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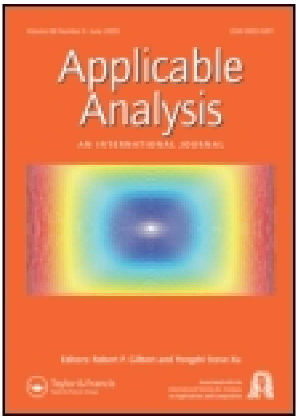

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Bifurcation analysis of the Gierer-Meinhardt system with a saturation in the activator production

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Bifurcation analysis of the Gierer–Meinhardt system with a saturation in the activator production

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The reaction–diffusion Gierer–Meinhardt system with a saturation in the activator production is considered. Stability of the unique positive constant steady state solution is analysed, and associated Hopf bifurcations and steady state bifurcations are obtained. A global bifurcation diagram of non-trivial periodic orbits and steady state solutions with respect to key system parameters is obtained, which improves the understanding of dynamics of Gierer–Meinhardt system with a saturation in different parameter regimes.

Keywords: Gierer–Meinhardt system with saturation; Hopf bifurcation; Steady state bifurcation; Heterogeneous case

AMS Subject Classifications: 35K57; 92E20

1. Introduction

In the pioneer work [1], Gierer and Meinhardt considered the following reaction–diffusion system of activator–inhibitor type with a saturation of activator production: (see [1] Equation (16))

$$\begin{cases} \dfrac{\partial a}{\partial t} = \rho_0\rho + c\rho\dfrac{a^2}{(1+\kappa a^2)h} - \mu a + D_a\dfrac{\partial^2 a}{\partial x^2}, \\ \dfrac{\partial h}{\partial t} = c'\rho' a^2 - \nu h + D_h\dfrac{\partial^2 h}{\partial x^2}. \end{cases} \tag{1.1}$$

Here, $a(x,t)$ and $h(x,t)$ are the concentration functions of the activator and inhibitor, respectively. In (1.1), the activator concentration is limited to a maximum value so that the activated area forms an approximately constant proportion of the total structure size. When $\kappa = 0$, (1.1) is reduced to the classical Gierer–Meinhardt system. Numerical simulations of concentration patterns of (1.1) were obtained in [1–3] for one- and two-dimensional spatial domains. In the last 40 years, the Gierer–Meindardt system was considered as one of most important spatiotemporal morphogenesis pattern formation reaction–diffusion model ([4,5]). There have been many research results on the non-homogeneous steady state solutions (such as multi-peak steady state solutions etc.) of the Gierer-Meindardt system

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(1.1) with $\kappa = 0$, (see [6–13]) and the Gierer–Meindardt system with saturation (1.1) with $\kappa > 0$, (see references [14–17]). There are also many results on the steady state solutions of other similar models, (see references [18–23]).

In this paper, we analyse the Gierer–Meinhardt system with saturation in the following form (see [16,24]):

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta u + \sigma - u + \dfrac{u^2}{(1+\kappa u^2)v}, & x \in \Omega,\ t > 0, \\ \tau \dfrac{\partial v}{\partial t} = d_2 \Delta v - v + u^2, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u(x,t)}{\partial \nu} = \dfrac{\partial v(x,t)}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) > 0, v(x,0) = v_0(x) > 0, & x \in \Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded connected open subset of $\mathbb{R}^n$ ($n \geq 1$), with a smooth boundary $\partial\Omega$ (if $n \geq 2$), and the Laplace operator $\Delta w(x,t) = \sum_{i=1}^{n} \dfrac{\partial^2 w(x,t)}{\partial x_i^2}$ for $w = u, v$ models the diffusion effect; $\dfrac{\partial w(x,t)}{\partial \nu}$ is the outer normal derivative of $w = u, v$ at $x \in \partial\Omega$, and a no-flux boundary condition is imposed for each of u and v; the parameters d_1, d_2, τ and σ are positive constants, whereas κ is a nonnegative constant. Morimoto [16] showed that system (1.2) admits a radially symmetric steady state solution when κ is small and $\Omega = (-1, 1)$ is one dimensional, and in [24] we obtained that when κ is large, the unique positive constant steady state solution of system (1.2) is globally asymptotically stable and hence (1.2) cannot have nontrivial steady state solution.

The existence of non-trivial steady state solution when κ is small and the non-existence of non-trivial steady state solution when κ is large make us to wonder the dynamical behaviour when κ is neither large nor small. In the first part of this paper, we follow the approach in [25–27] to consider the sequence of Hopf bifurcations and steady state bifurcations from the unique constant steady state solution for (1.2). While our techniques are similar to the ones in these previous work, here we face the difficulty that the constant steady state is not easily solvable as it is the root of a cubic polynomial. To overcome that, we use a new parameter λ (the u-coordinate of the constant steady state) as an equivalent parameter to κ to perform the bifurcation analysis for the case that the spatial domain $\Omega = (0, \ell\pi)$, with $\ell \in \mathbb{R}^+$. We discover that there are four different bifurcation scenarios with respect to λ or κ if the other four parameters d_1, d_2, τ and σ are in different regimes. To be more precise, our analysis shows that for two constants defined in terms of d_1, d_2, τ and σ:

$$M_1 = \frac{\tau - 1}{\tau + 1}, \quad \text{and} \quad M_2 = \frac{d_2 - d_1 - 2\sqrt{d_1 d_2}}{d_2 + d_1 + 2\sqrt{d_1 d_2}},$$

(1) If $\sigma > \max\{M_1, M_2\}$, then the constant steady state solution is locally asymptotically stable for all $\kappa \geq 0$; thus, there are not any bifurcations occurring from the constant steady state solution;

(2) If $M_1 \geq \sigma > M_2$, then the constant steady state solution is locally asymptotically stable for large $\kappa > 0$, and it loses stability via Hopf bifurcations for smaller κ (but, there are no steady state bifurcations);

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(3) If $M_2 > \sigma > M_1$, then the constant steady state solution is locally asymptotically stable for large $\kappa > 0$, and it loses stability via steady state bifurcations for smaller κ (but, there are no Hopf bifurcations);
(4) If $0 < \sigma < \min\{M_1, M_2\}$, then the constant steady state solution is locally asymptotically stable for large $\kappa > 0$, and both Hopf bifurcations and steady state bifurcations occur for smaller κ.

Temporally oscillatory patterns and non-constant stationary patterns have appeared in many reaction–diffusion pattern formation systems since the pioneer work [28], in which Turing identified six kinds of spatiotemporal patterns. Our classification of patterns in (1.2) according to two quantities M_1 and M_2 above demonstrates the importance of parameter values in non-linear pattern formation mechanism. Notice that M_1 depends on τ (time scale); M_2 depends on d_1 and d_2 (diffusion coefficients); and σ is the source term. Hence, the relative order of the three quantities M_1, M_2 and σ determines the occurrence (or non-occurrence) of patterns and also the types of patterns. It is also interesting to compare the role of activator source term σ in (1.2) to the one in CIMA reaction discussed in [25] (see also [29,30]). Here, for (1.2), a larger σ results in non-existence of patterns, while for CIMA reaction in [25], more patterns are possible for larger source term.

All pattern formation mentioned above is given by assuming a constant source σ. In the second part of this paper, we consider (1.2) with a non-constant source function $\sigma(x)$. Notice that in this case, there is no constant steady state solution. We prove the existence of a non-constant steady state solution as a "primary pattern", and this primary pattern is globally asymptotically stable for a large κ or a large σ. The research in this direction is still preliminary, and it is desirable to know the effect of feeding function $\sigma(x)$ on the primary and other patterns. In recent years, the effect of spatial heterogenous environment on diffusive competition and predator–prey models have been considered in for example, [31–38]).

The remaining part of the paper is organised as follows. In Section 2, stability and Hopf bifurcation analyses for system (1.2) are conducted. In Section 3, steady state bifurcations and the interaction between Hopf and steady state bifurcations are studied. In Section 4, we consider the case when the source function σ is spatially heterogeneous. Some numerical simulations and bifurcation diagrams are shown in Section 5. Throughout the paper, we denote by $\mathbb{N}$ the set of all the positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\mathbb{R}^+$ the set of all the positive real numbers.

2. Stability and Hopf bifurcations

In this section, we analyse the stability of the constant steady state solution of (1.2), and consider the related Hopf bifurcations for (1.2) with the spatial domain $\Omega = (0, \ell\pi)$, $\ell \in \mathbb{R}^+$, and a constant source term $\sigma > 0$, which is

$$\begin{cases} u_t = d_1 u_{xx} + \sigma - u + \dfrac{u^2}{(1+\kappa u^2)v}, & x \in (0, \ell\pi),\ t > 0, \\ v_t = \dfrac{1}{\tau}(d_2 v_{xx} - v + u^2), & x \in (0, \ell\pi),\ t > 0, \\ u_x(0,t) = v_x(0,t) = 0, \quad u_x(\ell\pi, t) = v_x(\ell\pi, t) = 0, & t > 0, \\ u(x,0) = u_0(x) > 0, \quad v(x,0) = v_0(x) > 0, & x \in (0, \ell\pi). \end{cases} \tag{2.1}$$

System (2.1) has a unique positive constant steady state solution (λ, λ^2), and λ is the unique positive root of

$$\frac{1}{1+\kappa\lambda^2} = \lambda - \sigma. \tag{2.2}$$

From elementary calculus, for any fixed κ, there exists a unique $\lambda = \lambda(\kappa) > 0$ satisfying Equation (2.2). Differentiating Equation (2.2) with respect to κ, we see that

$$2\lambda k\lambda'(\kappa) = -\frac{1}{(\lambda-\sigma)^2} - \lambda^2,$$

and hence $\lambda(\kappa)$ is a strictly decreasing function in κ for $\kappa \geq 0$, with $\lim_{\kappa\to 0}\lambda(\kappa) = \sigma+1$ and $\lim_{\kappa\to\infty}\lambda(\kappa) = \sigma$. Hence, throughout the paper, we always assume $\lambda \in (\sigma, \sigma+1]$, and use λ as a bifurcation parameter which is equivalent to κ.

As in [25,27], we define the real-valued Sobolev space

$$X := \left\{(u,v)^T \in H^2(0,\ell\pi) \times H^2(0,\ell\pi) : (u_x, v_x)|_{x=0,\,\ell\pi} = 0\right\}, \tag{2.3}$$

and also define the complexification of X to be $X_{\mathbb{C}} := X \oplus iX = \{x_1 + ix_2 |\, x_1, x_2 \in X\}$. The linearised operator of the steady state system of (2.1) at (λ, λ^2) is,

$$L(\lambda) := \begin{pmatrix} d_1\dfrac{\partial^2}{\partial x^2} + \dfrac{2(\lambda-\sigma)^2}{\lambda} - 1 & -\dfrac{\lambda-\sigma}{\lambda^2} \\ \dfrac{2\lambda}{\tau} & \dfrac{d_2}{\tau}\dfrac{\partial^2}{\partial x^2} - \dfrac{1}{\tau} \end{pmatrix}, \tag{2.4}$$

with the domain $D_{L(\lambda)} = X_{\mathbb{C}}$. It is known that the eigenvalue problem

$$-\varphi'' = \mu\varphi, \quad x \in (0,\ell\pi), \quad \varphi'(0) = \varphi'(\ell\pi) = 0,$$

has eigenvalues $\mu_n = \dfrac{n^2}{\ell^2}$ $(n = 0, 1, 2, \ldots)$, with corresponding eigenfunctions $\varphi_n(x) = \cos\dfrac{n}{\ell}x$ $(n = 0, 1, 2, \ldots)$. Let

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \sum_{n=0}^{\infty} \cos\frac{n}{\ell}x \begin{pmatrix} a_n \\ b_n \end{pmatrix} \tag{2.5}$$

be an eigenfunction for $L(\lambda)$ with eigenvalue $\beta(\lambda)$, that is, $L(\lambda)(\phi,\psi)^T = \beta(\lambda)(\phi,\psi)^T$. Then, we can obtain that (see [27])

$$L_n(\lambda)\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \beta(\lambda)\begin{pmatrix} a_n \\ b_n \end{pmatrix}, \quad n = 0, 1, 2, \ldots, \tag{2.6}$$

where

$$L_n(\lambda) := \begin{pmatrix} -\dfrac{d_1 n^2}{\ell^2} + \dfrac{2(\lambda-\sigma)^2}{\lambda} - 1 & -\dfrac{\lambda-\sigma}{\lambda^2} \\ \dfrac{2\lambda}{\tau} & -\dfrac{d_2 n^2}{\tau\ell^2} - \dfrac{1}{\tau} \end{pmatrix}. \tag{2.7}$$

Then, the characteristic equation of $L(\lambda)$ is

$$\beta^2 - T_n(\lambda)\beta + D_n(\lambda) = 0, \quad n = 0, 1, 2, \ldots, \tag{2.8}$$

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where

$$\begin{cases} T_n(\lambda) = -\left(d_1 + \dfrac{d_2}{\tau}\right)\dfrac{n^2}{\ell^2} - \dfrac{1}{\tau} - 1 + \dfrac{2(\lambda-\sigma)^2}{\lambda}, \\ D_n(\lambda) = \dfrac{1}{\tau}\left[1 - \dfrac{2(\lambda-\sigma)^2}{\lambda} + \dfrac{2(\lambda-\sigma)}{\lambda} + \left[d_1 + d_2\left(1 - \dfrac{2(\lambda-\sigma)^2}{\lambda}\right)\right]\dfrac{n^2}{\ell^2} + \dfrac{d_1 d_2 n^4}{\ell^4}\right]. \end{cases} \tag{2.9}$$

If the constant steady state (λ, λ^2) is locally asymptotically stable, then for each $n \in \mathbb{N}_0$, we have $T_n(\lambda) < 0$ and $D_n(\lambda) > 0$. Moreover, we identify Hopf bifurcation values λ_0 which satisfy the following Hopf bifurcation condition (see [25,27]):
($\mathbf{H}_1$): There exists $n \in \mathbb{N}_0$ such that

$$T_n(\lambda_0) = 0, \quad D_n(\lambda_0) > 0, \quad \text{and} \quad T_j(\lambda_0) \neq 0, \quad D_j(\lambda_0) \neq 0 \quad \text{for } j \neq n, \tag{2.10}$$

and a unique pair of complex eigenvalues near the imaginary axis $\alpha(\lambda) \pm i\omega(\lambda)$ satisfying

$$\alpha'(\lambda_0) \neq 0. \tag{2.11}$$

From (2.9), we have

$$T_n(\lambda) = -\left(d_1 + \frac{d_2}{\tau}\right)\frac{n^2}{\ell^2} - \frac{1}{\tau} - 1 - 4\sigma + 2\left(\lambda + \frac{\sigma^2}{\lambda}\right), \tag{2.12}$$

and hence $T_n(\lambda)$ is a strictly increasing function with respect to λ for $\lambda \in (\sigma, \sigma+1]$. So for a possible Hopf bifurcation value λ_0, we have $\alpha'(\lambda_0) = T_n'(\lambda_0)/2 > 0$. Moreover, the real part of one pair of complex eigenvalues of $L(\lambda)$ becomes positive when λ crosses λ_0 increasingly.

For the simplicity of notation, in the following, we denote

$$m = \frac{2}{\sigma+1} - 1, \quad \text{and} \quad d = \frac{d_1}{d_2}. \tag{2.13}$$

We consider the stability condition and the Hopf bifurcation condition of the positive constant steady state solution (λ, λ^2). Clearly, $T_0(\lambda) < 0$ for λ near σ. If $m \geq \dfrac{1}{\tau}$, then $T_0(\sigma+1) \geq 0$. Since $T_0(\lambda)$ is a strictly increasing function with respect to λ, then there exists a unique λ_* ($\sigma < \lambda_* \leq \sigma+1$), such that $T_0(\lambda) > 0$ for $\lambda > \lambda_*$, $T_0(\lambda) < 0$ for $\lambda < \lambda_*$, and $T_0(\lambda_*) = 0$. Apparently, if $m = \dfrac{1}{\tau}$, then $\lambda_* = \sigma+1$ satisfies the Hopf bifurcation condition ($\mathbf{H}_1$) and is the unique Hopf bifurcation value at which spatially homogeneous periodic orbits bifurcate out for any $\ell > 0$. In the following, we look for spatially non-homogeneous periodic orbits. So, we assume $m > \dfrac{1}{\tau}$, and define

$$\ell_n = n\sqrt{\frac{d_1\tau + d_2}{\tau T_0(\sigma+1)}} = n\sqrt{\frac{(\tau d_1 + d_2)(\sigma+1)}{(-1-\tau)(\sigma+1)+2\tau}}, \quad n \in \mathbb{N}. \tag{2.14}$$

Then, for $\ell_n \leq \ell < \ell_{n+1}$, ($n \in \mathbb{N}$), we define λ_j^H to be the unique root of $T_j(\lambda) = 0$ for $0 \leq j \leq n$. These points satisfy

$$\lambda_* \equiv \lambda_0^H < \lambda_1^H < \lambda_2^H < \cdots < \lambda_n^H \leq \sigma + 1.$$

Clearly, $T_j(\lambda_j^H) = 0$ and $T_i(\lambda_j^H) \neq 0$ for $i \neq j$. For $\ell \in [0, \ell_1)$, λ_* satisfies the Hopf bifurcation condition ($\mathbf{H}_1$) and is the unique Hopf bifurcation value at which spatially homogeneous periodic orbits bifurcate.

Now, we need to check whether $D_j(\lambda_j^H) > 0$ is satisfied. Define $p = \dfrac{n^2}{\ell^2}$, then

$$D_n(\lambda) = D(\lambda, p) = \frac{1}{\tau}\left[1 - \frac{2(\lambda-\sigma)^2}{\lambda} + \frac{2(\lambda-\sigma)}{\lambda} + \left[d_1 + d_2\left(1 - \frac{2(\lambda-\sigma)^2}{\lambda}\right)\right] p + d_1 d_2 p^2\right]. \tag{2.15}$$

Solving λ from equation $D(\lambda, p) = 0$ and choosing the one larger than σ, we have

$$\lambda(p) = \sigma + \frac{d_1 d_2 p^2 + (d_1 + d_2)p + 3}{4(d_2 p + 1)} + \frac{\sqrt{(d_1 d_2 p^2 + (d_1+d_2)p + 3)^2 + 8\sigma(d_2 p + 1)(d_1 d_2 p^2 + (d_1+d_2)p + 1)}}{4(d_2 p + 1)}. \tag{2.16}$$

The following lemma asserts that $D(\lambda, p) > 0$ for any possible (λ, p), hence $D_j(\lambda_j^H) > 0$ for any j.

LEMMA 2.1 *Let m and d be defined as in* (2.13)*, and let $D(\lambda, p)$ be defined as in* (2.15)*. If the parameters m and d satisfy*

$$m < 2\sqrt{d} + d, \tag{2.17}$$

then $D(\lambda, p) > 0$ for all $\lambda \in (\sigma, \sigma+1]$ and $p \in [0, \infty)$.

Proof If $m \leq d$, then

$$d_1 + d_2\left(1 - \frac{2(\lambda-\sigma)^2}{\lambda}\right) \geq 0$$

for all the $\lambda \in (\sigma, \sigma+1]$, and hence $D(\lambda, p) > 0$ for all $\lambda \in (\sigma, \sigma+1]$ and $p \in [0, \infty)$.

If m and d satisfy $d < m < 2\sqrt{d} + d$, then $(d-m)^2 - 4d < 0$. In this case, $D(\lambda, p) = 0$ has no positive roots with respect to p when $\lambda = \sigma + 1$. From

$$\lambda(0) = \sigma + \frac{3 + \sqrt{9 + 8\sigma}}{4} > \sigma + 1, \quad \text{and} \quad \lim_{p\to\infty} \lambda(p) = \infty, \tag{2.18}$$

we know that there exists $\tilde{p} \in [0, \infty)$ such that $\lambda(\tilde{p}) = \min_{p\in[0,\infty)} \lambda(p)$. Then, we have that $\lambda(\tilde{p}) > \sigma + 1$. If not, then there exist $p_1 \in [0, \tilde{p}]$ and $p_2 \in [\tilde{p}, \infty)$ such that $\lambda(p_1) = \lambda(p_2) = \sigma + 1$, which is a contradiction. Using the fact that the other root $\overline{\lambda}(p)$ of $D(\lambda, p) = 0$ is less than σ, then we have that $D(\lambda, p) > 0$ for all $\lambda \in (\sigma, \sigma+1]$ and $p \in [0, \infty)$.

From the two cases discussed above, we conclude that $D(\lambda, p) > 0$ for all $\lambda \in (\sigma, \sigma+1]$ and $p \in [0, \infty)$ if (2.17) is satisfied. □

Summarizing the above analysis, we have the following two results about the stability and Hopf bifurcation of the positive constant steady state solution.

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THEOREM 2.2 *Suppose that the parameters σ, τ, d_1 and d_2 satisfy*

$$m \equiv \frac{2}{\sigma+1} - 1 < \min\left\{\frac{d_1}{d_2} + 2\sqrt{\frac{d_1}{d_2}}, \frac{1}{\tau}\right\}. \tag{2.19}$$

Then, the positive constant steady state solution (λ, λ^2) *of system* (2.1) *is locally asymptotically stable for any* $\lambda \in (\sigma, \sigma+1]$.

Proof From $m < 1/\tau$ and $T_n(\lambda)$ is a strictly increasing function with respect to λ, we have $T_n(\lambda) < 0$ for all $\lambda \in (\sigma, \sigma+1]$ and $n \in \mathbb{N}_0$. Since $m < 2\sqrt{d} + d$, then from Lemma 2.1, $D_n(\lambda) > 0$ for all $\lambda \in (\sigma, \sigma+1]$ and $n \in \mathbb{N}_0$. So, the positive constant steady state solution (λ, λ^2) is locally asymptotically stable. □

THEOREM 2.3 *Suppose that the parameters σ, τ, d_1 and d_2 satisfy*

$$\frac{1}{\tau} \le m \equiv \frac{2}{\sigma+1} - 1 < \frac{d_1}{d_2} + 2\sqrt{\frac{d_1}{d_2}}, \tag{2.20}$$

and ℓ_n is defined as in Equation (2.14). *We define for each* $0 \le j \le n$,

$$\lambda_j^H = \frac{1}{4}\left[R_j + 4\sigma + \sqrt{R_j(R_j + 8\sigma)}\right], \tag{2.21}$$

where

$$R_j = \left(d_1 + \frac{d_2}{\tau}\right)\frac{j^2}{\ell^2} + \frac{1}{\tau} + 1.$$

(1) *For any m satisfying* (2.20) *and any $\ell > 0$, $\lambda = \lambda_0^H$ (defined in* (2.21)*) is the Hopf bifurcation point where spatially homogenous periodic orbits bifurcate from the curve* $\{(\lambda, u, v) = (\lambda, \lambda, \lambda^2) : \sigma < \lambda \le \sigma + 1\}$. *For $m = 1/\tau$, $\lambda_0^H = \sigma + 1$, and for $m > 1/\tau$, $\sigma < \lambda_0^H < \sigma + 1$.*

(2) *If $m > 1/\tau$, and $\ell \in [\ell_n, \ell_{n+1})$ for some $n \in \mathbb{N}$, there exist exactly n points λ_j^H (defined in* (2.21)*), $1 \le j \le n$, which satisfy*

$$\sigma < \lambda_0^H < \lambda_1^H < \lambda_2^H < \cdots < \lambda_n^H \le \sigma + 1, \tag{2.22}$$

such that system (2.1) *undergoes a Hopf bifurcation at $\lambda = \lambda_j^H$, and the bifurcating periodic orbits are spatially non-homogenous; the bifurcating periodic orbits near* $\left(\lambda_j^H, \lambda_j^H, (\lambda_j^H)^2\right)$ *can be parameterised as* $(\lambda(s), u(s), v(s))$ *so that* $\lambda(s) \in C^\infty$ *in the form* $\lambda(s) = \lambda_j^H + o(s)$ *for* $s \in (0, \delta)$ *for some small* $\delta > 0$, *and,*

$$\begin{cases} u(s)(x,t) = \lambda_j^H + s\left(a_n e^{2\pi i t/T(s)} + \overline{a_n} e^{-2\pi i t/T(s)}\right)\cos\frac{n}{\ell}x + o(s), \\ v(s)(x,t) = (\lambda_j^H)^2 + s\left(b_n e^{2\pi i t/T(s)} + \overline{b_n} e^{-2\pi i t/T(s)}\right)\cos\frac{n}{\ell}x + o(s), \end{cases} \tag{2.23}$$

where (a_n, b_n) *is the corresponding eigenvector of Equation* (2.6) *with respect to eigenvalue* $i\sqrt{D_j(\lambda_j^H)}$, $T(s) = 2\pi/\sqrt{D_j(\lambda_j^H)}$, *and D_j is defined in Equation* (2.9).

The proof of Theorem 2.3 is based on discussion above and arguments in [27], and we omit the details.

Remark 2.4

(1) The condition on parameter m in (2.19) is equivalent to

$$\sigma > \max\left\{\frac{\tau-1}{\tau+1}, \frac{d_2 - d_1 - 2\sqrt{d_1 d_2}}{d_2 + d_1 + 2\sqrt{d_1 d_2}}\right\}, \tag{2.24}$$

and similarly the condition on parameter m in (2.20) is equivalent to

$$\frac{\tau-1}{\tau+1} \geq \sigma > \frac{d_2 - d_1 - 2\sqrt{d_1 d_2}}{d_2 + d_1 + 2\sqrt{d_1 d_2}}. \tag{2.25}$$

Hence, the unique constant steady state solution (λ, λ^2) is locally asymptotically stable for any $\kappa > 0$ and σ, τ, d_1, d_2 satisfying (2.24). These two conditions are two of the four mentioned in the introduction, and the other two appear in Section 3.

(2) If (2.19) or (2.20) (or equivalently (2.24) or (2.25)) is satisfied, then there is no steady state solution bifurcation occurring from the constant steady state (λ, λ^2) for any $\lambda \in (\sigma, \sigma+1]$ (or equivalently $\kappa \in [0, \infty)$). However, it is not known whether non-trivial steady state solutions (not bifurcating from trivial solutions) exist for these parameter ranges.

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3. Steady state bifurcations

In Theorems 2.2 and 2.3, we have identified the ranges for parameter m in which the constant steady state solution (λ, λ^2) is locally asymptotically stable for any κ or λ (see Equation (2.19)), or it could lose stability to a limit cycle for small κ (see Equations (2.20) and (2.22)). In this section, we consider possible steady state bifurcations, and the steady state bifurcation values λ_0 satisfy the following condition (see [27]):
(**H**$_2$): There exists $n \in \mathbb{N}_0$ such that

$$T_n(\lambda_0) \neq 0, \quad D_n(\lambda_0) = 0, \quad \text{and} \quad T_j(\lambda_0) \neq 0, \quad D_j(\lambda_0) \neq 0 \quad \text{for } j \neq n; \tag{3.1}$$

and

$$D_n'(\lambda_0) \neq 0. \tag{3.2}$$

From Theorems 2.2 and 2.3, we know that if a steady state bifurcation occurs, then the parameters d and m must satisfy

$$m \geq 2\sqrt{d} + d. \tag{3.3}$$

Clearly (3.3) can be satisfied if d_1/d_2 is sufficiently small. Indeed, we have the following conclusion for the possible steady state bifurcation points.

LEMMA 3.1 *If the parameters d and m satisfy Equation* (3.3)*, and let $\lambda(p)$ be the function defined as in* (2.16)*, then there exists a unique $p^* > 0$, which is the global minimum of $\lambda(p)$, such that $\lambda(p)$ is strictly decreasing when $p \in [0, p^*)$ and is strictly increasing when $p \in [p^*, \infty)$. Moreover, if $m = 2\sqrt{d} + d$, then $\lambda(p^*) = \sigma + 1$, and if $m > 2\sqrt{d} + d$, then $\sigma < \lambda(p^*) < \sigma + 1$.*

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Proof From Equation (2.15), we see that for any fixed λ, $D(\lambda, p) = 0$ has at most two positive roots. Hence, from (2.18), we have that the monotonic behaviour of $\lambda(p)$ must belong to one of the following two cases:

(1) $\lambda(p)$ is a strictly increasing function for $p > 0$; or
(2) there exists a unique $p^* > 0$, which is the global minimum of $\lambda(p)$, such that $\lambda(p)$ is strictly decreasing when $p \in [0, p^*)$ and is strictly increasing when $p \in [p^*, \infty)$.

Since d and m satisfy Equation (3.3), then $D(\sigma + 1, p) = 0$ has positive real roots while $D(\sigma, p) = 0$ does not. Using the fact that $\lambda(0) > \sigma + 1$, we obtain that when d and m satisfy Equation (3.3), the second alternate for $\lambda(p)$ above occurs, which implies the conclusion of the lemma. □

In the case that $m > 2\sqrt{d} + d$, the equation $D(\sigma + 1, p) = 0$ has two positive roots, denoted by p_+ and p_- where $p_- < p_+$. Denote $\lambda^* = \lambda(p^*)$, where λ^* is the global minimum of $\lambda(p)$ as in Lemma 3.1. Then, for each $\lambda \in (\lambda^*, \sigma + 1]$, $D(\lambda, p) = 0$ has two positive roots $p_+(\lambda)$ and $p_-(\lambda)$ with respect to p, where $p_+(\lambda)$ is strictly increasing for $\lambda \in (\lambda^*, \sigma + 1]$ with $p_+(\lambda^*) = p^*$ and $p_+(\sigma + 1) = p_+$, and $p_-(\lambda)$ is strictly decreasing for $\lambda \in (\lambda^*, \sigma + 1]$ with $p_-(\lambda^*) = p^*$ and $p_-(\sigma + 1) = p_-$. Hence, if for some $n \in \mathbf{N}$, we have $p_- \leq n^2/\ell^2 \leq p_+$, then the value $\lambda_n^S = \lambda(n^2/\ell^2) \in [\lambda^*, \sigma + 1]$ satisfies $D_n(\lambda_n^S) = 0$. Define

$$\widetilde{\ell}_{n,\pm} = \frac{n}{\sqrt{p_\pm}}, \tag{3.4}$$

then for any $\ell \in \left[\widetilde{\ell}_{n,+}, \widetilde{\ell}_{n,-}\right]$, there exists a λ_n^S such that $D_n(\lambda_n^S) = 0$.

These points λ_n^S are potential steady state bifurcation points. We define

$$\begin{aligned} L^E = &\left\{\ell > 0 : D_i(\lambda, \ell) = D_j(\lambda, \ell) = 0, \quad \text{for some } \lambda \in [\lambda^*, \sigma + 1], i \neq j\right\} \\ &\bigcup \left\{\ell > 0 : D_i(\lambda, \ell) = T_j(\lambda, \ell) = 0, \quad \text{for some } \lambda \in [\lambda^*, \sigma + 1], i, j \in \mathbb{N}_0\right\}, \end{aligned} \tag{3.5}$$

and for any fixed $i, j (i \neq j)$, if $D_i(\lambda, \ell) = D_j(\lambda, \ell) = 0$, then $\lambda = \lambda(p)\Big|_{p=\frac{i^2}{\ell^2}} = \lambda(p)\Big|_{p=\frac{j^2}{\ell^2}}$, where $\lambda(p)$ is defined as in Equation (2.16). Hence, there is at most countable many points (ℓ, λ) satisfying $D_i(\lambda, \ell) = D_j(\lambda, \ell)$ for any fixed $i, j (i \neq j)$. Similarly, for any fixed i, j, there is at most countable many points (ℓ, λ) satisfying $D_i(\lambda, \ell) = T_j(\lambda, \ell)$. Hence, L^E has at most countably many points, (also see [25,27]). So, for a bifurcation from simple eigenvalue, we assume that $\ell \in \mathbb{R}^+ \backslash L^E$, and we consider the corresponding possible bifurcation points λ_n^S. To satisfy the bifurcation conditions, we only need to verify whether $D_n'(\lambda_n^S) \neq 0$, which is proved in the following lemma:

LEMMA 3.2 *Let λ_n^S and λ^* be defined as above. If $\lambda_n^S \neq \lambda^*$, then $D_n'(\lambda_n^S) \neq 0$.*

Proof From Equation (2.15), we have

$$D(\lambda, p) = \frac{1}{\tau}\left[F(\lambda) + [d_1 + d_2 E(\lambda))]\, p + d_1 d_2 p^2\right], \tag{3.6}$$

where $F(\lambda) = 1 - \frac{2(\lambda-\sigma)^2}{\lambda} + \frac{2(\lambda-\sigma)}{\lambda}$ and $E(\lambda) = 1 - \frac{2(\lambda-\sigma)^2}{\lambda}$. By differentiating $D(\lambda(p), p) = 0$ with respect to p, we have

$$\frac{\partial D}{\partial \lambda}\frac{d\lambda}{dp} + \frac{\partial D}{\partial p} = 0. \tag{3.7}$$

If we assume that $D_n'(\lambda_n^S) = 0$, then

$$\frac{\partial D}{\partial \lambda}\left(\lambda_n^S, \frac{n^2}{\ell^2}\right) = 0.$$

From $\lambda_n^S \neq \lambda^*$, it follows that $\frac{d\lambda}{dp}\left(\frac{n^2}{\ell^2}\right) \neq 0$. Hence, we have

$$\frac{\partial D}{\partial p}\left(\lambda_n^S, \frac{n^2}{\ell^2}\right) = 0,$$

that is,

$$\frac{n^2}{\ell^2} = -\frac{d_1 + d_2 E(\lambda_n^S)}{2d_1 d_2}.$$

Thus, $\lambda_n^S = \lambda^*$, which is a contradiction. □

Summarizing above analysis, we are ready to state a global bifurcation result of non-trivial steady state solutions of (2.1).

THEOREM 3.3 *Suppose that parameters d and m satisfy*

$$m > 2\sqrt{d} + d. \tag{3.8}$$

Let $\lambda(p)$ *be defined as in Equation* (2.16) *and let* $\tilde{\ell}_{n\pm}$ *be defined as in Equation* (3.4). *If for some* $n \in \mathbb{N}$, $\ell \in [\tilde{\ell}_{n,+}, \tilde{\ell}_{n,-}] \setminus L^E$, *then there exists exactly one point* $\lambda_n^S \in (\lambda_*, \sigma + 1]$ *such that* $\lambda(n^2/\ell^2) = \lambda_n^S$ *and* $\lambda = \lambda_n^S$ *is a bifurcation point where non-trivial steady state solutions with mode* $\cos(nx/\ell)$ *bifurcate from the trivial branch. To be more precise, there is a smooth curve* Γ_n *of positive steady state solutions of* (2.1) *bifurcating from* $\{(\lambda, u, v) = (\lambda_n^S, \lambda_n^S, (\lambda_n^S)^2) : \sigma < \lambda \leq \sigma + 1\}$, *with* Γ_n *contained in a global branch* $\mathcal{C}_n$ *of the positive steady state solutions of* (2.1). *Moreover,*

(1) *Near* $(\lambda, u, v) = (\lambda_n^S, \lambda_n^S, (\lambda_n^S)^2)$, $\Gamma_n = \{(\lambda(s), u(s), v(s)) : s \in (-\epsilon, \epsilon)\}$, *where*

$$u(s) = \lambda_n^S + s\boldsymbol{a}_n \cos(nx/\ell) + s\psi_1(s),$$
$$v(s) = (\lambda_n^S)^2 + s\boldsymbol{b}_n \cos(nx/\ell) + s\psi_2(s),$$

for some C^∞ *smooth functions* λ, ψ_1, ψ_2 *such that* $\lambda(0) = \lambda_n^S$ *and* $\psi_1(0) = \psi_2(0) = 0$. *Here* $\boldsymbol{a}_n$ *and* $\boldsymbol{b}_n$ *satisfy* $L_n(\lambda_0)(\boldsymbol{a}_n, \boldsymbol{b}_n)^T = (0, 0)^T$.

(2) *either* $\mathcal{C}_n$ *contains another* $(\lambda_j^S, \lambda_j^S, (\lambda_j^S)^2)$, *or the projection of* $\mathcal{C}_n$ *onto* λ*-axis contains the interval* $(\lambda_j^S, \sigma + 1]$.

The proof of Theorem 3.3 is again similar to the ones in [25,27], hence we omit here. We note that to obtain the results in Theorem 3.3, we need to prove that there exist two

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positive constants $\overline{C}$, $\underline{C} > 0$ such that any positive steady state solution $(u(x), v(x))$ of system (2.1) satisfies $\underline{C} \le u(x), v(x) \le \overline{C}$. We do not give a proof of that fact here as a more general result for the heterogeneous case will be given in the next section.

Remark 3.4

(1) From [24], we know that when κ is sufficiently large (or equivalently λ is near σ), then the positive steady state solution (λ, λ^2) is globally asymptotically stable. Hence in Theorem 3.3, there exists a $\hat{\lambda} > \sigma$ such that the projection of the global branch $\mathcal{C}_n$ does not contain the interval $(\sigma, \hat{\lambda})$. On the other hand, by using the calculations in [25], the bifurcation direction and stability of the bifurcating solutions can be calculated but very lengthy, hence we omit it.

(2) It is possible that the global branch $\mathcal{C}_n$ reaches $\lambda = \sigma + 1$ which corresponds to $\kappa = 0$, the classical Gierer–Meinhardt case.

(3) Similar to Remark 2.4.1, the condition (3.8) is equivalent to

$$0 < \sigma < \frac{d_2 - d_1 - 2\sqrt{d_1 d_2}}{d_2 + d_1 + 2\sqrt{d_1 d_2}}. \tag{3.9}$$

However, when the condition (3.8) (or equivalently (3.9)) is satisfied, Hopf bifurcation is still possible despite occurrence of steady state solution bifurcations, see the following proposition.

Here we describe the Hopf bifurcations when (3.8) (or equivalently (3.9)) is satisfied, which complements Theorem 2.3.

PROPOSITION 3.5 *Suppose that parameters d and m satisfy Equation* (3.8).

(1) *If* $m < 1/\tau$, *then for system* (2.1), *there is no* $\lambda \in (\sigma, \sigma + 1]$ *satisfying the Hopf bifurcation condition* $(\boldsymbol{H}_1)$, *and there is no Hopf bifurcations from the constant steady state solution* (λ, λ^2).

(2) *If* $m \ge 1/\tau$, *then system* (2.1) *undergoes a Hopf bifurcation at* $\lambda = \lambda_j^H$ *with* λ_j^H *defined in* (2.21), *and* $\lambda_j^H \notin [\lambda^*, \sigma + 1]$, *where* $\lambda^* = \lambda(p^*)$ *and* p^* *is defined in Lemma* 3.1.

The condition (3.9) and whether $m < 1/\tau$ together give the two other scenarios of bifurcations mentioned in the introduction. To be more precise, if

$$\frac{d_2 - d_1 - 2\sqrt{d_1 d_2}}{d_2 + d_1 + 2\sqrt{d_1 d_2}} > \sigma > \frac{\tau - 1}{\tau + 1}, \tag{3.10}$$

then steady state bifurcations can occur for $\lambda \in (\sigma, \sigma + 1]$ but not Hopf bifurcations; and if

$$\sigma < \min\left\{\frac{d_2 - d_1 - 2\sqrt{d_1 d_2}}{d_2 + d_1 + 2\sqrt{d_1 d_2}}, \frac{\tau - 1}{\tau + 1}\right\}, \tag{3.11}$$

then both steady state bifurcations and Hopf bifurcations occur for $\lambda \in (\sigma, \sigma + 1]$ from the constant steady state solution (λ, λ^2).

4. The heterogeneous source case

In this section, we suppose that the source function $\sigma(x)$ in (1.2) is heterogenous, that is, $\sigma(x)$ is a continuous non-negative function over $\overline{\Omega}$, but it is not necessarily a constant (as assumed in Sections 2 and 3). Then, system (1.2) becomes

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta u + \sigma(x) - u + \dfrac{u^2}{(1+\kappa u^2)v}, & x \in \Omega,\ t > 0, \\ \tau \dfrac{\partial v}{\partial t} = d_2 \Delta v - v + u^2, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) > 0,\, v(x,0) = v_0(x) > 0, & x \in \Omega. \end{cases} \tag{4.1}$$

In this section, we focus on the steady state solutions of system (4.1), which satisfy

$$\begin{cases} -d_1 \Delta u = \sigma(x) - u + \dfrac{u^2}{(1+\kappa u^2)v}, & x \in \Omega, \\ -d_2 \Delta v = -v + u^2, & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{4.2}$$

Since $\sigma(x)$ is not a constant, there is no constant steady state solution of (4.2). Our main result in this section is on the existence of a non-trivial steady state solution, which mainly follows the method in [35].

THEOREM 4.1 *Suppose that $d_1, d_2 > 0$ are fixed, $\kappa > 0$, and $\sigma(x)$ is a continuous non-negative function defined on $\overline{\Omega}$ and not identically zero. Then, system* (4.2) *has at least one positive solution. Moreover, for a sufficiently large κ, system* (4.2) *has a unique positive solution and this solution is linearly stable.*

Proof With a parameter $a \in [0, 1]$, we consider the following problem

$$\begin{cases} -d_1 \Delta u = \sigma(x) - u + \dfrac{au^2}{(1+\kappa u^2)v}, & x \in \Omega, \\ -d_2 \Delta v = -v + u^2, & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{4.3}$$

Clearly when $a = 1$, system (4.3) reduces to (4.2).

Let $(u_a(x), v_a(x))$ be a positive solution of (4.3) with $a \in [0, 1]$. From the comparison principal of elliptic equations, we have that

$$u_a(x) \geq \underline{u}(x), \quad \text{and} \quad v_a(x) \geq \min_{y \in \overline{\Omega}} \underline{u}^2(y),$$

where $\underline{u}(x)$ is the unique positive solution of

$$\begin{cases} -d_1 \Delta u = \sigma(x) - u, & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{4.4}$$

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Similarly, we also have

$$u_a(x) \le \overline{u}(x), \quad \text{and} \quad v_a(x) \le \max_{y\in\overline{\Omega}} \overline{u}^2(y),$$

where $\overline{u}(x)$ is the unique positive solution of

$$\begin{cases} -d_1\Delta u = \sigma(x) - u + \dfrac{1}{\kappa \min_{y\in\overline{\Omega}} \underline{u}^2(y)}, & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases}$$

We define

$$\mathcal{O} = \{(u(x), v(x)) \in C(\overline{\Omega}) \times C(\overline{\Omega}) : \underline{C} < u(x),\ v(x) < \overline{C}, x \in \overline{\Omega}\},$$

where

$$\underline{C} = \frac{1}{2}\min\left\{\min_{x\in\overline{\Omega}} \underline{u}(x), \min_{x\in\overline{\Omega}} \underline{u}^2(x)\right\}, \quad \text{and} \quad \overline{C} = 2\max\left\{\max_{x\in\overline{\Omega}} \overline{u}(x), \max_{x\in\overline{\Omega}} \overline{u}^2(x)\right\}. \tag{4.5}$$

Then, for any $a \in [0, 1]$, system (4.3) has no solution on ∂O. When $a = 0$ system (4.3) has a unique positive solution $(u_0, v_0) \in \mathcal{O}$, which is given by $u_0 = \underline{u}$, and v_0 is the unique positive solution of

$$\begin{cases} -d_2\Delta v = -v + \underline{u}^2, & x \in \Omega, \\ \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases}$$

As in [35], we define an operator

$$A(a, u, v) = (L_1 f(a, u, v), L_2 g(u, v)),$$

where $L_1 = (-d_1\Delta + I)^{-1},\ L_2 = (-d_2\Delta + I)^{-1}$, and

$$f(t, u, v) = \sigma(x) + \frac{tu^2}{(1+\kappa u^2)v}, \quad g(u, v) = u^2.$$

Then, A is completely continuous from $[0, 1]\times\mathcal{O}$ to $C(\overline{\Omega})\times C(\overline{\Omega})$, and (u_a, v_a) is a solution of system (4.3) if and only if $(u_a, v_a) = A(a, u_a, v_a)$. From the analysis above, we also have that $(u, v) \neq A(a, u, v)$ when $(u, v) \in \partial\mathcal{O}$ and $a \in [0, 1]$. So, we have that

$$\deg(I - A(0, u, v), \mathcal{O}, 0) = \deg(I - A(1, u, v), \mathcal{O}, 0).$$

When $a = 0$, system (4.3) has a unique positive solution $(u_0, v_0) \in \mathcal{O}$, and the linearised operator of system (4.3) at (u_0, v_0) for is

$$\tilde{L}(\lambda) := \begin{pmatrix} -d_1\Delta + 1 & 0 \\ -2u_0 & -d_2\Delta + 1 \end{pmatrix}.$$

So, it is clear that (u_0, v_0) is non-degenerate and linearly stable. Then, we have that

$$\deg(I - A(0, u, v), \mathcal{O}, 0) = 1,$$

and hence $\deg(I - A(1, u, v), \mathcal{O}, 0) = 1$. So, system (4.2) has a positive solution for any $d_1, d_2, \kappa > 0$.

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If (u_*, v_*) is a positive solution of system (4.2), then the eigenvalue problem associated with (u_*, v_*) is

$$\begin{cases} -d_1\Delta\phi + \phi - \dfrac{2u_*}{(1+\kappa u_*^2)^2 v_*}\phi + \dfrac{u_*^2}{(1+\kappa u_*^2)v_*^2}\psi = \lambda\phi, & x \in \Omega, \\ -d_2\Delta\psi - 2u_*\phi + \psi = \lambda\psi, & x \in \Omega, \\ \dfrac{\partial\phi}{\partial\nu} = \dfrac{\partial\psi}{\partial\nu} = 0, & x \in \partial\Omega, \end{cases} \tag{4.6}$$

where λ is an eigenvalue and (ϕ, ψ) is an associated eigenfunction satisfying $\|\phi\|_{L^2} = \|\psi\|_{L^2} = 1$. Multiplying $\bar{\phi}$ to the first equation of (4.6) and integrating, we have

$$d_1\|\nabla\phi\|_{L^2}^2 + 1 - \int_\Omega \frac{2u_*}{(1+\kappa u_*^2)^2 v_*}|\phi|^2 dx + \int_\Omega \frac{u_*^2}{(1+\kappa u_*^2)v_*^2}\psi\bar{\phi}dx = \lambda.$$

From Equation (4.5), we see that $\underline{C}$ depends on $\underline{u}(x)$, which is the unique positive solution of Equation (4.4), and hence $\underline{C}$ is independent of κ. Since $\underline{C} < u_*(x),\ v_*(x) < \overline{C}$, and $\underline{C}$ is independent of κ, it follows that

$$\begin{aligned} &\limsup_{\kappa\to\infty}\left|\int_\Omega \frac{2u_*}{(1+\kappa u_*^2)^2 v_*}|\phi|^2 dx\right| + \limsup_{\kappa\to\infty}\left|\int_\Omega \frac{u_*^2}{(1+\kappa u_*^2)v_*^2}\psi\bar{\phi}dx\right| \\ &\quad \le \lim_{\kappa\to\infty}\frac{2\left(\max_{x\in\overline{\Omega}}\sigma(x) + 1/(\kappa\underline{C})\right)}{(1+\kappa\underline{C}^2)^2\underline{C}} + \lim_{\kappa\to\infty}\frac{\left(\max_{x\in\overline{\Omega}}\sigma(x) + 1/(\kappa\underline{C})\right)^2}{(1+\kappa\underline{C}^2)\underline{C}^2} = 0. \end{aligned}$$

Thus, we conclude that for sufficient large κ, Re(λ) is positive. So, (u_*, v_*) is linearly stable and index$(A(1,\cdot), (u_*, v_*)) = 1$. Hence, for sufficiently large κ, system (4.2) has a unique positive solution and this solution is linearly stable. □

With a similar proof, we can also prove the following existence and stability result:

Proposition 4.2 *Suppose that $d_1, d_2 > 0$ are fixed, $\kappa > 0$, and $\sigma(x)$ is a continuous positive function defined on $\overline{\Omega}$. Then, there exists $\sigma_0 > 0$ such that system* (4.2) *has a unique positive solution which is linearly stable, if*

$$\min_{x\in\overline{\Omega}}\sigma(x) \ge \sigma_0.$$

We also remark that by using the methods in [24], under the conditions in Theorem 4.1 or Proposition 4.2, that is, either κ is large, or σ is large, then the unique positive steady state solution is not only linearly stable, but globally asymptotically stable if $\sigma(x)$ is a positive constant. We conjecture that if $\sigma(x)$ is not a constant, the global stability still holds when either κ is large, or σ is large.

5. Simulation examples

In this section, we use several sets of specific parameter values to illustrate our analytical results for (2.1). The parameters are chosen so that they belong to three regimes where three different non-trivial spatiotemporal patterns are predicted.

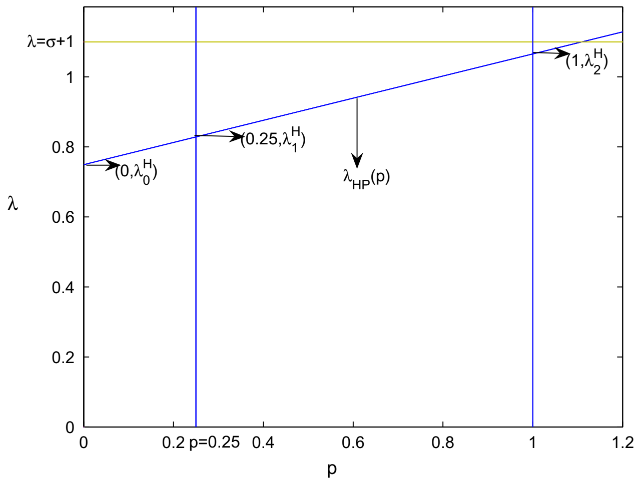

Figure 1. Hopf bifurcation points for parameters satisfying (5.1). Here the horizonal axis is p and the vertical axis is λ; the Hopf bifurcation curve $\lambda_{HP}(p) = \frac{1}{4}\left[R + 4\sigma + \sqrt{R(R+8\sigma)}\right]$ with $R = \left(d_1 + \frac{d_2}{\tau}\right)p + \frac{1}{\tau} + 1$. That is, $\lambda_{HP}(p)$ satisfies $T(\lambda_{HP}(p), p) = 0$, where $T(\lambda, p) = T_n(\lambda)$, with $p = n^2/\ell^2$ for $n = 0, 1, 2$.

Example 5.1 To visualize the results in Theorem 2.3, we choose

$$\sigma = 0.1, \quad \tau = 8, \quad d_1 = 0.5, \quad d_2 = 1, \quad \ell = 2. \tag{5.1}$$

Since this set of parameters satisfies Equation (2.20), then there is no λ satisfying the steady state bifurcation condition ($\mathbf{H}_2$) from Remark 2.4.2. We can also compute that in this case $\ell_n \approx 0.9495n$ for $n \in \mathbb{N}$. So $\ell = 2 \in [\ell_2, \ell_3)$, and there exist three Hopf bifurcation points, (see Figure 1):

$$\lambda_0^H < \lambda_1^H < \lambda_2^H,$$

where

$$\lambda_0^H = \lambda_{HP}(0) \approx 0.7492, \quad \lambda_1^H = \lambda_{HP}(0.25) \approx 0.8286, \quad \lambda_2^H = \lambda_{HP}(1) \approx 1.0656.$$

It is easy to see that if ℓ is larger, then there are more Hopf bifurcation points. The periodic orbits bifurcating from $\lambda = \lambda_0^H$ are spatially homogenous, hence they are also the periodic orbits of the ODE system corresponding to (2.1). Figure 2 shows the numerical bifurcation diagram of the periodic orbits from $\lambda = \lambda_0^H$ using κ as a parameter. Indeed, for this set of parameters, the direction of the Hopf bifurcation is forward. Since the equilibrium is unstable when κ is larger than the Hopf bifurcation point, the bifurcating periodic orbits are unstable, and hence the Hopf bifurcation is subcritical, (see Ref. [39]). But, the branch of cycles turns back at a fold bifurcation point $\tilde{\lambda}_0 < \lambda_0^H$. Finally, we give two numerical simulations for the occurrence of the Hopf bifurcation at $\lambda = \lambda_0^H$ (see Figures 3 and 4).

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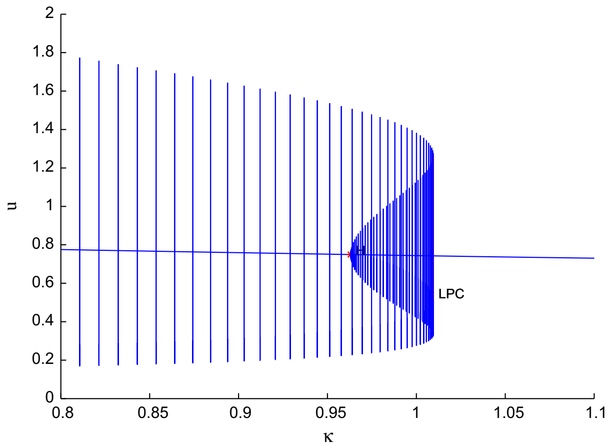

Figure 2. Bifurcation diagram of ODE system corresponding to (2.1). The horizontal axis is κ, and the vertical axis is u. A family of limit cycles bifurcates from the Hopf point H and LPC is a fold bifurcation of the cycles.

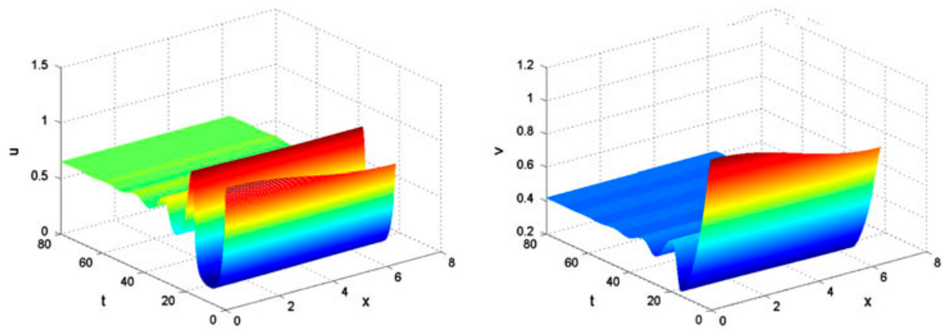

Figure 3. Convergence to the constant steady state solution. The parameters satisfy (5.1), $\kappa = 2$ and initial values: $u(x, 0) = 1 + 0.1\cos(x/2),\ x \in [0, 2\pi],\ v(x, 0) = 1 + 0.1\cos(x/2),\ x \in [0, 2\pi]$. Here $\lambda \approx 0.645473 < \lambda_0^H$.

Example 5.2 To visualize the results in Theorem 3.3, we choose

$$\sigma = 0.2, \quad \tau = 1, \quad d_1 = 0.03, \quad d_2 = 1, \quad \ell = 0.5. \tag{5.2}$$

Then, the parameters satisfy Equation (3.8) and $2/(\sigma + 1) - 1 < 1/\tau$, and $T_n(\lambda) > 0$ for $\lambda \in (\sigma, \sigma + 1]$ and $n \in \mathbb{N}_0$. So, there is no λ satisfying the Hopf bifurcation condition ($\mathbf{H}_1$). We can also compute that in this case $p_+ \approx 19.5141$ and $p_- \approx 1.7082$, and $\tilde{\ell}_{n,+} \approx 0.2264n$ and $\tilde{\ell}_{n,-} \approx 0.7651n$. So, $\ell_{n,\pm}$ satisfy

$$0 < \tilde{\ell}_{1,+} < \tilde{\ell}_{2,+} < \tilde{\ell}_{3,+} < \tilde{\ell}_{1,-} < \tilde{\ell}_{4,+} < \tilde{\ell}_{5,+} < \tilde{\ell}_{6,+} < \tilde{\ell}_{2,-} < \dots.$$

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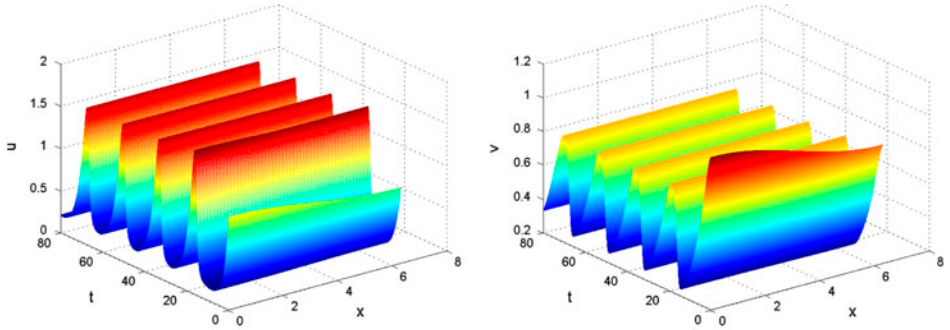

Figure 4. The constant steady state solution loses its stability through a Hopf bifurcation. The parameters satisfy (5.1), $\kappa = 0.92$, and initial values: $u(x,0) = 1 + 0.1\cos(x/2)$, $x \in [0, 2\pi]$, $v(x,0) = 1 + 0.1\cos(x/2)$, $x \in [0, 2\pi]$. Here $\lambda \approx 0.755617 > \lambda_0^H$.

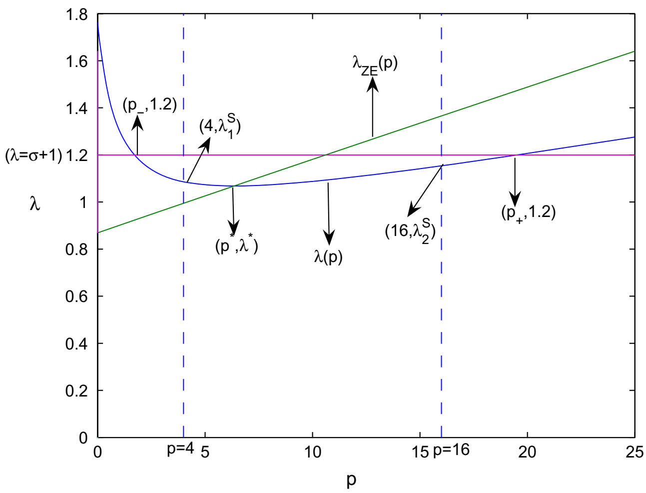

Figure 5. Steady state bifurcation points for parameters satisfying (5.2). Here, the horizonal axis is p and the vertical axis is λ; the steady state bifurcation curve $\lambda = \lambda(p)$ is given by (2.16), $\lambda_{ZE}(p) = \sigma + \frac{1}{4d_2}\left[R_Z + \sqrt{R_Z(R_Z + 8\sigma)}\right]$ where $R_Z = d_1 + d_2 + 2d_1 d_2 p^2$ satisfying $\frac{\partial D(\lambda_{ZE}(p), p)}{\partial p} = 0$; and p^* satisfies $\lambda'(p^*) = 0$.

Since $\ell = 0.5$ satisfies $\ell \in [\tilde{\ell}_{2,+}, \tilde{\ell}_{3,+})$, then $\ell \in [\tilde{\ell}_{j,+}, \tilde{\ell}_{j,-}]$ for $j = 1, 2$. Then, from Theorem 3.3, there exists two possible steady state bifurcation points λ_1^S and λ_2^S satisfying

$$p_- < \lambda_1^S < \lambda_2^S < p_+,$$

where

$$\lambda_1^S = \lambda\left(\frac{1}{\ell^2}\right) = \lambda(4) = 1.0864, \quad \lambda_2^S = \lambda\left(\frac{4}{\ell^2}\right) = \lambda(16) = 1.1540.$$

From Figure 5, we have that $\lambda_i^S \neq \lambda^*$ for $i = 1, 2$, and then they are the steady state bifurcation points.

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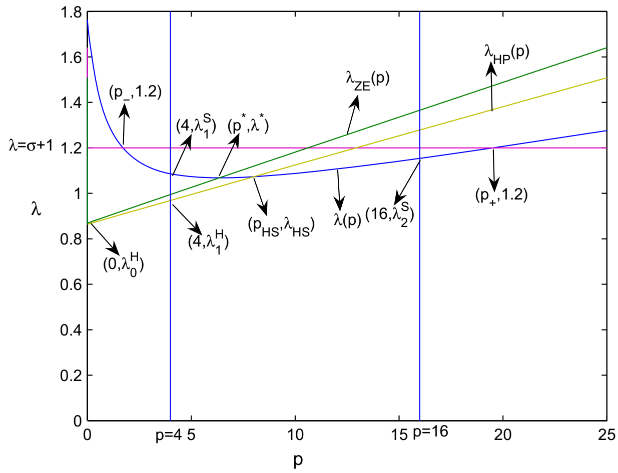

Figure 6. Steady state bifurcation points and Hopf bifurcation points for parameters satisfying (5.3). Here, the horizonal axis is p and the vertical axis is λ; the Hopf bifurcation curve $\lambda_{HP}(p) = \frac{1}{4}\left[R + 4\sigma + \sqrt{R(R+8\sigma)}\right]$ with $R = \left(d_1 + \frac{d_2}{\tau}\right)p + \frac{1}{\tau} + 1$; the steady state bifurcation curve $\lambda = \lambda(p)$ is given by (2.16), $\lambda_{ZE}(p) = \sigma + \frac{1}{4d_2}\left[R_Z + \sqrt{R_Z(R_Z+8\sigma)}\right]$ where $R_Z = d_1 + d_2 + 2d_1 d_2 p^2$ satisfying $\frac{\partial D(\lambda_{ZE}(p), p)}{\partial p} = 0$; and p^* satisfies $\lambda'(p^*) = 0$.

Example 5.3 In this example, there exist both Hopf bifurcation points and steady state bifurcation points, (see Figure 6). we choose

$$\sigma = 0.2, \quad \tau = 50, \quad d_1 = 0.03, \quad d_2 = 1, \quad \ell = 0.5. \tag{5.3}$$

Then, the parameters satisfy Equation (3.8) and $2/(\sigma+1) - 1 > 1/\tau$. From the results of Example 5.2, we still have $p_+ \approx 19.5141$ and $p_- \approx 1.7082$, and $\tilde{\ell}_{n,+} \approx 0.2264n$ and $\tilde{\ell}_{n,-} \approx 0.7651n$ for this case. So, $\tilde{\ell}_{n,\pm}$ satisfy

$$0 < \tilde{\ell}_{1,+} < \tilde{\ell}_{2,+} < \tilde{\ell}_{3,+} < \tilde{\ell}_{1,-} < \tilde{\ell}_{4,+} < \tilde{\ell}_{5,+} < \tilde{\ell}_{6,+} < \tilde{\ell}_{2,-} < \dots.$$

Since $\ell = 0.5$ satisfies $\ell \in [\tilde{\ell}_{2,+}, \tilde{\ell}_3)$, then there exists two steady state bifurcation points λ_1^S and λ_2^S satisfying

$$p_- < \lambda_1^S < \lambda_2^S < p^+,$$

where

$$\lambda_1^S = \lambda\left(\frac{1}{\ell^2}\right) = \lambda(4), \quad \lambda_2^S = \lambda\left(\frac{4}{\ell^2}\right) = \lambda(16).$$

Also, in this case, we compute that $\ell_n \approx 0.2781n$, then $\ell \in [\ell_1, \ell_2)$ and hence there may exist two Hopf bifurcation points: $\lambda_0^H < \lambda_1^H$. From Figure 6, we have $\lambda_0^H < \lambda_1^H < \lambda^*$, and hence λ_1^H and λ_0^H are the Hopf bifurcation points. So, in this case, there exist both Hopf bifurcation points and steady state bifurcation points, and

$$\lambda_0^H < \lambda_1^H < \lambda^* < \lambda_1^S < \lambda_2^S.$$

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We remark that in Figure 6, (P_{HS}, λ_{HS}) is the interaction point of curve $(p, \lambda_{HP}(p))$ and $(p, \lambda(p))$. So, when $\ell = \check{\ell}_n = n/\sqrt{P_{HS}}$, then $D_n(\lambda_0) = T_n(\lambda_0) = 0$, and a codimension-two bifurcation may occur there. This $\check{\ell}_n \in L^E$ where L^E is defined as in Equation (3.5). In general, the explicit value of λ_{HS} cannot be explicitly solved, and it may not exist for all parameter values. Further investigation of dynamics near the Turing-Hopf codimension-two bifurcation may be an interesting topic for future research.

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