MULTI-SPIKE STATIONARY SOLUTIONS OF THE CAHN-HILLIARD EQUATION IN HIGHER-DIMENSION AND INSTABILITY

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1. Introduction.

The nonlinear Cahn-Hilliard equation:

$$\begin{cases} u_t = \Delta(-\varepsilon^2 \Delta u + f(u)) & \text{in} \quad \Omega \times (0,\infty), \\ \frac{\partial \Delta u}{\partial n} = \frac{\partial u}{\partial n} = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\ u(\cdot,0) = u_0 & \text{in} \quad \Omega. \end{cases}$$

is a widely accepted model for some complicated behavior of the concentration $u(x,t)$ of a binary alloy.

In this paper, we study a certain class of stationary solutions to the Cahn-Hilliard equation characterized by being almost constant in the interior of the domain $\Omega \subset \mathbf{R}^n$, and having several "spikes" at points on the boundary. Since any constant is a stationary solution and average value is conserved by the dynamical Cahn-Hilliard equation, we foliate our function space with each leaf consisting of functions with a fixed average value, and then search for stationary spiked solutions in certain leaves. That is, we seek solutions to

$$\begin{cases} \varepsilon^2 \Delta u - f(u) = \sigma'_\varepsilon & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on} \quad \partial\Omega, \\ \int_\Omega u(x)dx = \overline{m}|\Omega|, \end{cases} \tag{1.1}$$

where $\sigma'_\varepsilon = -\frac{1}{|\Omega|}\int_\Omega f(u)dx$ and $\overline{m}$ is a constant. We assume that Ω is smooth (at least C^4) and $f : \mathbf{R} \to \mathbf{R}$ is of bistable type, that is, $f(u)u > 0$ for $|u|$ large enough, f has two zeros u_1 and u_2 such that $f'(u_i) > 0$ for $i = 1,2$. (see Figure 1) More specific conditions on f are given in a transformed form (g1-4) below. For bistable f, there exist two intervals (u_1, m_1) and (m_2, u_2) in which $f'(u) > 0$. These intervals are called *metastable regions* since homogeneous states with values here are stable with respect to Cahn-Hilliard dynamics, but for small ε, they are not the minimum energy solutions having the same average value (see [CGS].) We

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will actually find solutions which are on the common boundary of the basins of attraction of the least energy solutions and the homogeneous solutions.

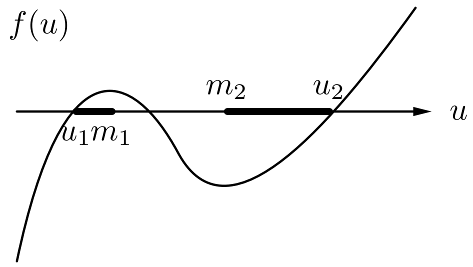

Figure 1: Metastable regions

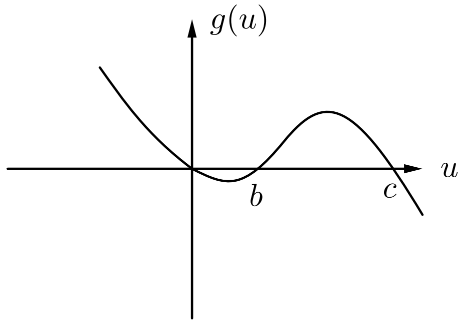

Figure 2: Graph of $g(u)$

Now we consider $\overline{m} \in (m_2, u_2)$ and make a transformation (for $\overline{m} \in (u_1, m_1)$, it is similar)

$$v = \overline{m} - u,$$
$$g(v) = f(\overline{m} - v) - f(\overline{m}).$$

Let

$$g'(0) = -\mu < 0, \qquad g(v) = -\mu v + h(v).$$

Then (1.1) becomes

$$\begin{cases} \varepsilon^2 \Delta v + g(v) - \sigma_\varepsilon = 0 & \text{in} \quad \Omega, \\ \frac{\partial v}{\partial n} = 0 & \text{on} \quad \partial\Omega, \\ \int_\Omega v(x)dx = 0, \end{cases}$$

where $\sigma_\varepsilon = \frac{1}{|\Omega|}\int_\Omega g(v)dx$ or

$$\begin{cases} \varepsilon^2 \Delta v - \mu v + h(v) - \frac{1}{|\Omega|}\int_\Omega h(v) = 0 & \text{in} \quad \Omega, \\ \frac{\partial v}{\partial n} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.2}$$

We assume that $g(v)$ satisfies (see Figure 2)

(g1) $g \in C^3(\mathbf{R}, \mathbf{R})$, $g(0) = 0$, $g'(0) < 0$,

(g2) There exists $b, c > 0$ such that $b < c$ and $g(b) = g(c) = 0$; $g(v) > 0$ in $(-\infty, 0) \cup (b, c)$ and $g(v) < 0$ in $(0, b) \cup (c, \infty)$,

(g3) $\int_0^c g(v)dv > 0$.

In addition to these assumptions on the shape of g, we assume the nondegeneracy of the ground state of a related equation in $\mathbf{R}^n$. Although, as we remark later, this can be weakened, here we will assume

(g4) The problem

$$\begin{cases} \Delta w + g(w) = 0 & \text{in} \quad \mathbf{R}^n, \\ w(0) = \max w(x), \quad w > 0, & \\ w(x) \to 0 & |x| \to \infty. \end{cases} \tag{1.3}$$

has a unique solution $w(x)$ which is radially symmetric and nondegenerate. (The solution w of (1.3) is called the *ground state* solution. w is said to be nondegenerate if the linearized operator $L_0 = \Delta + g'(w)$ on $L^2(\mathbf{R}^n)$ with domain $H^2(\mathbf{R}^n)$ has a bounded inverse when it is restricted to the subspace $L_r^2(\mathbf{R}^n) := \{u \in L^2(\mathbf{R}^n) : u(z) = u(|z|)\}$.)

Our main theorem is as follows:

Theorem 1.1. *Let $\overline{m}$ be in the metastable region (m_2, u_2), g satisfy (g1-4), and $P_i \in \partial\Omega$ $(i = 1, 2, \cdots, k,\ k \geq 1)$ be k points on the boundary of Ω satisfying*

(1) *$P_i \neq P_j$ if $i \neq j$,*

(2) *Let $H(P)$ be the mean curvature of $\partial\Omega$ at P, then*

$$DH(P_i) = 0, \qquad D^2H(P_i) \text{ is nondegenerate,}$$

where DH is the tangential gradient of H, and D^2H is the Hessian.

Then for $\varepsilon > 0$ small enough, there exists a solution u_ε of (1.1) such that as $\varepsilon \to 0$, $u_\varepsilon(x) \to \overline{m}$ for every $x \in \overline{\Omega}\backslash\{P_i\}$ and u_ε has exactly k local minimum points $Q_{\varepsilon,i} \in \partial\Omega$ for which $u_\varepsilon(Q_{\varepsilon,i}) < \overline{m}$ $(i = 1, 2, \cdots, k)$. $u_\varepsilon(Q_{\varepsilon,i}) \to \overline{m} - w(0)$ and may be enumerated so that $Q_{\varepsilon,i} \to P_i$ as $\varepsilon \to 0$ $(i = 1, 2, \cdots, k)$. A similar result holds for $\overline{m}$ in the metastable region (u_1, m_1).

We also can give some estimates of the degree to which these solutions are unstable. In addition to a weak instability which may derive from spikes moving along the boundary, there is a strong instability corresponding to nucleation events occurring at any spike.

Theorem 1.2. *Let u_ε be a multi-spike solution as in Theorem 1.1, with k spikes $Q_{\varepsilon,1}, \cdots, Q_{\varepsilon,k}$. Then the eigenvalue problem*

$$\begin{cases} \Delta(-\varepsilon^2\Delta\psi + f'(u)\psi) = -\lambda\psi & \text{in } \Omega, \\ \frac{\partial\psi}{\partial n} = \frac{\partial\Delta\psi}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

has at least k negative eigenvalues (counting the multiplicity) $\lambda_1^\varepsilon, \cdots, \lambda_k^\varepsilon$, and for $i = 1, \cdots, k$, $\lambda_i^\varepsilon \leq -C\varepsilon^{-2} < 0$ for some $C > 0$.

For the existence part in Theorem 1.1, the uniqueness of ground state and condition (2) in Theorem 1.1 can be weakened. We will discuss some weaker conditions in the remark at the end of Section 4. On the other hand, condition (g4) is not very restrictive. The existence and uniqueness of a ground state has been studied extensively. For g satisfying (g1-3), Ouyang and Shi[OS1,2] and Dancer[D4] proved that the ground state solution w is unique and nondegenerate if g also satisfies

(g5) Define $\theta > b$ to be the smallest positive number such that $G(u) = 0$, where $G(u) = \int_0^u g(s)ds$, and $\rho > b$ to be the smallest number such that $\frac{g(u)}{u-\rho}$ is nonincreasing for $u \in (\rho, c)$. Then either (i) $\theta \geq \rho$, or (ii) $\theta < \rho$ with $K_g(u)$ nonincreasing in (θ,ρ), $K_g(u) \geq K_g(\theta)$ for $u \in (b,\theta)$ and $K_g(u) \leq K_g(\rho)$ for $u \in (0,b) \cup (\rho,c)$, where $K_g(u) = \frac{ug'(u)}{g(u)}$.

In particular, the typical choice $f(u) = u^3 - u$ satisfies (g1-3) and (g5) for any $\overline{m}$ in the metastable region $(\frac{1}{\sqrt{3}}, 1)$, and in this case, $g(u) = u(u-b)(c-u)$ with $0 < 2b < c$. In [WW1], Wei and Winter require that g satisfies $\theta \geq \rho$ in (g5), which is true only when $\overline{m}$ is close to -1 or 1. The uniqueness of the ground state for this special case was proved by Peletier and Serrin [PS].

We summarize some facts about the ground state solution w as follows. Here, $L_0 = \Delta + g'(w) = \Delta - \mu + h'(w) : H^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$.

Proposition 1.3. *Let g satisfy (g1-3) and (g5), and let w be the unique solution of (1.3). Then w has the following properties:*

(i) $w \in C^2(\mathbf{R}^n) \cap H^2(\mathbf{R}^n)$,

(ii) *w is radially symmetric and $w(x) > 0$ in $\mathbf{R}^n$,*

(iii) *w and its derivatives up to order 2 decay exponentially at infinity, i.e., there exist $C, K > 0$ such that*

$$|D^\alpha w(z)| \leq Ce^{-K|z|} \quad \text{for } z \in \mathbf{R}^n \text{ and } |\alpha| \leq 2,$$

(iv) *The principle eigenvalue of L_0, $\lambda_1 = \lambda_1(L_0) < 0$ and is simple; the corresponding eigenfunction ϕ is positive and radially symmetric,*

(v) *$\lambda_2(L_0) = 0$, dim $Ker(L_0) = n$, and $Ker(L_0)$ is spanned by $\{\frac{\partial w}{\partial x_1}, \cdots, \frac{\partial w}{\partial x_n}\}$.*

These properties are well-known though it is hard to find a reference for a complete proof. Some related results can be found in [GNN], [NT2] and [D3]. For conveniance, we also sketch a proof in Appendix B. Note that even when (g5) is not satisfied, but (g4) is satisfied, most conclusions here are still ture, except 0 may not be the second eigenvalue (this fact is not essential in this paper).

Problem (1.1) has been investigated by numerous authors. One approach is to consider the solutions of (1.1) as critical points of an energy functional

$$I(\varepsilon,u) = \frac{\varepsilon}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{\varepsilon}\int_\Omega F(u)dx,$$

with $u \in H^1(\Omega)$ and satisfying $\int_\Omega u(x)dx = \overline{m}|\Omega|$. Carr, Gurtin and Slemrod [CGS] in one space dimension and Modica [M] in the higher dimensional case, showed that the global minimizer of $I(\varepsilon,u)$ exists, and has a transition layer for small $\varepsilon > 0$. In [M], the interface was shown to converge to a minimal surface as $\varepsilon \to 0$. Kohn and Sternberg [KS] showed the relation between $I(\varepsilon,u)$ and the perimeter of the interface, and proved the existence of local minimizers of I near local minimizers of the perimeter functional. Chen and Kowalczyk [CK] showed a type of local minimizer exists when $n = 2$, with the interface close to a circular arc intersecting the boundary orthogonally.

Another approach is the invariant manifold method initiated by Alikakos and Fusco [AF1,2]. They constructed a finite-dimensional manifold of states which have spherical interfaces within the domain and then decomposed the dynamical Cahn-Hilliard equation to a system comprised of a flow which is almost tangent to this manifold and a semiflow in the normal directions. Solutions to (1.1) are found very close to certain points on that manifold and hence have almost spherical interfaces, the location of the center being identified.

The spike layer solution of (1.1) was first studied by Bates and Fife [BFi2], in the case of $n = 1$. They showed the existence of a monotone spike layer solution with the spike at the endpoint of the interval. Rescaling, extending evenly and periodically the single spike solution give multiple spikes. They also estimated the principle eigenvalue of the linearized operator and discussed the dynamics associated with the attendant instability. In the higher dimension case, Bates and Fusco [BFu1,2] used the invariant manifold approach to show the existence of multi-spike solutions whose peaks are in the interior of Ω. We should mention that for the one dimensional case, the solutions of (1.1) for any $\overline{m}$ and ε has been completely classified for $f(u) = u^3 - u$, by Grinfeld and Novick-Cohen [GN] and independently by Bates, Lu and Ouyang [BLO].

A single-boundary-spike solution of (1.1) in higher dimensional space was obtained by Wei and Winter [WW1], by using a Lyapunov-Schmidt reduction method. They showed that for any nondegenerate critical point P of the mean curvature of the boundary, there is family of single-boundary-spike solutions with the peak location converging to P as $\varepsilon \to 0$. Our present paper mostly follows their line in showing the existence of multi-boundary-spike solutions. This reduction method was first used by Floer and Weinstein [FW] and Oh [O1,2] to show the existence of semi-classical bound state solutions of nonlinear Schödinger equation.

Spike layer solutions of semilinear elliptic equations have been studied extensively in recent years. For the equation

$$\begin{cases} \varepsilon^2 \Delta u - u + f(u) = 0 & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.5}$$

with $f(u) = u^p$, $1 < p < \frac{n+2}{n-2}$, or of a similar type, Ni and Takagi [NT1,2] (and [LNT] with Lin) showed the existence and asymptotic behavior of the least energy solution u_ε, which is a single-boundary-spike solution. In particular, they proved u_ε has exactly one local maximum point $P_\varepsilon \in \partial\Omega$, and when $\varepsilon \to 0$, $P_\varepsilon \to P_0 \in \partial\Omega$ at which the mean curvature $H(P)$ of $\partial\Omega$ achieves its global maximum. Later, Gui [G] showed the existence of multi-spike solutions of (1.5), with the peaks near the local maximum points of $H(P)$. The above all used variational methods to show the existence of solutions of Mountain-Pass type, and used energy methods and blowup arguments to determine the profile of solutions. Similar results were also obtained for $p = \frac{n+2}{n-2}$ by Adimuthi, Pacella and Yadava [APY1], Ni, Pan and Takagi [NPT], X. J. Wang [Wax], and Z. Q. Wang [Waz1]. Ni and Takagi [NT3] and Z. Q. Wang [Waz2,3] also obtained the existence of symmetric multi-spike solutions in certain symmetric domains.

Recently, the Lyapunov-Schmidt reduction method was employed by Wei [We2] for $1 < p < \frac{n+2}{n-2}$ and Adimuthi, Mancinni, Pacella and Yadava [AMY, APY2] for $p = \frac{n+2}{n-2}$, to prove the existence of single-boundary-spike solution of (1.5), with the peak near any nondegenerate critical point of $H(P)$. The invariant manifold approach was also applied to (1.5) by Kowalczyk [Ko]. He showed the existence of multi-interior-spike solutions. The corresponding Dirichlet problem of the equation in (1.5) has also been studied recently; see, for example, [NW], [We1], [CDNY], and [DFW]. See also [P1,2] and [R] for results on similar problems.

This work originated during the spring of 1994 when the first author visited the University of Sydney. At that time, Theorem 1.1 was stated and a proof outlined using an invariant manifold reduction, based on the ideas in [BFu2]. The current version abandons the invariant manifold approach in favor of the more straightforward Liapunov-Schmidt reduction used by Wei and Winter in [WW1].

After this work was completed, we became aware of the preprint by Y. Li [L]. There, he established the existence of multi-boundary-spike solutions to (1.5) with subcritical p, with spikes possible at locations where the mean curvature of $\partial\Omega$ has an isolated critical point which is C^1 stable (*i.e.* C^1 small perturbation of the mean curvature function have critical point nearby). This is slightly more general than requiring the critical point to be nondegenerate and is similar to our Brouwer degree condition mensioned in the remark at the end of the proof of Theorem 1.1. We would like to point out that the integral constraint included in (1.1) makes this problem significantly more complex than that for (1.5). We have also since received a preprint [WW2] by Wei and Winter, who have independently obtained Theorem 1.1, but do not include infomation about the stability of these multi-spike solutions.

Very little work has been done to examine the instability of the spike solutions. For $n = 1$, Bates and Fife [BFi2] showed the Morse index of the single-boundary-spike solution of (1.1) is 1, and the principle eigenvalue satisfies $\lambda_1^\varepsilon \le -C\varepsilon^{-2}$ for some $C > 0$. For the higher dimensional case, Bates and Fusco [BFu2] proved $\lambda_1^\varepsilon \le -C\varepsilon^{-2}$ for interior-spike solutions. Theorem 1.2 appears to be the most general instability result for multi-spike solutions so far. We conjecture that, for a multi-boundary-spike solution u_ε, all eigenvalues λ_i^ε of (1.4) satisfy $\varepsilon^2\lambda_i^\varepsilon \to C_j\lambda_j$, where λ_j is an eigenvalue of the ground state solution and C_j is a constant, and all "small" eigenvalues (for which, $\varepsilon^2\lambda_i^\varepsilon \to 0$, the second eigenvalue for the ground state) are determined by the curvatures of $\partial\Omega$ at the limit set of peaks. Thus the exact Morse indices of multi-spike solutions can be determined for u_ε.

Our assumptions (g1-3) allow h to be unbounded on $\mathbf{R}$. To make our proof simpler, we will assume that h is bounded in $C^3(\mathbf{R})$. In fact, we can define $\tilde{h} \in C^3(\mathbf{R})$ to satisfy

(1) $\tilde{h}(u) = h(u)$ for $u \in [-a, c]$ for some $a > 0$,
(2) $|\tilde{h}(u)|, |\tilde{h}'(u)|, |\tilde{h}''(u)|, |\tilde{h}'''(u)| \leq C$ for all $u \in \mathbf{R}$,
(3) $\tilde{g}(u) = -\mu u + \tilde{h}(u) > \delta_0$ for some $\delta_0 > 0$ and for any $u \in (-\infty, -a]$,
(4) $\tilde{g}(u) \leq 0$ for any $u \in (c, \infty)$.

If v is a solution of

$$\begin{cases} \varepsilon^2 \Delta v - \mu v + \tilde{h}(v) - \frac{1}{|\Omega|} \int_\Omega \tilde{h}(v) = 0 & \text{in} \quad \Omega, \\ \frac{\partial v}{\partial n} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.6}$$

and $v(x) \in [-a, c]$ for all $x \in \overline{\Omega}$, then v is also a solution of (1.2). After we obtain the spike solution v_ε of (1.6), we will show the range of v_ε is in $[-a, c]$, so v_ε is also a solution of (1.2) (see Lemma C.3.) Hence there is no loss of generality in assuming that h is bounded in the above sense and for simplicity of notation, we drop the tilde on h and g.

Using almost the same proof, we can also obtain the existence and instability of multi-spike stationary solutions of the Allen-Cahn equation:

$$\begin{cases} v_t = \varepsilon^2 \Delta v + g(v) & \text{in} \quad \Omega \times \mathbf{R}^+, \\ \frac{\partial v}{\partial n} = 0 & \text{on} \quad \partial\Omega \times \mathbf{R}^+, \end{cases} \tag{1.7}$$

or the conserving Allen-Cahn equation:

$$\begin{cases} v_t = \varepsilon^2 \Delta v + f(v) - \frac{1}{|\Omega|} \int_\Omega f(v) dx & \text{in} \quad \Omega \times \mathbf{R}^+, \\ \frac{\partial v}{\partial n} = 0 & \text{on} \quad \partial\Omega \times \mathbf{R}^+, \end{cases} \tag{1.8}$$

The results are:

Theorem 1.4. *Let g satisfy (g1-4), and $\mathbf{P}= (P_1, \cdots, P_k)$ satisfy the conditions in Theorem 1. Then for $\varepsilon > 0$ small enough, there exists a stationary solution w_ε of (1.7) such that as $\varepsilon \to 0$, $w_\varepsilon(x) \to 0$ for any $x \in \overline{\Omega} \backslash \{P_i\}$, w_ε has exactly k local maximum points $T_{\varepsilon,i} \in \partial\Omega$ $(i = 1, 2, \cdots, k)$ such that $T_{\varepsilon,i} \to P_i$ and $w_\varepsilon(T_{\varepsilon,i}) \to w(0)$. Moreover, the operator linearized at w_ε has at least k negative eigenvalues $\mu_1^\varepsilon, \cdots, \mu_k^\varepsilon$ and $\mu_i^\varepsilon \leq \lambda_1 + O(\varepsilon)$ for $i = 1, \cdots, k$, where λ_1 is the first eigenvalue of L_0.*

Theorem 1.5. *Let f be same as in Theorem 1.1, $\overline{m} \in (m_2, u_2)$ and $\mathbf{P}= (P_1, \cdots, P_k)$ satisfy the conditions in Theorem 1. Then for $\varepsilon > 0$ small enough, there exists a stationary solution w_ε of (1.8) such that as $\varepsilon \to 0$, $w_\varepsilon(x) \to \overline{m}$ for any $x \in \overline{\Omega} \backslash \{P_i\}$, w_ε has exactly k local maximum points $T_{\varepsilon,i} \in \partial\Omega$ for which $w_\varepsilon(T_{\varepsilon,i}) < \overline{m}$ $(i = 1, 2, \cdots, k)$ such that $T_{\varepsilon,i} \to P_i$ and $w_\varepsilon(T_{\varepsilon,i}) \to \overline{m} - w(0)$. Moreover, the operator* *linearized at w_ε has at least k negative eigenvalues $\mu_1^\varepsilon, \cdots, \mu_k^\varepsilon$ and $\mu_i^\varepsilon \le \lambda_1 + O(\varepsilon)$ for $i = 1, \cdots, k$, where λ_1 is the first eigenvalue of L_0.*

We organize our paper in the following way. In Section 2, we introduce some preliminaries, including the basic building block for approximate solutions. We give the reduction to the finite dimensional problem in Section 3. This is done by using a Lyapunov-Schmidt decomposition at an approximate solution. In Section 4, we solve the finite dimensional problem and give the proof to Theorem 1.1. In Sections 5 and 6, we study the instability of the solution of (1.1) and (1.7). In Appendix A, we give the proofs to some technical estimates. We also include a brief proof of Proposition 1.4 in Appendix B. In Appendix C, we give some a priori bounds associated with the Cahn-Hilliard equation (1.1).

2. Preliminaries.

Let $\mathbf{R}^n_+ = \{(x', x_n) \in \mathbf{R}^n : x_n > 0\}$, $B(p, r) = \{x \in \mathbf{R}^n : |x - p| < r\}$. If $p = 0$, we use $B(r) = B(0, r)$. Let Ω be a smooth domain with at least C^4 boundary $\partial\Omega$. For a point $P \in \partial\Omega$, let $n(P)$ denote the unit outer normal vector, and $\tau_i(P)$, $i = 1, 2, \cdots, n-1$, denote an orthogonal basis of the tangent space of $\partial\Omega$ at P. Correspondingly, let $\frac{\partial}{\partial n}$ and $\frac{\partial}{\partial \tau_i}$ denote the outer normal derivative and tangential derivative at P. The positive constant C in the paper can be different from line to line.

We will use standard Sobolev spaces $L^2(\Omega)$, $H^1(\Omega)$, and $H^2(\Omega)$ with the usual inner products and norms

$$\langle u, v\rangle = \int_\Omega uv dx,$$
$$||u||_0^2 = \langle u, u\rangle,$$
$$||u||_1^2 = ||u||_0^2 + \sum_{i=1}^n ||D_i u||_0^2,$$
$$\text{and} \qquad ||u||_2^2 = ||u||_0^2 + \sum_{i=1}^n ||D_i u||_0^2 + \sum_{i,j=1}^n ||D_{ij} u||_0^2.$$

In this paper, we will frequently rescale the domain, and the norm will also have rescaled forms:

$$\langle u, v\rangle_{0,\varepsilon} = \varepsilon^{-n}\langle u, v\rangle,$$
$$||u||_{0,\varepsilon}^2 = \varepsilon^{-n}||u||_0^2,$$
$$||u||_{1,\varepsilon}^2 = \varepsilon^{-n}\left[||u||_0^2 + \varepsilon^2\sum_{i=1}^n ||D_i u||_0^2\right],$$
$$\text{and} \qquad ||u||_{2,\varepsilon}^2 = \varepsilon^{-n}\left[||u||_0 + \varepsilon^2\sum_{i=1}^n ||D_i u||_0^2 + \varepsilon^4\sum_{i,j=1}^n ||D_{ij} u||_0^2\right].$$

We introduce a diffeomorphism which straightens the boundary in a neighborhood of $P \in \partial\Omega$. Through rotation of the coordinate system we may assume that the inner normal to $\partial\Omega$ at P is $(0, \cdots, 0, 1)$. Let $B'(\delta) = \{x' \in \mathbf{R}^{n-1} : |x'| < \delta\}$. Fix $P = (P', P_n) \in \partial\Omega$, then we can find $\delta_1 > 0$ and a smooth function $\rho : B'(\delta_1) \to \mathbf{R}$ such that for some neighborhood N of P

(1) $\rho(0) = 0$, $\nabla\rho(0) = 0$,

(2) $\Omega \cap N = \{(x', x_n) : x_n - P_n > \rho(x' - P')\}$ and $\partial\Omega \cap N = \{(x', x_n) : x_n - P_n = \rho(x' - P')\}$.

Now we define a mapping $y = \Psi(x) = (\Psi_1(x), \cdots, \Psi_n(x))$ for $x \in B(P, \delta_1)$, and

$$\Psi_i(x) = \begin{cases} x_i - P_i & i = 1, 2, \cdots, n-1, \\ x_n - P_n - \rho(x' - P') & i = n. \end{cases}$$

We also define a mapping which is a rescaling of Ψ:

$$\Psi_\varepsilon(x) = (\Psi_{\varepsilon 1}, \cdots, \Psi_{\varepsilon n}) = \varepsilon^{-1}\Psi(x).$$

We can calculate that $D\Psi(P) = I$, the identity mapping. Thus Ψ has an inverse mapping $x = \Phi(y) = \Psi^{-1}(y)$ for $y \in B(\delta_2) \subset \Psi(B(P, \delta_1))$, where δ_2 is some positive constant. Let $\Psi^{-1}(B(\delta_2)) \cap \Omega = \Omega_P$. Therefore we have defined a local diffeomorphism $\Psi : \Omega_P \to B(\delta_2)$ such that

(1) $\Psi(P) = 0$, $D\Psi(P) = I$,

(2) $\Psi(\Omega_P) = \mathbf{R}^n_+ \cap B(\delta_2)$ and $\Psi(\partial\Omega \cap \overline{\Omega_P}) = \partial\mathbf{R}^n_+ \cap \overline{B(\delta_2)}$.

We also define $\Phi_\varepsilon(y) = \Psi_\varepsilon^{-1}(y) = \Phi(\varepsilon y)$. Later in the paper, if we want to specify the location of the diffeomorphism, we will use notations Ψ_P and Φ_P, or $\Psi_{\varepsilon,P}$ and $\Phi_{\varepsilon,P}$.

We recall that the mean curvature of $\partial\Omega$ at P is $H(P) = \frac{1}{n-1}\sum_{i=1}^{n-1} \rho_{ii}(0)$, and we have the Taylor expansion of ρ

$$\begin{aligned} \rho(x' - P') = &\frac{1}{2}\sum_{i,j=1}^{n-1} \rho_{ij}(0)(x_i - P_i)(x_j - P_j) \\ &+ \frac{1}{6}\sum_{i,j,k=1}^{n-1} \rho_{ijk}(0)(x_i - P_i)(x_j - P_j)(x_k - P_k) + O(|x' - P'|^4), \end{aligned}$$

where the subscripts on ρ denote partial derivatives. We also observe that if $v(y) = u(\Phi_\varepsilon(y)) = u(x)$ then

$$\varepsilon^2 \Delta u(x) = \sum_{j,k=1}^{n} a_{jk}(y)\frac{\partial^2 v}{\partial y_j \partial y_k}(y) + \varepsilon \sum_{j=1}^{n} b_j(y)\frac{\partial v}{\partial y_j}(y),$$

where

$$a_{jk}(y) = \sum_{i=1}^{n} \frac{\partial \Psi_j}{\partial x_i}(\Phi(\varepsilon y)) \cdot \frac{\partial \Psi_k}{\partial x_i}(\Phi(\varepsilon y)), \qquad 1 \le j, k \le n, \tag{2.1}$$

$$\text{and} \qquad b_j(y) = (\Delta\Psi_j)(\Phi(\varepsilon y)), \qquad 1 \le j \le n. \tag{2.2}$$

By using the definition of Ψ and Φ, we obtain

$$\varepsilon^2\Delta_x = \Delta_y + |\nabla\rho|^2\frac{\partial^2}{\partial y_n^2} - 2\sum_{i=1}^{n-1}\rho_i\frac{\partial^2}{\partial y_i\partial y_n} - \varepsilon(\Delta\rho)\frac{\partial}{\partial y_n}. \tag{2.3}$$

Following [WW1], we introduce a trial function $W_{\varepsilon,P}\left(\frac{x-P}{\varepsilon}\right)$ as an approximation to a spike solution, taking it to be the unique solution of

$$\begin{cases} \varepsilon^2\Delta v - \mu v + h(w(\frac{x-P}{\varepsilon})) = 0 & \text{in} \quad \Omega, \\ \frac{\partial v}{\partial n} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{2.4}$$

It is clear that, $w\left(\frac{x-P}{\varepsilon}\right) - W_{\varepsilon,P} \to 0$ when $\varepsilon \to 0$. But the estimate for this $h_{\varepsilon,P}(x) = w(\frac{x-P}{\varepsilon}) - W_{\varepsilon,P}\left(\frac{x-P}{\varepsilon}\right)$ is important. $h_{\varepsilon,P}(x)$ satisfies

$$\begin{cases} \varepsilon^2\Delta v - \mu v = 0 & \text{in} \quad \Omega, \\ \frac{\partial v(x)}{\partial n} = \frac{\partial w(\frac{x-P}{\varepsilon})}{\partial n} & \text{on} \quad \partial\Omega. \end{cases} \tag{2.5}$$

First we notice that $W_{\varepsilon,P}$ and $h_{\varepsilon,P}$ are exponentially small when x is away from P.

Lemma 2.1. *Suppose that $\Omega_P \subset \Omega$ is any fixed neighborhood of $P \in \partial\Omega$, then there exists an $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$*

$$\left|W_{\varepsilon,P}\left(\frac{x-P}{\varepsilon}\right)\right| \le C_1 exp\left(-\frac{C_2|x-P|}{\varepsilon}\right),$$

$$|h_{\varepsilon,P}(x)| \le C_1 exp\left(-\frac{C_2|x-P|}{\varepsilon}\right),$$

$$\left|\frac{\partial W_{\varepsilon,P}\left(\frac{x-P}{\varepsilon}\right)}{\partial\tau_j}\right| \le C_1 exp\left(-\frac{C_2|x-P|}{\varepsilon}\right),$$

$$\left|\frac{\partial h_{\varepsilon,P}(x)}{\partial\tau_j}\right| \le C_1 exp\left(-\frac{C_2|x-P|}{\varepsilon}\right).$$

for $x \in \Omega\backslash\Omega_P$, where C_1, C_2 are constants only depending on Ω and δ_2.

Proof. The proof of the first inequality is the same as the proof in [NT] pg. 839-840 (see also Lemma 4.2 of [F].) The second inequality can be obtained by combining the first one and Proposition 1.4. The other two can be proved similarly.

□

Near a point $P \in \partial\Omega$, we need a more careful analysis. For $y \in B^+(\frac{\delta_2}{\varepsilon}) = \{y \in B(\frac{\delta_2}{\varepsilon}) : y_n > 0\}$, we define,

$$H_{\varepsilon,P}(y) = h_{\varepsilon,P}(\Phi_\varepsilon(y)).$$

Then $H_{\varepsilon,P}(y)$ satisfies

$$\begin{cases} \Delta v + |\nabla\rho|^2\frac{\partial^2 v}{\partial y_n^2} - 2\sum_{i=1}^{n-1}\rho_i\frac{\partial^2 v}{\partial y_i\partial y_n} - \varepsilon(\Delta\rho)\frac{\partial v}{\partial y_n} - \mu v = 0 & \text{in} \quad B^+(\frac{\delta_2}{\varepsilon}), \\ \sum_{i=1}^{n-1}\rho_i\frac{\partial v}{\partial y_i} - (1+|\nabla\rho|^2)\frac{\partial v}{\partial y_n} & \\ = \sqrt{1+|\nabla\rho|^2}w'(|\frac{\Phi_\varepsilon(y)-P}{\varepsilon}|)\frac{\langle\Phi_\varepsilon(y)-P, n(\Phi_\varepsilon(y))\rangle}{|\Phi_\varepsilon(y)-P|} & \text{on} \quad \{|y'| \le \frac{\delta_2}{\varepsilon}\}. \end{cases} \tag{2.6}$$

To find the desired solution v of the second order elliptic equation (2.6), we can formally expand v in the terms of ε. In fact, if we assume that

$$H_{\varepsilon,P}(y) = v_0(y) + \varepsilon v_1(y) + \varepsilon^2 v_2(y) + \cdots,$$

with each v_i extended to $\mathbf{R}^n_+$ and lying in $H^1(\mathbf{R}^n_+)$, then v_0 satisfies

$$\begin{cases} \Delta v - \mu v = 0 & \text{in} \quad \mathbf{R}^n_+, \\ \frac{\partial v}{\partial y_n} = 0 & \text{on} \quad \partial \mathbf{R}^n_+. \end{cases} \tag{2.7}$$

But the unique solution v_0 of (2.7) in $H^1(\mathbf{R}^n_+)$ is 0. So, v_1 satisfies

$$\begin{cases} \Delta v - \mu v = 0 & \text{in} \quad \mathbf{R}^n_+, \\ \frac{\partial v}{\partial y_n} = -\frac{w'(|y|)}{2|y|} \sum_{i,j=1}^{n-1} \rho_{ij}(0) y_i y_j & \text{on} \quad \partial \mathbf{R}^n_+. \end{cases} \tag{2.8}$$

Finally, v_2 satisfies

$$\begin{cases} \Delta v - \mu v - 2\sum_{i,j=1}^{n-1} \rho_{ij}(0) y_i \frac{\partial^2 v_1}{\partial y_j \partial y_n} - (\Delta \rho(0)) \frac{\partial v_1}{\partial y_n} = 0 & \text{in} \quad \mathbf{R}^n_+, \\ \frac{\partial v}{\partial y_n} = \sum_{i,j=1}^{n-1} \rho_{ij}(0) y_j \frac{\partial v_1}{\partial y_i} - \frac{w'(|y|)}{3|y|} \sum_{i,j,k=1}^{n-1} \rho_{ijk}(0) y_i y_j y_k & \text{on} \quad \partial \mathbf{R}^n_+. \end{cases} \tag{2.9}$$

We also need estimates for $\frac{\partial}{\partial \tau_j(P)} h_{\varepsilon,P}(x)$, which satisfies

$$\begin{cases} \varepsilon^2 \Delta v - \mu v = 0 & \text{in} \quad \Omega, \\ \frac{\partial v(x)}{\partial n} = \frac{\partial}{\partial \tau_j(P)} \frac{\partial}{\partial n} \left[w(\frac{x-P}{\varepsilon}) - W_{\varepsilon,P}\left(\frac{x-P}{\varepsilon}\right)\right] & \text{on} \quad \partial \Omega. \end{cases} \tag{2.10}$$

Similar to (2.6), the function $\frac{\partial}{\partial \tau_j(P)} h_{\varepsilon,P}(\Phi_\varepsilon(y))$ satisfies

$$\begin{cases} \Delta v + |\nabla \rho|^2 \frac{\partial^2 v}{\partial y_n^2} - 2\sum_{i=1}^{n-1} \rho_i \frac{\partial^2 v}{\partial y_i \partial y_n} - \varepsilon(\Delta\rho) \frac{\partial v}{\partial y_n} - \mu v = 0 & \text{in} \quad B^+(\frac{\delta_2}{\varepsilon}), \\ \sum_{i=1}^{n-1} \rho_i \frac{\partial v}{\partial y_i} - (1 + |\nabla \rho|^2) \frac{\partial v}{\partial y_n} & \\ = (1 + |\nabla\rho|^2) \frac{\partial}{\partial \tau_j(P)} \left[w'(|\frac{\Phi_\varepsilon(y) - P}{\varepsilon}|) \frac{\langle \Phi_\varepsilon(y) - P, n(\Phi_\varepsilon(y))\rangle}{|\Phi_\varepsilon(y) - P|}\right] & \text{on} \quad \{|y'| \le \frac{\delta_2}{\varepsilon}\}. \end{cases} \tag{2.11}$$

We also have consider the expansion

$$\frac{\partial}{\partial \tau_j(P)} h_{\varepsilon,P}(\Phi_\varepsilon(y)) = u_0(y) + \varepsilon u_1(y) + \cdots.$$

Then u_0 satisfies

$$\begin{cases} \Delta v - \mu v = 0 & \text{in} \quad \mathbf{R}^n_+, \\ \frac{\partial v}{\partial y_n} = -\frac{|y| w''(|y|) - w'(|y|)}{2|y|^3} \sum_{i,k=1}^{n-1} \rho_{ik}(0) y_i y_k y_j & \\ \qquad -\frac{w'(|y|)}{|y|} \sum_{i=1}^{n-1} \rho_{ij}(0) y_i & \text{on} \quad \partial \mathbf{R}^n_+. \end{cases} \tag{2.12}$$

All the above formal asymptotic expansions are obtained by using the Taylor expansions for ρ, ρ_i and $\Delta\rho$ and matching the corresponding terms. We shall make these expansions rigorous in the following.

Let Ω_P be the neighborhood of P introduced before, and let M and N also be neighborhoods of P satisfying $M \subset\subset N \subset \Omega_P$. Then we can define a cut-off function $\chi(x) : \mathbf{R}^n \to [0, 1]$ such that

(1) χ is smooth,

(2) $\chi(x) = 1$ for $x \in M$,

(3) $\chi(x) = 0$ for $x \in \overline{\Omega} \backslash N$.

Proposition 2.2. *Let $h_{\varepsilon,P}(x) = w(\frac{x-P}{\varepsilon}) - W_{\varepsilon,P}\left(\frac{x-P}{\varepsilon}\right)$ and $\tau_j(P)$ be a tangent vector of $\partial\Omega$ at P. Then there exist functions e_1 and e_2 such that*

$$h_{\varepsilon,P}(x) = \varepsilon v_1(\Psi_\varepsilon(x))\chi(x) + \varepsilon^2 v_2(\Psi_\varepsilon(x))\chi(x) + \varepsilon^3 e_1(x), \tag{2.13}$$

$$\frac{\partial h_{\varepsilon,P}}{\partial \tau_j(P)}(x) = u_0(\Psi_\varepsilon(x))\chi(x) + \varepsilon e_2(x), \tag{2.14}$$

where v_1, v_2 and u_0 are, respectively, the unique solutions to (2.8), (2.9) and (2.12) in $H^1(\mathbf{R}^n_+)$. Moreover, $||e_1||_{1,\varepsilon} \le C$ and $||e_2||_{1,\varepsilon} \le C$ with $C > 0$ independent of ε.

The proof of this proposition is technical, so we postpone it to Appendix A. Notice that the conclusion of this proposition is same as that of Propositions 2.1 and 2.3 in [WW1], but we correct the equations (2.7) and (2.8) of [WW1] to (2.9) of this paper. In (2.7) of [WW1], the term $(\Delta\rho(0))\frac{\partial v_1}{\partial y_n}$ is missed, though it does not affect the result in [WW1].

Finally, we prove that $W_{\varepsilon,P}\left(\frac{x-P}{\varepsilon}\right)$ is bounded in L^∞ as $\varepsilon \to 0$.

Proposition 2.3. *Suppose that g satisfies (g1-3), and $W_{\varepsilon,P}\left(\frac{x-P}{\varepsilon}\right)$ is the unique solution of (2.4). Then*

$$\mu^{-1}\min_{x\in\overline{\Omega}} h\left(w\left(\frac{x-P}{\varepsilon}\right)\right) \le W_{\varepsilon,P}\left(\frac{x-P}{\varepsilon}\right) \le \mu^{-1}\max_{x\in\overline{\Omega}} h\left(w\left(\frac{x-P}{\varepsilon}\right)\right).$$

Proof. For simplicity, we denote $W_{\varepsilon,P}\left(\frac{x-P}{\varepsilon}\right)$ by $v(x)$. Let $v(x_1) = \max_{x\in\overline{\Omega}} v(x)$, and $v(x_0) = \min_{x\in\overline{\Omega}} v(x)$. We first prove that $v(x_1) \le \mu^{-1}\max_{x\in\overline{\Omega}} h(w\left(\frac{x-P}{\varepsilon}\right))$. In fact, if $v(x_1) > \mu^{-1}\max_{x\in\overline{\Omega}} h(w\left(\frac{x-P}{\varepsilon}\right))$, then there exists a ball $B \subset \Omega$, such that $x_1 \in \partial B$, $v(x) > \mu^{-1}\max_{x\in\overline{\Omega}} h(w\left(\frac{x-P}{\varepsilon}\right))$ in B. Since $\varepsilon^2\Delta v(x) = \mu v(x) - h(w\left(\frac{x-P}{\varepsilon}\right)) > 0$ in B, by the Hopf Boundary Lemma, $\frac{\partial v}{\partial n} > 0$ at x_1. But this is impossible, since if $x_1 \in \Omega$, $\nabla v(x_1) = 0$, and if $x_1 \in \partial\Omega$, then $\frac{\partial v}{\partial n}(x_1) = 0$. A similar argument shows that $v(x_0) \ge \mu^{-1}\min_{x\in\overline{\Omega}} h(w\left(\frac{x-P}{\varepsilon}\right))$.

□

3. Reduction to a finite dimensional problem.

Let

$$H^2_N(\Omega) = \left\{u \in H^2(\Omega) : \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\right\}.$$

For any $u \in H^2_N(\Omega)$, set

$$S_\varepsilon(u) = \varepsilon^2\Delta u - \mu u + h(u) - \frac{1}{|\Omega|}\int_\Omega h(u). \tag{3.1}$$

Then u is a solution of (1.2) if and only if $S_\varepsilon(u) = 0$.

To get a multi-spike solution of (3.1), we define a trial solution $W_{\varepsilon,\mathbf{P}}(x)$. Fix a small constant $\delta > 0$, define

$$\Gamma^k_\delta = \{\mathbf{P} = (P_1, P_2, \cdots, P_k) \in (\partial\Omega) \times \cdots \times (\partial\Omega) : |P_i - P_j| \ge \delta \text{ if } i \ne j.\},$$

and for $\mathbf{P} \in \Gamma_\delta^k$,

$$W_{\varepsilon,\mathbf{P}}(x) = \sum_{i=1}^{k} W_{\varepsilon,P_i}\left(\frac{x-P_i}{\varepsilon}\right).$$

Whenever there is no confusion, we will use the abbreviated notation $W_{\varepsilon,\mathbf{P}} = W_{\varepsilon,\mathbf{P}}(x)$ and $W_{\varepsilon,P_i} = W_{\varepsilon,P_i}\left(\frac{x-P_i}{\varepsilon}\right)$. We will also use $w_{\varepsilon,P} = w\left(\frac{x-P}{\varepsilon}\right)$.

First, we give a linear analysis near $W_{\varepsilon,\mathbf{P}}$. We define a linear operator $\overline{S}_\varepsilon'(u) : H_N^2(\Omega) \to L^2(\Omega)$ for any $u \in H_N^2(\Omega)$,

$$\overline{S}_\varepsilon'(u)v = \varepsilon^2 \Delta v - \mu v + h'(u)v.$$

This is not the derivative of S_ε given by (3.1) but it is useful for our analysis.

Our reduction is motivated by the following lemma, which is a corollary of Proposition 1.3.

Lemma 3.1. *Let $L_0 : H_N^2(\mathbf{R}_+^n) \to L^2(\mathbf{R}_+^n)$ be given by*

$$L_0 = \Delta - \mu + h'(w).$$

Then

$$Ker(L_0) = span\left\{\frac{\partial w}{\partial y_1}, \cdots, \frac{\partial w}{\partial y_{n-1}}\right\}.$$

For $\overline{S}_\varepsilon'(u)$, define the following approximate kernels and associated projection operators

$$\begin{aligned}
K_{\varepsilon,P_i} &= span\left\{\frac{\partial W_{\varepsilon,P_i}}{\partial \tau_j(P_i)} : j = 1, 2, \cdots, n-1\right\},\\
K_{\varepsilon,\mathbf{P}} &= span\left\{\bigcup_{i=1}^{k} K_{\varepsilon,P_i}\right\},\\
K_0 &= span\left\{\frac{\partial w}{\partial y_1}, \cdots, \frac{\partial w}{\partial y_{n-1}}\right\} \subset H_N^2(\mathbf{R}_+^n),\\
K_{\varepsilon,P_i}^\perp &= \ L^2\text{-orthogonal complement of } K_{\varepsilon,P_i},\\
K_{\varepsilon,\mathbf{P}}^\perp &= \ L^2\text{-orthogonal complement of } K_{\varepsilon,\mathbf{P}},\\
K_0^\perp &= \ L^2\text{-orthogonal complement of } K_0,\\
\pi_{\varepsilon,P_i}^\perp, \pi_{\varepsilon,P_i} &: L^2(\Omega) \to K_{\varepsilon,P_i}^\perp, K_{\varepsilon,P_i},\ L^2\text{-orthogonal projections},\\
\pi_{\varepsilon,\mathbf{P}}^\perp, \pi_{\varepsilon,\mathbf{P}} &: L^2(\Omega) \to K_{\varepsilon,\mathbf{P}}^\perp, K_{\varepsilon,\mathbf{P}},\ L^2\text{-orthogonal projections},\\
\pi_0^\perp, \pi_0 &: L^2(\mathbf{R}_+^n) \to K_0^\perp, K_0,\ L^2\text{-orthogonal projections}.
\end{aligned}$$

We will look for a solution to (1.1) of the form $u(x) = W_{\varepsilon,\mathbf{P}}(x) + \psi_{\varepsilon,\mathbf{P}}(x)$, where $\psi_{\varepsilon,\mathbf{P}} \in K_{\varepsilon,\mathbf{P}}^\perp \cap H_N^2(\Omega)$. Our strategy is to solve following system

$$\begin{cases} \pi_{\varepsilon,\mathbf{P}}^\perp S_\varepsilon(W_{\varepsilon,\mathbf{P}} + \psi_{\varepsilon,\mathbf{P}}) = 0,\\ \pi_{\varepsilon,\mathbf{P}} S_\varepsilon(W_{\varepsilon,\mathbf{P}} + \psi_{\varepsilon,\mathbf{P}}) = 0. \end{cases}$$

In this section, we shall prove that for any $\mathbf{P}\in \Gamma_\delta^k$, there exists a unique $\psi_{\varepsilon,\mathbf{P}} \in K_{\varepsilon,\mathbf{P}}^\perp \cap H_N^2(\Omega)$ such that the first equation above holds, then in the next section, we locate $\mathbf{P}$ in Γ_δ^k to ensure the second equation is satisfied. To achieve that, we need to study the linear operators

$$\begin{aligned} L_{\varepsilon,\mathbf{P}} &= \pi_{\varepsilon,\mathbf{P}}^\perp \overline{S}'_\varepsilon(W_{\varepsilon,\mathbf{P}}), \\ L_{\varepsilon,P_i} &= \pi_{\varepsilon,P_i}^\perp \overline{S}'_\varepsilon(W_{\varepsilon,P_i}). \end{aligned}$$

We show that $L_{\varepsilon,\mathbf{P}}$ is one-to-one and surjective:

Lemma 3.2. *There exist $\varepsilon_1, \lambda > 0$ such that for all $\varepsilon \in (0, \varepsilon_1)$,*

$$||L_{\varepsilon,\mathbf{P}}\phi||_{0,\varepsilon} \geq \lambda ||\phi||_{2,\varepsilon}$$

for any $\phi \in H_N^2(\Omega) \cap K_{\varepsilon,\mathbf{P}}^\perp$ and $\mathbf{P}\in \Gamma_\delta^k$ with a fixed small $\delta > 0$.

Lemma 3.3. *There exists $\varepsilon_2 > 0$ such that for all $\varepsilon \in (0, \varepsilon_2)$, $L_{\varepsilon,\mathbf{P}} : H_N^2(\Omega) \cap K_{\varepsilon,\mathbf{P}}^\perp \to K_{\varepsilon,\mathbf{P}}^\perp$ is surjective for all $\mathbf{P}\in \Gamma_\delta^k$ with a fixed small $\delta > 0$.*

To prove the lemmas, we introduce a partition of unity associated with $\mathbf{P}$ which helps to localize the analysis. Let $P_i \in M_i \subset\subset N_i$, $i = 1, 2, \cdots, k$, where M_i and N_i are open neighborhoods of P_i such that

(1) $\overline{N_i} \cap \overline{N_j} = \emptyset$ for $i \neq j$,
(2) $N_i \subset \Omega_{P_i}$.

Then there exists a family of functions $\alpha_i : \overline{\Omega} \to [0, 1]$ $(1 \leq i \leq k)$ such that

(1) α_i is smooth,
(2) $\alpha_i(x) = 1$ if $x \in M_i$, and $\alpha_i(x) = 0$ if $x \in \overline{\Omega}\backslash N_i$, and consequently
(3) $\sum_{i=1}^k \alpha_i(x) \leq 1$ for all $x \in \overline{\Omega}$.

Define $\beta : \overline{\Omega} \to [0, 1]$ by $\beta(x) = 1 - \sum_{i=1}^k \alpha_i(x)$. Then α_i and β constitute a partition of unity in Ω associated with $\mathbf{P}$. Moreover, without loss of generality, we can assume that the choice of the cut-off function χ near P_i in Section 2 and α_i are coincident. Hence, Proposition 2.2 is also ture with χ replaced by α_i.

Proof of Lemma 3.2. Suppose it is not true, then there exists $\varepsilon_j \to 0$ $(j \to \infty)$, $\mathbf{P}_j = (P_{1j}, \cdots, P_{kj}) \in \Gamma_\delta^k$ and $\mathbf{P}= (P_1, \cdots, P_k) \in \Gamma_\delta^k$ such that $\mathbf{P}_j \to\mathbf{P}$ $(j \to \infty)$, $\phi_j \in K_{\varepsilon_j,\mathbf{P}_j}^\perp \cap H_N^2(\Omega)$ such that when $j \to \infty$

$$\begin{aligned} &||L_{\varepsilon_j,\mathbf{P}_j}\phi_j||_{0,\varepsilon_j} \to 0, \\ &||\phi_j||_{2,\varepsilon_j} = 1. \end{aligned}$$

We define $\phi_{ij}(x) = \alpha_i(x)\phi_j(x)$ for $i = 1, 2, \cdots, k$ and $j = 1, 2, \cdots$, and $\phi_{0j}(x) = \beta(x)\phi_j(x)$, where α_i and β is the partition of unity in Ω associated with $\mathbf{P}$. Then

$\phi_j(x) = \sum_{i=0}^k \phi_{ij}(x)$. Let $\Omega_\varepsilon = \varepsilon^{-1}\Omega = \{z = \varepsilon^{-1}x : x \in \Omega\}$. We define $\varphi_j(z) = \phi_j(\varepsilon_j z)$, and $\varphi_{ij}(z) = \phi_{ij}(\varepsilon_j z)$ for $i = 0, 1, 2, \cdots, k$, $j \in \mathbf{N}$ and $z \in \Omega_{\varepsilon_j}$. (Note the different symbols ϕ_j and φ_j.) By the assumptions, we have

$$||\varphi_j||_{H^2(\Omega_{\varepsilon_j})} = 1, \qquad ||\varphi_{ij}||_{H^2(\Omega_{\varepsilon_j})} \le C.$$

We can extend $\varphi_j(z)$, $\varphi_{ij}(z)$ to $z \in \mathbf{R}^n \backslash \Omega_\varepsilon$ such that

$$1 \le ||\varphi_j||_{H^2(\mathbf{R}^n)} \le C, \qquad ||\varphi_{ij}||_{H^2(\mathbf{R}^n)} \le C.$$

for some constant C independent of ε. Moreover, φ_{ij} still satisfy $supp(\varphi_{ij}) \cap supp(\varphi_{lj}) = \emptyset$ if $i \ne l$, $i, l = 1, 2, \cdots, k$ and $\varphi_j(z) = \sum_{i=0}^k \varphi_{ij}(z)$. This can be done since Ω_ε is a scaling of Ω. The constants in the extension theorem (see [GT] Lemma 6.37 and Theorem 7.25) can be chosen independent of ε whenever $\varepsilon < 1$. Therefore, there exists a subsequence of φ_j (still denoted by φ_j) which converges to a limit φ_∞ weakly in $H^2(\mathbf{R}^n)$. Similarly $\varphi_{ij} \to \varphi_{i\infty}$ in the same sense. Obviously $\sum_{i=0}^k \varphi_{i\infty} = \varphi_\infty$. The first part of proof is to show $\varphi_\infty \equiv 0$, and then we show $||\phi_j||_{2,\varepsilon_j} \to 0$, which contradicts our assumptions. We shall prove the lemma by several steps.

Step 1: As $j \to \infty$,

$$||L_{\varepsilon_j, P_{ij}} \phi_{ij}||_{0,\varepsilon_j} \to 0, \tag{3.2A}$$

$$||L_{\varepsilon_j, \mathbf{P}_j} \phi_{0j}||_{0,\varepsilon_j} \to 0. \tag{3.2B}$$

for $i = 1, 2, \cdots, k$.

We first prove (3.2B).

$$\begin{aligned} &||L_{\varepsilon_j,\mathbf{P}_j}\phi_{0j}||_{0,\varepsilon_j} = ||\pi^\perp_{\varepsilon_j,\mathbf{P}_j}\overline{S}'_{\varepsilon_j}(W_{\varepsilon_j,\mathbf{P}_j})\beta\phi_j||_{0,\varepsilon_j} \\ =&||\pi^\perp_{\varepsilon_j,\mathbf{P}_j}\beta\overline{S}'_{\varepsilon_j}(W_{\varepsilon_j,\mathbf{P}_j})\phi_j + \pi^\perp_{\varepsilon_j,\mathbf{P}_j}(\varepsilon_j^2\nabla\beta\cdot\nabla\phi_j) + \pi^\perp_{\varepsilon_j,\mathbf{P}_j}(\varepsilon_j^2\Delta\beta\cdot\phi_j)||_{0,\varepsilon_j} \\ \le&||\beta\overline{S}'_{\varepsilon_j}(W_{\varepsilon_j,\mathbf{P}_j})\phi_j||_{0,\varepsilon_j} + ||\varepsilon_j^2\nabla\beta\cdot\nabla\phi_j||_{0,\varepsilon_j} + ||\varepsilon_j^2\Delta\beta\cdot\phi_j||_{0,\varepsilon_j}. \end{aligned}$$

The latter two terms are bounded by $C\varepsilon_j||\phi_j||_{2,\varepsilon_j}$, hence converge to 0. Let $O = supp(\beta) \subset \overline{\Omega}\backslash(\cup_{i=1}^k M_i)$, then

$$||\beta\overline{S}'_{\varepsilon_j}(W_{\varepsilon_j,\mathbf{P}_j})\phi_j||_{0,\varepsilon_j} \le \left[\varepsilon_j^{-n}\int_O \left(\overline{S}'_{\varepsilon_j}(W_{\varepsilon_j,\mathbf{P}_j})\phi_j\right)^2 dx\right]^{\frac{1}{2}}.$$

We claim that

$$\varepsilon_j^{-n}\int_O \left(\overline{S}'_{\varepsilon_j}(W_{\varepsilon_j,\mathbf{P}_j})\phi_j\right)^2 dx \to 0,$$

as $j \to \infty$. We first prove that $||\overline{S}'_{\varepsilon_j}(W_{\varepsilon_j,\mathbf{P}_j})\phi_j||_{0,\varepsilon_j} \leq C$.

$$\begin{aligned}
&||\overline{S}'_{\varepsilon_j}(W_{\varepsilon_j,\mathbf{P}_j})\phi_j||_{0,\varepsilon_j} = ||\varepsilon_j^2\Delta\phi_j - \mu\phi_j + h'(W_{\varepsilon_j,\mathbf{P}_j})\phi_j||_{0,\varepsilon_j} \\
\leq &||\varepsilon_j^2\Delta\phi_j - \mu\phi_j||_{0,\varepsilon_j} + ||h'(W_{\varepsilon_j,\mathbf{P}_j})\phi_j||_{0,\varepsilon_j} \\
\leq &C||\phi_j||_{2,\varepsilon_j} \leq C.
\end{aligned}$$

$$\begin{aligned}
&\varepsilon_j^{-n}\int_O \left(\overline{S}'_{\varepsilon_j}(W_{\varepsilon_j,\mathbf{P}_j})\phi_j\right)^2 dx \\
\leq &C\varepsilon_j^{-n}\int_O \left(\pi^{\perp}_{\varepsilon_j,\mathbf{P}_j}\overline{S}'_{\varepsilon_j}(W_{\varepsilon_j,\mathbf{P}_j})\phi_j\right)^2 dx + C\varepsilon_j^{-n}\int_O \left(\pi_{\varepsilon_j,\mathbf{P}_j}\overline{S}'_{\varepsilon_j}(W_{\varepsilon_j,\mathbf{P}_j})\phi_j\right)^2 dx \\
\equiv &I_1 + I_2.
\end{aligned}$$

$$I_1 \leq ||L_{\varepsilon_j,\mathbf{P}_j}\phi_j||^2_{0,\varepsilon_j} \to 0.$$

For I_2, we consider that

$$\pi_{\varepsilon_j,\mathbf{P}_j}\overline{S}'_{\varepsilon_j}(W_{\varepsilon_j,\mathbf{P}_j})\phi_j = \sum_{i=1}^{k}\sum_{l=1}^{n-1} c_{ilj}\left\langle \overline{S}'_{\varepsilon_j}(W_{\varepsilon_j,\mathbf{P}_j})\phi_j, \frac{\partial W_{\varepsilon_j,P_{ij}}}{\partial\tau_l}\right\rangle_{0,\varepsilon_j}\frac{\partial W_{\varepsilon_j,P_{ij}}}{\partial\tau_l},$$

where $c_{ilj} = ||\frac{\partial W_{\varepsilon_j,P_{ij}}}{\partial\tau_l}||^{-2}_{0,\varepsilon_j}$. Since $||\frac{\partial W_{\varepsilon_j,P_{ij}}}{\partial\tau_l}||_{L^\infty(O)} = O(e^{-\frac{c}{\varepsilon_j}})$, $c_{ilj} \leq C\varepsilon_j^2$, and $||\overline{S}'_{\varepsilon_j}(W_{\varepsilon_j,\mathbf{P}_j})\phi_j||_{0,\varepsilon_j}$ is bounded, then $I_2 = \varepsilon_j^{-n+1}O(e^{-\frac{c}{\varepsilon_j}}) \to 0$. The claim is proved. Thus (3.2B) holds.

Let $\phi \in H^2_N(\Omega)$, and $\mathbf{P} \in (\partial\Omega)^k$, then

$$\begin{aligned}
\overline{S}'_\varepsilon(W_{\varepsilon,\mathbf{P}})\phi =&\varepsilon^2\Delta\phi - \mu\phi + h'(W_{\varepsilon,\mathbf{P}})\phi \\
=&\sum_{i=1}^{k}(\varepsilon^2\Delta(\alpha_i\phi) - \mu\alpha_i\phi + h'(W_{\varepsilon,P_i})\alpha_i\phi) \\
&+ \varepsilon^2\Delta(\beta\phi) - \mu\beta\phi + h'(W_{\varepsilon,\mathbf{P}})\beta\phi \\
&+ \sum_{i=1}^{k}\alpha_i[h'(W_{\varepsilon,\mathbf{P}}) - h'(W_{\varepsilon,P_i})] \\
=&\sum_{i=1}^{k}\overline{S}'_\varepsilon(W_{\varepsilon,P_i})\alpha_i\phi + \overline{S}'_\varepsilon(W_{\varepsilon,\mathbf{P}})\beta\phi + \eta_1(x),
\end{aligned} \tag{3.3}$$

where $\eta_1(x) := \sum_{i=1}^k \alpha_i[h'(W_{\varepsilon,\mathbf{P}}) - h'(W_{\varepsilon,P_i})]$.

Claim 1: $||\eta_1(x)||_{L^\infty(\Omega)} = O(e^{-\frac{c}{\varepsilon}})$.

In fact, $supp(\eta_1) \subset \cup_{i=1}^k N_i$. For $x \in N_i$,

$$\begin{aligned}
|\eta_1(x)| &= |\alpha_i(x)[h'(W_{\varepsilon,\mathbf{P}}) - h'(W_{\varepsilon,P_i})]| \\
&\leq \max|h''(u)| \cdot \sum_{l\neq i}|W_{\varepsilon,P_l}| \\
&\leq Ce^{-\frac{c}{\varepsilon}} \text{ by Lemma 2.1.}
\end{aligned}$$

Then from (3.3), we have

$$
\begin{aligned}
||L_{\varepsilon_j,\mathbf{P}_j}\phi_j||^2_{0,\varepsilon_j} =&||\pi^{\perp}_{\varepsilon_j,\mathbf{P}_j}\overline{S}'_{\varepsilon_j}(W_{\varepsilon_j,\mathbf{P}_j})\phi_j||^2_{0,\varepsilon_j}\\
=&\sum_{i=1}^{k}||\pi^{\perp}_{\varepsilon_j,\mathbf{P}_j}\overline{S}'_{\varepsilon_j}(W_{\varepsilon_j,P_{ij}})\phi_{ij}||^2_{0,\varepsilon_j}+||\pi^{\perp}_{\varepsilon_j,\mathbf{P}_j}\overline{S}'_{\varepsilon_j}(W_{\varepsilon_j,\mathbf{P}_j})\phi_{0j}||^2_{0,\varepsilon_j}\\
&+\sum_{i\neq l}\langle\pi^{\perp}_{\varepsilon_j,\mathbf{P}_j}\overline{S}'_{\varepsilon_j}(W_{\varepsilon_j,P_{ij}})\phi_{ij},\pi^{\perp}_{\varepsilon_j,\mathbf{P}_j}\overline{S}'_{\varepsilon_j}(W_{\varepsilon_j,P_{lj}})\phi_{lj}\rangle_{0,\varepsilon_j}\\
&+2\sum_{i=1}^{k}\langle\pi^{\perp}_{\varepsilon_j,\mathbf{P}_j}\overline{S}'_{\varepsilon_j}(W_{\varepsilon_j,P_{ij}})\phi_{ij},\pi^{\perp}_{\varepsilon_j,\mathbf{P}_j}\overline{S}'_{\varepsilon_j}(W_{\varepsilon_j,\mathbf{P}_j})\phi_{0j}\rangle_{0,\varepsilon_j}\\
&+O(e^{-\frac{c}{\varepsilon_j}}). \qquad (3.4)
\end{aligned}
$$

Claim 2: If $i\neq l$, then

$$
||\pi_{\varepsilon_j,P_{lj}}\overline{S}'_{\varepsilon_j}(W_{\varepsilon_j,P_{ij}})\phi_{ij}||_{0,\varepsilon_j}=O(e^{-\frac{c}{\varepsilon_j}}). \qquad (3.5)
$$

In fact, for $1\leq q\leq n-1$,

$$
\begin{aligned}
&\left\langle\overline{S}'_{\varepsilon_j}(W_{\varepsilon_j,P_{ij}})\phi_{ij},\frac{\partial W_{\varepsilon_j,P_{lj}}}{\partial\tau_q}\right\rangle_{0,\varepsilon_j}\\
=&\varepsilon_j^{-n}\int_{\Omega}\left[-\varepsilon_j^2\nabla\phi_{ij}\nabla\frac{\partial W_{\varepsilon_j,P_{lj}}}{\partial\tau_q}-\mu\phi_{ij}\frac{\partial W_{\varepsilon_j,P_{lj}}}{\partial\tau_q}+h'(W_{\varepsilon_j,P_{ij}})\phi_{ij}\frac{\partial W_{\varepsilon_j,P_{lj}}}{\partial\tau_q}\right]dx\\
=&O(e^{-\frac{c}{\varepsilon_j}}),
\end{aligned}
$$

because of $supp(\phi_{ij})\subset N_i$, $i\neq l$, and Lemma 2.1. Since $K_{\varepsilon_j,P_{lj}}$ is spanned by $\frac{\partial W_{\varepsilon_j,P_{lj}}}{\partial\tau_q}$, we obtain (3.5).

From (3.5), we have

$$
\begin{aligned}
&||\pi_{\varepsilon_j,\mathbf{P}_j}\overline{S}'_{\varepsilon_j}(W_{\varepsilon_j,P_{ij}})\phi_{ij}-\pi_{\varepsilon_j,P_{ij}}\overline{S}'_{\varepsilon_j}(W_{\varepsilon_j,P_{ij}})\phi_{ij}||_{0,\varepsilon_j}=O(e^{-\frac{c}{\varepsilon_j}}),\\
&||\pi^{\perp}_{\varepsilon_j,\mathbf{P}_j}\overline{S}'_{\varepsilon_j}(W_{\varepsilon_j,P_{ij}})\phi_{ij}-\pi^{\perp}_{\varepsilon_j,P_{ij}}\overline{S}'_{\varepsilon_j}(W_{\varepsilon_j,P_{ij}})\phi_{ij}||_{0,\varepsilon_j}=O(e^{-\frac{c}{\varepsilon_j}}). \qquad (3.6)
\end{aligned}
$$

Then, by (3.4) and (3.6), we obtain

$$
\begin{aligned}
||L_{\varepsilon_j,\mathbf{P}_j}\phi_j||^2_{0,\varepsilon_j} =&\sum_{i=1}^{k}||\pi^{\perp}_{\varepsilon_j,P_{ij}}\overline{S}'_{\varepsilon_j}(W_{\varepsilon_j,P_{ij}})\phi_{ij}||^2_{0,\varepsilon_j}+||\pi^{\perp}_{\varepsilon_j,\mathbf{P}_j}\overline{S}'_{\varepsilon_j}(W_{\varepsilon_j,\mathbf{P}_j})\phi_{0j}||^2_{0,\varepsilon_j}\\
&+\sum_{i\neq l}\langle\pi^{\perp}_{\varepsilon_j,P_{ij}}\overline{S}'_{\varepsilon_j}(W_{\varepsilon_j,P_{ij}})\phi_{ij},\pi^{\perp}_{\varepsilon_j,P_{lj}}\overline{S}'_{\varepsilon_j}(P_{\Omega,P_l}w)\phi_{lj}\rangle_{0,\varepsilon_j}\\
&+2\sum_{i=1}^{k}\langle\pi^{\perp}_{\varepsilon_j,P_{ij}}\overline{S}'_{\varepsilon_j}(W_{\varepsilon_j,P_{ij}})\phi_{ij},\pi^{\perp}_{\varepsilon_j,\mathbf{P}_j}\overline{S}'_{\varepsilon_j}(W_{\varepsilon_j,\mathbf{P}_j})\phi_{0j}\rangle_{0,\varepsilon_j}\\
&+O(e^{-\frac{c}{\varepsilon_j}}). \qquad (3.7)
\end{aligned}
$$

The third term in (3.7) is also in the order of $O(e^{-\frac{c}{\varepsilon_j}})$, and the fourth term converges to 0 because of (3.2B). Therefore (3.2A) holds.

Step 2: $\varphi_\infty \equiv 0$ if $P_i = 0$, for some $i = 1, 2, \cdots, k$.

Since $P_i = 0$, then there exists a ball $B(\delta_2)$ such that $supp(\phi_{lj}) \cap B(\delta_2) = \emptyset$ if $l \neq i$. Thus for $\varepsilon > 0$, $supp(\varphi_{lj}) \subset \{y : |y| > \varepsilon^{-1}\delta_2\}$, which implies $\varphi_{l\infty} \equiv 0$ for all $l \neq i$. So we only need to show that $\varphi_{i\infty} \equiv 0$.

Without loss of generality, we assume that $P_{ij} \in M_i$ for all $1 \leq i \leq k$ and all $j \in \mathbf{N}$. Since $supp(\phi_{ij}) \subset N_i \subset \Omega_{P_i}$, we can define $\psi_{ij} : \mathbf{R}^n_+ \to \mathbf{R}$

$$\psi_{ij}(y) = \begin{cases} \varphi_{ij}(\varepsilon^{-1}\Phi_{P_i}(\varepsilon y)) & y \in B^+(\varepsilon^{-1}\delta_2), \\ 0 & \text{otherwise.} \end{cases}$$

Since Φ and Φ^{-1} have bounded derivatives, then

$$||\psi_{ij}||_{H^2(\mathbf{R}^n_+)} \leq C$$

for all $j \in \mathbf{N}$. Therefore, there exists a subsequence (which we still denote by ψ_{ij}) which converges weakly to a limit $\psi_{i\infty}$ in $H^2(\mathbf{R}^n_+)$ as $j \to \infty$. From the proof of the extension theorem and the fact that φ_{ij} satisfies homogeneous Neumann boundary condition, we can assume that near $\partial\Omega_{\varepsilon_j}$, φ_{ij} are extended evenly. So to prove $\varphi_{i\infty} \equiv 0$, it suffices to prove $\psi_{i\infty} \equiv 0$.

Claim 3: $\psi_{i\infty} \in K_0^\perp$.

Let $P \in \partial\Omega$. We denote by $T(P)$ the tangent space of $\partial\Omega$ at P. Let $\tau_l(P_{ij}) \in T(P_{ij})$ and $\tau_l(P_i) \in T(P_i)$ satisfy $\tau_l(P_{ij}) \to \tau_l(P_i)$ as $j \to \infty$. We have

$$\begin{aligned}
&\int_\Omega \phi_{ij}(x) \frac{\partial W_{\varepsilon_j, P_{ij}}}{\partial \tau_l(P_{ij})} dx \\
=&\int_\Omega \phi_j \frac{\partial W_{\varepsilon_j, P_{ij}}}{\partial \tau_l(P_{ij})} dx - \int_\Omega (1 - \alpha_i)\phi_j \frac{\partial W_{\varepsilon_j, P_{ij}}}{\partial \tau_l(P_{ij})} dx \\
=&0 - O(e^{-\frac{c}{\varepsilon_j}}) = O(e^{-\frac{c}{\varepsilon_j}}).
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
\int_\Omega \phi_{ij}(x) \frac{\partial W_{\varepsilon_j, P_{ij}}}{\partial \tau_l(P_{ij})} dx =& \int_{\Omega_{P_i}} \phi_{ij}(x) \frac{\partial W_{\varepsilon_j, P_{ij}}}{\partial \tau_l(P_{ij})} \left(\frac{x - P_{ij}}{\varepsilon_j} \right) dx \\
=&\varepsilon_j^{n-1} \int_{\mathbf{R}^n_+} \psi_{ij}(y) \frac{\partial W_{\varepsilon_j, P_{ij}}}{\partial \tau_l(P_{ij})} \left(\frac{\Phi_{\varepsilon_j}(y) - P_{ij}}{\varepsilon_j} \right) |D\Phi(\varepsilon_j y)| dy \\
=&\varepsilon_j^{n-1} \int_{\mathbf{R}^n_+} \psi_{ij}(y) \frac{\partial W_{\varepsilon_j, P_{ij}}}{\partial \tau_l(P_{ij})} \left(\frac{\Phi_{\varepsilon_j}(y) - P_{ij}}{\varepsilon_j} \right) dy.
\end{aligned} \tag{3.9}$$

On the other hand,

$$\begin{aligned}
&\frac{\partial W_{\varepsilon_j, P_{ij}}}{\partial \tau_l(P_{ij})} \left(\frac{\Phi_{\varepsilon_j}(y) - P_{ij}}{\varepsilon_j} \right) \\
\to&\frac{\partial W_{\varepsilon_j, P_{ij}}}{\partial \tau_l(P_{ij})}(y) \to \frac{\partial w}{\partial \tau_l}(y) = \frac{\partial w}{\partial y_l}(y). \quad \text{as } j \to \infty
\end{aligned} \tag{3.10}$$

Therefore, by (3.9) and (3.10), for all $y_l \in \mathbf{R}^{n-1} \times \{0\}$,

$$\int_{\mathbf{R}^n_+} \psi_{i\infty} \frac{\partial w}{\partial y_l}(y)dy = 0.$$

Hence we have proved Claim 3.

Claim 4: $\psi_{i\infty} \in K_0$. (Hence $\psi_{i\infty} = 0$.)

To prove Claim 4, we first introduce the operators $\tilde{S}'_\varepsilon(u)$, $\tilde{\pi}^\perp_{\varepsilon,P}$ induced from $\overline{S}'_\varepsilon(u)$, $\pi^\perp_{\varepsilon,P}$ by the diffeomorphism Ψ_ε. We define $\tilde{S}'_\varepsilon(u) : H^2_N(\mathbf{R}^n_+) \to L^2(\mathbf{R}^n_+)$ by

$$\tilde{S}'_\varepsilon(u)v(y) = \sum_{j,k=1}^{n} a_{jk}(y)\frac{\partial^2 v}{\partial y_j \partial y_k}(y) + \sum_{j=1}^{n} b_j(y)\frac{\partial v}{\partial y_j}(y) - \mu v(y) + h'(u)v(y).$$

where a_{jk} and b_j are given by (2.1) and (2.2). For $P \in \partial\Omega$, we define $\tilde{K}_{\varepsilon,P}$ to be the a subspace of $H^2(\mathbf{R}^n_+)$ spanned by $\{\chi(\Phi_\varepsilon(y))\frac{\partial W_{\varepsilon,P}}{\partial \tau_l(P)}(\frac{\Phi_\varepsilon(y)-P}{\varepsilon})\}_{l=1}^{n-1}$, $\tilde{K}^\perp_{\varepsilon,P}$ the L^2-orthogonal complement of $\tilde{K}_{\varepsilon,P}$ in $L^2(\mathbf{R}^n_+)$ and $\tilde{\pi}^\perp_{\varepsilon,P} : L^2(\mathbf{R}^n_+) \to \tilde{K}^\perp_{\varepsilon,P}$ the L^2-orthogonal projection. Finally, we define

$$\tilde{L}_{\varepsilon,P} = \tilde{\pi}^\perp_{\varepsilon,P}\tilde{S}'_\varepsilon\left(\chi(\Phi_\varepsilon(y))W_{\varepsilon,P}\left(\frac{\Phi_\varepsilon(y)-P}{\varepsilon}\right)\right).$$

Since Φ and Φ^{-1} have bounded derivatives,

$$C_2||L_{\varepsilon_j,P_{ij}}\phi_{ij}||_{0,\varepsilon_j} \le ||\tilde{L}_{\varepsilon_j,P_{ij}}\psi_{ij}||_{L^2(\mathbf{R}^n_+)} \le C_1||L_{\varepsilon_j,P_{ij}}\phi_{ij}||_{0,\varepsilon_j}$$

for some constant $C_1 > C_2 > 0$. In particular, by (3.2),

$$||\tilde{L}_{\varepsilon_j,P_{ij}}\psi_{ij}||_{L^2(\mathbf{R}^n_+)} \to 0, \qquad \text{as } j \to \infty. \tag{3.11}$$

But when $j \to \infty$, $\tilde{\pi}^\perp_{\varepsilon_j,P_{ij}} \to \pi^\perp_0$ by (3.10), and in the space $L(H^2(\mathbf{R}^n_+), L^2(\mathbf{R}^n_+))$,

$$\tilde{S}'_{\varepsilon_j}\left(\chi(\Phi_{\varepsilon_j,P_{ij}}(y))W_{\varepsilon_j,P_{ij}}\left(\frac{\Phi_{\varepsilon_j,P_{ij}}(y)-P}{\varepsilon_j}\right)\right) \to \Delta - \mu + h'(w) = L_0.$$

(For detail of this convergence, see, for example, [NT1].) Therefore by (3.11),

$$\pi^\perp_0 L_0(\psi_{i\infty}) = 0.$$

Therefore $L_0(\psi_{i\infty}) \in K_0$. But on the other hand, for any $\xi \in K_0$, we have

$$\langle L_0(\psi_{i\infty}), \xi\rangle = \langle L_0(\xi), \psi_{i\infty}\rangle = 0.$$

Hence $L_0(\psi_{i\infty}) = 0$, which implies $\psi_{i\infty} \in K_0$ by Lemma 3.1. That in turn implies $\varphi_{i\infty} \equiv 0$. This completes Step 2.

Step 3: $\varphi_\infty \equiv 0$ if $0 \ne P_i$ for all $i = 1, 2, \cdots, k$.

Similar to the first paragraph of the proof of Step 2, $\varphi_{i\infty} \equiv 0$ for all $i = 1, 2, \cdots, k$. If $0 \notin \overline{\Omega}$, then $\varphi_{0\infty} \equiv 0$, too. If $0 \in \overline{\Omega}$ and $0 \neq P_i$ for all $i = 1, 2, \cdots, k$, then by (3.2B), we have

$$||L_{\varepsilon_j,\mathbf{P}_j}\phi_{0j}||_{0,\varepsilon_j} = ||\pi^{\perp}_{\varepsilon_j,\mathbf{P}}\overline{S}'_{\varepsilon_j}(W_{\varepsilon_j,\mathbf{P}_j})\phi_{0j}||_{0,\varepsilon_j} \to 0.$$

Also, because $supp(\phi_{0j}) \subset (\overline{\Omega} \setminus \cup_{i=1}^k N_i)$, we have

$$\pi_{\varepsilon_j,\mathbf{P}}\overline{S}'_{\varepsilon_j}(W_{\varepsilon_j,\mathbf{P}_j})\phi_{0j} = O(e^{-\frac{c}{\varepsilon_j}}).$$

Therefore, $\varphi_{0j}(z)$ satisfies

$$\begin{cases} \Delta v - \mu v + h'(W_{\varepsilon,\mathbf{P}})v = f_j & \text{in} \quad \Omega_{\varepsilon_j}, \\ \frac{\partial v}{\partial n} = \varepsilon_j^{-1}\frac{\partial \beta \phi_j}{\partial n} & \text{on} \quad \partial\Omega_{\varepsilon_j}. \end{cases}$$

where $||f_j||_{L^2(\mathbf{R}^n)} \to 0$ when $j \to \infty$. Therefore $\varphi_{0\infty}$ satisfies

$$\begin{cases} \Delta v - \mu v = 0 & \text{in} \quad \mathbf{R}^n, \\ v \in H^2(\mathbf{R}^n), \end{cases}$$

or

$$\begin{cases} \Delta v - \mu v = 0 & \text{in} \quad \mathbf{R}^n_+, \\ \frac{\partial v}{\partial n} = 0 & \text{on} \quad \partial\mathbf{R}^n_+, \\ v \in H^2(\mathbf{R}^n_+). \end{cases}$$

Thus $\varphi_{0\infty} \equiv 0$. Therefore, by Step 2 and Step 3, we have proved $\varphi_\infty \equiv 0$.

Step 4: $||\phi_j||_{2,\varepsilon_j} \to 0$ as $j \to \infty$.

First we claim that $||\pi_{\varepsilon_j,\mathbf{P}_j}\overline{S}'_{\varepsilon_j}(W_{\varepsilon_j,\mathbf{P}_j})\phi_j||_{0,\varepsilon_j} \to 0$. In fact,

$$\begin{aligned} &\langle \overline{S}'_{\varepsilon_j}(W_{\varepsilon_j,\mathbf{P}_j})\phi_j, \frac{\partial W_{\varepsilon_j,P_{ij}}}{\partial \tau_l}\rangle_{0,\varepsilon_j} \\ =&\int_{\Omega_{\varepsilon_j}} [\Delta\varphi_j - \mu\varphi_j + h'(W_{\varepsilon_j,\mathbf{P}_j})\varphi_j]\frac{\partial W_{\varepsilon_j,P_{ij}}}{\partial \tau_l}dy \\ =&\int_{\mathbf{R}^n} \varphi_j\left[\Delta\frac{\partial W_{\varepsilon_j,P_{ij}}}{\partial \tau_l} - \mu\frac{\partial W_{\varepsilon_j,P_{ij}}}{\partial \tau_l} + h'(W_{\varepsilon,\mathbf{P}})\frac{\partial W_{\varepsilon_j,P_{ij}}}{\partial \tau_l}\right]\chi_{\Omega_{\varepsilon_j}}dy \\ \to& 0, \end{aligned}$$

because that φ_j converges weakly to 0 in $H^2(\mathbf{R}^n)$ and the other factor converges strongly in $L^2(\mathbf{R}^n)$ as $j \to \infty$. Since $K_{\varepsilon_j,\mathbf{P}_j}$ is a finite dimensional space generated by $\frac{\partial W_{\varepsilon_j,P_{ij}}}{\partial \tau_l}$, then the claim follows from the convergence of the inner products.

By the claim and that

$$||L_{\varepsilon_j,\mathbf{P}_j}\phi_j||_{0,\varepsilon_j} = ||\pi^{\perp}_{\varepsilon_j,\mathbf{P}_j}\overline{S}'_{\varepsilon_j}(W_{\varepsilon_j,\mathbf{P}_j})\phi_j||_{0,\varepsilon_j} \to 0,$$

we have $||\overline{S}'_{\varepsilon_j}(W_{\varepsilon_j,\mathbf{P}_j})\phi_j||_{0,\varepsilon_j} \to 0$. On the other hand, by the weak convergence of φ_j, we also obtain $||h'(W_{\varepsilon_j,\mathbf{P}_j})\phi_j||_{0,\varepsilon_j} \to 0$. In fact, for any $\delta > 0$, by Lemma 2.1, there exists $R > 0$ such that

$$|h'(W_{\varepsilon_j,\mathbf{P}_j}(x))| < \frac{\delta}{4}, \quad \text{for } x \in \overline{\Omega} \text{ and } \min_{i=1}^k d(x, P_{ij}) > \varepsilon_j R.$$

Thus,

$$\varepsilon_j^{-n}||h'(W_{\varepsilon_j,\mathbf{P}_j})\phi_j||_{0,\varepsilon_j}^2 \leq \frac{\delta}{4}||\phi_j||_{0,\varepsilon_j}^2 + \sum_{i=1}^k \varepsilon_j^{-n}\int_{B(P_{ij},\varepsilon_j R)} |h'(W_{\varepsilon_j,\mathbf{P}_j}(x))|^2\phi_j^2(x)dx,$$

and for any $i = 1, 2, \cdots, k$,

$$\begin{aligned}
&\varepsilon_j^{-n}\int_{B(P_{ij},\varepsilon_j R)} |h'(W_{\varepsilon_j,\mathbf{P}_j}(x))|^2\phi_j^2(x)dx \\
&\leq \int_{B(R)} |h'(W_{\varepsilon_j,\mathbf{P}_j}(P_{ij}+\varepsilon_j y)|^2\phi_j^2(P_{ij}+\varepsilon_j y)dy \\
&\leq C\int_{B(R)} \varphi_j^2(y+\varepsilon_j^{-1}P_{ij})dy \\
&\to 0.
\end{aligned}$$

Note that we can prove that $\varphi_j^2(y+\varepsilon_j^{-1}P_{ij})$ converges weakly to 0 in $H^2(\mathbf{R}^n)$ by the same proof in Steps 2 and 3, and the weak convergence in $H^2(\mathbf{R}^n)$ implies strong convergence in $L^2(B(R))$. Therefore $||(\varepsilon_j^2\Delta - \mu)\phi_j||_{0,\varepsilon_j} \to 0$, and

$$\begin{aligned}
C_1||\phi_j||_{1,\varepsilon_j}^2 \leq& \varepsilon_j^{-n}\left[\varepsilon_j^2\int_\Omega |\nabla\phi_j|^2dx + \mu\int_\Omega \phi_j^2dx\right] \\
=& -\varepsilon_j^{-n}\int_\Omega [(\varepsilon_j^2\Delta - \mu)\phi_j]\phi_j dx \\
\leq& C_2||(\varepsilon_j^2\Delta - \mu)\phi_j||_{0,\varepsilon_j}||\phi_j||_{0,\varepsilon_j} \to 0.
\end{aligned}$$

So we have $||\varepsilon_j^2\Delta\phi_j||_{0,\varepsilon_j} \to 0$ and $||\phi_j||_{1,\varepsilon_j} \to 0$. By the regularity estimate in [WW1] Appendix B, we have

$$||\phi_j||_{2,\varepsilon_j} \leq C(||\varepsilon_j^2\Delta\phi_j||_{0,\varepsilon_j} + ||\phi_j||_{1,\varepsilon_j}),$$

with $C > 0$ independent of $\varepsilon > 0$. Thus $||\phi_j||_{2,\varepsilon_j} \to 0$ as $j \to \infty$, which contradicts the assumption that $||\phi_j||_{2,\varepsilon_j} = 1$ for all j. This completes the proof of Lemma 3.2.

□

Proof of Lemma 3.3. Lemma 3.2 implies that $Ker(L_{\varepsilon,\mathbf{P}}) = \emptyset$, so it suffices to show that $L_{\varepsilon,\mathbf{P}}$ is a Fredholm operator with index 0. In fact, $L_{\varepsilon,\mathbf{P}} : D(L_{\varepsilon,\mathbf{P}}) \to K_{\varepsilon,\mathbf{P}}^\perp$ is a densely defined closed operator with $D(L_{\varepsilon,\mathbf{P}}) = H_N^2(\Omega) \cap K_{\varepsilon,\mathbf{P}}^\perp \subset K_{\varepsilon,\mathbf{P}}^\perp$. Then $Range(L_{\varepsilon,\mathbf{P}})$ is closed. Moreover

$$L_{\varepsilon,\mathbf{P}} = \overline{S}_\varepsilon'(W_{\varepsilon,\mathbf{P}}) - \pi_{\varepsilon,\mathbf{P}}\overline{S}_\varepsilon'(W_{\varepsilon,\mathbf{P}}).$$

It is standard to show that $\overline{S}_\varepsilon'(W_{\varepsilon,\mathbf{P}})$ is a Fredholm operator with index 0, and $\pi_{\varepsilon,\mathbf{P}}\overline{S}_\varepsilon'(W_{\varepsilon,\mathbf{P}})$ is $\overline{S}_\varepsilon'(W_{\varepsilon,\mathbf{P}})$-compacted in the sense of Kato[K] pg.194, since $K_{\varepsilon,\mathbf{P}}$ is finite dimensional. Therefore, by Chapter 4 Theorem 5.26 in [K] pg.238, $L_{\varepsilon,\mathbf{P}}$

is also Fredholm with same index as $\overline{S}'_\varepsilon(W_{\varepsilon,\mathbf{P}})$. Therefore $codimRange(L_{\varepsilon,\mathbf{P}}) = dimKer(L_{\varepsilon,\mathbf{P}}) = 0$ and $Range(L_{\varepsilon,\mathbf{P}}) = K^{\perp}_{\varepsilon,\mathbf{P}}$.

□

Finally, to use the Contraction Mapping Theorem near 0, we need following estimate.

Lemma 3.4. *$||S_\varepsilon(W_{\varepsilon,\mathbf{P}})||_{0,\varepsilon} = O(\varepsilon)$ for $\varepsilon > 0$ small.*

Proof.

$$\begin{aligned}S_\varepsilon(W_{\varepsilon,\mathbf{P}})(x) &= S_\varepsilon(\sum_{i=1}^{k} W_{\varepsilon,P_i})(x)\\ &= \sum_{i=1}^{k} S_\varepsilon(W_{\varepsilon,P_i})(x) + \eta_2(x) - \frac{1}{|\Omega|}\int_\Omega \eta_2(x)dx,\end{aligned}$$

where $\eta_2(x) = h(W_{\varepsilon,\mathbf{P}}) - \sum_{i=1}^{k} h(W_{\varepsilon,P_i})$. Similar to in the proof of Claim 1 in Lemma 3.2, we have

$$\begin{aligned}|\eta_2(x)| &\le \left|h(\sum_{i=1}^{k} W_{\varepsilon,P_i}) - h(W_{\varepsilon,P_i})\right| + \sum_{j\neq i}\left|h(W_{\varepsilon,P_j})\right|\\ &\le C\max|h'(u)|\cdot\sum_{j\neq i}\left|W_{\varepsilon,P_j}\right|\\ &\le Ce^{-\frac{c}{\varepsilon}}.\ \text{(by Lemma 2.1)}\end{aligned}$$

Therefore

$$\begin{aligned}||S_\varepsilon(W_{\varepsilon,\mathbf{P}})||_{0,\varepsilon} \le &\sum_{i=1}^{k} ||S_\varepsilon(W_{\varepsilon,P_i})||_{0,\varepsilon} + O(e^{-\frac{c}{\varepsilon}})\\ \le &k\max_i ||S_\varepsilon(W_{\varepsilon,P_i})||_{0,\varepsilon} + O(e^{-\frac{c}{\varepsilon}}).\end{aligned}$$

$$\begin{aligned}S_\varepsilon(W_{\varepsilon,P_i}) &= \varepsilon^2\Delta W_{\varepsilon,P_i} - \mu W_{\varepsilon,P_i} + h(W_{\varepsilon,P_i}) - \frac{1}{|\Omega|}\int_\Omega h(W_{\varepsilon,P_i})dx\\ &= h(W_{\varepsilon,P_i}) - h(w_{\varepsilon,P_i}) - \frac{1}{|\Omega|}\int_\Omega h(W_{\varepsilon,P_i})dx.\end{aligned}$$

Since $\frac{1}{|\Omega|}\int_\Omega h(W_{\varepsilon,P_i})dx = O(\varepsilon^n)$, then $||\frac{1}{|\Omega|}\int_\Omega h(W_{\varepsilon,P_i})dx||_{0,\varepsilon} = O(\varepsilon^{\frac{n}{2}}) = O(\varepsilon)$, since $n \ge 2$. On the other hand, using Proposition 2.2,

$$\begin{aligned}&||h(W_{\varepsilon,P_i}) - h(w_{\varepsilon,P_i})||_{0,\varepsilon}\\ \le&||h'||_\infty \cdot ||h_{\varepsilon,P_i}||_{0,\varepsilon}\\ \le&C||\varepsilon v_1(\Psi_{\varepsilon,P_i}(x))\alpha_i(x) + \varepsilon^2 v_2(\Psi_{\varepsilon,P_i}(x))\alpha_i(x) + \varepsilon^3 e_1(x)||_{0,\varepsilon}\\ \le&C[\varepsilon||v_1||_{L^2(\mathbf{R}^n_+)} + \varepsilon^2||v_2||_{L^2(\mathbf{R}^n_+)} + \varepsilon^3||e_1||_{0,\varepsilon}]\\ =&O(\varepsilon).\end{aligned}$$

Therefore $||S_\varepsilon(W_{\varepsilon,P_i})||_{0,\varepsilon} = O(\varepsilon)$.

□

Now we are ready to reduce the original problem to a finite dimensional problem.

Proposition 3.5. *There exists an $\varepsilon_3 > 0$ such that for any $\varepsilon \in (0, \varepsilon_3)$ and $\mathbf{P} \in \Gamma_\delta^k$, there exists a unique $\psi_{\varepsilon,\mathbf{P}} \in K_{\varepsilon,\mathbf{P}}^\perp \cap H_N^2(\Omega)$ satisfying $S_\varepsilon(W_{\varepsilon,\mathbf{P}} + \psi_{\varepsilon,\mathbf{P}}) \in K_{\varepsilon,\mathbf{P}}$ and $||\psi_{\varepsilon,\mathbf{P}}||_{2,\varepsilon} \leq C\varepsilon$.*

This is Lemma 3.3 in [WW1]. For the sake of completeness, we reproduce a proof here.

Proof. For any $\psi \in K_{\varepsilon,\mathbf{P}}^\perp \cap H_N^2(\Omega)$, we have

$$\pi_{\varepsilon,\mathbf{P}}^\perp S_\varepsilon(W_{\varepsilon,\mathbf{P}} + \psi) = \pi_{\varepsilon,\mathbf{P}}^\perp S_\varepsilon(W_{\varepsilon,\mathbf{P}}) + \pi_{\varepsilon,\mathbf{P}}^\perp \overline{S}_\varepsilon'(W_{\varepsilon,\mathbf{P}})\psi + \pi_{\varepsilon,\mathbf{P}}^\perp N_{\varepsilon,\mathbf{P}}(\psi),$$

where

$$N_{\varepsilon,\mathbf{P}}(\psi) = h(W_{\varepsilon,\mathbf{P}}+\psi) - h(W_{\varepsilon,\mathbf{P}}) - h'(W_{\varepsilon,\mathbf{P}})\psi + \frac{1}{|\Omega|}\int_\Omega [h(W_{\varepsilon,\mathbf{P}}+\psi) - h(W_{\varepsilon,\mathbf{P}})]dx.$$

So we shall look for a $\psi \in K_{\varepsilon,\mathbf{P}}^\perp \cap H_N^2(\Omega)$ satisfying

$$\psi = -L_{\varepsilon,\mathbf{P}}^{-1}\left[\pi_{\varepsilon,\mathbf{P}}^\perp S_\varepsilon(W_{\varepsilon,\mathbf{P}}) + \pi_{\varepsilon,\mathbf{P}}^\perp N_{\varepsilon,\mathbf{P}}(\psi)\right] \equiv T_{\varepsilon,\mathbf{P}}(\psi).$$

Note that $L_{\varepsilon,\mathbf{P}}$ is invertible from Lemmas 3.2 and 3.3, and $L_{\varepsilon,\mathbf{P}}^{-1} : K_{\varepsilon,\mathbf{P}}^\perp \to K_{\varepsilon,\mathbf{P}}^\perp \cap H_N^2(\Omega)$ is bounded with $||L_{\varepsilon,\mathbf{P}}^{-1}|| \leq \lambda^{-1}$. For $\delta > 0$, define

$$X_{\varepsilon,\delta} = \left\{\psi \in K_{\varepsilon,\mathbf{P}}^\perp \cap H_N^2(\Omega) : ||\psi||_{2,\varepsilon} \leq \delta\right\},$$

then $T_{\varepsilon,\mathbf{P}} : X_{\varepsilon,\delta} \to K_{\varepsilon,\mathbf{P}}^\perp \cap H_N^2(\Omega)$ is well defined. Since $h \in C^3(\mathbf{R})$ and has bounded derivatives, considered as an operator in $L^2(\Omega)$, h has a Frechét derivative and for any $\psi \in X_{\varepsilon,\delta}$,

$$||f_{\varepsilon,\mathbf{P}}(\psi)||_{0,\varepsilon} \equiv ||h(W_{\varepsilon,\mathbf{P}} + \psi) - h(W_{\varepsilon,\mathbf{P}}) - h'(W_{\varepsilon,\mathbf{P}})\psi||_{0,\varepsilon} \leq C(\delta)||\psi||_{0,\varepsilon},$$

with $C(\delta) \to 0$ as $\delta \to 0$. Therefore, since

$$N_{\varepsilon,\mathbf{P}}(\psi) = f_{\varepsilon,\mathbf{P}}(\psi) + \frac{1}{|\Omega|}\int_\Omega f_{\varepsilon,\mathbf{P}}(\psi)dx + \frac{1}{|\Omega|}\int_\Omega h'(W_{\varepsilon,\mathbf{P}})\psi dx,$$

$$\begin{aligned}
||N_{\varepsilon,\mathbf{P}}(\psi)||_{0,\varepsilon} &\leq ||f_{\varepsilon,\mathbf{P}}(\psi)||_{0,\varepsilon} + C\varepsilon^{-\frac{n}{2}}\int_\Omega |f_{\varepsilon,\mathbf{P}}(\psi)|dx + C\varepsilon^{-\frac{n}{2}}\int_\Omega |h'(W_{\varepsilon,\mathbf{P}})\psi|dx \\
&\leq C(\delta)||\psi||_{0,\varepsilon} + C||h'(W_{\varepsilon,\mathbf{P}})||_0||\psi||_{0,\varepsilon} \\
&\leq C(\delta)\delta + C||h'(W_{\varepsilon,\mathbf{P}})||_0\delta = [C(\delta) + C\varepsilon^{\frac{n}{2}}]\delta.
\end{aligned}$$

In the above, the definition of $C(\delta)$ has changed by a constant multiple. From the above, for any $\psi \in X_{\varepsilon,\delta}$, by Lemmas 3.2 and 3.4,

$$\begin{aligned}||T_{\varepsilon,\mathbf{P}}(\psi)||_{2,\varepsilon} \leq &\lambda^{-1}\left(||\pi^{\perp}_{\varepsilon,\mathbf{P}} S_{\varepsilon}(W_{\varepsilon,\mathbf{P}})||_{0,\varepsilon} + ||\pi^{\perp}_{\varepsilon,\mathbf{P}} N_{\varepsilon,\mathbf{P}}(\psi)||_{0,\varepsilon}\right)\\ \leq &\lambda^{-1}[C\varepsilon + C(\delta)\delta + C\varepsilon^{\frac{n}{2}}\delta)].\end{aligned}$$

So, with fixed $\delta > 0$ small enough, for ε sufficiently small, $T_{\varepsilon,\mathbf{P}}(X_{\varepsilon,\delta}) \subset X_{\varepsilon,\delta}$. On the other hand, for any $\psi_1, \psi_2 \in X_{\varepsilon,\delta}$, we have

$$\begin{aligned}||g_{\varepsilon,\mathbf{P}}(\psi_1,\psi_2)||_{0,\varepsilon} &\equiv ||h(W_{\varepsilon,\mathbf{P}}+\psi_1) - h(W_{\varepsilon,\mathbf{P}}+\psi_2) - h'(W_{\varepsilon,\mathbf{P}})(\psi_1-\psi_2)||_{0,\varepsilon}\\ &\leq C(\delta)||\psi_1-\psi_2||_{0,\varepsilon},\end{aligned}$$

where $C(\delta) \to 0$ as $\delta \to 0$. Hence, by Lemma 3.2,

$$\begin{aligned}&||T_{\varepsilon,\mathbf{P}}(\psi_1) - T_{\varepsilon,\mathbf{P}}(\psi_2)||_{2,\varepsilon} \leq \lambda^{-1}||N_{\varepsilon,\mathbf{P}}(\psi_1) - N_{\varepsilon,\mathbf{P}}(\psi_2)||_{0,\varepsilon}\\ \leq &C||g_{\varepsilon,\mathbf{P}}(\psi_1,\psi_2)||_{0,\varepsilon} + C\varepsilon^{-\frac{n}{2}}\int_{\Omega}|g_{\varepsilon,\mathbf{P}}(\psi_1,\psi_2)|dx + C\varepsilon^{-\frac{n}{2}}\int_{\Omega}|h'(W_{\varepsilon,\mathbf{P}})(\psi_1-\psi_2)|dx\\ \leq &C(\delta)||\psi_1-\psi_2||_{0,\varepsilon} + C\varepsilon^{\frac{n}{2}}||\psi_1-\psi_2||_{0,\varepsilon}.\end{aligned}$$

Choose $\delta > 0$ such that $C(\delta) < \frac{1}{2}$ and ε sufficiently small so that $T_{\varepsilon,\mathbf{P}}$ is a contraction on $X_{\varepsilon,\delta}$. By the Contraction Mapping Principle, there exists a unique fixed point $\psi_{\varepsilon,\mathbf{P}} \in K^{\perp}_{\varepsilon,\mathbf{P}}$ of $T_{\varepsilon,\mathbf{P}}$. Moreover,

$$\begin{aligned}||\psi_{\varepsilon,\mathbf{P}}||_{2,\varepsilon} \leq &\lambda^{-1}\left(||S_{\varepsilon}(W_{\varepsilon,\mathbf{P}})||_{0,\varepsilon} + ||N_{\varepsilon,\mathbf{P}}(\psi_{\varepsilon,\mathbf{P}})||_{0,\varepsilon}\right)\\ \leq &C\varepsilon + [C(\delta) + C\varepsilon^{\frac{n}{2}}]||\psi_{\varepsilon,\mathbf{P}}||_{2,\varepsilon}.\end{aligned}$$

So $||\psi_{\varepsilon,\mathbf{P}}||_{2,\varepsilon} \leq C\varepsilon$.

□

For future purposes, we need a further asymptotic expansion of $\psi_{\varepsilon,\mathbf{P}}$ in term of ε, which is similar to the expansion of $h_{\varepsilon,P}$ in Section 2.

Lemma 3.6.

$$\psi_{\varepsilon,\mathbf{P}}(x) = \varepsilon\sum_{i=1}^{k}\alpha_i(x)\psi_0(\Psi_{\varepsilon,P_i}(x)) + \varepsilon^2 e_3(x),$$

where ψ_0 is the unique solution of

$$\begin{cases}\Delta v - \mu v + h'(w(y))v - h'(w(y))v_1 = 0 & \text{in } \mathbf{R}^n_+,\\ \frac{\partial v}{\partial n} = 0 & \text{on } \partial\mathbf{R}^n_+,\\ v \in K_0^{\perp}.\end{cases} \tag{3.12}$$

(Recall that v_1 is the solution to (2.8)), and $||e_3||_{2,\varepsilon} \leq C$ for some C not depending on ε.

The proof of this lemma is given in Appendix A.

4. Finite dimensional problem.

Let $u_\varepsilon(x) = W_{\varepsilon,\mathbf{P}}(x) + \psi_{\varepsilon,\mathbf{P}}(x)$, where $\psi_{\varepsilon,\mathbf{P}}$ is obtained in Proposition 3.5 for any $\mathbf{P}\in \Gamma_\delta^k$. In this section, we are going to find $\mathbf{P}\in \Gamma_\delta^k$ such that

$$\pi_{\varepsilon,\mathbf{P}} S_\varepsilon(W_{\varepsilon,\mathbf{P}} + \psi_{\varepsilon,\mathbf{P}}) = 0.$$

It suffices to find $\mathbf{P}\in \Gamma_\delta^k$ such that

$$\left\langle S_\varepsilon(W_{\varepsilon,\mathbf{P}} + \psi_{\varepsilon,\mathbf{P}}), \frac{\partial W_{\varepsilon,P_i}}{\partial \tau_j} \right\rangle = 0$$

for all P_i and τ_j, $1 \le i \le k$, $1 \le j \le n-1$. In the following proof, we shall call a function $f : \mathbf{R}_+^n \to \mathbf{R}$ an *even* function, if $f(-y', y_n) = f(y', y_n)$ for any $y = (y', y_n) \in \mathbf{R}_+^n$; and f an *odd* function, if $f(-y', y_n) = -f(y', y_n)$. Also we will frequently use the estimates (2.13), (2.14), (3.12) and

$$\begin{aligned} \frac{\partial W_{\varepsilon,P_i}}{\partial \tau_j} =& \frac{1}{\varepsilon}\frac{\partial w(y)}{\partial y_j} - \alpha_i(x)\Bigg[\frac{|y|w''(|y|) - w'(|y|)}{2|y|^3}\sum_{l,k=1}^{n-1} \rho_{lk}(0) y_l y_k y_j y_n \\ &+ \sum_{k=1}^{n-1} \rho_{kj}(0) y_k \frac{\partial w(y)}{\partial y_n} + u_0(y)\Bigg] + \varepsilon e_4(x), \end{aligned} \tag{4.1}$$

$$\begin{aligned} h^{(k)}(w_{\varepsilon,P_i}) =& h^{(k)}(w(y)) - \varepsilon h^{(k+1)}(w(y)) \frac{w'(|y|)}{|y|} \sum_{l,p=1}^{n-1} \rho_{lp}(0) y_l y_p y_n \\ &+ O(\varepsilon^2 e^{-C|y|}), \end{aligned} \tag{4.2}$$

where in (4.1), $x \in \Omega$, $y = \Psi_{\varepsilon,P_i}(x)$ and $||e_4||_{0,\varepsilon} \le C$; in (4.2), $x \in \Omega_{P_i}$, $y = \Psi_{\varepsilon,P_i}(x)$ and $h^{(k)}(u)$ is the k-th derivative of $h(u)$, with $k = 0, 1, 2$ and $h^{(0)} = h$. We prove these two estimates at the end of Appendix A.

Proposition 4.1. *Let $u_\varepsilon(x) = W_{\varepsilon,\mathbf{P}}(x) + \psi_{\varepsilon,\mathbf{P}}(x)$. Then for fixed small $\delta > 0$, $\mathbf{P}= (P_1, P_2, \cdots, P_k) \in \Gamma_\delta^k$, and tangent vector $\tau_j(P_i)$, $1 \le i \le k$, $1 \le j \le n-1$, we have*

$$\left\langle S_\varepsilon(W_{\varepsilon,\mathbf{P}} + \psi_{\varepsilon,\mathbf{P}}), \frac{\partial W_{\varepsilon,P_i}}{\partial \tau_j} \right\rangle = \varepsilon^{n+1}\gamma D_j H(P_i) + O(\varepsilon^{n+2}),$$

where

$$\gamma = \frac{1}{3}\int_{\partial \mathbf{R}_+^n} \left[\frac{w'(|y|)}{|y|}\right]^2 y_j^4 dy \ne 0.$$

Proof. In this proof, we always assume that $y = \Psi_{\varepsilon,P_i}(x)$, for $x \in \Omega_{P_i}$, and *h.o.t.* stands for higher order terms.

$$\begin{aligned} &\left\langle S_\varepsilon(W_{\varepsilon,\mathbf{P}} + \psi_{\varepsilon,\mathbf{P}}), \frac{\partial W_{\varepsilon,P_i}}{\partial \tau_j} \right\rangle \\ =& \int_\Omega \left[\varepsilon^2 \Delta u_\varepsilon - \mu u_\varepsilon + h(u_\varepsilon) - \frac{1}{|\Omega|}\int_\Omega h(u_\varepsilon)dx\right] \frac{\partial W_{\varepsilon,P_i}}{\partial \tau_j} dx. \end{aligned}$$

To estimate the last term, we note, by (A.17) and (4.1),

$$
\begin{aligned}
&\int_\Omega \left(\int_\Omega h(u_\varepsilon)dx\right) \frac{\partial W_{\varepsilon,P_i}}{\partial \tau_j} dx = C\varepsilon^n \int_\Omega \frac{\partial W_{\varepsilon,P_i}}{\partial \tau_j} dx \\
=& C\varepsilon^n \int_\Omega \Bigg\{ \frac{1}{\varepsilon}\frac{\partial w(y)}{\partial y_j} - \alpha_i(x)\Bigg[\frac{|y|w''(|y|) - w'(|y|)}{2|y|^3} \sum_{l,k=1}^{n-1} \rho_{lk}(0) y_l y_k y_j y_n \\
&+ \sum_{k=1}^{n-1} \rho_{kj}(0) y_k \frac{\partial w(y)}{\partial y_n} + u_0(y)\Bigg] + \varepsilon e_4(x)\Bigg\} dx \\
=& C\varepsilon^{2n-1} \int_{\mathbf{R}^n_+} \frac{\partial w(y)}{\partial y_j} dy + O(\varepsilon^{2n}) \\
=& O(\varepsilon^{2n}) \qquad \left(\text{since } \frac{\partial w(y)}{\partial y_j} \text{ is odd}\right) \\
=& O(\varepsilon^{n+2}). \qquad (\text{since } n \geq 2)
\end{aligned}
$$

The other part may be written

$$
\begin{aligned}
&\int_\Omega [\varepsilon^2 \Delta u_\varepsilon - \mu u_\varepsilon + h(u_\varepsilon)] \frac{\partial W_{\varepsilon,P_i}}{\partial \tau_j} dx \\
=& \int_\Omega \Bigg[u_\varepsilon \left(\varepsilon^2 \Delta \frac{\partial W_{\varepsilon,P_i}}{\partial \tau_j} - \mu \frac{\partial W_{\varepsilon,P_i}}{\partial \tau_j} + h'(w_{\varepsilon,P_i}) \frac{\partial w_{\varepsilon,P_i}}{\partial \tau_j} \right) \\
&+ h(u_\varepsilon) \frac{\partial W_{\varepsilon,P_i}}{\partial \tau_j} - h'(w_{\varepsilon,P_i}) \frac{\partial w_{\varepsilon,P_i}}{\partial \tau_j} u_\varepsilon \Bigg] dx \\
=& \int_\Omega \left[h(u_\varepsilon) \frac{\partial W_{\varepsilon,P_i}}{\partial \tau_j} - h'(w_{\varepsilon,P_i}) \frac{\partial w_{\varepsilon,P_i}}{\partial \tau_j} u_\varepsilon \right] dx \\
=& \int_\Omega [h(W_{\varepsilon,\mathbf{P}} + \psi_{\varepsilon,\mathbf{P}}) - h(W_{\varepsilon,\mathbf{P}}) - h'(W_{\varepsilon,\mathbf{P}})\psi_{\varepsilon,\mathbf{P}}] \frac{\partial W_{\varepsilon,P_i}}{\partial \tau_j} dx \\
&+ \int_\Omega \left[h'(W_{\varepsilon,\mathbf{P}}) \frac{\partial W_{\varepsilon,P_i}}{\partial \tau_j} - h'(w_{\varepsilon,P_i}) \frac{\partial w_{\varepsilon,P_i}}{\partial \tau_j} \right] \psi_{\varepsilon,\mathbf{P}} dx \\
&+ \int_\Omega \left[h(w_{\varepsilon,P_i}) \frac{\partial W_{\varepsilon,P_i}}{\partial \tau_j} - h'(w_{\varepsilon,P_i}) \frac{\partial w_{\varepsilon,P_i}}{\partial \tau_j} W_{\varepsilon,\mathbf{P}} \right] dx \\
&+ \int_\Omega [h(W_{\varepsilon,\mathbf{P}}) - h(w_{\varepsilon,P_i})] \frac{\partial W_{\varepsilon,P_i}}{\partial \tau_j} dx \\
\equiv& I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

Note that

$$\begin{aligned}
I_3 =& \int_\Omega \left[h(w_{\varepsilon,P_i}) \frac{\partial W_{\varepsilon,P_i}}{\partial \tau_j} - h'(w_{\varepsilon,P_i}) \frac{\partial w_{\varepsilon,P_i}}{\partial \tau_j} W_{\varepsilon,P_i} \right] dx \\
&- \int_\Omega h'(w_{\varepsilon,P_i}) \frac{\partial w_{\varepsilon,P_i}}{\partial \tau_j} \sum_{l\neq i} W_{\varepsilon,P_l} dx \\
=& \int_\Omega \frac{\partial W_{\varepsilon,P_i}}{\partial \tau_j} [\varepsilon^2 \Delta W_{\varepsilon,P_i} - \mu W_{\varepsilon,P_i} + h(w_{\varepsilon,P_i})] dx \\
&- \int_\Omega W_{\varepsilon,P_i} \left[\varepsilon^2 \Delta \frac{\partial W_{\varepsilon,P_i}}{\partial \tau_j} - \mu \frac{\partial W_{\varepsilon,P_i}}{\partial \tau_j} + h'(w_{\varepsilon,P_i}) \frac{\partial w_{\varepsilon,P_i}}{\partial \tau_j} \right] dx \\
&- \int_\Omega h'(w_{\varepsilon,P_i}) \frac{\partial w_{\varepsilon,P_i}}{\partial \tau_j} \sum_{l\neq i} W_{\varepsilon,P_l} dx \\
=& - \int_\Omega h'(w_{\varepsilon,P_i}) \frac{\partial w_{\varepsilon,P_i}}{\partial \tau_j} \sum_{l\neq i} W_{\varepsilon,P_l} dx.
\end{aligned}$$

So $I_3 = O(e^{-\frac{c}{\varepsilon}})$. Next we decompose I_1 into managable parts:

$$\begin{aligned}
I_1 =& \frac{1}{2} \int_\Omega h''(W_{\varepsilon,\mathbf{P}}) \psi_{\varepsilon,\mathbf{P}}^2 \frac{\partial W_{\varepsilon,P_i}}{\partial \tau_j} dx + \frac{1}{6} \int_\Omega h'''(W_{\varepsilon,\mathbf{P}} + \xi(x)) \psi_{\varepsilon,\mathbf{P}}^3 \frac{\partial W_{\varepsilon,P_i}}{\partial \tau_j} dx \\
=& \frac{1}{2} \int_\Omega h''(w_{\varepsilon,P_i}) \psi_{\varepsilon,\mathbf{P}}^2 \frac{\partial W_{\varepsilon,P_i}}{\partial \tau_j} dx + \frac{1}{2} \int_\Omega [h''(W_{\varepsilon,P_i}) - h''(w_{\varepsilon,P_i})] \psi_{\varepsilon,\mathbf{P}}^2 \frac{\partial W_{\varepsilon,P_i}}{\partial \tau_j} dx \\
&+ \frac{1}{2} \int_\Omega [h''(W_{\varepsilon,\mathbf{P}}) - h''(W_{\varepsilon,P_i})] \psi_{\varepsilon,\mathbf{P}}^2 \frac{\partial W_{\varepsilon,P_i}}{\partial \tau_j} dx \\
&+ \frac{1}{6} \int_\Omega h'''(W_{\varepsilon,\mathbf{P}} + \xi(x)) \psi_{\varepsilon,\mathbf{P}}^3 \frac{\partial W_{\varepsilon,P_i}}{\partial \tau_j} dx \\
\equiv& I_{11} + I_{12} + I_{13} + I_{14},
\end{aligned}$$

where $0 \leq \xi(x) \leq \psi_{\varepsilon,\mathbf{P}}(x)$. By Proposition 2.2, Lemma 3.6 and (4.1),

$$\begin{aligned}
|I_{12}| \leq& C \max |h'''(u)| \int_\Omega \left| \varepsilon^2 \alpha_i(x)^3 v_1(y) \psi_0(y) \frac{\partial w(y)}{\partial y_j} \right| dx + h.o.t. \\
=& O(\varepsilon^{n+2}).
\end{aligned}$$

Similarly,

$$|I_{13}| \leq C \max |h'''(u)| \int_\Omega \sum_{l\neq i} \left| W_{\varepsilon,P_l} \psi_{\varepsilon,\mathbf{P}}^2 \frac{\partial W_{\varepsilon,P_i}}{\partial \tau_j} \right| dx = O(e^{-\frac{c}{\varepsilon}}).$$

Similarly,

$$\begin{aligned}
|I_{14}| \leq& C \max |h'''(u)| \int_\Omega \left| \varepsilon \sum_{l=1}^k \alpha_l(x) \psi_0(y) + \varepsilon^2 e_3(x) \right|^3 \cdot \left| \frac{\partial W_{\varepsilon,P_i}}{\partial \tau_j} \right| dx \\
=& O(\varepsilon^{n+2}).
\end{aligned}$$

By Proposition 2.2, Lemma 3.6 and (4.1),

$$\begin{aligned}|I_{11}| \leq& \frac{1}{2}\left|\int_\Omega \left[h''(w(y)) + h.o.t.\right] \cdot \left[\varepsilon \sum_{l=1}^{k} \alpha_l(x)\psi_0(y) + h.o.t.\right]^2 \cdot \left[\frac{1}{\varepsilon}\frac{\partial w(y)}{\partial y_j} + h.o.t.\right] dx\right| \\ \leq& \frac{1}{2}\varepsilon\left|\int_{\mathbf{R}^n_+} h''(w(y))\psi_0^2(y)\frac{\partial w(y)}{\partial y_j}dx\right| + O(\varepsilon^{n+2}) \\ =& O(\varepsilon^{n+2}).\end{aligned}$$

The last equality is because $h''(w(y))$ and $\psi_0^2(y)$ are even, while $\frac{\partial w}{\partial y_j}(y)$ is odd. So $I_1 = O(\varepsilon^{n+2})$. Next

$$\begin{aligned}I_2 =& \int_\Omega [h'(W_{\varepsilon,\mathbf{P}}) - h'(W_{\varepsilon,P_i})]\frac{\partial W_{\varepsilon,P_i}}{\partial \tau_j}\psi_{\varepsilon,\mathbf{P}}dx \\ &+ \int_\Omega h''(w_{\varepsilon,P_i})(W_{\varepsilon,P_i} - w_{\varepsilon,P_i})\frac{\partial W_{\varepsilon,P_i}}{\partial \tau_j}\psi_{\varepsilon,\mathbf{P}}dx \\ &+ \frac{1}{2}\int_\Omega h'''(w_{\varepsilon,P_i} + \xi(x))(W_{\varepsilon,P_i} - w_{\varepsilon,P_i})^2\frac{\partial W_{\varepsilon,P_i}}{\partial \tau_j}\psi_{\varepsilon,\mathbf{P}}dx \\ &+ \int_\Omega h'(w_{\varepsilon,P_i})\left[\frac{\partial W_{\varepsilon,P_i}}{\partial \tau_j} - \frac{\partial w_{\varepsilon,P_i}}{\partial \tau_j}\right]\psi_{\varepsilon,\mathbf{P}}dx \\ \equiv& I_{21} + I_{22} + I_{23} + I_{24}.\end{aligned}$$

By Lemma 2.1,

$$\begin{aligned}|I_{21}| \leq& \max|h''(u)|\int_\Omega \sum_{l\neq i}|W_{\varepsilon,P_l}| \cdot \left|\frac{\partial W_{\varepsilon,P_i}}{\partial \tau_j}\psi_{\varepsilon,\mathbf{P}}\right|dx \\ =& O(e^{-\frac{c}{\varepsilon}}).\end{aligned}$$

Also,

$$\begin{aligned}|I_{22}| \leq& \varepsilon\left|\int_\Omega \alpha_i^2(x)h''(w(y))\frac{\partial w(y)}{\partial y_j}v_1(y)\psi_0(y)dx\right| + h.o.t. \\ =& \varepsilon^{n+1}\left|\int_{\mathbf{R}^n_+} h''(w(y))\frac{\partial w(y)}{\partial y_j}v_1(y)\psi_0(y)dy\right| + O(\varepsilon^{n+2}) \\ =& O(\varepsilon^{n+2}).\end{aligned}$$

The last equality is true because $h''(w(y))$, $v_1(y)$ and $\psi_0(y)$ are even function, while $\frac{\partial w}{\partial y_j}(y)$ is odd. Similarly,

$$\begin{aligned}|I_{23}| \leq& C\varepsilon^2 \max|h'''(u)|\int_\Omega \left|\alpha_i^2(x)\frac{\partial w(y)}{\partial y_j}v_1^2(y)\psi_0(y)\right|dx + h.o.t. \\ =& O(\varepsilon^{n+2}).\end{aligned}$$

Similarly,

$$
\begin{aligned}
|I_{24}| \leq & \left| \int_\Omega \alpha_i^2(x) h'(w(y)) u_0(y) \psi_0(y) dx \right| + h.o.t. \\
\leq & \varepsilon^{n+1} \left| \int_{\mathbf{R}_+^n} h'(w(y)) u_0(y) \psi_0(y) dy \right| + O(\varepsilon^{n+2}) \\
= & O(\varepsilon^{n+2}).
\end{aligned}
$$

The last equality is true because $h'(w(y))$ and $\psi_0(y)$ are even function, while $u_0(y)$ is odd. Therefore $I_2 = O(\varepsilon^{n+2})$.

Similarly,

$$
\begin{aligned}
I_4 = & \int_\Omega [h(W_{\varepsilon,\mathbf{P}}) - h(W_{\varepsilon,P_i})] \frac{\partial W_{\varepsilon,P_i}}{\partial \tau_j} dx + \int_\Omega [h(W_{\varepsilon,P_i}) - h(w_{\varepsilon,P_i})] \frac{\partial W_{\varepsilon,P_i}}{\partial \tau_j} dx \\
= & \int_\Omega h'(w_{\varepsilon,P_i})(W_{\varepsilon,P_i} - w_{\varepsilon,P_i}) \frac{\partial W_{\varepsilon,P_i}}{\partial \tau_j} dx + \int_\Omega [h(W_{\varepsilon,\mathbf{P}}) - h(W_{\varepsilon,P_i})] \frac{\partial W_{\varepsilon,P_i}}{\partial \tau_j} dx \\
& + \frac{1}{2} \int_\Omega h''(w_{\varepsilon,P_i})(W_{\varepsilon,P_i} - w_{\varepsilon,P_i})^2 \frac{\partial W_{\varepsilon,P_i}}{\partial \tau_j} dx \\
& + \frac{1}{6} \int_\Omega h'''(w_{\varepsilon,P_i} + \xi(x))(W_{\varepsilon,P_i} - w_{\varepsilon,P_i})^3 \frac{\partial W_{\varepsilon,P_i}}{\partial \tau_j} dx \\
\equiv & I_{41} + I_{42} + I_{43} + I_{44}.
\end{aligned}
$$

By Lemma 2.1,

$$
\begin{aligned}
|I_{42}| \leq & \max |h'(u)| \int_\Omega \sum_{l \neq i} \left| W_{\varepsilon,P_l} \cdot \frac{\partial W_{\varepsilon,P_i}}{\partial \tau_j} \right| dx \\
= & O(e^{-\frac{c}{\varepsilon}}).
\end{aligned}
$$

As with I_{22},

$$
\begin{aligned}
|I_{43}| \leq & \frac{1}{2} \varepsilon \left| \int_\Omega \alpha_i(x) h''(w(y)) v_1^2(y) \frac{\partial w(y)}{\partial y_j} dx \right| + h.o.t. \\
\leq & \frac{1}{2} \varepsilon^{n+1} \left| \int_{\mathbf{R}_+^n} h''(w(y)) v_1^2(y) \frac{\partial w(y)}{\partial y_j} dy \right| + O(\varepsilon^{n+2}) \\
= & O(\varepsilon^{n+2}).
\end{aligned}
$$

The last equality is true because $h''(w(y))$ and $v_1^2(y)$ are even function, while $\frac{\partial w}{\partial y_j}$ is odd. Similarly,

$$
|I_{44}| \leq \frac{1}{6} \varepsilon^2 \max |h'''(u)| \int_\Omega \left| \alpha_i^3(x) v_1^3(y) \frac{\partial w(y)}{\partial y_j} \right| dx + h.o.t. = O(\varepsilon^{n+2}).
$$

Finally,

$$
\begin{aligned}
I_{41} =& \int_\Omega \alpha_i(x)\Big[h'(w(y)) - \varepsilon h''(w(y))\frac{w'(|y|)}{|y|}\sum_{l,k=1}^{n-1}\rho_{lk}(0)y_l y_k y_n\Big]\cdot\Big[-\varepsilon v_1(y) - \varepsilon^2 v_2(y)\Big] \\
&\cdot\Big\{\frac{1}{\varepsilon}\frac{\partial w(y)}{\partial y_j} - \alpha_i(x)\Big[\frac{|y|w''(|y|) - w'(|y|)}{2|y|^3}\sum_{l,k=1}^{n-1}\rho_{lk}(0)y_l y_k y_j y_n \\
&+ \sum_{k=1}^{n-1}\rho_{kj}(0)y_k\frac{\partial w(y)}{\partial y_n} + u_0(y)\Big]\Big\}dx + h.o.t. \\
=& -\varepsilon^n\int_{\mathbf{R}^n_+} h'(w(y))v_1(y)\frac{\partial w(y)}{\partial y_j}dy + \varepsilon^{n+1}\int_{\mathbf{R}^n_+} h'(w(y))v_1(y)\sum_{k=1}^{n-1}\rho_{kj}(0)y_k\frac{\partial w(y)}{\partial y_n}dy \\
&+ \varepsilon^{n+1}\int_{\mathbf{R}^n_+} h'(w(y))v_1(y)u_0(y)dy - \varepsilon^{n+1}\int_{\mathbf{R}^n_+} h'(w(y))v_2(y)\frac{\partial w(y)}{\partial y_j}dy \\
&+ \varepsilon^{n+1}\int_{\mathbf{R}^n_+} h'(w(y))v_1(y)\frac{|y|w''(|y|) - w'(|y|)}{2|y|^3}\sum_{l,k=1}^{n-1}\rho_{lk}(0)y_l y_k y_j y_n dy \\
&+ \varepsilon^{n+1}\int_{\mathbf{R}^n_+} h''(w(y))\frac{w'(|y|)}{|y|}\sum_{l,k=1}^{n-1}\rho_{lk}(0)y_l y_k y_n v_1(y)\frac{\partial w(y)}{\partial y_j}dy + O(\varepsilon^{n+2}) \\
=& -\varepsilon^{n+1}\int_{\mathbf{R}^n_+} h'(w(y))v_2(y)\frac{\partial w(y)}{\partial y_j}dy + O(\varepsilon^{n+2}),
\end{aligned}
$$

because $h'(w(y))$, $h''(w(y))$ and $v_1(y)$ are even, while $\frac{\partial w}{\partial y_j}$ and $u_0(y)$ are odd.

Now we further decompose v_2 into the odd part v_{21} and the even part v_{22}. By (2.9), v_{21} satisfies

$$
\begin{cases} \Delta v - \mu v = 0 & \text{in } \mathbf{R}^n_+, \\ \frac{\partial v}{\partial y_n} = -\frac{w'(|y|)}{3|y|}\sum_{i,j,k=1}^{n-1}\rho_{ijk}(0)y_i y_j y_k & \text{on } \partial\mathbf{R}^n_+. \end{cases} \tag{4.3}
$$

and v_{22} satisfies

$$
\begin{cases} \Delta v - \mu v - 2\sum_{i,j=1}^{n-1}\rho_{ij}(0)y_i\frac{\partial^2 v_1}{\partial y_j\partial y_n} - (\Delta\rho(0))\frac{\partial v_1}{\partial y_n} = 0 & \text{in } \mathbf{R}^n_+, \\ \frac{\partial v}{\partial y_n} = \sum_{i,j=1}^{n-1}\rho_{ij}(0)y_j\frac{\partial v_1}{\partial y_i} & \text{on } \partial\mathbf{R}^n_+. \end{cases}
$$

$$
\begin{aligned}
&\int_{\mathbf{R}^n_+} h'(w)v_{21}\frac{\partial w}{\partial y_j}dy = -\int_{\mathbf{R}^n_+}\left(\Delta\frac{\partial w}{\partial y_j}-\mu\frac{\partial w}{\partial y_j}\right)v_{21}dy\\
=&\int_{\partial\mathbf{R}^n_+}\left(\frac{\partial v_{21}}{\partial y_n}\frac{\partial w}{\partial y_j}-v_{21}\frac{\partial}{\partial y_n}\frac{\partial w}{\partial y_j}\right)dy+\int_{\mathbf{R}^n_+}(\Delta v_{21}-\mu v_{21})\frac{\partial w}{\partial y_j}dy\\
=&\int_{\partial\mathbf{R}^n_+}\frac{\partial v_{21}}{\partial y_n}\frac{\partial w}{\partial y_j}dy=-\frac{1}{3}\int_{\mathbf{R}^n_+}\left[\frac{w'(|y|)}{|y|}\right]^2\sum_{l,p,k=1}^{n-1}\rho_{lpk}(0)y_ly_py_jy_kdy\\
=&-\frac{1}{3}\rho_{jjj}(0)\int_{\mathbf{R}^n_+}\left[\frac{w'(|y|)}{|y|}\right]^2y_j^4dy-\sum_{l\neq j}\rho_{llj}(0)\int_{\mathbf{R}^n_+}\left[\frac{w'(|y|)}{|y|}\right]^2y_j^2y_l^2dy \qquad (4.4)\\
=&-\gamma\sum_{l=1}^{n-1}\rho_{llj}(0) \qquad (4.5)\\
=&-\gamma D_jH(P_i). \qquad (4.6)
\end{aligned}
$$

(4.4) is true because $\rho\in C^4$, $\rho_{jll}(0)=\rho_{ljl}(0)=\rho_{llj}(0)$, thus the term containing $\rho_{jll}(0)$ is counted three times. On the other hand, any other term is zero because it is an odd function in $\mathbf{R}^{n-1}$. (4.5) depends on following equality:

$$
\int_{\mathbf{R}^{n-1}}\left[\frac{w'(|y|)}{|y|}\right]^2y_j^4dy=3\int_{\mathbf{R}^{n-1}}\left[\frac{w'(|y|)}{|y|}\right]^2y_j^2y_l^2dy, \qquad (4.7)
$$

where $1\leq l\leq n-1$ and $l\neq j$. The proof of (4.7) is elementary, and we postpone it in Appendix A. At last, (4.6) comes from

$$
D_jH(P_i)=\sum_{l=1}^{n-1}\rho_{llj}(0). \qquad (4.8)
$$

A proof can be found in [APY] pg.63 Lemma 4.1. Therefore

$$
I_{41}=\varepsilon^{n+1}\gamma D_jH(P_i)+O(\varepsilon^{n+2}).
$$

By all the above estimates, we have

$$
\left\langle S_\varepsilon(u_\varepsilon),\frac{\partial W_{\varepsilon,P_i}}{\partial\tau_j}\right\rangle=\varepsilon^{n+1}\gamma D_jH(P_i)+O(\varepsilon^{n+2}).
$$

□

Now we are in the position to give the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $\mathbf{P}=(P_1,P_2,\cdots,P_k)$ satisfy the conditions in the theorem. Let $\mathbf{Q}=(Q_1,Q_2,\cdots,Q_k)\in\Gamma_\delta^k$, where $\delta<\frac{1}{2}\min_{i\neq j}d(P_i,P_j)$. We define a mapping $F : \Gamma_\delta^k \to \mathbf{R}^{(n-1)k}$, $F(\mathbf{Q}) = (F_1, F_2, \cdots, F_k)$, $F_i = (F_{i1}, F_{i2}, \cdots, F_{i(n-1)})$ where $i = 1, 2, \cdots, k$, and for $1 \le i \le k$, $1 \le j \le n-1$,

$$F_{ij} = \varepsilon^{-n-1} \left\langle S_\varepsilon(W_{\varepsilon,\mathbf{Q}} + \psi_{\varepsilon,\mathbf{Q}}), \frac{\partial W_{\varepsilon,Q_i}}{\partial \tau_j} \right\rangle.$$

By Proposition 4.1, we have

$$F_i(\mathbf{Q}) = \gamma DH(Q_i) + O(\varepsilon),$$

$$F(\mathbf{Q}) = \gamma\big(DH(Q_1), DH(Q_2), \cdots, DH(Q_k)\big) + e(\varepsilon, \mathbf{Q}),$$

where $|e(\varepsilon, \mathbf{Q})| \le C\varepsilon$ uniformly for $\mathbf{Q} \in \Gamma_\delta^k$. We restrict the definition of F to a neighborhood $N = N_1 \times N_2 \times \cdots \times N_k$ of $\mathbf{P}$ in Γ_δ^k, where N_i is an open neighborhood of P_i in $\partial\Omega$ and N_i is homeomorphic to an $(n-1)$ dimensional ball. Since P_i is a nondegenerate critical point of $H(P)$, then for N_i small enough, $|DH(Q)| \ge C_i > 0$ for $Q \in \partial N_i$. We choose $\varepsilon > 0$ small enough, such that $F(\mathbf{Q}) \ne 0$ for $\mathbf{Q} \in \partial N$, and

$$deg(F, N) = deg(F_1, N) \ne 0,$$

where $F_1(\mathbf{Q}) = \big(DH(Q_1), DH(Q_2), \cdots, DH(Q_k)\big)$. Therefore there exists a $\mathbf{P}_\varepsilon$ near $\mathbf{P}$ such that $F(\mathbf{P}_\varepsilon) = 0$. That implies $u_\varepsilon = W_{\varepsilon,\mathbf{P}_\varepsilon} + \psi_{\varepsilon,\mathbf{P}_\varepsilon}$ is a solution to (1.6). Since $\int_\Omega h(u_\varepsilon)dx = O(\varepsilon^n)$, then by Lemma C.3, u_ε is a solution to (1.2).

We prove the other statements in Theorem 1.1 in several steps.

Step 1: We prove that for any $p > 2$,

$$\int_\Omega |u_\varepsilon|^p dx \le C(p)\varepsilon^n,$$

where $C(p)$ is a constant which only depends on $p > 2$. In fact, $-C\varepsilon^n \le u_\varepsilon(x) < c$ for all $x \in \overline{\Omega}$ (by Lemma C.3). Therefore for $\varepsilon > 0$ small enough,

$$\int_\Omega |u_\varepsilon|^p dx \le c^{p-2} \int_\Omega |u_\varepsilon|^2 dx \le c^{p-2} C\varepsilon^n.$$

Step 2: If Q_ε is a local maximum point of u_ε and $u_\varepsilon(Q_\varepsilon) > 0$, then $u_\varepsilon(Q_\varepsilon) > b$. (Recall that $b > 0$ is the first positive zero of $g(u)$.) Since u_ε satisfies $\Delta u_\varepsilon + g(u_\varepsilon) = \sigma_\varepsilon$, where $\sigma_\varepsilon > 0$ by Lemma C.1, and $u_\varepsilon(Q_\varepsilon) > 0$, then $u_\varepsilon(Q_\varepsilon)$ can not lie in $(0, b)$ by the maximum principle.

Step 3: We prove some H^1 energy estimates. If O is a subdomain of Ω, we define $H_\varepsilon^1(O)$ to be the function space $H^1(O)$ with the rescaled norm

$$||u||_{1,\varepsilon,O}^2 = \varepsilon^{-n}\Big[||u||_{L^2(O)} + \varepsilon^2 \sum_{i=1}^n ||D_i u||_{L^2(O)}\Big].$$

We claim for any $1 > \delta > 0$, there exists $\varepsilon_3, C_\delta > 0$ such that for $\varepsilon \in (0, \varepsilon_3)$,

$$||u_\varepsilon||^2_{1,\varepsilon,N_{\varepsilon,i}} \geq \left(\frac{1}{2} - \delta\right) ||w||^2_{H^1(\mathbf{R}^n)}, \tag{4.9}$$

$$||u_\varepsilon||^2_{1,\varepsilon,O_\varepsilon} \leq k\delta ||w||^2_{H^1(\mathbf{R}^n)} + O(\varepsilon), \tag{4.10}$$

$$||u_\varepsilon||^2_{1,\varepsilon} = \frac{k}{2}||w||^2_{H^1(\mathbf{R}^n)} + O(\varepsilon), \tag{4.11}$$

where $N_{\varepsilon,i} = \{x \in \Omega : |x - P_i| \leq C_\delta \varepsilon\}$ for $i = 1, \cdots, k$ and $O_\varepsilon = \Omega \backslash (\cup_{i=1}^k N_{\varepsilon,i})$. Since

$$||\psi_{\varepsilon,\mathbf{P}}||_{2,\varepsilon} \leq C\varepsilon, \qquad ||W_{\varepsilon,P_i} - w_{\varepsilon,P_i}||_{1,\varepsilon} \leq C\varepsilon, \tag{4.12}$$

by Lemma 3.6 and Proposition 2.2, we have

$$||u_\varepsilon||^2_{1,\varepsilon} = ||W_{\varepsilon,\mathbf{P}_\varepsilon} + \psi_{\varepsilon,\mathbf{P}_\varepsilon}||^2_{1,\varepsilon} = \sum_{i=1}^k ||W_{\varepsilon_j,P_{ij}}||^2_{1,\varepsilon} + O(\varepsilon)$$

$$= \sum_{i=1}^k ||w_{\varepsilon,P_{ij}}||^2_{1,\varepsilon} + O(\varepsilon) = \frac{k}{2}||w||^2_{H^1(\mathbf{R}^n)} + O(\varepsilon).$$

(4.12) can be proved similarly, because of $D\Psi_{P_i}(P_i) = I$, the exponentially decaying of w and (4.12). (4.13) is obtained by combining the first and the third one.

Step 4: If Q_ε is a local maximum point of u_ε and $u_\varepsilon(Q_\varepsilon) > 0$, then

$$\min\{d(Q_\varepsilon, P_i) : 1 \leq i \leq k\} \leq C_2\varepsilon, \tag{4.13}$$

for $\varepsilon > 0$ small enough. Suppose to the contrary that there exists a decreasing $\varepsilon_j \to 0$ such that

$$\rho_j = \frac{1}{2}\min\{d(Q_j, P_i)\} \to \infty, \qquad \text{as } j \to \infty,$$

where Q_j is the abbreviation for Q_{ε_j}. Define a function v_j on $B(\rho_j)$ by

$$v_j(z) = u_{\varepsilon_j}(Q_j + \varepsilon_j z), \qquad \text{for } z \in B(\rho_j).$$

Now we can repeat the arguments in [NT1] pg. 830-832, due to the L^p estimates in Step 1 and the maximum principle in Step 2, to show that

$$v_j \to w \qquad \text{in } C^2_{loc}(\mathbf{R}^n),$$

where w is the solution of (1.3). Moreover, let R be an arbitrarily large number, then there is an integer j_R such that if $j \geq j_R$ then $\rho_j \geq 2R$ and

$$||v_j - w||_{C^2(\overline{B(2R)})} \leq Cexp\left(-\frac{R}{2}\right).$$

Similar to the proof in [NT1] pg. 832-834, we then can show that

$$||u_{\varepsilon_j}||^2_{1,\varepsilon_j,B(Q_j,\rho_j)} \geq ||w||^2_{H^1(\mathbf{R}^n)} - Cexp(-\mu R), \tag{4.14}$$

for $C, \mu > 0$ independent of j and R. Now we choose $\delta > 0$ such that $k\delta < \frac{1}{2}$ and j large enough such that $\rho_j > 2C_\delta$, then $B(Q_j, \rho_j) \subset O_{\varepsilon_j}$. From (4.10) and (4.11), we have

$$||w||^2_{H^1(\mathbf{R}^n)} - Cexp(-\mu R) \leq k\delta ||w||^2_{H^1(\mathbf{R}^n)} + O(\varepsilon),$$

which is a contradiction. Therefore (4.13) holds true.

Step 5: We shall show that there is exactly one positive local maximum point $Q_{\varepsilon,i}$ of u_ε in $N_{\varepsilon,i} = \{x \in \overline{\Omega} : |x - P_i| \leq C_2\varepsilon\}$ for $\varepsilon > 0$ small enough, where C_2 is the same as the one in (4.13). Let $\delta_2 > 0$ be the constant defined in Section 2. We define $\tilde{U}_\varepsilon(y) = u_\varepsilon(\Phi_{P_i}(y))$ for $y \in \overline{B^+(\delta_2)}$ and extend it to $\overline{B(\delta_2)}$ by reflection: $\tilde{U}_\varepsilon(y) = \tilde{U}_\varepsilon(y', -y_n)$ for $y \in \overline{B^-(\delta_2)} = \{y \in \overline{B(\delta_2)} : y_n \leq 0\}$. Moreover we define $U_\varepsilon(z) = \tilde{U}_\varepsilon(\varepsilon z)$ for $z \in \overline{B(\varepsilon^{-1}\delta_2)}$, then we can repeat the argument in [NT1] pg. 834-836 to conclude that

$$U_\varepsilon \to U \qquad \text{in } C^2_{loc}(\mathbf{R}^n).$$

The limit U belongs to $C^2(\mathbf{R}^n) \cap W^{2,p}(\mathbf{R}^n)$ for all $p > 0$, and U is a solution of

$$\Delta U - \mu U + h(U) = 0 \qquad \text{in } \mathbf{R}^n.$$

So U is either 0 or a translation of w. By (4.9), $U \not\equiv 0$. Then $U(z) = w(z - z^0)$. $z^0_n = 0$, since $U_\varepsilon(z)$ is symmetric with respect to $z_n = 0$, so is the limit U, hence the unique maximum point of U must lie in $\{z_n = 0\}$. Moreover, let R be an arbitrarily large number, then there is an integer ε_R such that if $\varepsilon \in (0, \varepsilon_R)$ then $\varepsilon^{-1}\delta_2 \geq 2R$ and

$$||U_\varepsilon - U||_{C^2(\overline{B(2R)})} \leq Cexp\left(-\frac{R}{2}\right).$$

By the same proof as in [NT1] pg. 836-837, we can show that U_ε has only one local maximum point $\tilde{Q}_\varepsilon$ in $B(R)$. By (4.13), $d(\tilde{Q}_\varepsilon, 0) \leq C$, where C is independent of ε. Therefore, for $\varepsilon > 0$ small enough, u_ε has exactly one positive local maximum point $Q_{\varepsilon,i}$ in N_i, and in fact $Q_{\varepsilon,i} \in N_{\varepsilon,i} \cap \partial\Omega$.

Summarizing above proof, we have proved for $\varepsilon > 0$ small enough, u_ε has exactly k positive local maximum points $Q_{\varepsilon,i}$, $1 \leq i \leq k$, $Q_{\varepsilon,i} \in \partial\Omega$, and $d(Q_{\varepsilon,i}, P_i) \leq C\varepsilon$. From the proof of Step 5, we also have $u_\varepsilon(Q_\varepsilon) \to w(0)$. Using similar arguments in Step 4 and 5, we can also easily show that $u_\varepsilon(x) \to 0$ for $x \in \overline{\Omega} \backslash \{P_i\}$. This completes the proof of Theorem 1.1. □

Remark. One can prove the existence of solutions under much weaker conditions. It suffices to assume that each point P_i is an isolated critical point of non-zero topological degree for the map $x \to \nabla H(x)$ on $\partial\Omega$. Since our reduction respects the variational structure, it is enough to assume each critical point has some Morse number nontrivial in the sense of Chang [C] or Dancer [D1]. Here we need to choose coefficients in a field in the definition of the Morse numbers to ensure the validity of the product theorem for the Morse numbers (as on pg. 15 of [D1].) In fact, we do not need to assume that we have isolated points but rather disjoint isolated sets of critical points of the mean curvature whose Conley index is non-trivial. Using this, one can show that if H and $\partial\Omega$ are real analytic and H is not constant on $\partial\Omega$, there is always a two-peak solution with one peak near $\{x \in \partial\Omega : H(x) = \max_{\partial\Omega} H(z)\}$, and one near $\{x \in \partial\Omega : H(x) = \min_{\partial\Omega} H(z)\}$. In addition, one can also obtain multiplicity results from these ideas. If H has only finitely many critical points, , and assume that $p = cuplength(\partial\Omega)$, then there are at least $\frac{1}{2}p(p+1)$ two-peak solutions, since there must be at least p critical points of H with a non-trivial Morse number. Once again we need to use $cuplength(\partial\Omega)$ for coefficients in a field and we need to use the ideas in the proof of Theorem 5 in [D2].

Finally, note that the uniqueness of the ground state solution of (1.3) can be avoided in the construction of the multi-peak solutions, provided we assume this solution of (1.3) is nondegenerate in a suitable sense.

□

5. Instability of multi-spike solutions under Allen-Cahn flow.

In this and the next sections, we study the instability of the multi-spike solutions which we obtained in Section 4. In this section, we construct k approximate eigenfunctions of the Allen-Cahn equation, orthonormal in $L^2(\Omega)$. This construction and the variational characterization of the eigenvalue of Allen-Cahn equation imply the existence of k negative eigenvalues for a $k-$peak solution. Here the $k-$peak solution can be a solution of either Allen-Cahn equation or Cahn-Hilliard equation. This also implies the existence of $k-1$ or k negative eigenvalues for Cahn-Hilliard flow, by Theorem 8 of [BFi1]. We shall modify the construction in this section in the next section, to prove there are indeed at least k negative eigenvalues for Cahn-Hilliard flow. Our main result in this section is

Proposition 5.1. *Let u_ε be a multi-spike solution as in Theorem 1.1 or Theorem 1.4, with k spikes $Q_{\varepsilon,1}, \cdots, Q_{\varepsilon,k}$. Then the problem*

$$\begin{cases} \varepsilon^2\Delta\psi - \mu\psi + h'(u_\varepsilon)\psi = -\lambda\psi & in \quad \Omega, \\ \frac{\partial\psi}{\partial n} = 0 & on \quad \partial\Omega. \end{cases} \tag{5.1}$$

has at least k negative eigenvalues (counting the multiplicity) $\mu_1^\varepsilon, \cdots, \mu_k^\varepsilon$, and for $i = 1, \cdots, k$, $\mu_i^\varepsilon \leq \lambda_1 + O(\varepsilon) < 0$, where λ_1 is the first eigenvalue of L_0.

The proof of Proposition 5.1 need some preliminaries. First we recall that ϕ is the eigenfunction of the first eigenvalue λ_1 of $L_0 = \Delta - \mu + h'(w) : H^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$.

Let P be a point on $\partial\Omega$, then $\phi_{\varepsilon,P} = \phi(\frac{x-P}{\varepsilon})$ satisfies

$$\begin{cases} \varepsilon^2\Delta v - \mu v + h'(w(\frac{x-P}{\varepsilon}))v = -\lambda_1 v & \text{in} \quad \Omega, \\ \frac{\partial v}{\partial n} = \frac{\partial\phi(\frac{x-P}{\varepsilon})}{\partial n} & \text{on} \quad \partial\Omega. \end{cases}$$

Similar to $W_{\varepsilon,P}$, we define $V_{\varepsilon,P} = V_{\varepsilon,P}\left(\frac{x-P}{\varepsilon}\right)$ to be the unique solution of

$$\begin{cases} \varepsilon^2\Delta v - \mu v + h'(w(\frac{x-P}{\varepsilon}))\phi(\frac{x-P}{\varepsilon}) = -\lambda_1\phi(\frac{x-P}{\varepsilon}) & \text{in} \quad \Omega, \\ \frac{\partial v}{\partial n} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{5.2}$$

Then $J_{\varepsilon,P}(x) = \phi(\frac{x-P}{\varepsilon}) - V_{\varepsilon,P}\left(\frac{x-P}{\varepsilon}\right)$ satisfies

$$\begin{cases} \varepsilon^2\Delta v - \mu v = 0 & \text{in} \quad \Omega, \\ \frac{\partial v}{\partial n} = \frac{\partial\phi(\frac{x-P}{\varepsilon})}{\partial n} & \text{on} \quad \partial\Omega. \end{cases} \tag{5.3}$$

Comparing (2.5) and (5.3), we find the only difference is that w is replaced by ϕ in the latter equation. Therefore, $J_{\varepsilon,P}$ has an exactly same asymptotic expansion as $h_{\varepsilon,P}$ in Proposition 2.2. Also we have an estimate similar to (4.1). So we have

Proposition 5.2.

$$J_{\varepsilon,P}(x) = \varepsilon\tilde{v_1}(y)\chi(x) + \varepsilon^2\tilde{v_2}(y)\chi(x) + \varepsilon^3\tilde{e_1}(x),$$

$$\frac{\partial}{\partial\tau_j(P)}J'_{\varepsilon,P}(x) = \tilde{u_0}(y)\chi(x) + \varepsilon\tilde{e_2}(x),$$

$$V_{\varepsilon,P} = \phi(y) - \varepsilon\chi(x)\left[\frac{\phi'(|y|)}{2|y|}\sum_{l,p=1}^{n-1}\rho_{lp}(0)y_l y_p y_n + \tilde{v_1}(y)\right] + \varepsilon^2 e_5(x),$$

where $\tilde{v_1}$, $\tilde{v_2}$ and $\tilde{u_0}$ are respectively the unique solutions to (2.8), (2.9) and (2.12) in $H^1(\mathbf{R}^n_+)$ where all w terms are replaced by ϕ. Moreover $||\tilde{e_1}||_{1,\varepsilon} \le C$, $||\tilde{e_2}||_{1,\varepsilon} \le C$, $||e_5||_{0,\varepsilon} \le C$ with $C > 0$ independent of ε.

The proof of Proposition 6.1 is based on the variational characterization of the eigenvalues. We define

$$\mathcal{L}_\varepsilon(u)v = -\varepsilon^2\Delta v + \mu v - h'(u)v,$$

$$\text{and} \quad B_u(v,v) = \int_\Omega (\varepsilon^2|\nabla v|^2 + \mu v^2 - h'(u)v^2)dx.$$

Then it is well-known that the eigenvalues μ_n^ε of $\mathcal{L}_\varepsilon(u)v = \lambda v$ are given by

$$\mu_n^\varepsilon = Min_n\ Max_n\frac{B_u(v,v)}{\langle v,v\rangle},$$

where Max_n is over all $v \in T_n$, and Min_n is over all subspaces T_n of $H^1(\Omega)$ of dimension n.

Let $u_\varepsilon = W_{\varepsilon,\mathbf{P}} + \psi_{\varepsilon,\mathbf{P}}$ be a multi-spike solution as in Theorem 1.1 or 1.4, with k spikes. In the next lemma, we show that V_{ε,P_i}, $i = 1,\cdots,k$, the solution of (5.3) with $P = P_i$, span an approximate eigenspace of dimension k. Without loss of generality, we assume that $\langle V_{\varepsilon,P_i}, V_{\varepsilon,P_i}\rangle = \varepsilon^n$ so that $||V_{\varepsilon,P_i}||_{L^\infty} \le C$ for $i = 1,\cdots,k$ and $\varepsilon > 0$.

Lemma 5.3.

$$\frac{B_{u_\varepsilon}(V_{\varepsilon,P_i}, V_{\varepsilon,P_i})}{\langle V_{\varepsilon,P_i}, V_{\varepsilon,P_i}\rangle} = \lambda_1 + \kappa_i\varepsilon + O(\varepsilon^2),$$
$$\langle V_{\varepsilon,P_i}, V_{\varepsilon,P_j}\rangle = O(e^{-\frac{c}{\varepsilon}}) \text{ for } i \neq j,$$
$$B_{u_\varepsilon}(V_{\varepsilon,P_i}, V_{\varepsilon,P_j}) = O(e^{-\frac{c}{\varepsilon}}) \text{ for } i \neq j,$$

where $i, j = 1, \cdots, k$ and the constant

$$\kappa_i = \int_{\mathbf{R}^n_+}\Big[\lambda_1\tilde{v_1} - h''(w)\psi_0\phi + h''(w)v_1\phi + h'(w)\tilde{v_1}\Big]\phi dy.$$

Proof. From (6.2) and (6.3), we have

$$\begin{aligned}
&B_{u_\varepsilon}(V_{\varepsilon,P_i}, V_{\varepsilon,P_i})\\
=&\langle -\varepsilon^2\Delta V_{\varepsilon,P_i} + \mu V_{\varepsilon,P_i} - h'(u_\varepsilon)V_{\varepsilon,P_i}, V_{\varepsilon,P_i}\rangle\\
=&\langle \lambda_1\phi_{\varepsilon,P_i} - h'(u_\varepsilon)V_{\varepsilon,P_i} + h'(w_{\varepsilon,P_i})\phi_{\varepsilon,P_i}, V_{\varepsilon,P_i}\rangle\\
=&\lambda_1\langle V_{\varepsilon,P_i}, V_{\varepsilon,P_i}\rangle + \lambda_1\int_\Omega(\phi_{\varepsilon,P_i} - V_{\varepsilon,P_i})V_{\varepsilon,P_i}dx\\
&-\int_\Omega[h'(u_\varepsilon)V_{\varepsilon,P_i} - h'(w_{\varepsilon,P_i})\phi_{\varepsilon,P_i}]V_{\varepsilon,P_i}dx\\
=&\lambda_1\langle V_{\varepsilon,P_i}, V_{\varepsilon,P_i}\rangle + \lambda_1\int_\Omega(\phi_{\varepsilon,P_i} - V_{\varepsilon,P_i})V_{\varepsilon,P_i}dx\\
&-\int_\Omega[h'(u_\varepsilon) - h'(W_{\varepsilon,P_i})]V^2_{\varepsilon,P_i}dx - \int_\Omega[h'(W_{\varepsilon,P_i}) - h'(w_{\varepsilon,P_i})]V^2_{\varepsilon,P_i}dx\\
&-\int_\Omega h'(w_{\varepsilon,P_i})(V_{\varepsilon,P_i} - \phi_{\varepsilon,P_i})V_{\varepsilon,P_i}dx\\
\equiv&\lambda_1\langle V_{\varepsilon,P_i}, V_{\varepsilon,P_i}\rangle + J_1 - J_2 - J_3 - J_4.
\end{aligned} \tag{5.4}$$

$$\begin{aligned}
J_1 =&\lambda_1\varepsilon\int_\Omega \alpha_i(x)\tilde{v_1}(y)\phi(y)dx + h.o.t.\\
=&\lambda_1\varepsilon^{n+1}\int_{\mathbf{R}^n_+}\tilde{v_1}\phi dy + O(\varepsilon^{n+2}).
\end{aligned}$$

since $\tilde{v_1}$ and ϕ are both even.

$$
\begin{aligned}
J_2 = &\int_\Omega [h'(W_{\varepsilon,\mathbf{P}} + \psi_{\varepsilon,\mathbf{P}}) - h'(W_{\varepsilon,P_i})] V_{\varepsilon,P_i}^2 dx \\
= &\int_\Omega [h'(W_{\varepsilon,\mathbf{P}} + \psi_{\varepsilon,\mathbf{P}}) - h'(W_{\varepsilon,\mathbf{P}}) - h''(W_{\varepsilon,\mathbf{P}})\psi_{\varepsilon,\mathbf{P}}] V_{\varepsilon,P_i}^2 dx \\
&+ \int_\Omega [h'(W_{\varepsilon,\mathbf{P}}) - h'(W_{\varepsilon,P_i})] V_{\varepsilon,P_i}^2 dx \\
&+ \int_\Omega [h''(W_{\varepsilon,\mathbf{P}}) - h''(W_{\varepsilon,P_i})] \psi_{\varepsilon,\mathbf{P}} V_{\varepsilon,P_i}^2 dx \\
&+ \int_\Omega [h''(W_{\varepsilon,P_i}) - h''(w_{\varepsilon,P_i})] \psi_{\varepsilon,\mathbf{P}} V_{\varepsilon,P_i}^2 dx \\
&+ \int_\Omega h''(w_{\varepsilon,P_i}) \psi_{\varepsilon,\mathbf{P}} V_{\varepsilon,P_i}^2 dx \\
\equiv &J_{21} + J_{22} + J_{23} + J_{24} + J_{25}.
\end{aligned}
$$

As in the proof of Proposition 4.1,

$$
\begin{aligned}
|J_{21}| \leq &C\varepsilon^2 \max |h'''(u)| \int_\Omega |\alpha_i(x)\psi_0(y)\phi(y)|^2 \, dx + h.o.t. \\
= &O(\varepsilon^{n+2}).
\end{aligned}
$$

Also

$$
J_{22} = O(e^{-\frac{c}{\varepsilon}}).
$$
$$
\text{and } J_{23} = O(e^{-\frac{c}{\varepsilon}}).
$$

Again

$$
\begin{aligned}
|J_{24}| \leq &\varepsilon^2 \max |h'''(u)| \int_\Omega |\alpha_i(x) v_1(y) \psi_0(y) \phi^2(y)| dx \\
= &O(\varepsilon^{n+2}),
\end{aligned}
$$

and

$$
\begin{aligned}
J_{25} = &\varepsilon \int_\Omega \alpha_i(x) h''(w(y)) \psi_0(y) \phi^2(y) dx \\
= &\varepsilon^{n+1} \int_{\mathbf{R}_+^n} h''(w) \psi_0 \phi^2 dy + O(\varepsilon^{n+2}).
\end{aligned}
$$

Therefore,

$$
J_2 = \varepsilon^{n+1} \int_{\mathbf{R}_+^n} h''(w) \psi_0 \phi^2 dy + O(\varepsilon^{n+2}).
$$

Similarly,

$$
\begin{aligned}
J_3 = &\int_\Omega h''(w_{\varepsilon,P_i})(W_{\varepsilon,P_i} - w_{\varepsilon,P_i}) V_{\varepsilon,P_i}^2 dx \\
&+ \int_\Omega [h'(W_{\varepsilon,P_i}) - h'(w_{\varepsilon,P_i}) - h''(w_{\varepsilon,P_i})(W_{\varepsilon,P_i} - w_{\varepsilon,P_i})] V_{\varepsilon,P_i}^2 dx \\
\equiv &J_{31} + J_{32}.
\end{aligned}
$$

We have

$$J_{31} = -\int_\Omega \alpha_i(x)h''(w(y))v_1(y)\phi^2(y)dy + h.o.t.$$
$$= -\varepsilon^{n+1}\int_{\mathbf{R}^n_+} h''(w(y))v_1(y)\phi^2(y)dy + O(\varepsilon^{n+2}),$$

and

$$|J_{32}| \le c\varepsilon^2 \max|h'''(u)|\int_\Omega |\alpha_i(x)v_1^2(y)\phi(y)|^2dx + h.o.t.$$
$$= O(\varepsilon^{n+2}).$$

Finally

$$J_4 = -\varepsilon\int_\Omega \alpha_i(x)\tilde{v_1}(y)\phi(y)dx + h.o.t.$$
$$= -\varepsilon^{n+1}\int_{\mathbf{R}^n_+} h'(w(y))\tilde{v_1}(y)\phi(y)dy + O(\varepsilon^{n+2}).$$

Putting together all the above estimates, we have the first equation of Lemma 5.3. The other two equations are simply from the exponentially decay of ϕ and that P_i, P_j are bounded away from each other.

□

Proof of Proposition 5.1. We define

$$\psi_i^\varepsilon(x) = \frac{\alpha_i(x)V_{\varepsilon,P_i}}{||\alpha_i(x)V_{\varepsilon,P_i}||_{0,\varepsilon}},$$

for $i = 1, 2, \cdots, k$, and $\mathcal{M}_i = span\{\psi_j^\varepsilon : j = 1, \cdots, i\}$, then $\mathcal{M}_i$ is a subspace of $H^1(\Omega)$ of dimension i for $i = 1, 2, \cdots, k$. Moreover, since $||V_{\varepsilon,P_i}||_{0,\varepsilon} = 1$ and V_{ε,P_i} is exponentially small outside of $supp(\alpha_i)$, and the supports of ψ_i^εs are disjoint, so $\langle\psi_i^\varepsilon, \psi_j^\varepsilon\rangle = \delta_{ij}\varepsilon^n$, and

$$\frac{B_{u_\varepsilon}(\psi_i^\varepsilon, \psi_i^\varepsilon)}{\langle\psi_i^\varepsilon, \psi_i^\varepsilon\rangle} = \lambda_1 + \kappa_i\varepsilon + O(\varepsilon^2),$$
$$B_{u_\varepsilon}(\psi_i^\varepsilon, \psi_j^\varepsilon) = O(e^{-\frac{c}{\varepsilon}}) \text{ for } i \ne j.$$

Recall that

$$\mu_n^\varepsilon = Min_n\ Max_n \frac{B_{u_\varepsilon}(v, v)}{\langle v, v\rangle}$$

where Max_n is over all $v \in T_n$, and Min_n is over all subspaces T_n of $H^1(\Omega)$ of dimension n. So, to prove $\mu_i^\varepsilon \le \lambda_1 + O(\varepsilon)$, it suffices to prove

$$Max_{v\in\mathcal{M}_i} \frac{B_{u_\varepsilon}(v, v)}{\langle v, v\rangle} = \lambda_1 + O(\varepsilon)$$

For any $v \in \mathcal{M}_i$, $v = \sum_{j=1}^{i} k_j \psi_j^\varepsilon$ for some k_j. Choose the k_j's so that $\sum_{j=1}^{i} k_j^2 = 1$, and so $\langle v, v\rangle = \varepsilon^n$. Then

$$\begin{aligned} B_{u_\varepsilon}(v,v) =& \sum_{j=1}^{i} k_j^2 B_{u_\varepsilon}(\psi_j^\varepsilon, \psi_j^\varepsilon) + 2\sum_{j\neq l} k_j k_l B_{u_\varepsilon}(\psi_j^\varepsilon, \psi_l^\varepsilon) \\ =& \lambda_1 \varepsilon^n + \sum_{j=1}^{i} \kappa_j k_j^2 \varepsilon^{n+1} + O(\varepsilon^{n+2}) + O(e^{-\frac{c}{\varepsilon}}) \end{aligned}$$

Therefore

$$\frac{B_{u_\varepsilon}(v,v)}{\langle v, v\rangle} = \lambda_1 + O(\varepsilon)$$

for any $v \in \mathcal{M}_i$. Therefore $\mu_i^\varepsilon \leq \lambda_1 + O(\varepsilon)$, for $i = 1, \cdots, k$.

□

6. Instability of multi-spikes solutions under Cahn-Hilliard flow.

In this section we prove Theorem 1.2. First we recall the variational characterization of the eigenvalues of Cahn-Hilliard equation ([BFi1] Theorem 5):

$$\lambda_n^\varepsilon = Min_n \ Max_n \frac{B_{u_\varepsilon}(v,v)}{\langle(-\Delta)^{-1} v, v\rangle},$$

where Max_n is over all $v \in T_n$, and Min_n is over all subspaces T_n of $\overline{H}^1(\Omega)$ of dimension n. Here $\overline{H}^1(\Omega)$ is the subspace of $H^1(\Omega)$ in which $\int_\Omega u dx = 0$.

To modify our construction, we define a function $\theta : \mathbf{R}^n \to \mathbf{R}$ such that

(1) θ is smooth, and radially symmetric,

(2) $supp(\theta) \subset B(2) \backslash B(1)$,

(3) $\int_{\mathbf{R}^n} \theta(x) dx = \int_{\mathbf{R}^n} \phi(x) dx$,

and $\theta_\beta(x) = \beta^n \theta(\beta x)$. It is easy to verify that

$$\int_{\mathbf{R}^n} \theta_\beta(x) dx = \int_{\mathbf{R}^n} \theta(x) dx, \ \ supp(\theta_\beta) \subset B(2\beta^{-1}) \backslash B(\beta^{-1}), \tag{6.1}$$

$$||\theta_\beta||_{L^2(\mathbf{R}^n)} = \beta^n ||\theta||_{L^2(\mathbf{R}^n)}, \ \ ||\nabla\theta_\beta||_{L^2(\mathbf{R}^n)} = \beta^{3n} ||\nabla\theta||_{L^2(\mathbf{R}^n)}. \tag{6.2}$$

Now we define our new new approximate eigenfunction to be

$$\tilde{V}_{\varepsilon,P_i}\left(\frac{x-P_i}{\varepsilon}\right) = \alpha_i(x) V_{\varepsilon,P_i}\left(\frac{x-P_i}{\varepsilon}\right) - \tau_{\varepsilon,P_i} \theta_\beta \left(\frac{x-P_i}{\varepsilon}\right), \tag{6.3}$$

where τ_{ε,P_i} is a constant such that

$$\int_\Omega \tilde{V}_{\varepsilon,P_i}\left(\frac{x-P_i}{\varepsilon}\right) dx = 0.$$

By the definition of V_{ε,P_i} and θ_β, if we choose β small enough (but independent of ε), then $\tau_{\varepsilon,P_i} = 1 + O(\varepsilon)$. From the estimates in the last section and (6.2), it is easy to obtain the following estimates:

Lemma 6.1.

$$B_{u_\varepsilon}(\tilde{V}_{\varepsilon,P_i},\tilde{V}_{\varepsilon,P_i}) = [\lambda_1+\beta^n+o(\beta^n)]\varepsilon^n + O(\varepsilon^{n+1}), \tag{6.4}$$
$$B_{u_\varepsilon}(\tilde{V}_{\varepsilon,P_i},\tilde{V}_{\varepsilon,P_j}) = 0 \textit{ for } i \neq j, \tag{6.5}$$

where $i,j = 1,\cdots,k$.

Here we introduce a bilinear form $\langle\cdot,\cdot\rangle_A$. For $u,v \in \overline{H}^1_N(\Omega)$, define

$$\langle u,v\rangle_A = \langle(-\Delta)^{-1}u,v\rangle$$
$$\text{and } ||u||_A^2 = \langle u,u\rangle_A$$

To estimate $\langle\tilde{V}_{\varepsilon,P_i},\tilde{V}_{\varepsilon,P_i}\rangle_A$, we assume that $\eta(x)$ is the unique meanvalue zero solution of

$$\begin{cases} -\Delta\eta = \tilde{V}_{\varepsilon,P_i} & \text{in } \Omega, \\ \frac{\partial\eta}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.6}$$

and $U(y)$ is the unique meanvalue zero solution of

$$\begin{cases} -\Delta U(y) = \phi(y) - \theta_\beta(y) & \text{in } \mathbf{R}^n_+, \\ \frac{\partial U}{\partial y_n} = 0 & \text{on } \partial\mathbf{R}^n_+. \end{cases} \tag{6.7}$$

Note that the right hand sides of the equations have meanvalue zero.

Lemma 6.2.

$$\eta(x) = \varepsilon^2\alpha_i(x)U\left(\frac{x-P_i}{\varepsilon}\right) + \varepsilon^3 e_6(x), \tag{6.8}$$

where $\|\nabla e_6\|_{0,\varepsilon} \leq C$.

The proof of Lemma 6.2 is similar to that of Proposition 2.2, so we omit it. Notice that if we replace the operator $\Delta - \mu$ by Δ in Lemma A.1, then we should replace the H^1 norm by the L^2 norm of the gradiant in the conclusion of Lemma A.1.

From (6.8), we obtain

$$\nabla\eta(x) = \varepsilon\alpha_i(x)\nabla U\left(\frac{x-P_i}{\varepsilon}\right) + \varepsilon^2 U\left(\frac{x-P_i}{\varepsilon}\right)\nabla\alpha_i(x) + \varepsilon^3\nabla e_6(x).$$

Therefore,

$$\begin{aligned}
&\langle\tilde{V}_{\varepsilon,P_i},\tilde{V}_{\varepsilon,P_i}\rangle_A = \langle\eta,\tilde{V}_{\varepsilon,P_i}\rangle \\
=&-\int_\Omega \eta(x)\Delta\eta(x)dx = \int_\Omega |\nabla\eta(x)|^2 dx \\
=&\varepsilon^2\int_\Omega \alpha_i^2(x)\left|\nabla U\left(\frac{x-P_i}{\varepsilon}\right)\right|^2 dx + \varepsilon^4\int_\Omega |\nabla\alpha_i(x)|^2 U^2\left(\frac{x-P_i}{\varepsilon}\right)dx \\
&+ 2\varepsilon^3\int_\Omega \alpha_i(x)U\left(\frac{x-P_i}{\varepsilon}\right)\nabla\alpha_i(x)\cdot\nabla U\left(\frac{x-P_i}{\varepsilon}\right)dx + O(\varepsilon^4) \\
=&\varepsilon^{n+2}\int_{\mathbf{R}^n_+} |\nabla U(y)|^2 dy + O(\varepsilon^{n+2}).
\end{aligned} \tag{6.9}$$

On the other hand, for $\varepsilon > 0$ small enough,

$$\langle \tilde{V}_{\varepsilon,P_i}, \tilde{V}_{\varepsilon,P_j} \rangle_A = 0, \tag{6.10}$$

for $i \neq j$. By (6.4),(6.7),(6.9) and (6.10), we obtain that

$$\frac{B_{u_\varepsilon}(\tilde{V}_{\varepsilon,P_i}, \tilde{V}_{\varepsilon,P_i})}{\langle (-\Delta)^{-1} \tilde{V}_{\varepsilon,P_i}, \tilde{V}_{\varepsilon,P_i} \rangle} = C_0[\lambda_1 + O(\beta^n)]\varepsilon^{-2} + O(\varepsilon^{-1}),$$

where

$$C_0 = \left[\int_{\mathbf{R}^n_+} |\nabla U(y)|^2 dy \right]^{-1}.$$

Now Theorem 1.2 can be proved by using exactly the same argument as that of Proposition 5.1. We omit the details.

Appendix A: Proofs of Proposition 2.2, Lemma 3.6 and other estimates.

The proof of Proposition 2.2 relies on the following lemma.

Lemma A.1. *Define z by $x - P = \varepsilon z$, and let $\Omega_{\varepsilon,P} = \{z : x = P + \varepsilon z \in \Omega\}$. Let u be the solution of*

$$\begin{cases} \Delta u - \mu u = f(x) & \text{in } \ \Omega_{\varepsilon,P}, \\ \frac{\partial u}{\partial n} = g(x) & \text{on } \ \partial\Omega_{\varepsilon,P}. \end{cases} \tag{A.1}$$

Then

$$||u||_{H^1(\Omega_{\varepsilon,P})} \leq C[||f||_{L^2(\Omega_{\varepsilon,P})} + ||g||_{L^2(\partial\Omega_{\varepsilon,P})}],$$

where C does not depend on ε.

Proof. Multiply the equation by u and integrate over $\Omega_{\varepsilon,P}$,

$$-\int_{\Omega_{\varepsilon,P}} |\nabla u|^2 - \mu \int_{\Omega_{\varepsilon,P}} u^2 + \int_{\partial\Omega_{\varepsilon,P}} ug = \int_{\Omega_{\varepsilon,P}} uf.$$

Then for any $\kappa > 0$,

$$\begin{aligned} ||u||^2_{H^1(\Omega_{\varepsilon,P})} \leq & C \left[\int_{\Omega_{\varepsilon,P}} |uf| + \int_{\partial\Omega_{\varepsilon,P}} |ug| \right] \\ \leq & C[\kappa ||u||^2_{L^2(\Omega_{\varepsilon,P})} + \kappa^{-1} ||f||^2_{L^2(\Omega_{\varepsilon,P})}] \\ & + C[\kappa ||u||^2_{L^2(\partial\Omega_{\varepsilon,P})} + \kappa^{-1} ||g||^2_{L^2(\partial\Omega_{\varepsilon,P})}]. \end{aligned} \tag{A.2}$$

Let $u_1(x) = u(z)$, with $x \in \Omega$ and $z \in \Omega_{\varepsilon,P}$. Then $||u||^2_{L^2(\partial\Omega_{\varepsilon,P})} = \varepsilon^{1-n} ||u_1||^2_{L^2(\partial\Omega)}$, $||u||^2_{L^2(\Omega_{\varepsilon,P})} = \varepsilon^{-n} ||u_1||^2_{L^2(\Omega)}$, and $||\nabla u||^2_{L^2(\Omega_{\varepsilon,P})} = \varepsilon^{2-n} ||\nabla u_1||^2_{L^2(\Omega)}$. By the trace inequality, for any $0 < \varepsilon < 1$, we have

$$||u_1||^2_{L^2(\partial\Omega)} \leq C_1(\varepsilon ||\nabla u_1||^2_{L^2(\Omega)} + \varepsilon^{-1} ||u_1||^2_{L^2(\Omega)}),$$

where C_1 only depends on Ω, not ε. Therefore

$$||u||^2_{L^2(\partial\Omega_{\varepsilon,P})} \leq C_2||u||^2_{H^1(\Omega_{\varepsilon,P})}$$

for some constant C_2 not depending on ε. Therefore we can choose a suitably small $\kappa > 0$ in (A.2) to get

$$||u||^2_{H^1(\Omega_{\varepsilon,P})} \leq C(||f||^2_{L^2(\Omega_{\varepsilon,P})} + ||g||^2_{L^2(\partial\Omega_{\varepsilon,P})}).$$

□

For the proof of Proposition 2.2, we notice that v_1, v_2 and u_0 are exponentially decaying. More precisely, let v be one of v_1, v_2 and u_0, then

$$|v(y)|, |Dv(y)|, |D^2v(y)| \leq Ce^{-k|y|}$$

for $0 < k < \sqrt{m}$. In particular, if $N \subset \Omega_P$ is a neighborhood of $P \in \partial\Omega$, such that $\Psi_\varepsilon(N) \supset B(\delta\varepsilon^{-1})$ for some $\delta > 0$. Then

$$\begin{aligned} &|v(\Psi_\varepsilon(x))|, |Dv(\Psi_\varepsilon(x))|, |D^2v(\Psi_\varepsilon(x))| \leq Ce^{\frac{-k|x-P|}{\varepsilon}} \qquad x \in N \\ &|v(\Psi_\varepsilon(x))|, |Dv(\Psi_\varepsilon(x))|, |D^2v(\Psi_\varepsilon(x))| \leq Ce^{\frac{-c}{\varepsilon}} \qquad x \in \Omega_P \backslash N \end{aligned}$$

for some $c, C > 0$. Also, let ρ and Ψ_ε be defined as in Section 2. For $x \in \Omega_P$, we have the following elementary properties:

$$n(x) = \frac{1}{\sqrt{1+|\nabla\rho|^2}}(\nabla\rho(x'-P'), -1), \tag{A.3}$$

$$\tau_j(x) = (0, \cdots, 1, \cdots, 0, \rho_j(x'-P')), \qquad j = 1, \cdots, n-1, \tag{A.4}$$

$$\frac{\partial}{\partial n}(x) = \frac{1}{\sqrt{1+|\nabla\rho|^2}}\left(\sum_{j=1}^{n-1}\rho_j(x'-P')\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_n}\right), \tag{A.5}$$

$$\frac{\partial}{\partial \tau_j}(x) = \frac{\partial}{\partial x_j} + \rho_j(x'-P')\frac{\partial}{\partial x_n}, \qquad j = 1, \cdots, n-1. \tag{A.6}$$

With the change of variables $y = \Psi_\varepsilon(x)$, we have

$$\varepsilon^2\Delta_x = \Delta_y + |\nabla\rho|^2\frac{\partial^2}{\partial y_n^2} - 2\sum_{i=1}^{n-1}\rho_i\frac{\partial^2}{\partial y_i\partial y_n} - \varepsilon(\Delta\rho)\frac{\partial}{\partial y_n}, \tag{A.7}$$

$$\text{and} \quad \frac{\partial}{\partial n_x} = \frac{1}{\varepsilon\sqrt{1+|\nabla\rho|^2}}\left(\sum_{k=1}^{n-1}\rho_k\frac{\partial}{\partial y_k} - (1+|\nabla\rho|^2)\frac{\partial}{\partial y_n}\right). \tag{A.8}$$

Finally, we have

Lemma A.2. *Let* $x \in \overline{\Omega_P} \cap \partial\Omega$, $y = \Psi_\varepsilon(x)$, $z = \frac{x-P}{\varepsilon}$, *then*

$$\frac{w'(|z|)}{|z|} = \frac{w'(|y|)}{|y|} + \varepsilon^2 \frac{|y|w''(|y|) - w'(|y|)}{8|y|^3} \left(\sum_{i,j=1}^{n-1} \rho_{ij}(0) y_i y_j \right)^2 + O(\varepsilon^3 e^{-k|y|}), \quad (A.9)$$

for some $k > 0$.

Proof. By Taylor expansion, we have

$$\frac{w'(|z|)}{|z|} = \frac{w'(|y|)}{|y|} + \frac{|y|w''(|y|) - w'(|y|)}{|y|^2} (|z| - |y|) + O\left((|z| - |y|)^2 e^{-k|y|} \right). \quad (A.10)$$

And since $x \in \partial\Omega$, $y_n = [x_n - P_n - \rho(x' - P')]/\varepsilon = 0$, then

$$\begin{aligned} |z| - |y| =& \frac{1}{\varepsilon}[|x - P| - |\Psi(x)|] = \frac{\rho(x' - P')^2}{\varepsilon[|x - P| + |\Psi(x)|]} \\ =& \frac{\varepsilon^2}{8|y|} \left(\sum_{i,j=1}^{n-1} \rho_{ij}(0) y_i y_j \right)^2 + O(\varepsilon^3 |y|^4). \end{aligned} \quad (A.11)$$

Therefore (A.9) can be obtained by combining (A.10) and (A.11).

□

Proof of Proposition 2.2. We first prove that $||e_1||_{1,\varepsilon} \leq C$:

$$\begin{aligned} & - [\varepsilon^2 \Delta e_1(x) - \mu e_1(x)] \\ =& \frac{1}{\varepsilon^3} \left[-\varepsilon^2 \Delta h_{\varepsilon,P} + \varepsilon^3 \Delta(v_1 \chi) + \varepsilon^4 \Delta(v_2 \chi) + \mu h_{\varepsilon,P} - \varepsilon \mu v_1 \chi - \varepsilon^2 \mu v_2 \chi \right] \\ =& \frac{1}{\varepsilon^2} [\chi(\varepsilon^2 \Delta v_1 - \mu v_1 + \varepsilon^3 \Delta v_2 - \mu v_2) \\ & + 2\nabla\chi \cdot (\varepsilon^2 \nabla v_1 + \varepsilon^3 \nabla v_2) + \Delta\chi(\varepsilon^2 v_1 + \varepsilon^3 v_2)]. \end{aligned}$$

For the terms involving $\nabla\chi$ and $\Delta\chi$, we have

$$\begin{aligned} & \frac{1}{\varepsilon^2} ||[2\nabla\chi \cdot (\varepsilon^2 \nabla v_1 + \varepsilon^3 \nabla v_2) + \Delta\chi(\varepsilon^2 v_1 + \varepsilon^3 v_2)]||_{L^\infty} \\ =& ||2\nabla\chi \cdot (\nabla v_1 + \varepsilon \nabla v_2) + \Delta\chi(v_1 + \varepsilon v_2)]||_{L^\infty} \\ =& O(e^{-\frac{c}{\varepsilon}}). \end{aligned}$$

On the other hand, with $y=\Psi_\varepsilon(x)$,

$$
\begin{aligned}
&\frac{1}{\varepsilon^2}\chi(\varepsilon^2\Delta v_1-\mu v_1+\varepsilon^3\Delta v_2-\mu v_2)\\
=&\frac{1}{\varepsilon^2}\chi\Bigg[\Bigg(\Delta_y v_1+|\nabla\rho|^2\frac{\partial^2 v_1}{\partial y_n^2}-2\sum_{i=1}^{n-1}\rho_i\frac{\partial^2 v_1}{\partial y_i\partial y_n}-\varepsilon(\Delta\rho)\frac{\partial v_1}{\partial y_n}-\mu v_1\Bigg)\\
&+\varepsilon\Bigg(\Delta_y v_2+|\nabla\rho|^2\frac{\partial^2 v_2}{\partial y_n^2}-2\sum_{i=1}^{n-1}\rho_i\frac{\partial^2 v_2}{\partial y_i\partial y_n}-\varepsilon(\Delta\rho)\frac{\partial v_2}{\partial y_n}-\mu v_2\Bigg)\Bigg]\\
=&\frac{1}{\varepsilon^2}\chi\Bigg[\Bigg(|\nabla\rho|^2\frac{\partial^2 v_1}{\partial y_n^2}-2\sum_{i=1}^{n-1}\rho_i\frac{\partial^2 v_1}{\partial y_i\partial y_n}-\varepsilon(\Delta\rho)\frac{\partial v_1}{\partial y_n}\Bigg)\\
&+\varepsilon\Bigg(|\nabla\rho|^2\frac{\partial^2 v_2}{\partial y_n^2}-2\sum_{i=1}^{n-1}\rho_i\frac{\partial^2 v_2}{\partial y_i\partial y_n}-\varepsilon(\Delta\rho)\frac{\partial v_2}{\partial y_n}\\
&+2\sum_{i,j=1}^{n-1}\rho_{ij}(0)y_i\frac{\partial^2 v_1}{\partial y_j\partial y_n}-(\Delta\rho(0))\frac{\partial v_1}{\partial y_n}\Bigg)\Bigg]\\
=&\frac{1}{\varepsilon^2}\chi\Bigg[\Bigg(|\nabla\rho|^2\frac{\partial^2 v_1}{\partial y_n^2}-2\sum_{i=1}^{n-1}\Big[\rho_i-\varepsilon\sum_{j=1}^{n-1}\rho_{ij}(0)y_j\Big]\frac{\partial^2 v_1}{\partial y_i\partial y_n}\Bigg)\\
&+\varepsilon\Bigg(|\nabla\rho|^2\frac{\partial^2 v_2}{\partial y_n^2}-\varepsilon(\Delta\rho)\frac{\partial v_2}{\partial y_n}-2\sum_{i=1}^{n-1}\rho_i\frac{\partial^2 v_2}{\partial y_i\partial y_n}-[\Delta\rho-\Delta\rho(0)]\frac{\partial v_1}{\partial y_n}\Bigg)\Bigg]\\
\equiv&f_1(x).
\end{aligned}
$$

By the Taylor expansions of $\nabla\rho$ and $\Delta\rho$ and the exponential decaying property of Dv_i and D^2v_i, we have that $|f_1(x)|\le C\chi(x)e^{-\frac{k|x-P|}{\varepsilon}}$. Then

$$||f_1||_0^2\le C\int_{supp\chi}e^{-\frac{2k|x-P|}{\varepsilon}}dx\le C\varepsilon^n.$$

Hence,

$$||\varepsilon^2\Delta e_1(x)-\mu e_1(x)||_0^2\le C\varepsilon^n. \tag{A.12}$$

Next we estimate $\frac{\partial e_1}{\partial n}$.

$$
\begin{aligned}
\frac{\partial e_1}{\partial n}&=\frac{1}{\varepsilon^3}\left[\frac{\partial h_{\varepsilon,P}}{\partial n}-\varepsilon\frac{\partial(v_1\chi)}{\partial n}-\varepsilon^2\frac{\partial(v_2\chi)}{\partial n}\right]\\
=&\frac{\chi}{\varepsilon^3}\left[\frac{\partial w(\frac{x-P}{\varepsilon})}{\partial n}-\varepsilon\frac{\partial v_1}{\partial n}-\varepsilon^2\frac{\partial v_2}{\partial n}\right]+\frac{1-\chi}{\varepsilon^3}\frac{\partial w(\frac{x-P}{\varepsilon})}{\partial n}-\frac{\partial\chi}{\partial n}(\varepsilon v_1+\varepsilon^2 v_2).
\end{aligned}
$$

By the exponential decay of Dw, v_1 and v_2, we have

$$\left|\left|\frac{1-\chi}{\varepsilon^3}\frac{\partial w(\frac{x-P}{\varepsilon})}{\partial n}\right|\right|_{L^\infty}=O(e^{-\frac{c}{\varepsilon}}),\qquad\left|\left|\frac{\partial\chi}{\partial n}(\varepsilon v_1+\varepsilon^2 v_2)\right|\right|_{L^\infty}=O(e^{-\frac{c}{\varepsilon}}).$$

On the other hand,

$$\frac{\chi}{\varepsilon^3}\left[\frac{\partial w(\frac{x-P}{\varepsilon})}{\partial n}-\varepsilon\frac{\partial v_1}{\partial n}-\varepsilon^2\frac{\partial v_2}{\partial n}\right]$$

$$=\frac{\chi}{\varepsilon^3}\left[\frac{w'(|z|)}{|z|}\frac{\langle x-P,n(x)\rangle}{\varepsilon^2}-\frac{1}{\sqrt{1+|\nabla\rho|^2}}\left(\sum_{k=1}^{n-1}\rho_k\frac{\partial v_1}{\partial y_k}-(1+|\nabla\rho|^2)\frac{\partial v_1}{\partial y_n}\right)\right.$$

$$\left.-\frac{\varepsilon}{\sqrt{1+|\nabla\rho|^2}}\left(\sum_{k=1}^{n-1}\rho_k\frac{\partial v_2}{\partial y_k}-(1+|\nabla\rho|^2)\frac{\partial v_2}{\partial y_n}\right)\right].$$

Since

$$\langle x-P,n(x)\rangle=\frac{1}{\sqrt{1+|\nabla\rho|^2}}\left[\sum_{k=1}^{n-1}(x_i-P_i)\rho_i(x'-P')-(x_n-P_n)\right]$$

$$=\frac{1}{\sqrt{1+|\nabla\rho|^2}}\left[\frac{\varepsilon^2}{2}\sum_{i,j=1}^{n-1}\rho_{ij}(0)y_iy_j+\frac{\varepsilon^3}{3}\sum_{i,j,k=1}^{n-1}\rho_{ijk}(0)y_iy_jy_k+O(\varepsilon^4|y|^4)\right],$$

then, together with Lemma A.2, (2.8) and (2.9), we have

$$\frac{\chi}{\varepsilon^3}\left[\frac{\partial w(\frac{x-P}{\varepsilon})}{\partial n}-\varepsilon\frac{\partial v_1}{\partial n}-\varepsilon^2\frac{\partial v_2}{\partial n}\right]=\frac{\chi}{\varepsilon^3\sqrt{1+|\nabla\rho|^2}}O(\varepsilon^2e^{-k|y|}).$$

Hence,

$$\left|\varepsilon\frac{\partial e_1}{\partial n}(x)\right|\le C\chi(x)e^{-\frac{k|x-P|}{\varepsilon}},$$

$$\left\|\varepsilon\frac{\partial e_1}{\partial n}\right\|^2_{L^2(\partial\Omega)}\le C\varepsilon^{n-1}. \tag{A.13}$$

As in Lemma A.1, let $x-P=\varepsilon z$ and $\Omega_{\varepsilon,P}=\{z|x=P+\varepsilon z\in\Omega\}$, and write $\tilde{e}_1(z)=e_1(x)$. Then $\tilde{e}_1$ satisfies

$$\begin{cases}\Delta u-\mu u=\varepsilon^2\Delta e_1-\mu e_1:=f(z) & \text{in } \Omega_{\varepsilon,P},\\ \frac{\partial u}{\partial n}=\varepsilon\frac{\partial e_1}{\partial n}:=g(z) & \text{on } \partial\Omega_{\varepsilon,P}.\end{cases} \tag{A.14}$$

By (A.12), (A.13),

$$||f||^2_{L^2(\Omega_{\varepsilon,P})}=\varepsilon^{-n}||\varepsilon^2\Delta e_1(x)-\mu e_1(x)||_0^2\le C,$$

$$\text{and }\ ||g||^2_{L^2(\partial\Omega_{\varepsilon,P})}=\varepsilon^{1-n}\left\|\varepsilon\frac{\partial e_1}{\partial n}\right\|^2_{L^2(\partial\Omega)}\le C.$$

Therefore by Lemma A.1,

$$||e_1||_{1,\varepsilon}=||\tilde{e}_1||_{H^1(\Omega_{\varepsilon,P})}\le C(||f||_{L^2(\Omega_{\varepsilon,P})}+||g||_{L^2(\Omega_{\varepsilon,P})})\le C.$$

The proof of $||e_2||_{1,\varepsilon} \leq C$ is similar. First

$$\begin{aligned}
&-[\varepsilon^2\Delta e_2(x) - \mu e_2(x)] \\
=&-\frac{1}{\varepsilon}\left[\varepsilon^2\Delta\frac{\partial}{\partial\tau_j(P)}h_{\varepsilon,P} - \mu\frac{\partial}{\partial\tau_j(P)}h_{\varepsilon,P} - \varepsilon^2\Delta(\chi u_0) + \mu\chi u_0\right] \\
=&\frac{1}{\varepsilon}[\varepsilon^2\Delta(\chi u_0) - \mu\chi u_0] \\
=&\frac{\chi}{\varepsilon}(\varepsilon^2\Delta u_0 - \mu u_0) + \varepsilon(2\nabla\chi\cdot\nabla u_0 + \Delta\chi\cdot u_0).
\end{aligned}$$

Note that

$$||\varepsilon(2\nabla\chi\cdot\nabla u_0 + \Delta\chi\cdot u_0)||_{L^\infty} \leq Ce^{-\frac{c}{\varepsilon}},$$

and

$$\begin{aligned}
&\frac{\chi}{\varepsilon}(\varepsilon^2\Delta u_0 - \mu u_0) \\
=&\frac{\chi}{\varepsilon}\left[\Delta_y u_0 + |\nabla\rho|^2\frac{\partial^2 u_0}{\partial y_n^2} - 2\sum_{i=1}^{n-1}\rho_i\frac{\partial^2 u_0}{\partial y_i\partial y_n} - \varepsilon(\Delta\rho)\frac{\partial u_0}{\partial y_n} - \mu u_0\right] \\
=&\frac{\chi}{\varepsilon}\left[|\nabla\rho|^2\frac{\partial^2 u_0}{\partial y_n^2} - 2\sum_{i=1}^{n-1}\rho_i\frac{\partial^2 u_0}{\partial y_i\partial y_n} - \varepsilon(\Delta\rho)\frac{\partial u_0}{\partial y_n}\right] \\
\leq&C\chi(x)e^{-\frac{k|x-P|}{\varepsilon}}.
\end{aligned}$$

Therefore,

$$||\varepsilon^2\Delta e_2(x) - \mu e_2(x)||_0^2 \leq C\varepsilon^n.$$

On the other hand, by (2.13), for $x \in \partial\Omega$,

$$\begin{aligned}
\frac{\partial e_2}{\partial n} =&\frac{1}{\varepsilon}\left[\frac{\partial}{\partial\tau_j(P)}\frac{\partial h_{\varepsilon,P}}{\partial n} - \frac{\partial(u_0\chi)}{\partial n}\right] \\
=&\frac{1}{\varepsilon}\left[\frac{\varepsilon\partial}{\partial\tau_j(P)}\frac{\partial(v_1\chi)}{\partial n} - \frac{\partial(u_0\chi)}{\partial n}\right] + \varepsilon\left[\frac{\partial}{\partial\tau_j(P)}\frac{\partial(v_2\chi)}{\partial n} + \varepsilon\frac{\partial}{\partial\tau_j(P)}\frac{\partial e_1}{\partial n}\right]
\end{aligned}$$

By (2.8) and (2.12), for $y \in\mathbf{R}^n_+$, we have

$$\frac{\partial v_1(y)}{\partial y_j} = u_0(y).$$

Hence

$$\begin{aligned}
&\frac{1}{\varepsilon}\left[\frac{\varepsilon\partial}{\partial\tau_j(P)}\frac{\partial(v_1\chi)}{\partial n} - \frac{\partial(u_0\chi)}{\partial n}\right] \\
=&\frac{\chi}{\varepsilon}\frac{\partial}{\partial n}\left[\frac{\partial v_1(y)}{\partial y_j} - u_0(y)\right] + \frac{1}{\varepsilon}\frac{\partial\chi}{\partial n}\left[\frac{\partial v_1(y)}{\partial y_j} - u_0(y)\right] \\
=&0,
\end{aligned}$$

and

$$\left|\left|\varepsilon\frac{\partial e_2}{\partial n}\right|\right|^2_{L^2(\partial\Omega)} = \left|\left|\varepsilon^2\left[\frac{\partial}{\partial\tau_j(P)}\frac{\partial(v_2\chi)}{\partial n} + \varepsilon\frac{\partial}{\partial\tau_j(P)}\frac{\partial e_1}{\partial n}\right]\right|\right|^2_{L^2(\partial\Omega)} \leq C\varepsilon^{n-1}.$$

Then by Lemma A.1, we have $||e_2||_{1,\varepsilon} \leq C$.

□

Proof of Lemma 3.6. First by Proposition 3.5, we have $\pi^{\perp}_{\varepsilon,\mathbf{P}}(S_\varepsilon(W_{\varepsilon,\mathbf{P}} + \psi_{\varepsilon,\mathbf{P}})) = 0$. Then

$$\begin{aligned}
0 =&\pi^{\perp}_{\varepsilon,\mathbf{P}}\Bigg[\varepsilon^2\Delta W_{\varepsilon,\mathbf{P}} - \mu W_{\varepsilon,\mathbf{P}} \\
&+ \varepsilon^2\Delta\psi_{\varepsilon,\mathbf{P}} - \mu\psi_{\varepsilon,\mathbf{P}} + h(W_{\varepsilon,\mathbf{P}} + \psi_{\varepsilon,\mathbf{P}}) - \frac{1}{|\Omega|}\int_\Omega h(W_{\varepsilon,\mathbf{P}} + \psi_{\varepsilon,\mathbf{P}})dx\Bigg] \\
=&\pi^{\perp}_{\varepsilon,\mathbf{P}}\Bigg\{\sum_{i=1}^k[\varepsilon^2\Delta(W_{\varepsilon,P_i} - w_{\varepsilon,P_i}) - \mu(W_{\varepsilon,P_i} - w_{\varepsilon,P_i})] \\
&+ \sum_{i=1}^k[\varepsilon^2\Delta w_{\varepsilon,P_i} - \mu w_{\varepsilon,P_i} + h(w_{\varepsilon,P_i})] + \Bigg[\varepsilon^2\Delta\psi_{\varepsilon,\mathbf{P}} - \mu\psi_{\varepsilon,\mathbf{P}} \\
&+ h(W_{\varepsilon,\mathbf{P}} + \psi_{\varepsilon,\mathbf{P}}) - \sum_{i=1}^k h(w_{\varepsilon,P_i}) - \frac{1}{|\Omega|}\int_\Omega h(W_{\varepsilon,\mathbf{P}} + \psi_{\varepsilon,\mathbf{P}})dx\Bigg]\Bigg\} \\
=&\pi^{\perp}_{\varepsilon,\mathbf{P}}\Bigg[\varepsilon^2\Delta\psi_{\varepsilon,\mathbf{P}} - \mu\psi_{\varepsilon,\mathbf{P}} + h(W_{\varepsilon,\mathbf{P}} + \psi_{\varepsilon,\mathbf{P}}) \\
&- \sum_{i=1}^k h(w_{\varepsilon,P_i}) - \frac{1}{|\Omega|}\int_\Omega h(W_{\varepsilon,\mathbf{P}} + \psi_{\varepsilon,\mathbf{P}})\Bigg] \qquad \text{(A.15)}
\end{aligned}$$

Let $e_3(x) = e_{31}(x) + e_{32}(x) + e_{33}(x)$, where e_{3i}, $i = 1, 2, 3$ are specified as follows. Let e_{31} be the unique solution of

$$\begin{cases} \varepsilon^2\Delta v - \mu v = 0 & \text{in} \quad \Omega, \\ \frac{\partial v}{\partial n} = -\frac{1}{\varepsilon}\frac{\partial}{\partial n}\sum_{i=1}^k \alpha_i(x)\psi_0(\Psi_{\varepsilon,P_i}(x)) & \text{on} \quad \partial\Omega. \end{cases} \qquad \text{(A.16)}$$

And

$$\begin{aligned}
e_{32}(x) &= -\frac{1}{\varepsilon}\pi_{\varepsilon,\mathbf{P}}\sum_{i=1}^k \alpha_i(x)\psi_0(\Psi_{\varepsilon,P_i}(x)) - \pi_{\varepsilon,\mathbf{P}}e_{31}(x), \\
e_{33}(x) &= e_3(x) - e_{31}(x) - e_{32}(x).
\end{aligned}$$

Since $\psi_{\varepsilon,\mathbf{P}} \in K^{\perp}_{\varepsilon,\mathbf{P}}$, then we have

$$\begin{cases} 0 & = \varepsilon\pi_{\varepsilon,\mathbf{P}}(\sum_{i=1}^k \alpha_i\psi_0) + \varepsilon^2\pi_{\varepsilon,\mathbf{P}}e_{31} + \varepsilon^2\pi_{\varepsilon,\mathbf{P}}e_{32} + \varepsilon^2\pi_{\varepsilon,\mathbf{P}}e_{33}, \\ \psi_{\varepsilon,\mathbf{P}} & = \varepsilon\pi^{\perp}_{\varepsilon,\mathbf{P}}(\sum_{i=1}^k \alpha_i\psi_0) + \varepsilon^2\pi^{\perp}_{\varepsilon,\mathbf{P}}e_{31} + \varepsilon^2\pi^{\perp}_{\varepsilon,\mathbf{P}}e_{32} + \varepsilon^2\pi^{\perp}_{\varepsilon,\mathbf{P}}e_{33}. \end{cases}$$

By the definition, $e_{33} \in K^{\perp}_{\varepsilon,\mathbf{P}}$.

Claim : $||e_{31}||_{2,\varepsilon} \le C$, $||e_{32}||_{2,\varepsilon} \le C$.

From (A.16), and ψ_0 is exponentially decaying,

$$\left|\left|\varepsilon \frac{\partial e_{31}}{\partial n}\right|\right|^2_{L^2(\partial\Omega)} \le C\varepsilon^{n-1}.$$

Then by Lemma A.1,

$$||e_{31}||_{1,\varepsilon} \le C.$$

Also by (A.16), $\varepsilon^2||\Delta e_{31}||_{0,\varepsilon} = \mu||e_{31}||_{0,\varepsilon}$, by Appendix B of [WW1],

$$||e_{31}||_{2,\varepsilon} \le C(\varepsilon^2||\Delta e_{31}||_{0,\varepsilon} + ||e_{31}||_{1,\varepsilon}) \le C.$$

Since $\pi_{\varepsilon,\mathbf{P}}$ is an L^2 orthonormal projection, then

$$e_{32} = -\sum_{i=1}^{k}\sum_{j=1}^{n-1}\sum_{l=1}^{k}\left\langle \varepsilon^{-1}\alpha_l\psi_0(\Psi_{l,\varepsilon}(x)) + e_{31}, \frac{\partial W_{\varepsilon,P_i}}{\partial\tau_j}\right\rangle \frac{\partial W_{\varepsilon,P_i}}{\partial\tau_j}.$$

Since $||\frac{\partial W_{\varepsilon,P_i}}{\partial\tau_j}||_{2,\varepsilon} \le C$ for all i,j, then we only need to show that

$$\left|\left\langle \varepsilon^{-1}\alpha_l\psi_0(\Psi_{l,\varepsilon}(x)) + e_{31}, \frac{\partial W_{\varepsilon,P_i}}{\partial\tau_j}\right\rangle\right| \le C.$$

We have

$$\begin{aligned}
&\left\langle \varepsilon^{-1}\alpha_l\psi_0(\Psi_{l,\varepsilon}(x)) + e_{31}, \frac{\partial W_{\varepsilon,P_i}}{\partial\tau_j}\right\rangle \\
\le&\varepsilon^{-1}||\alpha_l\psi_0(\Psi_{l,\varepsilon}(x))||_0\left|\left|\frac{\partial W_{\varepsilon,P_i}}{\partial\tau_j}\right|\right|_0 + ||e_{31}||_0\left|\left|\frac{\partial W_{\varepsilon,P_i}}{\partial\tau_j}\right|\right|_0 \\
=&\varepsilon^{-1}O(\varepsilon^{\frac{n}{2}})O(\varepsilon^{\frac{n-2}{2}}) + O(\varepsilon^{\frac{n}{2}})O(\varepsilon^{\frac{n-2}{2}}) \\
=&O(\varepsilon^{n-2}).
\end{aligned}$$

It remains to prove that $||e_{33}||_{2,\varepsilon} \le C$. Note that $e_{33} \in K^{\perp}_{\varepsilon,\mathbf{P}}$ satisfies

$$L_{\varepsilon,\mathbf{P}}e_{33} = -\frac{1}{\varepsilon^2}f_0,$$

where $f_0(x) = L_{\varepsilon,\mathbf{P}}[\psi_{\varepsilon,\mathbf{P}} - \varepsilon\sum_{i=1}^{k}\alpha_i\psi_0(\Psi_{\varepsilon,P_i}) - \varepsilon^2 e_{31} - \varepsilon^2 e_{32}]$. By Lemma 3.2,

$$||L_{\varepsilon,\mathbf{P}}e_{33}||_{0,\varepsilon} \ge \lambda||e_{33}||_{2,\varepsilon}.$$

Let $f(x) = \overline{S}'_\varepsilon(W_{\varepsilon,\mathbf{P}})[\psi_{\varepsilon,\mathbf{P}} - \varepsilon\sum_{i=1}^{k}\alpha_i\psi_0(\Psi_{\varepsilon,P_i}) - \varepsilon^2 e_{31} - \varepsilon^2 e_{32}]$, so that $\pi^{\perp}_{\varepsilon,\mathbf{P}}f = f_0$. It suffices to prove that $||f_0||_{0,\varepsilon} \le C\varepsilon^2$.

$$
\begin{aligned}
f(x) =&\overline{S}_\varepsilon'(W_{\varepsilon,\mathbf{P}})(\psi_{\varepsilon,\mathbf{P}}) - \varepsilon\overline{S}_\varepsilon'(W_{\varepsilon,\mathbf{P}})\left(\sum_{i=1}^k \alpha_i \psi_0(\Psi_{\varepsilon,P_i})\right) \\
& - \varepsilon^2\overline{S}_\varepsilon'(W_{\varepsilon,\mathbf{P}})(e_{31}) - \varepsilon^2\overline{S}_\varepsilon'(W_{\varepsilon,\mathbf{P}})(e_{32}) \\
\equiv& I - \varepsilon II - \varepsilon^2 III - \varepsilon^2 IV.
\end{aligned}
$$

$$
\begin{aligned}
III =&\varepsilon^2\Delta e_{31} - \mu e_{31} + h'(W_{\varepsilon,\mathbf{P}})e_{31} \\
=&h'(W_{\varepsilon,\mathbf{P}})e_{31}.
\end{aligned}
$$

$$
||\pi^\perp_{\varepsilon,\mathbf{P}}\varepsilon^2 III||_{0,\varepsilon} \le ||\varepsilon^2 III||_{0,\varepsilon} \le C\varepsilon^2.
$$

Since $e_{32} \in K_{\varepsilon,\mathbf{P}}$, then there exist $k_{ij} \in \mathbf{R}$, such that

$$
e_{32}(x) = \sum_{i=1}^k \sum_{j=1}^{n-1} k_{ij} \frac{\partial W_{\varepsilon,P_i}}{\partial \tau_j}.
$$

Then

$$
\begin{aligned}
IV =&\varepsilon^2\Delta e_{32} - \mu e_{32} + h'(W_{\varepsilon,\mathbf{P}})e_{32} \\
=&\sum_{i=1}^k \sum_{j=1}^{n-1} k_{ij}\left(\varepsilon^2\Delta\frac{\partial W_{\varepsilon,P_i}}{\partial \tau_j} - \mu\frac{\partial W_{\varepsilon,P_i}}{\partial \tau_j} + h'(w_{\varepsilon,P_i})\frac{\partial w_{\varepsilon,P_i}}{\partial \tau_j}\right) \\
& + \sum_{i=1}^k \sum_{j=1}^{n-1} k_{ij}\left(h'(W_{\varepsilon,\mathbf{P}})\frac{\partial W_{\varepsilon,P_i}}{\partial \tau_j} - h'(w_{\varepsilon,P_i})\frac{\partial w_{\varepsilon,P_i}}{\partial \tau_j}\right) \\
=&\sum_{i=1}^k \sum_{j=1}^{n-1} k_{ij}\left(h'(W_{\varepsilon,\mathbf{P}})\frac{\partial W_{\varepsilon,P_i}}{\partial \tau_j} - h'(w_{\varepsilon,P_i})\frac{\partial w_{\varepsilon,P_i}}{\partial \tau_j}\right).
\end{aligned}
$$

So by Lemma 2.1 and Proposition 2.2,

$$
||\pi^\perp_{\varepsilon,\mathbf{P}}\varepsilon^2 IV||_{0,\varepsilon} \le ||\varepsilon^2 IV||_{0,\varepsilon} \le C\varepsilon^2.
$$

$$
\begin{aligned}
I =&\varepsilon^2\Delta\psi_{\varepsilon,\mathbf{P}} - \mu\psi_{\varepsilon,\mathbf{P}} + h'(W_{\varepsilon,\mathbf{P}})\psi_{\varepsilon,\mathbf{P}} \\
=&\Bigg[\varepsilon^2\Delta\psi_{\varepsilon,\mathbf{P}} - \mu\psi_{\varepsilon,\mathbf{P}} + h(W_{\varepsilon,\mathbf{P}} + \psi_{\varepsilon,\mathbf{P}}) - \sum_{i=1}^k h(w_{\varepsilon,P_i}) \\
& - \frac{1}{|\Omega|}\int_\Omega h(W_{\varepsilon,\mathbf{P}} + \psi_{\varepsilon,\mathbf{P}})dx\Bigg] - h(W_{\varepsilon,\mathbf{P}} + \psi_{\varepsilon,\mathbf{P}}) + \sum_{i=1}^k h(w_{\varepsilon,P_i}) \\
& + \frac{1}{|\Omega|}\int_\Omega h(W_{\varepsilon,\mathbf{P}} + \psi_{\varepsilon,\mathbf{P}})dx + h'(W_{\varepsilon,\mathbf{P}})\psi_{\varepsilon,\mathbf{P}}.
\end{aligned}
$$

$$
\begin{aligned}
II =& \varepsilon^2 \sum_{i=1}^{k} \big[\Delta\big(\alpha_i \psi_0(y)\big) - \mu \alpha_i \psi_0(y) + h'(W_{\varepsilon,\mathbf{P}}) \alpha_i \psi_0(y)\big] \\
=& \sum_{i=1}^{k} \alpha_i \big[\varepsilon^2 \Delta \psi_0(y) - \mu \psi_0(y) + h'(W_{\varepsilon,\mathbf{P}}) \psi_0(y)\big] \\
& + 2\varepsilon^2 \sum_{i=1}^{k} \nabla \alpha_i \cdot \nabla \psi_0(y) + \varepsilon^2 \sum_{i=1}^{k} \Delta \alpha_i \cdot \psi_0(y) \\
=& \sum_{i=1}^{k} \alpha_i \bigg[\Delta \psi_0(y) + |\nabla \rho|^2 \frac{\partial^2 \psi_0(y)}{\partial y_n^2} - 2 \sum_{j=1}^{n-1} \rho_j \frac{\partial^2 \psi_0(y)}{\partial y_j \partial y_n} \\
& - \varepsilon (\Delta \rho) \frac{\partial \psi_0(y)}{\partial y_n} - \mu \psi_0(y)\bigg] + h'(W_{\varepsilon,\mathbf{P}}) \sum_{i=1}^{k} \alpha_i \psi_0(\Psi_{\varepsilon,P_i}) + O(\varepsilon^2) \\
=& \sum_{i=1}^{k} \alpha_i [\Delta \psi_0(y) - \mu \psi_0(y) + h'(w) \psi_0(y) - h'(w) v_1] \\
& - \sum_{i=1}^{k} \alpha_i h'(w) \psi_0(y) + \sum_{i=1}^{k} \alpha_i h'(w) v_1 + h'(W_{\varepsilon,\mathbf{P}}) \sum_{i=1}^{k} \alpha_i \psi_0(\Psi_{\varepsilon,P_i}) + O(\varepsilon) \\
=& - \sum_{i=1}^{k} \alpha_i h'(w) \psi_0(\Psi_{\varepsilon,P_i}) + \sum_{i=1}^{k} \alpha_i h'(w) v_1 + h'(W_{\varepsilon,\mathbf{P}}) \sum_{i=1}^{k} \alpha_i \psi_0(\Psi_{\varepsilon,P_i}) + O(\varepsilon).
\end{aligned}
$$

So by (A.15),

$$
||\pi^{\perp}_{\varepsilon,\mathbf{P}}(I - \varepsilon II)||_{0,\varepsilon} = ||\pi^{\perp}_{\varepsilon,\mathbf{P}} f_1||_{0,\varepsilon} \leq ||f_1||_{0,\varepsilon},
$$

where

$$
\begin{aligned}
f_1 =& - h(W_{\varepsilon,\mathbf{P}} + \psi_{\varepsilon,\mathbf{P}}) + \sum_{i=1}^{k} h(w_{\varepsilon,P_i}) + \frac{1}{|\Omega|} \int_\Omega h(W_{\varepsilon,\mathbf{P}} + \psi_{\varepsilon,\mathbf{P}}) \\
& + h'(W_{\varepsilon,\mathbf{P}}) \psi_{\varepsilon,\mathbf{P}} - \varepsilon \sum_{i=1}^{k} \alpha_i h'(w) \psi_0(\Psi_{\varepsilon,P_i}) + \varepsilon \sum_{i=1}^{k} \alpha_i h'(w) v_1 \\
& + \varepsilon h'(W_{\varepsilon,\mathbf{P}}) \sum_{i=1}^{k} \alpha_i \psi_0(\Psi_{\varepsilon,P_i}) + O(\varepsilon^2).
\end{aligned}
$$

Note that

$$
\begin{aligned}
& \frac{1}{|\Omega|} \int_\Omega h(W_{\varepsilon,\mathbf{P}} + \psi_{\varepsilon,\mathbf{P}}) dx \\
\leq & C \max |h''(u)| \int_\Omega \big(|W_{\varepsilon,\mathbf{P}}|^2 + |\psi_{\varepsilon,\mathbf{P}}|^2\big) dx \\
\leq & C\varepsilon^n + C\varepsilon^{n+2} \\
= & O(\varepsilon^n).
\end{aligned} \tag{A.17}
$$

Hence

$$
\begin{aligned}
f_1 = & - [h(W_{\varepsilon,\mathbf{P}} + \psi_{\varepsilon,\mathbf{P}}) - h(W_{\varepsilon,\mathbf{P}}) - h'(W_{\varepsilon,\mathbf{P}})\psi_{\varepsilon,\mathbf{P}}] \\
& - \sum_{i=1}^{k} \alpha_i [h(W_{\varepsilon,\mathbf{P}}) - h(W_{\varepsilon,P_i})] \\
& - \sum_{i=1}^{k} \alpha_i [h(W_{\varepsilon,P_i}) - h(w_{\varepsilon,P_i}) - \varepsilon h'(w) v_1] \\
& + \varepsilon \sum_{i=1}^{k} \alpha_i \psi_0(\Psi_{\varepsilon,P_i})[h'(W_{\varepsilon,\mathbf{P}}) - h'(W_{\varepsilon,P_i})] \\
& + \varepsilon \sum_{i=1}^{k} \alpha_i \psi_0(\Psi_{\varepsilon,P_i})[h'(W_{\varepsilon,P_i}) - h'(w_{\varepsilon,P_i})] + O(\varepsilon^2).
\end{aligned}
$$

So $||f_1||_{0,\varepsilon} = O(\varepsilon^2)$ because the $||\cdot||_{0,\varepsilon}$ norm of each term above is of order $O(\varepsilon^2)$.

□

Finally, we prove the estimates (4.1), (4.2) and (4.7). First, similar to (A.11), let $x \in \Omega_P$, $z = \frac{x-P}{\varepsilon}$ and $y = \Psi_\varepsilon(x)$, then

$$
\begin{aligned}
|z| - |y| = & \frac{|x-P|^2 - |\Psi(x)|^2}{\varepsilon(|x-P| + |\Psi(x)|)} = \frac{\varepsilon^2 y_n^2 - [\varepsilon y_n + \rho(x'-P')]^2}{\varepsilon(|x-P| + |\Psi(x)|)} \\
= & \frac{-2\varepsilon y_n \rho(x'-P') - \rho^2(x'-P')}{\varepsilon^2(|y| + |z|)} \\
= & - \frac{\varepsilon}{2|y|} \sum_{i,j=1}^{n-1} \rho_{ij}(0) y_i y_j y_n + O(\varepsilon^2 |y|^2).
\end{aligned} \tag{A.18}
$$

By (2.14), we have

$$
\frac{\partial W_{\varepsilon,P_i}}{\partial \tau_j} = \frac{\partial w_{\varepsilon,P_i}}{\partial \tau_j} - \alpha_i(x) u_0(\Psi_{\varepsilon,P_i}(x)) - \varepsilon e_2(x).
$$

By (A.6) and (A.18),

$$
\begin{aligned}
\frac{\partial w_{\varepsilon,P_i}}{\partial \tau_j} = & \frac{\partial w\left(\frac{x-P_i}{\varepsilon}\right)}{\partial x_j} = \frac{1}{\varepsilon}\frac{\partial w(z)}{\partial y_j} - \frac{\rho_j(x'-P')}{\varepsilon}\frac{\partial w(z)}{\partial y_n} \\
= & \frac{1}{\varepsilon}\frac{\partial w(y)}{\partial y_j} + \frac{1}{\varepsilon}\frac{|y|w''(|y|) - w'(|y|)}{|y|^2} y_j(|z| - |y|) - \sum_{l=1}^{n-1} \rho_{lj}(0) y_l \frac{\partial w(y)}{\partial y_n} + O(\varepsilon e^{-k|y|}) \\
= & \frac{1}{\varepsilon}\frac{\partial w(y)}{\partial y_j} - \frac{|y|w''(|y|) - w'(|y|)}{2|y|^3} \sum_{l,k=1}^{n-1} \rho_{lk}(0) y_l y_k y_j y_n \\
& - \sum_{l=1}^{n-1} \rho_{lj}(0) y_l \frac{\partial w(y)}{\partial y_n} + O(\varepsilon e^{-k|y|}).
\end{aligned} \tag{A.19}
$$

Then (4.1) follows. (4.2) can be obtained by using Taylor expansion and (A.18). Note that $h \in C^3$, so we can use the Taylor expansion up to the third term.

Proof of (4.7). Without loss of generality, we assume that $y_1 = y_j$ and $y_2 = y_l$. We introduce the polar coordinates for $\mathbf{R}^{n-1}$:

$$\begin{cases} y_1 & = r\sin\theta_{n-2}\sin\theta_{n-3}\sin\theta_2\sin\theta_1, \\ y_2 & = r\sin\theta_{n-2}\sin\theta_{n-3}\sin\theta_2\cos\theta_1, \\ y_3 & = r\sin\theta_{n-2}\sin\theta_{n-3}\cos\theta_2, \\ \cdots, & \\ y_{n-1} & = r\cos\theta_{n-2}. \end{cases}$$

Note that $\mathbf{R}^{n-1} = \{(r, \theta_1, \cdots, \theta_{n-2}) : r > 0, 0 \le \theta_1 < 2\pi, 0 \le \theta_k \le \pi \text{ for } k = 2, \cdots, n-2\}$ and that

$$dy = r^{n-2}\sin\theta_2\sin^2\theta_3\cdots\sin^{n-3}\theta_{n-2}drd\theta_1\cdots d\theta_{n-2}.$$

Therefore

$$\int_{\mathbf{R}^{n-1}}\left[\frac{w'(|y|)}{|y|}\right]^2 y_1^4 dy = A\int_0^{2\pi}\sin^4\theta_1 d\theta_1,$$
$$\text{and} \quad \int_{\mathbf{R}^{n-1}}\left[\frac{w'(|y|)}{|y|}\right]^2 y_1^2 y_2^2 dy = A\int_0^{2\pi}\sin^2\theta_1\cos^2\theta_1 d\theta_1,$$

where

$$A = I_2\cdots I_{n-2}\int_0^\infty [w'(r)]^2 r^n dr,$$

and $I_k = \int_0^\pi \sin^{k+3}\theta d\theta$ for $k = 2, \cdots, n-2$. Then (4.7) can be easily followed from that

$$\int_0^{2\pi}\sin^4\theta d\theta = \frac{3}{4}\pi, \qquad \int_0^{2\pi}\sin^2\theta\cos^2\theta d\theta = \frac{1}{4}\pi.$$

□

Appendix B. Proof of Proposition 1.3.

Proof of Proposition 1.3. (i), (ii), (iii) are well-known, we refer to [GNN] pg. 370 Theorem 2 and [BL] pg. 317 Theorem 1 for proofs. It is easy to verify that $\{\frac{\partial w}{\partial x_1}, \cdots, \frac{\partial w}{\partial x_n}\} \subset Ker(L_0)$, but $\lambda_1(L_0)$ must be simple, therefore $\lambda_1 = \lambda_1(L_0) < 0$ and the corresponding eigenfunction ϕ is positive. Since ϕ is positive, then it must be radial by [GNN]. Thus (iv) is proved. To prove (v), we use an argument in [NT2]. Let S^{n-1} be the standard unit sphere in $\mathbf{R}^n$. It is well-known that the eigenvalues of the Laplace-Beltrami operator $-\Delta$ on S^{n-1} are $0 = \nu_0 < \nu_1 = \nu_2 = \cdots = \nu_n = (n-1) < \nu_{n+1} < \cdots$. We denote the k-th eigenfunction by e_k, $k = 0, 1, 2, \cdots$, then $\{e_k : k = 0, 1, 2, \cdots\}$ is a complete orthogonal basis of $L^2(S^{n-1})$. Let ψ be a eigenfunction of L_0 with eigenvalue $\lambda \le 0$, *i.e.*,

$$\begin{cases} \Delta\psi - \mu\psi + h'(w)\psi = -\lambda\psi & \text{in} \quad \mathbf{R}^n, \\ \psi \to 0 & x \to \infty. \end{cases}$$

Define

$$\psi_k(r) = \int_{S^{n-1}} \psi(r,\theta) e_k(\theta) d\theta$$

Then for any $k > 0$, $\psi_k(0) = 0$, and $\psi_k(r)$ satisfies

$$|D^\alpha \psi_k(r)| \le Ce^{-Kr} \quad \text{for } r \in (0,\infty) \tag{B.1}$$

with $|\alpha| \le 2$. By direct calculation, ψ_k satisfies

$$\psi_k''(r) + \frac{n-1}{r}\psi_k'(r) - \mu\psi + \left[h'(w) - \frac{\nu_k}{r^2}\right]\psi = -\lambda\psi_k \tag{B.2}$$

for all $r \in (0,\infty)$. On the other hand, w_r satisfies

$$w_r''(r) + \frac{n-1}{r}w_r'(r) - mw_r + \left[h'(w) - \frac{n-1}{r^2}\right] w_r = 0 \tag{B.3}$$

for all $r \in (0,\infty)$ and $w_r(r) < 0$. Multiplying (B.2) by $r^{n-1}w_r$ and multiplying (B.3) by $r^{n-1}\psi_k$, then substracting and integrating over $(0,\rho)$, we obtain

$$\rho^{n-1}[\psi_k'(\rho)w_r(\rho) - w_r'(\rho)\psi_k(\rho)] + (n-1-\nu_k)\int_0^\rho r^{n-3}w_r\psi_k dr$$
$$= -\lambda\int_0^\rho r^{n-1}w_r\psi_k dr. \tag{B.4}$$

We claim that for $k > n$, $\psi_k \equiv 0$. Suppose not, we can assume that $\psi_k > 0$ near 0. If there is a $\rho > 0$ such that $\psi_k(\rho) = 0$ and $\psi_k(r) > 0$ in $(0,\rho)$, then the left hand side of (B.4) is $\rho^{n-1}[\psi_k'(\rho)u_r(\rho) + (n-1-\nu_k)\int_0^\rho r^{n-3}u_r\psi_k dr > 0$, and the right hand side is $-\lambda\int_0^\rho r^{n-1}w_r\psi_k dr \le 0$. Therefore $\psi_k > 0$ for all $r > 0$. Since w_r, w_{rr}, ψ_k and ψ_k' are all exponentially decay at infinity, we have

$$\rho^{n-1}[\psi_k'(\rho)w_r(\rho) - w_r'(\rho)\psi_k(\rho)] \to 0$$

when $\rho \to 0$. Let ρ in (B.4) be large enough, then the left hand side is $(n-1-\nu_k)\int_0^\rho r^{n-3}u_r\psi_k dr + O(r^m e^{-r}) > 0$, while the right hand side is $-\lambda\int_0^\rho r^{n-1}w_r\psi_k dr \le 0$. That is again a contradiction. Hence, for $k > n$, $\psi_k \equiv 0$.

Since $\{e_k : k = 0,1,2,\cdots\}$ is a complete orthogonal basis of $L^2(S^{n-1})$, we have

$$\psi(x) = \psi(r,\theta) = \sum_{k=0}^n \psi_k(r)e_k(\theta).$$

Moreover, if $\lambda < 0$, a similar argument as above shows $\psi_k \equiv 0$ for $k = 1,\cdots,n$, hence $\psi(x) = \psi_0(r)e_0(\theta) = \psi_0(r)$, which implies ψ is radial if $\lambda < 0$. But, as showed in [D4] and [OS1], if g, h satisfies (g1-3) and (g5), then the solution ξ of variational equation

$$\begin{cases} \xi'' + \frac{n-1}{r}\xi' - \mu\xi + h'(w)\xi = 0 & \text{in} \quad (0,\infty), \\ \xi(0) = 1, \qquad \xi'(0) = 0. \end{cases}$$

has exactly one zero and $\lim_{r\to\infty}\xi(0) < 0$, which implies L_0 has only one negative eigenvalue if we restrict the domain of L_0 to $H_r^2(\mathbf{R}^n) = \{u \in H^2(\mathbf{R}^n) : u(x) = u(|x|)\}$. Therefore, 0 must be the second eigenvalue of L_0. At last, we can similarly prove that $Ker(L_0)$ is spaned by $\{\frac{\partial w}{\partial x_1},\cdots,\frac{\partial w}{\partial x_n}\}$, for detail see [NT2] Lemma 4.2.

□

Appendix C. Apriori estimates.

Here we will give some results involving the L^∞ norm of the solutions of (1.2) or (1.7), by using the maximal principle. First, the second positive zero c is always the upper bound of all solutions of (1.2).

Lemma C.1. *Suppose that g satisfies (g1-3). Let v be a non-constant solution of (1.2) and $\sigma_\varepsilon = \frac{1}{|\Omega|}\int_\Omega h(v)dx$. Then $\sigma_\varepsilon > 0$, $v(x) < c$ for any $x \in \overline{\Omega}$.*

Proof. Since $\int_\Omega v(x)dx = 0$ and v is not constant, then $v(x_0) = \min_{x\in\Omega} v(x) < 0$ and $\Delta v(x_0) \geq 0$. Therefore

$$\sigma_\varepsilon = \varepsilon^2 \Delta v(x_0) + g(v(x_0)) > 0 \tag{C.1}$$

Suppose that $v(x_1) = \max_{x\in\Omega} v(x) \geq c$, then $\sigma_\varepsilon = \varepsilon^2\Delta v(x_1) + g(v(x_1)) \leq 0$, that contradicts $\sigma_\varepsilon > 0$. Hence $v(x) < c$ for all x.

□

For g only satisfying (g1-3), a lower bound usually does not exist, especially when g has many "zig-zags" in $(-\infty, 0)$. But if we assume that g also satisfies
(g6) There exists $u_0 > 0$ such that $g' \leq 0$ for any $u < u_0$,
then there is also a lower bound. Note that (g6) is equivelent to a convex condition on potential $F(u)$.

Lemma C.2. *Suppose that g satisfies (g1-3) and (g6). Let $m = \max_{0\leq u\leq c} g(u)$, and $u_1 = \min\{u \leq u_0 : g(u) \geq m\}$. Then for any solution v of (1.2), $v(x) > u_1$ for any $x \in \overline{\Omega}$.*

Proof. Suppose not, then $v(x_0) = \min_{x\in\Omega} v(x) \leq u_1$. Therefore

$$\varepsilon^2 \Delta v(x_0) = \frac{1}{|\Omega|}\int_\Omega [g(v(x)) - g(v(x_0))] < 0,$$

that is a contradiction.

□

Lemma C.2 holds for any $\varepsilon > 0$ and any type of solutions of (1.2). But if we have certain information of σ_ε as $\varepsilon \to 0$, conditions like (g6) are not necessary to obtain a lower bound.

Lemma C.3. *Suppose that g satisfies (g1-3), and $\tilde{h}, \tilde{g}$ are as defined in Section 1. If $\sigma_\varepsilon = \frac{1}{|\Omega|}\int_\Omega \tilde{h}(v_\varepsilon)dx \to 0$ as $\varepsilon \to 0$ where v_ε is a solution of (1.7), then for any given $a > 0$, there exists an $\varepsilon_0 > 0$, such that for all $\varepsilon \in (0, \varepsilon_0)$, $v_\varepsilon(x) \in [-a, c]$.*

Proof. Since v_ε satisfies

$$\begin{cases} \varepsilon^2 \Delta v + \tilde{g}(v) - \sigma_\varepsilon = 0 & \text{in} \quad \Omega, \\ \frac{\partial v}{\partial n} = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

$\sigma_\varepsilon \to 0$ as $\varepsilon \to 0$ and $\tilde{g}(u) > \delta_4$ for $u \in (-\infty, -a]$, then there exists a $\varepsilon_0 > 0$, such that $\tilde{g}(u) - \sigma_\varepsilon > 0$ for any $\varepsilon \in (0, \varepsilon_0)$ and $u \leq -a$. We claim that $v_\varepsilon(x) \geq -a$ for any $\varepsilon \in (0, \varepsilon_0)$ and any $x \in \overline{\Omega}$. In fact, if $v_\varepsilon(x_1) = \min_{x \in \overline{\Omega}} v_\varepsilon(x) < -a$, then there exists a ball $B \subset \Omega$, such that $x_1 \in \partial B$, $v_\varepsilon(x) < -a$ in B. Since $\varepsilon^2 \Delta v_\varepsilon(x) = -\tilde{g}(v_\varepsilon(x)) + \sigma_\varepsilon < 0$ in B. By Hopf Boundary Lemma, $\frac{\partial v_\varepsilon}{\partial n} < 0$ at x_1. But this is impossible, since if $x_1 \in \Omega$, $\nabla v(x_1) = 0$, and if $x_1 \in \partial\Omega$, then $\frac{\partial v}{\partial n}(x_1) = 0$ by the boundary condition. So $v_\varepsilon(x) \geq -a$. Similarly, since $\tilde{g}(u) \leq 0$ for any $u \in (c, \infty)$, then same as that of Lemma C.1, we can show that $v_\varepsilon(x) \leq c$.

□

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