\ proj\Sigma_2 = [\lambda_1^0 + \delta_4, \lambda_2^0 - \delta_5],\ proj\Sigma_3 = [\lambda_2^0 + \delta_6, \lambda_2^0 + \delta_1],$$

where $proj\Sigma_i$ is the projection of Σ_i into $\mathbf{R} = (\lambda)$.

The bifurcation diagrams in Theorems 3.1 and 3.2 are both of exact shape (there is no wiggling of the diagrams), and hold for very general domains. (In Theorem 3.1, there is no restriction on Ω; and in Theorem 3.2, (13) is needed. When the $<$ sign in (13) is replaced by $>$, a similar diagram can be shown, see details in [SS]; and when $<$ sign is $=$, some more interesting bifurcation occurs, also see details in [SS].)

4 Fold and Cusp

In this section, we apply Theorem 2.4 to a few application problems. Our first example is

$$\Delta u + \lambda(1 + u + u^2 - \varepsilon u^3) = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{14}$$

which was studied by Crandall and Rabinowitz [CR2], and Brown, Ibrahim and Shivaji [BIS].

Theorem 4.1. *The bifurcation diagram of* (14) *is exactly S-shaped when* $\varepsilon > 0$ *is small and* Ω *is a ball* B^n *in* $\mathbf{R}^n$ *with* $1 \le n \le 6$.

We sketch a proof of Theorem 4.1. The proof is helped by two auxiliary equations: (A) $\Delta u + \lambda(1 + u + u^2) = 0$ in Ω, $u = 0$ on $\partial\Omega$; and (B) $\Delta v + \mu(v^2 - v^3) = 0$ in Ω, $u = 0$ on $\partial\Omega$. The second equation can be obtained by letting $v = \varepsilon u$, $\mu = \varepsilon^{-1}\lambda$, and then setting $\varepsilon = 0$. The basic idea is to use solutions of (A) to approximate solutions of (14) when $u(0)$ is small, and to use (B) when $u(0)$ is in the scale of $O(\varepsilon^{-1})$. The exact multiplicity of solutions of (A) and (B) are known from previous results: the bifurcation diagram of (A) is exactly $\supset$-shaped (see for example [OS2], Fig. 8a); and that of (B) is exactly $\subset$-shaped (see [OS1], Fig. 8b.) To get $\supset$-shaped curve in (A), we need the condition $n \le 6$ to make u^2 to be subcritical. We can show that when $0 < u(0) < (3\varepsilon)^{-1}$, the portion of the bifurcation diagram is exactly $\supset$-shaped since for $f(u) = 1 + u + u^2 - \varepsilon u^3$, $f''(u) > 0$ for $u \in (0, (3\varepsilon)^{-1})$, then a standard bifurcation argument in [OS2] can be used. When $((3 + \delta)\varepsilon)^{-1} < u(0) < ((1 + \delta)\varepsilon)^{-1}$, for a small $\delta > 0$, the portion of the bifurcation diagram is exactly $\subset$-shaped since in the (μ, v) coordinate, it is a perturbation of a compact portion of bifurcation diagram of (B) which is of that shape, and there is exactly one degenerate solution on that portion (see [Da] for perturbation of a compact portion of a curve of solutions). Finally we can prove that when $u(0) > ((1 + \delta)\varepsilon)^{-1}$, the diagram is monotone with respect to λ. Thus the bifurcation diagram is exactly S-shaped as in Fig. 8c.

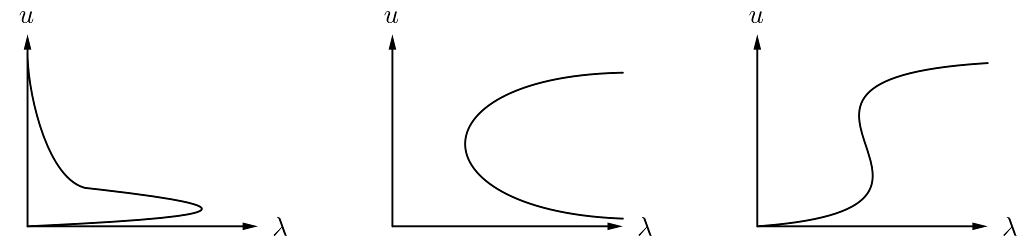


Figure 8: (a) Equation (A); (b) Equation (B); (c) Equation (14)

Our second example is the perturbed Gelfand equation:

$$\Delta u + \lambda \exp[-1/(u + \varepsilon)] = 0 \;\text{ in } B^n,\; u = 0 \;\text{ on } \partial B^n, \tag{15}$$

where B^n is a ball in $\mathbf{R}^n$. The original perturbed Gelfand equation is $\Delta v + \mu \exp[v/(\varepsilon v + 1)] = 0$ in B^n, $v = 0$ on ∂B^n. (15) is obtained by a change of variables: $u = \varepsilon^2 v$ and $\lambda = \varepsilon^2 \exp(1/\varepsilon)\mu$, which was introduced by Du and Lou [DL]. Let Σ_ε be the solution set of (15). It has been a long-time conjecture that when $n = 1$ or 2, there exists $\varepsilon_0 > 0$ such that when $\varepsilon \in (0, \varepsilon_0)$, Σ_ε is exactly S-shaped; when $\varepsilon \in (\varepsilon_0, \infty)$, Σ_ε is monotone increasing; and when $\varepsilon = \varepsilon_0$, there is a cusp type degenerate solution. (See Fig. 2.)

It is easy to show that when $\varepsilon \ge 1/4$, Σ_ε is monotone increasing. The proof of exact S-shaped curve for small $\varepsilon > 0$ has been studied by several authors. Dancer [Da] first proved (15) has exactly three solutions for $\lambda \in (\lambda_0, \lambda_1(\varepsilon))$, $\varepsilon \in (0, \varepsilon_1)$ and $n = 1, 2$. Hastings and McLeod [HM] proved Σ_ε is exactly S-shaped for $n = 1$, $\varepsilon \in (0, \varepsilon_1)$ using quadratures. Recently, Wang [W2] showed that for $n = 1$ and $\varepsilon \in (0, 1/4.4967]$, Σ_ε is exactly S-shaped using a different quadrature method, and the upper bound was improved to 1/4.35 by Korman and Li [KL] using the bifurcation method. Du and Lou [DL] proved Σ_ε is exactly S-shaped for $n = 2$ and $\varepsilon \in (0, \varepsilon_1)$ for some small $\varepsilon_1 > 0$ using a combination of bifurcation method and a perturbation argument.

Using the proof of Theorem 4.1 above, we can prove the result of [DL] in a more direct way. We point out that, to completely resolve the conjecture for all $\varepsilon > 0$, we need to prove the following estimate: at a cusp degenerate solution of (15),

$$3\int_{B^n} f_{uu}(\varepsilon, u(x))w^2(x)\theta(x)dx + \int_{B^n} f_{uuu}(\varepsilon, u(x))w^4(x)dx < 0. \tag{16}$$

where θ is the solution of

$$\Delta\theta + \lambda_0 f'(\varepsilon_0, u_0)\theta + \lambda_0 f''(\varepsilon_0, u_0)w_0^2 = 0 \text{ in } B^n, \quad u = 0 \text{ on } \partial B^n. \tag{17}$$

Finally we mention that another long-standing open conjecture is the bifurcation diagram of

$$u'' + \lambda(u-\varepsilon)(u-b)(c-u) = 0, \text{ in } (0,1), \quad u(0) = u(1) = 0, \tag{18}$$

where $c > b > \varepsilon > 0$, and $c - b > b - \varepsilon$. The equation arises from the studies of dynamics of FitzHugh-Nagumo equation and population biology. When $\varepsilon = 0$, Smoller and Wasserman [SW] proved that the bifurcation diagram is exactly $\subset$-shaped (see Fig. 9a) by quadrature methods, and later it was also proved by Korman, Li and Ouyang [KLO1] using bifurcation method. (The higher dimensional analog for radially symmetric solutions was proved by Korman, Li and Ouyang [KLO2] for two-dimensional ball, and all dimensions by Ouyang and Shi [OS1].

Figure 9: (a) Bifurcation diagram when $\varepsilon = 0$; (b) Bifurcation diagram when $\varepsilon > 0$.

For $\varepsilon > 0$, Smoller and Wasserman [SW] claimed that the bifurcation diagram is as Fig. 9b, but Wang [W1] found an error in their proof. The exact multiplicity result as in Fig. 9b was proved by Wang [W1], Ye and Li [YL] and Korman, Li and Ouyang [KLO1] but all of them need some extra conditions on $f(u)$. An approach using the ideas in this paper is possible for a complete solution for the problem: here one need to show a cusp bifurcation will not occur when ε increases from 0 to ∞. This was pursued by Korman, Li and Ouyang [KLO3] recently, but still some extra conditions are needed.

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