Journal of Functional Analysis **169**, 494–531 (1999)
Article ID jfan.1999.3483, available online at http://www.idealibrary.com on IDEAL®

Persistence and Bifurcation of Degenerate Solutions

Junping Shi

Department of Mathematics, Tulane University, New Orleans, Louisiana 70118
E-mail: shij@math.tulane.edu

Communicated by R. B. Melrose

Received April 28, 1999; accepted July 20, 1999

We consider a nonlinear equation $F(\varepsilon, \lambda, u) = 0$, where F is a differentiable mapping from $\mathbf{R} \times \mathbf{R} \times X$ to Y and X, Y are Banach spaces. When ε varies from a fixed $\varepsilon = \varepsilon_0$, bifurcation occurs to the solution curve $(\lambda(s), u(s))$. We study the degenerate solutions of the equation, and we obtain several bifurcation theorems on the degenerate solutions, which can be applied in many nonlinear problems to obtain precise global bifurcation diagrams.

1. INTRODUCTION

Bifurcation phenomena occur frequently in solving nonlinear equations. Here we consider an equation

$$F(\varepsilon, \lambda, u) = 0, \tag{1.1}$$

where $F: \mathbf{R} \times \mathbf{R} \times X \to Y$ is a nonlinear differentiable map and X, Y are Banach spaces.

First we consider the problem for a fixed $\varepsilon = \varepsilon_0$. Let (λ_0, u_0) be a solution of $F(\varepsilon_0, \cdot, \cdot) = 0$. If $F_u(\varepsilon_0, \lambda_0, u_0)$ is a linear homeomorphism (so $(\varepsilon_0, \lambda_0, u_0)$ is a *nondegenerate solution*), then by the implicit function theorem, for λ close to λ_0, $F(\varepsilon_0, \cdot, \cdot) = 0$ has a unique solution $(\lambda, u(\lambda))$, and $u(\lambda)$ is continuously differentiable with respect to λ. Thus λ_0 is a bifurcation point only if $F_u(\varepsilon_0, \lambda_0, u_0)$ is singular. If $(\varepsilon_0, \lambda_0, u_0)$ is a solution of $F = 0$ with $N(F_u(\varepsilon_0, \lambda_0, u_0)) \neq \emptyset$, then we call $(\varepsilon_0, \lambda_0, u_0)$ a *degenerate solution*. (From now on, we denote $\mathbf{R} \times \mathbf{R} \times X$ by M, and whenever there is no confusion, we will use F_u, *etc.*, instead of $F_u(\varepsilon_0, \lambda_0, u_0)$, *etc.*) Very commonly and also generically, at a degenerate solution $(\varepsilon_0, \lambda_0, u_0)$, 0 is a *simple eigenvalue* of F_u, that is,

0022-1236/99 $30.00

(F1) $\dim N(F_u(\varepsilon_0, \lambda_0, u_0)) = \operatorname{codim} R(F_u(\varepsilon_0, \lambda_0, u_0)) = 1$, and $N(F_u(\varepsilon_0, \lambda_0, u_0)) = span\{w_0\}$,

where $N(F_u)$ and $R(F_u)$ are the null space and the range of linear operator F_u. In [CR2], Crandall and Rabinowitz proved

THEOREM 1.1 [CR2, Theorem 3.2]. *Let $F: M \to Y$ be continuously differentiable. At $(\varepsilon_0, \lambda_0, u_0) \in M$, $F(\varepsilon_0, \lambda_0, u_0) = 0$, F satisfies* (F1) *and*

(F2) $F_\lambda(\varepsilon_0, \lambda_0, u_0) \notin R(F_u(\varepsilon_0, \lambda_0, u_0))$.

Then for fixed $\varepsilon = \varepsilon_0$, the solutions of (1.1) *near (λ_0, u_0) form a C^1 curve $(\lambda(s), u(s))$, $\lambda(0) = \lambda_0$, $u(0) = u_0$, $\lambda'(0) = 0$ and $u'(0) = w_0$. Moreover, if F is k-times continuously differentiable, so are $\lambda(s)$, $u(s)$.*

If $\lambda''(0) \neq 0$, then the solutions near $(\varepsilon_0, \lambda_0, u_0)$ form a curve with a turning point. Depending on the sign of $\lambda''(0)$, $F(\varepsilon_0, \cdot, \cdot) = 0$ has zero or two solutions on the left or right hand side of λ_0. For this reason, sometimes we also call (λ_0, u_0) a *turning point* on the solution curve for fixed ε_0.

Another situation is when $F(\varepsilon_0, \lambda, 0) \equiv 0$ for all $\lambda \in \mathbf{R}$, a bifurcation from the trivial solution $u = 0$ occurs if $F_u(\varepsilon_0, \lambda_0, 0)$ is not invertible. In [CR1], Crandall and Rabinowitz proved:

THEOREM 1.2 [CR1, Theorem 1.7]. *Let $F: M \to Y$ be continuously differentiable. Suppose that $F(\varepsilon_0, \lambda, u_0) = 0$ for $\lambda \in \mathbf{R}$, the partial derivative $F_{\lambda u}$ exists and is continuous. At $(\varepsilon_0, \lambda_0, u_0) \in M$, F satisfies* (F1) *and*

(F3) $F_{\lambda u}(\varepsilon_0, \lambda_0, u_0)[w_0] \notin R(F_u(\varepsilon_0, \lambda_0, u_0))$.

Then for fixed $\varepsilon = \varepsilon_0$, the solutions of (1.1) *near (λ_0, u_0) consists precisely of the curves $u = u_0$ and $(\lambda(s), u(s))$, $s \in I = (-\delta, \delta)$, where $(\lambda(s), u(s))$ are C^1 functions such that $\lambda(0) = \lambda_0$, $u(0) = u_0$, $u'(0) = w_0$.*

There is a degenerate solution $(\varepsilon_0, \lambda_0, u_0)$ of F in each of these two results. This paper is mainly concerned with the persistence and the bifurcation of degenerate solutions when ε varies near ε_0. Such problems arise naturally when we have a two-parameter equation $F(\varepsilon, \lambda, u) = 0$, and such information is important in determining the global bifurcation diagram of (1.1). We summarize our results here: (The precise statements are given in Section 2.)

(1) Let $(\varepsilon_0, \lambda_0, u_0)$ be the degenerate solution in Theorem 1.1. If it also satisfies

(F4) $F_{uu}(\varepsilon_0, \lambda_0, u_0)[w_0, w_0] \notin R(F_u(\varepsilon_0, \lambda_0, u_0))$,

then (1.1) has a unique degenerate solution $(\varepsilon, \lambda(\varepsilon), u(\varepsilon))$ near $(\varepsilon_0, \lambda_0, u_0)$ for ε near ε_0;

(2) Let $(\varepsilon_0, \lambda_0, u_0)$ be the degenerate solution in Theorem 1.1. If it also satisfies

$$\text{(F4}'\text{)} \quad F_{uu}(\varepsilon_0, \lambda_0, u_0)[w_0, w_0] \in R(F_u(\varepsilon_0, \lambda_0, u_0)),$$

and (2.5), then the degenerate solutions near $(\varepsilon_0, \lambda_0, u_0)$ form a curve $(\varepsilon(s), \lambda(s), u(s))$ for $s \in (-\delta, \delta)$ with $\varepsilon'(0) = \lambda'(0) = 0$;

(3) Let $(\varepsilon_0, \lambda_0, u_0)$ be the degenerate solution in Theorem 1.2. If it also satisfies

$$\text{(F5)} \quad F_\varepsilon(\varepsilon_0, \lambda_0, u_0) \notin R(F_u(\varepsilon_0, \lambda_0, u_0)),$$

then the degenerate solutions near $(\varepsilon_0, \lambda_0, u_0)$ form a curve $(\varepsilon(s), \lambda(s), u(s))$ for $s \in (-\delta, \delta)$ with $\varepsilon'(0) = 0$.

The tools to prove the results above are the implicit function theorem and Theorem 1.1. Our results can be applied to a wide class of nonlinear equations, and our main motivation of these abstract results comes from the study of nonlinear eigenvalue problems like

$$\Delta u + \lambda f(u) = 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{1.2}$$

The solution set of (1.2) is often a union of curves on (λ, u) space. While many tools are available for proving the existence of one or multiple solutions for various nonlinearities f and certain parameters λ, there are very few examples that the solution set can be precisely determined. When the domain Ω in (1.2) is the unit ball in $\mathbf{R}^n$, all positive solutions are radially symmetric and can be globally parameterized by $\|u\|_\infty$. So it is possible to determine the precise global bifurcation diagram. In [OS1, OS2], Ouyang and the author use a unified approach to show that the exact global bifurcation diagrams can be obtained for a very wide class of nonlinearities f. (See also many references in [OS2] for extensive works on this direction.)

The nonlinearity f in [OS1, OS2] satisfies $(f(u)/u)'$ change sign at most once in $(0, \infty)$, and the corresponding solution curve is connected with at most one turning point. This phenomena was observed earlier by P. L. Lions [L], and he conjectured that the solution curve of (1.2) in (λ, u) space resembles the graph of $\lambda = u/f(u)$. This conjecture is not true if the domain Ω is too complicated or there exist solutions with higher ($\geqslant 2$) Morse index. But it is also verified to be true for all results in [OS2]. In this paper, we use the abstract bifurcation results above combining the techniques in [OS2] to show that for some examples of f for which $(f(u)/u)'$ changes sign twice, the solution curve has exactly two turning points, and it is exactly S-shaped or reversed S-shaped. The basic idea here is to add a parameter ε in the nonlinearity f, perturb the solution curve obtained in [OS2] when f is perturbed, and track the persistence and the bifurcation of turning points.

The abstract results can be applied in two different situations. First we can study perturbation problem. For example, if we know the precise global bifurcation diagram for (1.2) when $\varepsilon=\varepsilon_0$, then a question is whether the same diagram will persist near $\varepsilon=\varepsilon_0$, or some parts will persist while a local bifurcation occurs in other parts. We find the results (1) and (3) above are very efficient when applied to such occasions. Another situation is the homotopy problem. If the bifurcation for certain nonlinearity f_1 is difficult, but the bifurcation for another one f_2 is easier, then it is possible to find a one parameter family of nonlinear function $\varepsilon \to f(\varepsilon, \cdot)$ such that $f(0,u)=f_1(u)$ and $f(1,u)=f_2(u)$, and we can study the persistence or variation of bifurcation diagrams when the parameter ε goes from 0 to 1. The result (2) can serve in such an approach. However, this seems to be more difficult since the estimates involved are one level more complicated than the ones in perturbation problems. In Section 6, we use the results (1), (2) and (3) to classify the global bifurcation diagrams of one-sign solutions of (1.2) for $f(\varepsilon,u)=(u+\varepsilon)^3-b(u+\varepsilon)^2+c(u+\varepsilon)$ with $b, c>0$, $3c>4b^2$, $\Omega=(-1,1)$ for all $\varepsilon\in\mathbf{R}$ and $\lambda>0$. In particular, an evolution of monotone curve to curve with two turning points is shown. (See Fig. 6.)

Our work is partially motivated by an earlier work of Dancer [Da] and recent works of Du and Lou [DL1, DL2] on the S-shaped solution curves. In fact, The result (1) (see Theorem 2.1 in Section 2) was implicitly included in Theorem 2 and Remark after it in [Da]. A related discussion can be found in the appendix of [DL1]. S-shaped solution curve for perturbed Gelfand equation $\Delta u+\lambda\exp[u/(1+\varepsilon u)]=0$, in Ω and $u=0$ on $\partial\Omega$ and its variants have been the subject of many previous studies, for example, [BIS, Da, Du, DL2, HM, KL, W, WL]. An ultimate goal of this study is to completely classify the bifurcation diagrams for perturbed Gelfand equation and any $\varepsilon>0$ and at least Ω being a ball. An evolution from a monotone curve to S-shaped curve is expected when ε decreases from 1/4 to 0. (See Fig. 7.) Though this is still not achieved in this paper, but the problem has been reduced to prove certain integral is always negative. (See Subsection 6.5 for a more detailed discussion of this equation.)

Our results also relate to the infinite dimensional singularity theory. (See the survey by Church and Timourian [CT].) From that view, a degenerate solution is a singularity of the map $F(\varepsilon,\lambda,u)$, the degenerate solution in (1) is called a *fold*, and the one in (2) is called *cusp*. The singularity in (3) somehow differs from the ones in [CT]. A systematic study of bifurcation theory can also be found in Chow and Hale [CH].

We organize our paper in the following way: in Section 2, we give the precise statements of our main abstract theorems, and the proof of the results on degenerate solutions are given in Section 3. In Section 4, some computations of turning direction of the curve of degenerate solutions are given, and the results on solution curves of (1.1) are proved. In Section 5, we apply the abstract results to semilinear elliptic problems, and in Section 6, several examples of both local and global bifurcation are presented. In the paper, we use $\|\cdot\|_X$ (or $\|\cdot\|$ when no confusion) as the norm of Banach space X, $\langle\cdot,\cdot\rangle_X$ (or $\langle\cdot,\cdot\rangle$ when no confusion) as the duality pair of a Banach space X and its dual space X^* or the inner-product in a Hilbert space X for whichever is appropriate in the context. We will use $\|\cdot\|_2$ and $\|\cdot\|_\infty$ to denote the norm of $L^2(\Omega)$ and $L^\infty(\Omega)$ respectively. For a nonlinear operator F, we use either F_u as the partial derivative of F with respect to argument u. For a linear operator L, we use $N(L)$ as the null space of L and $R(L)$ as the range of L.

2. STATEMENTS OF MAIN RESULTS

To study the turning points, we consider a system of equations

$$\begin{cases} F(\varepsilon, \lambda, u) = 0, \\ F_u(\varepsilon, \lambda, u)[w] = 0. \end{cases} \tag{2.1}$$

However, the solution of (2.1) is not deterministic since w can be substituted by kw, $k \neq 0$, and the equation is still satisfied. A natural restriction on w is assuming $\|w\| = 1$. However, to seek the solution w on the unit sphere of a Banach space, the geometry of Banach space has to play a role here, which we try to avoid. In fact, the unit ball in a Banach space is a differentiable (Banach) manifold only if X is uniformly convex. So instead we restrict w belonging to a hyperplane of codimension one.

Our setting of the problem is as follows: let $(\varepsilon_0, \lambda_0, u_0, w_0)$ be a solution of (2.1) which satisfies (F1), where $(\varepsilon_0, \lambda_0, u_0) \in M$, and $w_0 \in X_1 \equiv \{x \in X : \|x\| = 1\}$. By the Hahn–Banach Theorem (see Lemma 7.1), there exists a closed subspace X_3 of X with codimension 1 such that $X = L(w_0) \oplus X_3$, where $L(w_0) = span\{w_0\}$, and $d(w_0, X_3) = \inf\{\|w - x\| : x \in X_3\} > 0$. Let $X_2 = w_0 + X_3 = \{w_0 + x : x \in X_3\}$. Then X_2 is a closed hyperplane of X with codimension 1. Since X_3 is a closed subspace of X, then X_3 is also a Banach space in the subspace topology. Hence we can regard $M_1 = M \times X_2$ as a Banach space with product topology. Moreover, the tangent space of M_1 is homeomorphic to $M \times X_3$.

We shall look for the solution of (2.1) in M_1. Define

$$H(\varepsilon, \lambda, u, w) = \begin{pmatrix} F(\varepsilon, \lambda, u) \\ F_u(\varepsilon, \lambda, u)[w] \end{pmatrix}, \tag{2.2}$$

where $(\varepsilon, \lambda, u, w) \in M_1$. First we study the situation in Theorem 1.1. The turning point persists under perturbation if (F4) is satisfied.

THEOREM 2.1. *Let F be twice continuously differentiable. For* $T_0 = (\varepsilon_0, \lambda_0, u_0, w_0) \in M_1$, $H(T_0) = (0, 0)$, *and T_0 satisfies* (F1), (F2), *and* (F4). *Then there exists $\delta > 0$ such that all the solutions of $H(\varepsilon, \lambda, u, w) = (0, 0)$ near T_0 are in a form*

$$\{T_\varepsilon = (\varepsilon, \lambda(\varepsilon), u(\varepsilon), w(\varepsilon)) : \varepsilon \in (\varepsilon_0 - \delta, \varepsilon_0 + \delta)\}, \tag{2.3}$$

where $u(\varepsilon_0) = u_0$, $w(\varepsilon_0) = w_0$ and $\lambda(\varepsilon_0) = \lambda_0$. And,

$$\lambda'(\varepsilon_0) = -\lambda_0 \frac{\langle l, F_\varepsilon(\varepsilon_0, \lambda_0, u_0) \rangle}{\langle l, F_\lambda(\varepsilon_0, \lambda_0, u_0) \rangle}, \tag{2.4}$$

where $l \in Y^*$ satisfying $N(l) = R(F_u(\varepsilon_0, \lambda_0, u_0))$. Moreover, if $F \in C^k$, so are $u(\varepsilon)$, $\lambda(\varepsilon)$, and $w(\varepsilon)$.

When (F4) fails, a bifurcation of turning points occurs at T_0.

THEOREM 2.2. *Let F be twice continuously differentiable. For* $T_0 = (\varepsilon_0, \lambda_0, u_0, w_0) \in M_1$, $H(T_0) = (0, 0)$, *and T_0 satisfies* (F1), (F2), (F4′), *and*

$$H_\varepsilon(\varepsilon_0, \lambda_0, u_0, w_0) \notin R(H_{(\lambda, u, w)}(\varepsilon_0, \lambda_0, u_0, w_0)). \tag{2.5}$$

Then there exists $\delta > 0$ such that all the solutions of $H(\varepsilon, \lambda, u, w) = (0, 0)$ near T_0 are in a form

$$\{T_s = (\varepsilon(s), \lambda(s), u(s), w(s)) : s \in (-\delta, \delta)\}, \tag{2.6}$$

where $u(s) = u_0 + sw_0 + z_1(s)$, $w(s) = w_0 + s\theta + z_2(s)$, $\lambda(s) = \lambda_0 + z_3(s)$ and $\varepsilon(s) = \varepsilon_0 + \tau(s)$, $\theta \in X_3$ is the unique solution of

$$F_{uu}(\varepsilon_0, \lambda_0, u_0)[w_0, w_0] + F_u(\varepsilon_0, \lambda_0, u_0)[\theta] = 0, \tag{2.7}$$

and $z_i(0) = z_i'(0) = 0$ for $i = 1, 2, 3$, $\tau(0) = \tau'(0) = 0$.

Using these two theorems and a result by Dancer [Da, Theorem 2], we are able to obtain the precise structure of solution set of (1.1) near a degenerate solution as in Theorem 2.1 (a fold) or in Theorem 2.2 (a cusp).

THEOREM 2.3. (1) *Assume the conditions in Theorem* 2.1 *are satisfied, and $\{T_\varepsilon\}$ is defined as in Theorem* 2.1. *Then there exists $\rho_1 > 0$ such that for $\varepsilon \in (\varepsilon_0 - \rho_1, \varepsilon_0 + \rho_1)$, all the solutions of* (1.1) *near T_0 are on a curve $\Sigma_\varepsilon = (\bar{\lambda}(t), \bar{u}(t))$, where $t \in (-\eta, \eta)$ for $\eta = \eta(\varepsilon) > 0$, $\bar{\lambda}(0) = \lambda(\varepsilon)$, $\bar{u}(0) = u(\varepsilon)$, $\bar{\lambda}'(0) = 0$ and $\bar{\lambda}''(0) \neq 0$.* (*See Fig.* 1.)

FIG. 1. Persistence of degenerate solution at a fold.

(2) *Assume the conditions in Theorem* 2.2 *are satisfied, and* $\{T_s\}$ *is defined as in Theorem* 2.2. *In addition we assume that* $F \in C^3(M)$ *and*

(F6) $F_{uuu}(\varepsilon_0, \lambda_0, u_0)[w_0, w_0, w_0] + 3F_{uu}(\varepsilon_0, \lambda_0, u_0)[\theta, w_0] \notin R(F_u(\varepsilon_0, \lambda_0, u_0))$.

Then $\varepsilon''(0) \neq 0$, $\lambda''(0) \neq 0$, *and there exists* $\rho_1 > 0$ *such that for* $\varepsilon \in (\varepsilon_0 - \rho_1, \varepsilon_0 + \rho_1)$, *all the solutions of* (1.1) *near* T_0 *are on a curve* $\Sigma_\varepsilon = (\bar{\lambda}(t), \bar{u}(t))$, *where* $t \in I = (-\eta, \eta)$ *for* $\eta = \eta(\varepsilon)$. *If* $\varepsilon''(0) > 0$, $\langle l, F_\lambda \rangle > 0$ *and* $\langle l, F_{uuu}[w_0, w_0, w_0] + 3F_{uu}[\theta, w_0] \rangle < 0$, *then*

(A) *for* $\varepsilon \in (\varepsilon_0 - \rho_1, \varepsilon_0)$, $\bar{\lambda}'(t) > 0$ *for* $t \in I$;

(B) *for* $\varepsilon = \varepsilon_0$, $\bar{\lambda}(0) = \lambda_0$, $\bar{\lambda}'(0) = \bar{\lambda}''(0) = 0$, $\bar{\lambda}'''(0) > 0$ *and* $\bar{\lambda}'(t) > 0$ *for* $t \in I \backslash \{0\}$;

(C) *for* $\varepsilon \in (\varepsilon_0, \varepsilon_0 + \rho_1)$, *there exists* $t_1, t_2 \in I$ *such that* $\bar{\lambda}'(t_1) = \bar{\lambda}'(t_2) = 0$, $\bar{\lambda}''(t_1) < 0$, $\bar{\lambda}''(t_2) > 0$, $\bar{\lambda}'(t) > 0$ *in* $(-\eta, t_1) \cup (t_2, \eta)$ *and* $\bar{\lambda}'(t) < 0$ *in* (t_1, t_2). (*See Fig.* 2.)

Thus the perturbed solution curve near a fold type of degenerate solution basically keeps the same shape: a parabola-like curve. But near a cusp type of degenerate solution, a bifurcation occurs: the curve is monotone before the cusp point, while it becomes a *S*-shaped curve with two fold type degenerate solutions after the the cusp point. In the latter case, if $\bar{\lambda}'''(0) < 0$, then locally the curve is reversed *S*-shaped (Like Fig. 6(C)).

Next we consider the situation as in Theorem 1.2.

THEOREM 2.4. *Let* F *be twice continuously differentiable, and* $F(\varepsilon_0, \lambda, u_0) \equiv 0$ *for* $\lambda \in \mathbf{R}$. *For* $T_0 = (\varepsilon_0, \lambda_0, u_0, w_0) \in M_1$, $H(T_0) = (0, 0)$, *and* T_0 *satisfies* (F1), (F3), *and* (F5). *Then there exists* $\delta > 0$ *such that all the solutions of* $H(\varepsilon, \lambda, u, w) = (0, 0)$ *near* T_0 *are in a form of* (2.6), $\varepsilon(s) = \varepsilon_0 + \tau(s)$, *and*

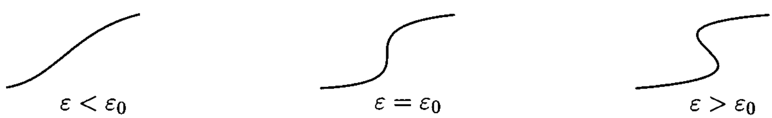


FIG. 2. Bifurcation of degenerate solutions at a cusp.

$\tau(0)=\tau'(0)=0$. If (F4) is satisfied, then $u(s)=u_0+skw_0+z_1(s)$, $w(s)=w_0+s\psi+z_2(s)$, $\lambda(s)=\lambda_0+s+z_3(s)$, where k is the unique number such that $\langle l, F_{\lambda u}[w_0]+kF_{uu}[w_0, w_0]\rangle=0$, $\psi\in X_3$ is the unique solution of

$$F_{\lambda u}(\varepsilon_0, \lambda_0, u_0)[w_0]+kF_{uu}(\varepsilon_0, \lambda_0, u_0)[w_0, w_0]+F_u(\varepsilon_0, \lambda_0, u_0)[\psi]=0, \tag{2.8}$$

and $z_i(0)=z_i'(0)=0$ for $i=1, 2, 3$. If (F4′) is satisfied, then $u(s)=u_0+sw_0+z_1(s)$, $w(s)=w_0+s\theta+z_2(s)$, $\lambda(s)=\lambda_0+z_3(s)$, $\theta\in X_3$ is the unique solution of (2.7), and $z_i(0)=z_i'(0)=0$ for $i=1, 2, 3$.

In Theorem 1.2, if (F4) is satisfied, the solutions of (1.1) has a transcritical bifurcation, and if (F4′) is satisfied, it is a pitch-fork bifurcation. From Theorem 2.4, we obtain the precise structure of the solution set of (1.1) near $(\varepsilon_0, \lambda_0, 0)$ if (F4) is also satisfied. In particular, we can observe how a transcritical bifurcation changes.

THEOREM 2.5. *Assume the conditions in Theorem 2.4 are satisfied, and $\{T_s\}$ is defined as in Theorem 2.4. In addition we assume that (F4) is satisfied. Then $\varepsilon''(0)\neq 0$. If $\varepsilon''(0)>0$, $\langle l, F_{\lambda u}\rangle>0$ and $\langle l, F_{uu}[w_0, w_0]\rangle>0$, then there exists $\rho_1, \delta_1, \delta_2>0$, such that for $N=\{(\lambda, u)\in \mathbf{R}\times X : |\lambda-\lambda_0|\leqslant\delta_1, \|u\|\leqslant\delta_2\}$,*

(A) *for $\varepsilon\in(\varepsilon_0-\rho_1, \varepsilon_0)$,*

$$F^{-1}(0)\cap N=\Sigma_\varepsilon^1\cup\Sigma_\varepsilon^2, \quad \Sigma_\varepsilon^i=\{(\lambda, \overline{u_i}(\lambda)): \lambda\in[\lambda_0-\delta_1, \lambda_0+\delta_1]\}, \quad i=1, 2; \tag{2.9}$$

(B) *for $\varepsilon=\varepsilon_0$,*

$$\begin{gathered} F^{-1}(0)\cap N=\{(\lambda, 0): \lambda\in[\lambda_0-\delta_1, \lambda_0+\delta_1]\}\cup\Sigma_0, \\ \Sigma_0=\{(\bar\lambda(t), \bar u(t)): t\in[-\eta, \eta]\}, \end{gathered} \tag{2.10}$$

and $\bar\lambda'(t)>0$ for $t\in[-\eta, \eta]$;

(C) *for $\varepsilon\in(\varepsilon_0, \varepsilon_0+\rho_1)$,*

$$F^{-1}(0)\cap N=\Sigma_\varepsilon^+\cup\Sigma_\varepsilon^-, \qquad \Sigma_\varepsilon^\pm=\{(\bar\lambda_\pm(t), \bar u_\pm(t)): t\in[-\eta, \eta]\}, \tag{2.11}$$

$\bar\lambda_+(\pm\eta)=\lambda_0+\delta_1$, $\bar\lambda_-(\pm\eta)=\lambda_0-\delta_1$, $\bar\lambda_\pm'(0)=0$, $\bar\lambda_+''(0)<0$, $\bar\lambda_-''(0)>0$, and there is exactly one turning point on each component $\Sigma_\varepsilon^\pm$. (See Fig. 3.)

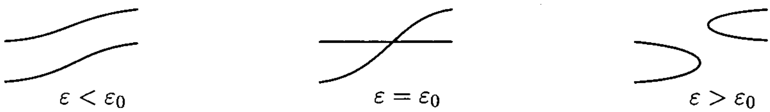


FIG. 3. Bifurcation of degenerate solutions in a transcritical bifurcation.

It is possible to obtain part of the result in Theorem 2.4 without the condition $F(\varepsilon_0, \lambda, u_0) \equiv 0$. Instead, we can consider it under the opposite of (F2):

(F2′) $F_\lambda(\varepsilon_0, \lambda_0, u_0) \in R(F_u(\varepsilon_0, \lambda_0, u_0))$.

We have

THEOREM 2.6. *Let F be twice continuously differentiable. For $T_0 = (\varepsilon_0, \lambda_0, u_0, w_0) \in M_1$, $H(T_0) = (0, 0)$, and T_0 satisfies* (F1), (F2′), *and* (F4). *Then there exists $\delta > 0$ such that all the solutions of $H(\varepsilon, \lambda, u, w) = (0, 0)$ near T_0 are in a form of* (2.6), *where $u(s) = u_0 + sv + z_1(s)$, $w(s) = w_0 + s\psi + z_2(s)$, $\lambda(s) = \lambda_0 + s + z_3(s)$ and $\varepsilon(s) = \varepsilon_0 + \tau(s)$, $(v, \psi) \in X \times X_3$ is the unique solution of*

$$\begin{cases} F_\lambda(\varepsilon_0, \lambda_0, u_0) + F_u(\varepsilon_0, \lambda_0, u_0)[v] = 0, \\ F_{\lambda u}(\varepsilon_0, \lambda_0, u_0)[w_0] + F_{uu}(\varepsilon_0, \lambda_0, u_0)[v, w_0] + F_u(\varepsilon_0, \lambda_0, u_0)[\psi] = 0, \end{cases} \tag{2.12}$$

and $z_i(0) = z_i'(0) = 0$ for $i = 1, 2, 3$, $\tau(0) = \tau'(0) = 0$.

The proofs of Theorems 2.1, 2.2, 2.4, and 2.6 are given in Section 3, and the proofs of Theorems 2.3 and 2.5 are given in Section 4 after some necessary calculations are done.

3. PROOFS OF MAIN RESULTS (1)

Proof of Theorem 2.1. We define a differential operator $K: \mathbf{R} \times X \times X_3 \to Y \times Y$,

$$K[\tau, v, \psi] = H_{(\lambda, u, w)}(\varepsilon, \lambda, u, w)[\tau, v, \psi]$$
$$= \begin{pmatrix} \tau F_\lambda(\varepsilon_0, \lambda_0, u_0) + F_u(\varepsilon_0, \lambda_0, u_0)[v] \\ \tau F_{\lambda u}(\varepsilon_0, \lambda_0, u_0)[w_0] + F_{uu}(\varepsilon_0, \lambda_0, u_0)[v, w_0] + F_u(\varepsilon_0, \lambda_0, u_0)[\psi] \end{pmatrix}. \tag{3.1}$$

We prove that K is an isomorphism.

First we prove that K is injective. Suppose there exists (τ, v, ψ) such that $K(\tau, v, \psi) = (0, 0)$. Let $l \in Y^*$ satisfying $N(l) = R(F_u(\varepsilon_0, \lambda_0, u_0))$. Then by (F2),

$$\langle l, F_\lambda(\varepsilon_0, \lambda_0, u_0) \rangle \neq 0. \tag{3.2}$$

From the first equation in (3.1), we obtain $\tau \langle l, F_\lambda(\varepsilon_0, \lambda_0, u_0) \rangle = 0$. So by (3.2), $\tau = 0$. And, because of the first equation in (3.1) and (F1), $v = kw_0$ for some $k \in \mathbf{R}$. Substituting it into the second equation in (3.1), we have

$$F_u(\varepsilon_0, \lambda_0, u_0)\psi + kF_{uu}(\varepsilon_0, \lambda_0, u_0)[w_0, w_0] = 0. \tag{3.3}$$

Thus $k\langle l, F_{uu}(\varepsilon_0, \lambda_0, u_0)[w_0, w_0] \rangle = 0$, and $k = 0$ because of (F4). Hence $v = 0$ and $\psi \in N(F_u(\varepsilon_0, \lambda_0, u_0)) = L(w_0)$. On the other hand, $\psi \in X_3$ by definition. Since $X = X_3 \oplus L(w_0)$, then $\psi = 0$. Therefore, $(\tau, v, \psi) = (0, 0, 0)$ and K is injective.

Next we prove K is surjective. Let $(h, g) \in Y \times Y$, then we need to find $(\tau, v, \psi) \in \mathbf{R} \times X \times X_3$ such that

$$\tau F_\lambda(\varepsilon_0, \lambda_0, u_0) + F_u(\varepsilon_0, \lambda_0, u_0)[v] = h, \tag{3.4}$$

$$\tau F_{\lambda u}(\varepsilon_0, \lambda_0, u_0)[w_0] + F_{uu}(\varepsilon_0, \lambda_0, u_0)[v, w_0] + F_u(\varepsilon_0, \lambda_0, u_0)[\psi] = g. \tag{3.5}$$

Applying l to (3.4), we get

$$\tau = \frac{\langle l, h \rangle}{\langle l, F_\lambda(\varepsilon_0, \lambda_0, u_0) \rangle}. \tag{3.6}$$

Let $v = kw_0 + v_0$ where $v_0 \in X_3$. By (F1), $K_1 \equiv F_u(\varepsilon_0, \lambda_0, u_0)|_{X_3}$ is an isomorphism. Thus $v_0 = (K_1)^{-1}(h - \tau F_\lambda(\varepsilon_0, \lambda_0, u_0))$ is uniquely determined where τ is defined in (3.6). $k \in \mathbf{R}$ can be uniquely determined by applying l to (3.5),

$$\begin{aligned} &\tau \langle l, F_{\lambda u}(\varepsilon_0, \lambda_0, u_0)[w_0] \rangle + k \langle l, F_{uu}(\varepsilon_0, \lambda_0, u_0)[w_0, w_0] \rangle \\ &\qquad + \langle l, F_{uu}(\varepsilon_0, \lambda_0, u_0)[v_0, w_0] \rangle = \langle l, g \rangle \end{aligned} \tag{3.7}$$

because of (F4). Finally,

$$\psi = (K_1)^{-1}(g - \tau F_{\lambda u}(\varepsilon_0, \lambda_0, u_0)[w_0] + F_{uu}(\varepsilon_0, \lambda_0, u_0)[v_0 + kw_0, w_0]).$$

Therefore, $(h, g) \in R(K)$, and K is a bijection. On the other hand, since F is twice differentiable, then K is continuous, and K^{-1} is also continuous by the open mapping theorem of Banach [Y, p. 75]. Thus K is a linear homeomorphism, and by the implicit function theorem, (see, for example, [CR1]), the solutions of $H(\varepsilon, \lambda, u, w)=0$ can be written as the form in (2.3). For the last statement, we differentiate $F(\varepsilon, \lambda(\varepsilon), u(\varepsilon))=0$ with respect to ε, and apply l to the resulting equation, then (2.4) follows. ▮

Proof of Theorem 2.2. We apply Theorem 1.1. Recall that $X=L(w_0)\oplus X_3$ and K is the differential operator defined in (3.1).

(1) $\dim N(K)=1$. Suppose $(\tau, v, \psi)\in N(K)$ and $(\tau, v, \psi)\neq 0$. Then by the same proof of injectivity in Theorem 2.1, we obtain that $\tau=0$, $v=kw_0$ and ψ satisfies (3.3), thus $\psi=k\theta$. Hence $(\tau, v, \psi)=k(0, w_0, \theta)$ and $\dim N(K)=1$.

(2) $\operatorname{codim} R(K)=1$. Let $(h, g)\in R(K)$, and its pre-image be (τ, v, ψ). Thus (h, g) and (τ, v, ψ) satisfy (3.4) and (3.5). Applying l to (3.4), we have

$$\tau \langle l, F_\lambda(\varepsilon_0, \lambda_0, u_0)\rangle = \langle l, h\rangle. \tag{3.8}$$

Since $\langle l, F_\lambda(\varepsilon_0, \lambda_0, u_0)\rangle \neq 0$, then τ can be uniquely determined by (3.8) for any given $h\in Y$. We denote this τ by τ_h, and substitute it back in (3.4), then equation $F_u[v]=h-\tau_h\langle l, F_\lambda(\varepsilon_0, \lambda_0, u_0)\rangle$ has a unique solution $v\in X_3$, which we denote by v_h.

We substitute (τ_h, v_h) into (3.5), and apply l, then we obtain

$$\tau_h \langle l, F_{\lambda u}(\varepsilon_0, \lambda_0, u_0)[w_0]\rangle + \langle l, F_{uu}(\varepsilon_0, \lambda_0, u_0)[v_h, w_0]\rangle = \langle l, g\rangle. \tag{3.9}$$

Therefore, $(h, g)\in R(K)$ if and only if (h, g) satisfies (3.9).

We notice that $T_1\colon Y\to \mathbf{R}\times X_3$, $T_1(h)=(\tau_h, v_h)$ is a linear continuous operator, and $T_2\colon \mathbf{R}\times X_3\to\mathbf{R}$, $T_2(\tau, v)=\tau\langle l, F_{\lambda u}(\varepsilon_0, \lambda_0, u_0)[w_0]\rangle+\langle l, F_{uu}(\varepsilon_0, \lambda_0, u_0)[v, w_0]\rangle$ is a linear continuous functional. Thus $l_1=T_2\circ T_1\colon Y\to\mathbf{R}$ is a linear continuous functional on Y, and $(h, g)\in R(K)$ if and only if $\langle l_1, h\rangle-\langle l, g\rangle=0$. Let $l_2=(l_1, -l)\in Y^*\times Y^*$, then $N(l_2)=R(K)$. Therefore $\operatorname{codim} R(K)=1$.

Finally, we have $H_\varepsilon(\varepsilon_0, \lambda_0, u_0, w_0)\notin R(K)$ by (2.5). Therefore we can apply Theorem 1.1, and we obtain the solution curve $(\varepsilon_0+\tau(s), C(s))$, where $C(s)=(\lambda_0, u_0, w_0)+s(0, w_0, \theta)+z(s)$ and $z(s)=(z_1(s), z_2(s), z_3(s))$ satisfies the conclusions in Theorem 2.2. ▮

Proof of Theorem 2.4. Similar to the proof of Theorem 2.2, we apply Theorem 1.1.

(1) $\dim N(K)=1$. Suppose $(\tau, v, \psi) \in N(K)$ and $(\tau, v, \psi) \neq 0$. First we consider the case of $\tau=0$. From (3.4) with $h=0$, $F_u(\varepsilon_0, \lambda_0, u_0)[v]=0$, thus $v=kw_0$. We substitute $v=kw_0$ and $\tau=0$ into (3.5) with $g=0$, then $kF_{uu}(\varepsilon_0, \lambda_0, u_0)[w_0, w_0]+F_u(\varepsilon_0, \lambda_0, u_0)[\psi]=0$. So if (F4) is satisfied, then $k=\psi=0$, hence $\tau \neq 0$ if (F4) is satisfied. If (F4′) is satisfied, then $(0, w_0, \theta) \in N(K)$. Next we consider the case of $\tau \neq 0$. Without loss of generality, we assume that $\tau=1$. Then $F_u(\varepsilon_0, \lambda_0, u_0)[v]=0$ and $v=kw_0$. By substitution, (3.5) becomes

$$F_{\lambda u}(\varepsilon_0, \lambda_0, u_0)[w_0]+kF_{uu}(\varepsilon_0, \lambda_0, u_0)[w_0, w_0]+F_u(\varepsilon_0, \lambda_0, u_0)[\psi]=0. \tag{3.10}$$

By applying l to (3.10), we obtain

$$\langle l, F_{\lambda u}(\varepsilon_0, \lambda_0, u_0)[w_0]\rangle + k\langle l, F_{uu}(\varepsilon_0, \lambda_0, u_0)[w_0, w_0]\rangle = 0. \tag{3.11}$$

If (F4) is satisfied, there exists a unique $k=k_1$ such that (3.11) holds. Moreover, with $k=k_1$ in (3.10), we can solve a unique $\psi=\psi_1 \in X_3$ satisfying (3.10). Therefore $N(K)=span\{(1, k_1 w_0, \psi_1)\}$ if (F4) is satisfied. If (F4′) is satisfied, we obtain $\langle l, F_{\lambda u}(\varepsilon_0, \lambda_0, u_0)[w_0]\rangle=0$, which contradicts with (F3). Thus when (F4′) is satisfied, then $N(K)=span\{(0, w_0, \theta)\}$.

(2) $\operatorname{codim} R(K)=1$. Let $(h, g) \in R(K)$, and its pre-image be (τ, v, ψ). Thus (h, g) and (τ, v, ψ) satisfy (3.4) and (3.5). By applying l to (3.4), we find that $\langle l, h\rangle=0$. Thus $h \in N(l)=R(F_u(\varepsilon_0, \lambda_0, u_0))$, there exists a unique $v_1 \in X_3$ such that $F_u(\varepsilon_0, \lambda_0, u_0)[v_1]=h$. Let $v=v_1+kw_0$. If (F4) is satisfied, we can take $\tau=0$, k is determined by

$$\langle l, F_{uu}(\varepsilon_0, \lambda_0, u_0)[v_1, w_0]\rangle + k\langle l, F_{uu}(\varepsilon_0, \lambda_0, u_0)[w_0, w_0]\rangle = \langle l, g\rangle, \tag{3.12}$$

and ψ can also be uniquely determined. If (F4′) is satisfied, then we can take $k=0$, τ is determined by

$$\tau\langle l, F_{\lambda u}(\varepsilon_0, \lambda_0, u_0)[w_0]\rangle + \langle l, F_{uu}(\varepsilon_0, \lambda_0, u_0)[w_0, v_1]\rangle = \langle l, g\rangle, \tag{3.13}$$

and ψ can also be uniquely determined. Therefore, in both cases, $R(K)= R(F_u(\varepsilon_0, \lambda_0, u_0)) \times Y$, which is codimension one.

(3) $H_\varepsilon(\varepsilon_0, \lambda_0, u_0, w_0) \notin R(K)$. This directly comes from (F5) and $R(K)=R(F_u(\varepsilon_0, \lambda_0, u_0)) \times Y$. So the result follows from Theorem 1.1. ∎

The proof of Theorem 2.6 is similar to that of Theorem 2.4, so we omit it.

4. SOME CALCULATIONS AND PROOFS OF MAIN RESULTS (2)

In a bifurcation problem, it is important to determine the direction or the orientation of the bifurcation. Roughly speaking, if ρ is a real bifurcation parameter, $\rho(s)$, $s \in I = (-\delta, \delta)$, is a bifurcation curve near a bifurcation point $\rho(0) = \rho_0$, then the bifurcation is *supercritical* if $\rho(s) > \rho_0$ for $s \in I$, is *subcritical* if $\rho(s) < \rho_0$ for $s \in I$, and is *transcritical* if $\rho(s) > \rho_0$ for $s \in (0, \delta)$ and $\rho(s) < \rho_0$ for $s \in (-\delta, 0)$ or vice versa. The direction is determined by the first non-vanishing derivative of $\rho(s)$ at $s = 0$. In this section we are going to compute the direction of bifurcation in all the occasions which we have discussed.

First we consider the case when $\varepsilon = \varepsilon_0$ is fixed, and the bifurcation in Theorems 1.1 and 1.2. In Theorem 1.1, we differentiate $F(\lambda(s), u(s)) = 0$ and apply l, then we obtain

$$\lambda'(0) = 0, \qquad \lambda''(0) = -\frac{\langle l, F_{uu}(\lambda_0, u_0)[w_0, w_0] \rangle}{\langle l, F_\lambda(\lambda_0, u_0) \rangle}. \tag{4.1}$$

So if (F4) is satisfied, then $\lambda''(0) \neq 0$. If (F4′) is satisfied, then $\lambda''(0) = 0$ and

$$\lambda'''(0) = -\frac{\langle l, F_{uuu}(\lambda_0, u_0)[w_0, w_0, w_0] \rangle + 3 \langle l, F_{uu}(\lambda_0, u_0)[w_0, \theta] \rangle}{\langle l, F_\lambda(\lambda_0, u_0) \rangle}, \tag{4.2}$$

where θ is the solution of (2.7).

In the situation of Theorem 1.2, following the original proof by Crandall and Rabinowitz [CR1], we define

$$G(s, \lambda, z) = \begin{cases} s^{-1} F(\lambda, s w_0 + s z) & \text{if } s \neq 0, \\ F_u(\lambda, 0)(w_0 + z) & \text{if } s = 0, \end{cases} \tag{4.3}$$

where $(s, \lambda, z) \in \mathbf{R} \times \mathbf{R} \times X_3$ such that $s(w_0 + z) \in V$, a neighborhood of 0 in X, and $s \in (-\delta, \delta)$. From the proof of Theorem 1.7 in [CR1], the solution curve $(\lambda(s), u(s))$ is actually the solution curve $(s, \lambda(s), z(s))$ of $G(s, \lambda, z) = 0$. So we differentiate $G(s, \lambda(s), z(s)) = 0$ with respect to s at $s = 0$, and we get

$$\begin{aligned} 0 &= \frac{d}{ds} G(s, \lambda(s), z(s)) \Big|_{s=0} \\ &= \frac{1}{2} F_{uu}(\lambda_0, u_0)[w_0, w_0] + \lambda'(0) F_{\lambda u}(\lambda_0, u_0) w + F_u(\lambda_0, 0) z'(0). \end{aligned} \tag{4.4}$$

Applying l to (4.4), we obtain

$$\lambda'(0) = -\frac{\langle l, F_{uu}(\lambda_0, u_0)[w_0, w_0] \rangle}{2 \langle l, F_{\lambda u}(\lambda_0, u_0) \rangle}. \tag{4.5}$$

So if (F4) is satisfied, then $\lambda'(0) \neq 0$. If (F4′) is satisfied, then $\lambda'(0) = 0$ and

$$\lambda''(0) = -\frac{\langle l, F_{uuu}(\lambda_0, u_0)[w_0, w_0, w_0] \rangle + 3 \langle l, F_{uu}(\lambda_0, u_0)[w_0, \theta] \rangle}{3 \langle l, F_{\lambda u}(\lambda_0, u_0) \rangle}, \tag{4.6}$$

where θ is the solution of (2.7).

Now let us turn to the bifurcation of degenerate solutions. Let $\{ T_s = (\varepsilon(s), \lambda(s), u(s), w(s)) : s \in (-\delta, \delta) \}$ be a curve of degenerate solutions which we obtained in Theorem 2.2. Then $\varepsilon'(0) = \lambda'(0) = 0$, and $u'(0) = w_0$, $w'(0) = \theta$. Here we determine $\varepsilon''(0)$ and $\lambda''(0)$.

Differentiating (2.1) with respect to s, we obtain

$$F_\varepsilon \varepsilon'(s) + F_\lambda \lambda'(s) + F_u[u'(s)] = 0, \tag{4.7}$$

$$F_{\varepsilon u}[w(s)]\, \varepsilon'(s) + F_{\lambda u}[w(s)]\, \lambda'(s) + F_{uu}[w(s), u'(s)] + F_u[w'(s)] = 0. \tag{4.8}$$

When $s = 0$, we have

$$F_u[w_0] = 0, \qquad \text{and} \qquad F_u[\theta] + F_{uu}[w_0, w_0] = 0. \tag{4.9}$$

Differentiating (4.7) and (4.8) again, we obtain

$$\begin{aligned} &F_{\varepsilon\varepsilon}[\varepsilon'(s)]^2 + F_\varepsilon \varepsilon''(s) + F_{\lambda\lambda}[\lambda'(s)]^2 + F_\lambda \lambda''(s) + F_{uu}[u'(s), u'(s)] \\ &\quad + F_u[u''(s)] + 2F_{\varepsilon\lambda} \varepsilon'(s)\, \lambda'(s) + 2F_{\varepsilon u}[u'(s)]\, \varepsilon'(s) + 2F_{\lambda u}[u'(s)]\, \lambda'(s) = 0, \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} &F_{\varepsilon\varepsilon u}[w(s)][\varepsilon'(s)]^2 + F_{\varepsilon u}[w(s)]\, \varepsilon''(s) + F_{\lambda\lambda u}[w(s)][\lambda'(s)]^2 + F_{\lambda u}[w(s)]\, \lambda''(s) \\ &\quad + F_{uuu}[u'(s), u'(s), w(s)] + F_{uu}[w(s), u''(s)] + F_u[w''(s)] \\ &\quad + 2F_{\varepsilon\lambda u}[w(s)]\, \varepsilon'(s)\, \lambda'(s) + 2F_{\varepsilon uu}[u'(s), w(s)]\, \varepsilon'(s) \\ &\quad + 2F_{\lambda uu}[u'(s), w(s)]\, \lambda'(s) + 2F_{\varepsilon u}[w'(s)]\, \varepsilon'(s) \\ &\quad + 2F_{\lambda u}[w'(s)]\, \lambda'(s) + 2F_{uu}[w'(s), u'(s)] = 0. \end{aligned} \tag{4.11}$$

When $s=0$, we have

$$\varepsilon''(0)\, F_\varepsilon + \lambda''(0)\, F_\lambda + F_{uu}[w_0, w_0] + F_u[u''(0)] = 0, \tag{4.12}$$

and

$$\begin{aligned}\varepsilon''(0)\, F_{\varepsilon u}[w_0] + \lambda''(0)\, F_{\lambda u}[w_0] + F_{uuu}[w_0, w_0, w_0] + F_{uu}[w_0, u''(0)]\\ + F_u[w''(0)] + 2F_{uu}[\theta, w_0] = 0.\end{aligned} \tag{4.13}$$

By applying l to (4.12) and (4.13), and combining (4.9), we get

$$\begin{cases}\varepsilon''(0)\langle l, F_\varepsilon\rangle + \lambda''(0)\langle l, F_\lambda\rangle = 0\\ \varepsilon''(0)\langle l, F_{\varepsilon u}[w_0]\rangle + \lambda''(0)\langle l, F_{\lambda u}[w_0]\rangle = -\langle l, F_{uuu}[w_0, w_0, w_0]\rangle\\ \qquad -3\,\langle l, F_{uu}[\theta, w_0]\rangle - \langle l, F_{uu}[w_0, u''(0)-\theta]\rangle\\ \varepsilon''(0)\, F_\varepsilon + \lambda''(0)\, F_\lambda = -F_u[u''(0)-\theta]\end{cases} \tag{4.14}$$

Define k_1, k_2,

$$k_1 = \varepsilon''(0), \qquad \text{and} \qquad k_2 = \frac{\lambda''(0)}{\varepsilon''(0)} = -\frac{\langle l, F_\varepsilon\rangle}{\langle l, F_\lambda\rangle}.$$

Let $v = F_u|_{X_3}^{-1}\,[F_\varepsilon + k_2 F_\lambda]$, then $u''(0)-\theta = -k_1 v$. We substitute v into the second equation of (4.14), then

$$\begin{aligned}k_1\,\langle l, F_{\varepsilon u}[w_0] + k_2 F_{\lambda u}[w_0] + F_{uu}[w_0, v]\rangle\\ = -\langle l, F_{uuu}[w_0, w_0, w_0]\rangle - 3\,\langle l, F_{uu}[\theta, w_0]\rangle,\end{aligned}$$

for which k_1 is solvable if

$$F_{\varepsilon u}[w_0] + k_2 F_{\lambda u}[w_0] + F_{uu}[w_0, v] \notin R(F_u(\varepsilon_0, \lambda_0, u_0)). \tag{4.15}$$

Therefore $\varepsilon''(0)$ and $\lambda''(0)$ are solvable if (4.15) is satisfied. Relation (4.15) will be easier to check if X, Y are Hilbert space.

Next we consider the special case of X and Y being Hilbert spaces. Let Y be a Hilbert space, with inner-product $\langle \cdot, \cdot \rangle_Y$, $X \subset Y$ be a closed dense subspace, $X_3 = X \cap L(w_0)^\perp$. Let

$$F_u(\varepsilon_0, \lambda_0, u_0)\colon X \to Y,$$

and (4.16)

$$F_{uu}(\varepsilon_0, \lambda_0, u_0)[w_0, \cdot]\colon X \to Y$$

be linear self-adjoint operators. Then (2.5) and (4.15) will become more transparent in this case. In fact, in the proof of Theorem 2.2, we have shown that $(h, g) \in R(K)$ if and only if (h, g) satisfies (3.9). For Hilbert spaces and self-adjoint operators, we have a more explicit representation of

$R(K)$. First, by Riesz presentation theorem $l \in Y^*$ can be represented as $\langle l, x\rangle = \langle w_0, x\rangle_Y$ for $x \in Y$. From (3.4) and (3.5), we have

$$\tau \langle w_0, F_\lambda\rangle_Y = \langle w_0, h\rangle_Y \tag{4.17}$$

$$\tau \langle w_0, F_{\lambda u}[w_0]\rangle_Y + \langle w_0, F_{uu}[w_0, v]\rangle_Y = \langle w_0, g\rangle_Y. \tag{4.18}$$

From $F_u[\theta] + F_{uu}[w_0, w_0] = 0$ and $F_u[v] + \tau F_\lambda = h$, and $\langle v, F_u[\theta]\rangle_Y = \langle \theta, F_u[v]\rangle_Y$ because F_u is self-adjoint, then

$$\langle v, F_{uu}[w_0, w_0]\rangle_Y - \tau\langle \theta, F_\lambda\rangle_Y = -\langle \theta, h\rangle_Y. \tag{4.19}$$

Notice that $\langle w_0, F_{uu}[w_0, v]\rangle_Y = \langle v, F_{uu}[w_0, w_0]\rangle_Y$ because that $F_{uu}(\varepsilon_0, \lambda_0, u_0)[w_0, \cdot]$ is self-adjoint. Therefore, from (4.17), (4.18), and (4.19), we have

$$\begin{aligned}[\langle w_0, F_{\lambda u}[w_0]\rangle_Y + \langle \theta, F_\lambda(\varepsilon_0, \lambda_0, u_0)\rangle_Y]\langle w_0, h\rangle_Y \\ = \langle w_0, F_\lambda\rangle_Y [\langle w_0, g\rangle_Y + \langle \theta, h\rangle_Y],\end{aligned} \tag{4.20}$$

which implies $(h, g) \in R(K)$ if and only if

$$\det\begin{bmatrix} \langle w_0, h\rangle_Y & \langle w_0, F_\lambda\rangle_Y \\ \langle w_0, g\rangle_Y + \langle \theta, h\rangle_Y & \langle w_0, F_{\lambda u}[w_0]\rangle_Y + \langle \theta, F_\lambda\rangle_Y \end{bmatrix} = 0. \tag{4.21}$$

Consequently, Eq. (2.5) becomes

$$\det\begin{bmatrix} \langle w_0, F_\varepsilon\rangle_Y & \langle w_0, F_\lambda\rangle_Y \\ \langle w_0, F_{\varepsilon u}[w_0]\rangle_Y + \langle \theta, F_\varepsilon\rangle_Y & \langle w_0, F_{\lambda u}[w_0]\rangle_Y + \langle \theta, F_\lambda\rangle_Y \end{bmatrix} \neq 0. \tag{4.22}$$

We derive a similar formula for $\varepsilon''(0)$ and $\lambda''(0)$. Since $\langle u''(0), F_u[\theta]\rangle_Y = \langle \theta, F_u[u''(0)]\rangle_Y$, then by applying θ to (4.12), we get

$$\begin{aligned}&\langle u''(0), F_{uu}[w_0, w_0]\rangle_Y \\ &\quad = \varepsilon''(0)\langle \theta, F_\varepsilon\rangle_Y + \lambda''(0)\langle \theta, F_\lambda\rangle_Y + \langle \theta, F_{uu}[w_0, w_0]\rangle_Y.\end{aligned} \tag{4.23}$$

On the other hand, by applying w_0 to (4.13), we get

$$\begin{aligned}&\varepsilon''(0)\langle w_0, F_{\varepsilon u}[w_0]\rangle_Y + \lambda''(0)\langle w_0, F_{\lambda u}[w_0]\rangle_Y + \langle w_0, F_{uuu}[w_0, w_0, w_0]\rangle_Y \\ &\quad + \langle w_0, F_{uu}[w_0, u''(0)]\rangle_Y + 2\langle w_0, F_{uu}[\theta, w_0]\rangle_Y = 0.\end{aligned} \tag{4.24}$$

Since $\langle w_0, F_{uu}[w_0, u''(0)]\rangle_Y = \langle u''(0), F_{uu}[w_0, w_0]\rangle_Y$, then combining the first equation in (4.14), we have

$$\begin{bmatrix} \langle w_0, F_\varepsilon\rangle_Y & \langle w_0, F_\lambda\rangle_Y \\ \langle w_0, F_{\varepsilon u}[w_0]\rangle_Y + \langle \theta, F_\varepsilon\rangle_Y & \langle w_0, F_{\lambda u}[w_0]\rangle_Y + \langle \theta, F_\lambda\rangle_Y \end{bmatrix} \begin{pmatrix} \varepsilon''(0) \\ \lambda''(0) \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ -\langle w_0, F_{uuu}[w_0, w_0, w_0]\rangle_Y - 3\langle w_0, F_{uu}[\theta, w_0]\rangle_Y \end{pmatrix}. \tag{4.25}$$

Because of (4.22), (4.25) has a unique solution $(\varepsilon''(0), \lambda''(0))$. The solution is non-zero if (F6) is satisfied.

Next we compute the bifurcation direction of curve in Theorem 2.4. First we assume that (F4) is satisfied. Since $\lambda'(0)=1$ and $\varepsilon'(0)=0$, then we need to determine $\varepsilon''(0)$. In the context of Theorem 2.4, we set $s=0$ in (4.7), (4.8), and (4.10), and apply l to these equations, then we obtain

$$\langle l, F_{\lambda u}[w_0]\rangle + \langle l, F_{uu}[w_0, v]\rangle = 0, \tag{4.26}$$

and

$$\varepsilon''(0)\langle l, F_\varepsilon\rangle + \langle l, F_{\lambda\lambda}\rangle + \langle l, F_{uu}[v, v]\rangle + 2\langle l, F_{\lambda u}[v]\rangle = 0. \tag{4.27}$$

Since $F(\varepsilon, \lambda, 0) \equiv 0$, then $v = kw_0$, where

$$k = -\frac{\langle l, F_{\lambda u}[w_0]\rangle}{\langle l, F_{uu}[w_0, w_0]\rangle}.$$

Then (4.27) becomes

$$\varepsilon''(0) = -k\frac{\langle l, F_{\lambda u}[w_0]\rangle}{\langle l, F_\varepsilon\rangle} = \frac{\langle l, F_{\lambda u}[w_0]\rangle^2}{\langle l, F_{uu}[w_0, w_0]\rangle\langle l, F_\varepsilon\rangle}. \tag{4.28}$$

If (F4′) is satisfied, then $\varepsilon'(0) = \lambda'(0) = 0$. From (4.10), $\varepsilon''(0) = 0$. $\lambda''(0)$ and $\varepsilon'''(0)$ can be determined if further smoothness is assumed, but they are not needed in this paper, so we will not go into that.

Now we are ready to prove Theorems 2.3 and 2.5.

Proof of Theorem 2.3. Part (1) is obvious by Theorem 1.1 and Theorem 2.1. For part (2), we apply Theorem 2 in [Da], then the solution set Σ_ε for $\varepsilon \in (\varepsilon_0 - \delta_1, \varepsilon_0 + \delta_1)$ is a curve $(\bar{\lambda}(t), \bar{u}(t))$ with $t \in I$. Moreover, $\varepsilon''(0) > 0$ so there are exactly two degenerate solutions for $\varepsilon \in (\varepsilon_0, \varepsilon_0 + \delta_1)$ and there is no degenerate solution for $\varepsilon \in (\varepsilon_0 - \delta_1, \delta_1)$.

When $\varepsilon = \varepsilon_0$, $\bar{\lambda}'(0) = 0$ by Theorem 1.1, $\bar{\lambda}''(0) = 0$ by (4.1) and $\bar{\lambda}'''(0) > 0$ by (4.2) and our assumptions. And there exists $\eta_1 > 0$ such that for $t \in (-\eta_1, \eta_1)\backslash\{0\}$, $\bar{\lambda}'(t) > 0$. When $\varepsilon \in (\varepsilon_0 - \delta_1, \varepsilon_0)$, there is no degenerate solution near $(\varepsilon_0, \lambda_0, u_0)$, so $\bar{\lambda}'(t) > 0$ for all $t \in (-\eta_2, \eta_2)$ for some $\eta_2 \in (0, \eta_1)$ since Σ_ε is a perturbation of Σ_0.

When $\varepsilon \in (\varepsilon_0, \varepsilon_0 + \delta_1)$, there are exactly two degenerate solutions $(\lambda(s_1), u(s_1))$ and $(\lambda(s_2), u(s_2))$ on Σ_ε, where $s_1 < 0$ and $s_2 > 0$. For a degenerate solution $(\varepsilon(s), \lambda(s), u(s), w(s))$, define

$$A(s) = \langle l(s), F_{uu}(\varepsilon(s), \lambda(s), u(s))[w(s), w(s)] \rangle,$$

where $l(s) \in Y^*$ satisfying $N(l(s)) = R(F_u(\varepsilon(s), \lambda(s), u(s)))$. Then A is differentiable and

$$\begin{aligned} A'(0) &= 2 \langle l, F_{uu}[w_0, \theta] \rangle + \langle l, F_{uuu}[w_0, w_0, w_0] \rangle + \langle l'(0), F_{uu}[w_0, w_0] \rangle \\ &= 3 \langle l, F_{uu}(\varepsilon_0, \lambda_0, u_0)[w_0, \theta] \rangle + \langle l, F_{uuu}(\varepsilon_0, \lambda_0, u_0)[w_0, w_0, w_0] \rangle < 0, \end{aligned} \tag{4.29}$$

since $\varepsilon'(0) = \lambda'(0) = 0$, $u'(0) = w_0$ and $w'(0) = \theta$. In (4.29), we obtain $\langle l'(0), F_{uu}[w_0, w_0] \rangle = \langle l, F_{uu}[w_0, \theta] \rangle$ by differentiating $\langle l(s), F_u(\varepsilon(s), \lambda(s), u(s))[w_0] \rangle = 0$ twice, and using (4.13). In particular, $A(s_1) > 0$ and $A(s_2) < 0$. On the other hand, $\langle l(s), F_\lambda(\varepsilon(s), \lambda(s), u(s)) \rangle > 0$ for $|s|$ small, so $\bar{\lambda}''(s_1) < 0$ and $\bar{\lambda}''(s_2) > 0$ from the formula (4.1). Therefore, $\bar{\lambda}(t)$ is a connected curve with a local minimum at $t_1 = \lambda(s_1)$ and a local maximum at $t_2 = \lambda(s_2)$. ∎

Proof of Theorem 2.5. Because of (4.28) and (F4), $\varepsilon''(0) \neq 0$. Define $N = \{(\lambda, u) \in \mathbf{R} \times X : |\lambda - \lambda_1| \leqslant \delta_1,\ \|u\| \leqslant \delta_2\}$, such that (2.10) holds and $\bar{\lambda}(-\eta) = \lambda_0 - \delta_1$, $\bar{\lambda}(\eta) = \lambda_0 + \delta_1$, and for $t \in [-\eta, \eta]$, $\|\bar{u}(t)\| \leqslant \delta_2/2$.

We denote by Σ_ε the solution set of (1.1) in N for fixed ε. If $\varepsilon''(0) > 0$, then there exists $\rho_2 > 0$ such that for $\varepsilon \in (\varepsilon_0, \varepsilon_0 + \rho_2)$, (1.1) has exactly two degenerate solutions (λ_+, u_+) and (λ_-, u_-), where $\lambda_+ = \lambda(s_+) > \lambda_0$ and $\lambda_- = \lambda(s_-) < \lambda_0$, $s_+ > 0 > s_-$, $u_\pm = s_\pm w_0 + o(|s|)$ for $|s|$ small. Moreover, $u_\pm$ are degenerate solutions which satisfy the conditions of Theorem 1.1. In fact, we only need to check that $F_\lambda(\varepsilon, \lambda_\pm, u_\pm) \notin R(F_u(\varepsilon, \lambda_\pm, u_\pm))$. Define $B(s) = \langle l(s), F_\lambda(\varepsilon(s), \lambda(s), u(s)) \rangle$, where $l(s) \in Y^*$ satisfying $N(l(s)) = R(F_u(\varepsilon(s), \lambda(s), u(s)))$. Then $B'(0) = \langle l, F_{\lambda u}(\varepsilon_0, \lambda_0, 0)[w_0] \rangle > 0$, and $B(0) = 0$ by the assumptions, so $B(s_+) > 0$ and $B(s_-) < 0$. On the other hand, since $\langle l, F_{uu}(\varepsilon_0, \lambda_0, 0)[w_0, w_0] \rangle < 0$, we have

$$\langle l(s), F_{uu}(\varepsilon(s), \lambda(s), u(s))[w(s), w(s)] \rangle < 0$$

for $|s|$ small. By (4.1) and Theorem 1.1, near (λ_+, u_+), the solutions forms a curve $\Sigma_\varepsilon^+ = \{(\lambda_+(t), u_+(t)) : t \in [-\eta, \eta]\}$ with $\lambda_+(0) = \lambda_+$, $u_+(0) = u_+$, $\lambda_+'(0) = 0$, $\lambda_+''(0) < 0$ and $\lambda_+(\pm\eta) = \lambda_0 + \delta_1$. And similarly, the solutions near (λ_-, u_-) also form a curve $\Sigma_\varepsilon^- = \{(\lambda_-(t), u_-(t)) : t \in [-\eta, \eta]\}$ with $\lambda_-''(0) > 0$ and $\lambda_-(\pm\eta) = \lambda_0 - \delta_1$.

We claim that Σ_ε consists of only Σ_ε^+ and Σ_ε^- for $\varepsilon \in (\varepsilon_0, \varepsilon_0+\rho_1)$ with $\rho_1 \in (0, \rho_2)$. By the definition of N, there exists $\rho_3>0$ such that for $\varepsilon \in [\varepsilon_0, \varepsilon_0+\rho_3]$, (1.1) has no solution u with $\|u\|=\delta_2$. And there exists $\rho_4>0$ such that for $\varepsilon \in (\varepsilon_0, \varepsilon_0+\rho_4]$, there is no degenerate solution of (1.1) with $|\lambda-\lambda_0| \geqslant \delta_1/2$. For $\varepsilon=\varepsilon_0$, (1.1) has a unique nontrivial solution $(\lambda_*, u_*) \in \partial N$ such that $\lambda_*=\lambda_0+\delta_1$ and $\|u_*\| \leqslant \delta_2/2$. Since u_* is non-degenerate, for fixed $\lambda=\lambda_*$, by the implicit function theorem, there exists $\rho_5>0$ such that for $\varepsilon \in (\varepsilon_0-\rho_5, \varepsilon_0+\rho_5)$, (1.1) has a unique solution $u_*(\varepsilon)$ near u_*, and a unique solution $u_0(\varepsilon)$ near the trivial solution $(\lambda_*, 0)$. By choosing ρ_5 smaller, we can assume that (1.1) has exactly these two solutions with $\lambda=\lambda_*$ and $u \in \partial N$. Let $\rho_1=\min(\rho_3, \rho_4, \rho_5)$. Suppose that for some $\varepsilon \in (\varepsilon_0, \varepsilon_0+\rho_1)$, there is another solution (λ, u) of (1.1) which is not on $\Sigma_\varepsilon^\pm$. Then (λ, u) is non-degenerate since the only nondegenerate solutions are (λ_+, u_+) and (λ_-, u_-). So by the implicit function theorem, (λ, u) is on a solution curve $\Sigma_1=(\tilde{\lambda}(s), \tilde{u}(s))$ which can be extended to ∂N. But there is no solution can be on $\|u\|=\delta_2$, so there is a solution (λ_*, u^*) on Σ_1. On the other hand, Σ_ε^+ has another two solutions on $\lambda=\lambda_*$, thus there are at least three solutions on $\lambda=\lambda_*$, which contradicts with the definition of ρ_1.

For $\varepsilon \in (\varepsilon_0-\rho_1, \varepsilon_0)$, we have shown there is no degenerate solution. By a similar argument as the last paragraph, we can show that the solution set in N consists of two disjoint curves with no degenerate solutions. ∎

5. APPLICATIONS TO SEMILINEAR ELLIPTIC EQUATIONS

In this section, we apply the abstract results to a semilinear elliptic equation:

$$\mathscr{L}u+\lambda f(\varepsilon, u)=0 \quad \text{in } \Omega, \qquad u=0 \quad \text{on } \partial\Omega, \tag{5.1}$$

where

$$\mathscr{L}=\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left[a_{ij}(x)\frac{\partial}{\partial x_j}\right]+b(x)=\sum_{i,j=1}^{n} \partial_i(a_{ij}(x)\,\partial_j)+b(x) \tag{5.2}$$

is a uniformly elliptic, formally self-adjoint linear second order differential operator, with real-valued coefficient functions $a_{ij}=a_{ji}, b \in C^1(\bar{\Omega})$, Ω is a bounded smooth domain in $\mathbf{R}^n$, $n \geqslant 1$, ε is a real parameter and λ is a *positive* parameter. (That does not lose any generality, if $\lambda<0$, then we can consider $-f$; if $\lambda=0$, the equation is trivial.) We also assume that $f \in C^2(\mathbf{R}\times\mathbf{R})$ unless other conditions are specified.

Let L denote the linear operator induced by $\mathscr{L}$ in $Y=L^2(\Omega)$, with domain $X=D(L)=H_0^1(\Omega) \cap H^2(\Omega)$. Then L is self-adjoint, and the spectrum of L consists of a sequence $(\lambda_k)_{k\in\mathbf{N}}: 0<\lambda_1<\lambda_2\leqslant\lambda_3\leqslant\cdots$ of eigenvalues. Most results here are also true if the nonlinearity f depends on x, and some of them can also be generalized to the case of non-self-adjoint elliptic operators and more general Sobolev spaces $W^{k,p}(\Omega)$. But in the present setting, the results will be in a clearest and simplest form, which is best for illustrating our abstract results.

We define

$$F(\varepsilon, \lambda, u) = \mathscr{L}u + \lambda f(\varepsilon, u), \tag{5.3}$$

$$G(\varepsilon, \lambda, u, w) = F_u(\varepsilon, \lambda, u)[w] = \mathscr{L}w + \lambda f_u(\varepsilon, u) w, \tag{5.4}$$

where $F: \mathbf{R}\times\mathbf{R}\times X\to Y$, $G: \mathbf{R}\times\mathbf{R}\times X\times X_1\to Y$, $X_1=\{u\in X: \|u\|_2=1\}$. Let $(\varepsilon_0, \lambda_0, u_0, w_0)$ be a solution of $F(\varepsilon, \lambda, u)=0$ and $G(\varepsilon, \lambda, u, w)=0$. Define $X_3=\{u\in X: \int_\Omega u(x)\, w_0(x)\, dx=0\}$, and $X_2=w_0+X_3=\{w_0+u: u\in X_3\}$. Then X_3 is a complement subspace of $L(w_0)$ in X, and X_2 is a closed hyperplane in X with codimension 1. We consider the problem

$$H(\varepsilon, \lambda, u, w) = \begin{pmatrix} F(\varepsilon, \lambda, u) \\ G(\varepsilon, \lambda, u, w) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{5.5}$$

where $H: \mathbf{R}\times\mathbf{R}\times X\times X_3\to Y\times Y$. (For simplicity, we denote $\mathbf{R}\times\mathbf{R}\times X\times X_3$ by M_1.)

The application of Theorem 2.1 to (5.1) is

THEOREM 5.1. *Suppose* $H(T_0)=0$ *for* $T_0=(\varepsilon_0, \lambda_0, u_0, w_0)\in M_1$, T_0 *satisfy* (F1),

$$\int_\Omega f(\varepsilon_0, u_0)\, w_0\, dx \neq 0, \tag{5.6}$$

and

$$\int_\Omega f_{uu}(\varepsilon_0, u_0)\, w_0^3\, dx \neq 0. \tag{5.7}$$

Then there exists $\delta>0$ *such that all the solutions to* $H(\varepsilon, \lambda, u, w)=0$ *near* T_0 *are in a form of* (2.3), *where* $u(\varepsilon_0)=u_0$, $w(\varepsilon_0)=w_0$ *and* $\lambda(\varepsilon_0)=\lambda_0$. *And,*

$$\lambda'(\varepsilon_0) = -\lambda_0 \frac{\int_\Omega f_\varepsilon(\varepsilon_0, u_0)\, w_0\, dx}{\int_\Omega f(\varepsilon_0, u_0)\, w_0\, dx}. \tag{5.8}$$

Moreover, if $f\in C^k$, *so is* $u(\varepsilon)$, $w(\varepsilon)$ *and* $\lambda(\varepsilon)$.

Proof. We show F satisfies (F2) and (F4). It is easy to see that $F_\lambda(\varepsilon_0, \lambda_0, u_0) = f(\varepsilon_0, u_0)$. If $v \in R(F_u(\varepsilon_0, \lambda_0, u_0))$, then there exists $\psi \in X$ such that $\mathscr{L}\psi + \lambda_0 f_u(\varepsilon_0, u_0)\psi = v$. On the other hand, $w_0 \in X$ satisfies $\mathscr{L}w_0 + \lambda_0 f_u(\varepsilon_0, u_0) w_0 = 0$. Using integration by parts and Green's formula, we obtain $\int_\Omega vw_0\, dx = 0$. So $v \in R(F_u)$ if and only if $\int_\Omega vw_0\, dx = 0$, Thus (5.6) implies (F2). Similarly, $F_{uu}[w_0, w_0] = \lambda_0 f_{uu}(\varepsilon_0, u_0) w_0^2$, and (5.7) implies (F4). (Note that $\lambda_0 > 0$.) Then the result follows from Theorem 2.1. The formula (5.8) also follows from (2.4) since $l(v) = \int_\Omega vw_0\, dx$ for $v \in Y^* = Y$. ∎

Similarly, we have the application of Theorem 2.2 to (5.1):

THEOREM 5.2. *Suppose* $H(T_0) = 0$ *for* $T_0 = (\varepsilon_0, \lambda_0, u_0, w_0) \in M_1$, T_0 *satisfies* (F1), (5.6),

$$\int_\Omega f_{uu}(\varepsilon_0, u_0)\, w_0^3\, dx = 0, \tag{5.9}$$

and

$$D = \det \begin{bmatrix} \lambda_0 \int_\Omega f_\varepsilon w_0\, dx & \int_\Omega f w_0\, dx \\ \lambda_0 \int_\Omega f_\varepsilon \theta\, dx + \lambda_0 & \int_\Omega f\theta\, dx + \int_\Omega f_u w_0^2\, dx \end{bmatrix} \neq 0, \tag{5.10}$$

where $f_i = f_i(\varepsilon_0, u_0)$ *and* $i = \varnothing, \varepsilon, u, \varepsilon u$. *Then there exists* $\delta > 0$ *such that all the solutions to* $H(\varepsilon, \lambda, u, w) = 0$ *near* T_0 *are in a form of* (2.6), *where* $u(s) = u_0 + sw_0 + z_1(s)$, $w(s) = w_0 + s\theta + z_2(s)$, $\lambda(s) = \lambda_0 + z_3(s)$ *and* $\varepsilon(s) = \varepsilon_0 + \tau(s)$, θ *is the solution to*

$$\begin{cases} \mathscr{L}\theta + \lambda_0 f_u(\varepsilon_0, u_0)\theta + \lambda_0 f_{uu}(\varepsilon_0, u_0)\, w_0^2 = 0, & \text{in } \Omega, \\ \theta = 0, \quad \text{on } \partial\Omega, \quad \int_\Omega \theta w_0\, dx = 0, \end{cases} \tag{5.11}$$

and $z_i(0) = z_i'(0) = 0$ *for* $i = 1, 2, 3$, $\tau(0) = \tau'(0) = 0$.

Proof. By the proof of Theorem 5.1, T_0 satisfies (F2) and (F4′). To prove T_0 satisfies (2.5), we use the explicit form of (2.5) in Section 3, (4.22). Since $Y = L^2(\Omega)$, then $\langle u, v\rangle_Y = \int_\Omega uv\, dx$. So (4.22) follows from (5.10). ∎

Under the conditions of Theorem 5.2, the turning direction $(\varepsilon''(0), \lambda''(0))$ can be determined by the computation in in Section 3. Suppose all conditions in Theorem 5.2 are satisfied and $\{(\varepsilon(s), \lambda(s), u(s), w(s)) : s \in (-\delta, \delta)\}$ is the curve of the degenerate solutions. Then by (4.25), $(\varepsilon''(0), \lambda''(0))$ satisfies

$$\begin{bmatrix} \lambda_0 \int_\Omega f_\varepsilon w_0\, dx & \int_\Omega f w_0\, dx \\ \lambda_0 \int_\Omega f_\varepsilon \theta\, dx + \lambda_0 \int_\Omega f_{\varepsilon u} w_0^2\, dx & \int_\Omega f\theta\, dx + \int_\Omega f_u w_0^2\, dx \end{bmatrix} \begin{pmatrix} \varepsilon''(0) \\ \lambda''(0) \end{pmatrix} = \begin{pmatrix} 0 \\ -e \end{pmatrix}, \tag{5.12}$$

where

$$e = 3\lambda_0 \int_\Omega f_{uu}(\varepsilon_0, u_0)\, \theta w_0^2\, dx + \lambda_0 \int_\Omega f_{uuu}(\varepsilon_0, u_0)\, w_0^4\, dx. \tag{5.13}$$

Thus

$$\varepsilon''(0) = \frac{e}{D} \int_\Omega f w_0\, dx, \qquad \text{and} \qquad \lambda''(0) = -\frac{e}{D} \lambda_0 \int_\Omega f_\varepsilon w_0\, dx. \tag{5.14}$$

Finally, Theorem 2.4 in the context of (5.1) is

THEOREM 5.3. *Suppose* $f(\varepsilon_0, 0) = 0$, $f_u(\varepsilon_0, 0) > 0$, $H(T_0) = 0$ *for* $T_0 = (\varepsilon_0, \lambda_0, 0, w_0) \in M_1$. *Assume* T_0 *satisfy* (F1), (5.7), *and*

$$\int_\Omega f_\varepsilon(\varepsilon_0, 0)\, w_0\, dx \neq 0. \tag{5.15}$$

Then there exists $\delta > 0$ *such that for* $\varepsilon \in (\varepsilon_0 - \delta, \varepsilon_0 + \delta)$, *all the solutions to* $H(\varepsilon, \lambda, u, w) = 0$ *near* T_0 *are in a form of* (2.6), *and* $\varepsilon(s) = \varepsilon_0 + \tau(s)$, $\tau(0) = \tau'(0) = 0$. *If* (5.7) *is also satisfied,* $u(s) = u_0 + skw_0 + z_1(s)$, $w(s) = w_0 + s\psi + z_2(s)$, $\lambda(s) = \lambda_0 + s + z_3(s)$, ψ *is the unique solution of*

$$\begin{cases} \mathscr{L}\psi + \lambda_0 f_u(\varepsilon_0, 0)\psi + f_u(\varepsilon_0, 0)\, w_0 \\ \qquad + k\lambda_0 f_{uu}(\varepsilon_0, 0)\, w_0^2 = 0, \text{ in } \Omega, \\ \psi = 0, \text{ on } \partial\Omega, \ \int_\Omega \psi w_0\, dx = 0, \end{cases} \tag{5.16}$$

and $z_i(0) = z_i'(0) = 0$ *for* $i = 1, 2, 3$. *If* (5.9) *is satisfied,* $u(s) = u_0 + sw_0 + z_1(s)$, $w(s) = w_0 + s\theta + z_2(s)$, $\lambda(s) = \lambda_0 + z_3(s)$, θ *is the unique solution of* (5.11), *and* $z_i(0) = z_i'(0) = 0$ *for* $i = 1, 2, 3$.

The proof is straightforward so we omit it. We also mention that Theorems 2.3 and 2.5 have corresponding applications in the context of Eq. (5.1) in an obvious way, and we would not repeat the results here, but we will see these results in specific examples in the next section.

6. EXAMPLES OF GLOBAL BIFURCATION DIAGRAMS

In this section, we consider (5.1) with $\mathscr{L} = \Delta$:

$$\Delta u + \lambda f(\varepsilon, u) = 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{6.1}$$

for several special nonlinearities $f(\varepsilon, u)$. We will only consider the positive solutions and negative solutions of (6.1) for $\lambda > 0$. In this section λ_1 is the first eigenvalue of $-\Delta$ on $H_0^1(\Omega)$. We will always use $(\varepsilon(s), \lambda(s), u(s), w(s))$ as the curve of the degenerate solutions, and $(\bar{\lambda}(t), \bar{u}(t))$ as the curve of the solutions of (6.1).

6.1. *A Perturbed Logistic Equation*

Here we consider $f(\varepsilon, u) = u - bu^2 - \varepsilon$, where $b > 0$. When $\varepsilon = 0$, then Eq. (6.1) is the Logistic equation, and it is well known that, for $\lambda > \lambda_1$, there is a unique positive solution. We consider the bifurcation of solutions of (6.1) near $(\varepsilon, \lambda, u) = (0, \lambda_1, 0)$ for small ε. Since this is a local bifurcation near $u = 0$, so we can actually work on more general conditions. We assume that $f(\varepsilon, u)$ satisfies

$$
\begin{aligned}
&f \in C^2(\mathbf{R} \times \mathbf{R}), \qquad f(\varepsilon, u) = f(0, u) - \varepsilon, \qquad f(0, 0) = 0,\\
&f_u(0, 0) = 1, \qquad f_{uu}(0, 0) = -2b < 0.
\end{aligned}
\tag{6.2}
$$

When $\varepsilon = 0$, $\lambda = \lambda_1$ is a point where a bifurcation from the trivial solution occurs: there exists $\delta > 0$ such that all the nontrivial solutions near $(\lambda_1, 0)$ form a curve $\Sigma_0 = (\bar{\lambda}(t), \bar{u}(t))$, $t \in (-\eta, \eta)$, $\bar{\lambda}(0) = \lambda_1$, $\bar{u}(0) = 0$, $\bar{u}(t) = tw_0 + z(s)$ and by (4.1)

$$
\bar{\lambda}'(0) = -\frac{\lambda_1 f_{uu}(0, 0) \int_\Omega w_0^3 \, dx}{2 f_u(0, 0) \int_\Omega w_0^2 \, dx} > 0, \tag{6.3}
$$

where w_0 is the eigenfunction corresponding to λ_1 and $z(s)$ is a higher order term. By the maximal principle, we can show that for $t \in (0, \eta)$, $\bar{u}(t)$ is a positive solution, and for $t \in (-\eta, 0)$, $\bar{u}(t)$ is a negative solution. Note that $(\lambda, u) = (\lambda_1, 0)$ is the only degenerate solution on the curve.

We verify the conditions in Theorems 5.3 and (2.5). First (F1) is satisfied since λ_1 is a simple eigenvalue and $\Delta + \lambda_1$ is a Fredholm operator with index 0, $\int_\Omega f_{uu}(0, 0)\, w_0^3 \, dx = -2b \int_\Omega w_0^3 \, dx \neq 0$ so (5.7) is satisfied. Next $\int_\Omega f(0, 0) w \, dx = 0$ and $\int_\Omega f_\varepsilon(0, 0)\, w_0 \, dx = -\int_\Omega w_0 \, dx \neq 0$. Thus Theorem 5.3 can be applied, and the degenerate solutions near $(0, \lambda_1, 0, w_0)$ forms a curve $\{(\varepsilon(s), \lambda(s), u(s), w(s)) : s \in (-\delta, \delta)\}$. Moreover, $\lambda'(0) = 1$, $\varepsilon'(0) = 0$ and by (4.28),

$$
\begin{aligned}
\varepsilon''(0) &= \frac{\left(\int_\Omega f_u(0, 0)\, w_0^2 \, dx\right)^2}{\lambda_1^2 \left(\int_\Omega f_{uu}(0, 0)\, w_0^3 \, dx\right)\left(\int_\Omega f_\varepsilon(0, 0)\, w_0 \, dx\right)}\\
&= \frac{\left(\int_\Omega w_0^2 \, dx\right)^2}{2b\lambda_1^2 \left(\int_\Omega w_0^3 \, dx\right)\left(\int_\Omega w_0 \, dx\right)} > 0.
\end{aligned}
$$

Therefore, $\varepsilon=\varepsilon(s)$ is a $\subset$-shaped curve near $s=0$, and there exists $\varepsilon_1>0$ such that for $\varepsilon\in(0,\varepsilon_1)$, there are exactly two turning points near $(\lambda,u)=(\lambda_1,0)$; for $\varepsilon\in(-\varepsilon_1,0)$, there is no turning point near $(\lambda_1,0)$. For Theorem 2.5, we have $\langle l, F_\lambda(0,\lambda_1,0)\rangle=\int_\Omega f_u(0,0)\,w_0^2\,dx>0$ and $\langle l, F_{uu}(0,\lambda_1,0)[w_0,w_0]\rangle=\int_\Omega f_{uu}(0,0)\,w_0^3\,dx<0$. So Theorem 2.5 can also be applied; the solution set near $(\lambda_1,0)$ for $\varepsilon\in(-\varepsilon_1,\varepsilon_1)$ is just like Fig. 3. When $\varepsilon>0$, since $w_0>0$, u_+ is a positive solution, and similarly, u_- is a negative solution. By choosing ρ_2 smaller, we can also show that the solutions on Σ_ε^+ are all positive, and the solutions on Σ_ε^- are all negative. Similarly, when $\varepsilon<0$, one component consists of positive solutions, and the other consists of negative solutions. It is also easy to see the convergence of Σ_ε to Σ_0 as $\varepsilon\to 0$. In fact, when $\varepsilon>0$, the upper branch of Σ_ε^+ (which can be characterized as the stable branch: the solution with Morse index 0) converges to the positive branch of Σ_0, while the lower branch (the solutions with Morse indices 1) converges to $\{(\lambda,0):\lambda>\lambda_1\}$ as $\varepsilon\to 0^+$. Similar for Σ_ε^-. When $\varepsilon<0$, Σ_ε^+ converges to the positive branch of Σ_0 and $\{(\lambda,0):\lambda<\lambda_1\}$, and Σ_ε^- converges to the negative branch of Σ_0 and $\{(\lambda,0):\lambda>\lambda_1\}$ as $\varepsilon\to 0^+$. So there is a switch of convergence when ε changes from positive to negative.

Now we go back to our original example $f(\varepsilon,u)=u-bu^2-\varepsilon$, where $b>0$. For Ω being the unit ball, and $n\leqslant 4$, the local picture in Fig. 3 can be extended to a precise global bifurcation diagram. Note that in this case, all positive or negative solutions are radially symmetric and can be parameterized by $t=u(0)$. In fact, for $\varepsilon>0$, by Theorem 6.16 of [OS2] (note that f is concave and $f(\varepsilon,0)<0$), Σ_+ is a solution curve with only one turning point (λ_+,u_+), the upper branch of Σ_ε^+ continues to $(\lambda,t)=(\infty,z_+)$, where z_+ is the second positive zero of $f(\varepsilon,\cdot)=0$; the lower branch stops at some (λ_*,u_*) since the solution will become a sign-changing solution for $\lambda>\lambda_*$. (This part is actually true for balls of any dimension.) On the other hand, by Theorem 6.21 of [OS2], Σ_ε^- starts from $(\lambda,u)=(0,0)$, continues to the right up to the unique turning point (λ_-,u_-), then bends back and continues to $(0,\infty)$. There is no any other one-sign solution. For $\varepsilon<0$, by Theorem 6.11 and Theorem 6.2 of [OS2], Σ_ε^+ and Σ_ε^- are both monotone. (See Fig. 4.)

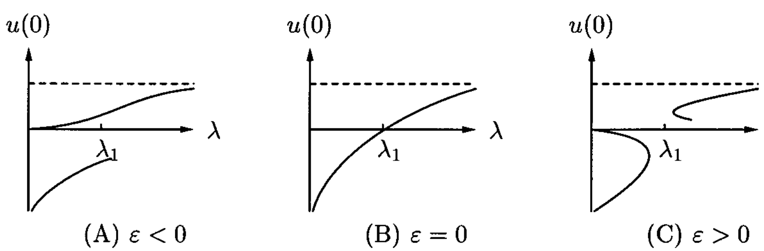


FIG. 4. Global diagram for $f(\varepsilon,u)=u-bu^2-\varepsilon$, $b>0$.

6.2. *Properties of Ordinary Differential Equations*

In our remaining examples, we are going to show some precise global bifurcation diagrams of an ordinary differential equation, which is Eq. (6.1) with spatial-dimension $n=1$,

$$u_{xx}+\lambda f(\varepsilon, u)=0, \qquad x\in(0,1), \qquad u_x(0)=u(1)=0. \tag{6.4}$$

In this subsection, we recall some basic results on (6.4), and we assume that $f\in C^1(\mathbf{R}\times\mathbf{R})$ and $\lambda>0$.

We consider (6.4) instead of

$$u_{xx}+\lambda f(\varepsilon, u)=0, \qquad x\in(-1,1), \qquad u(-1)=u(1)=0, \tag{6.5}$$

because the solution u of (6.5) is always an even function with respect to $x=0$, thus u is also the solution of (6.4). Also we only consider the monotone increasing and monotone decreasing solutions, since any non-monotone solution of (6.4) can be obtained by rescaling and periodically extending a monotone solution. Define $\Sigma_\varepsilon^+=\{(\lambda,u)$: (λ,u) satisfies (6.4), and $u_x(x)<0$, $x\in(0,1)\}$, and $\Sigma_\varepsilon^-=\{(\lambda,u)$: (λ,u) satisfies (6.4), and $u_x(x)>0$, $x\in(0,1)\}$. For $(\lambda,u)\in\Sigma_\varepsilon^+$, $u(x)>0$, and similarly, for $(\lambda,u)\in\Sigma_\varepsilon^-$, $u(x)<0$. Moreover, Σ_ε^+ and Σ_ε^- can be parameterized by $s=u(0)$:

Lemma 6.1. *For fixed $\varepsilon\in\mathbf{R}$ and any $s>0$, there is at most one $\bar\lambda(s)>0$ such that (6.4) has a solution $u(s)$ such that $u(0)=s$ and $(\bar\lambda(s),u(s))\in\Sigma_\varepsilon^+$. Let $T=\{s>0:$ (6.4) has a monotone solution with $u(0)=s\}$, then $s\mapsto\bar\lambda(s)$ is a well-defined continuous function from T to $\mathbf{R}^+$. If $f\in C^k(\mathbf{R}\times\mathbf{R})$, then $\bar\lambda(\cdot)\in C^k(T)$. Similar results also hold for Σ_ε^-.*

The lemma is well known, see [OS2]. From Lemma 6.1, $\Sigma_\varepsilon^\pm$ can be represented in a graph $(\bar\lambda(s),s)$ in $\mathbf{R}^+\times\mathbf{R}=\{(\lambda,u(0))$: $\lambda>0$, $u(0)\in\mathbf{R}\}$. So we will call $\Sigma_\varepsilon=\{(\bar\lambda(s),s)$: $s\in T\}$ the *bifurcation curve* of (6.4), where T is defined as in Lemma 6.1, and $\Sigma_\varepsilon=\Sigma_\varepsilon^+\cup\Sigma_\varepsilon^-$. Note that Σ_ε includes all the nontrivial monotone solutions, which excludes $u\equiv 0$. When $f(\varepsilon,0)=0$, $u\equiv 0$ is a trivial solution for any $\lambda>0$.

$(\bar\lambda(s),s)$ is a nondegenerate solution if $\bar\lambda'(s)\neq 0$, and it is degenerate if $\bar\lambda'(s)=0$. Equivalently, a solution (λ,u) of (6.4) is nondegenerate if

$$w_{xx}+\lambda f_u(\varepsilon,u)w=0, \qquad x\in(0,1), \quad w_x(0)=w(1)=0, \tag{6.6}$$

has only trivial solution, and it is degenerate if (6.6) has a nontrivial solution. For the degenerate solutions, we have (for the proof, see Proposition 4.4, Lemma 5.3 in [OS2], and Lemma 2.3 in [OS1])

LEMMA 6.2. *Suppose that* $u(\cdot)$ *is a degenerate solution of* (6.4). *Then the solution set of* (6.6) *is a one-dimensional linear space* $L(w)=\{kw\colon k\in\mathbf{R}\}$, w *can be chosen as positive and* $u_x(1)\neq 0$. *Moreover,*

$$\int_0^1 f(\varepsilon, u) w\,dx = \frac{1}{2\lambda} w_x(1)\, u_x(1). \tag{6.7}$$

Finally, the following result shows when a solution is not degenerate.

LEMMA 6.3. *Suppose* $f\in C^1(\mathbf{R}\times\mathbf{R})$, *and we define* $\theta_f(\varepsilon, u)=f(\varepsilon, u)-uf_u(\varepsilon, u)$. *If there exists* $u_0\in(0,\infty)$ *such that for* $u\in[0, u_0)$, $\theta_f(\varepsilon, u)>(<)\,\theta_f(\varepsilon, u_0)$, *and* $(\bar{\lambda}(u_0), u_0)\in\Sigma_\varepsilon$, *then* $\bar{\lambda}'(u_0)<(>)0$.

Proof. We use the time-mapping method. (See, for example, [BIS, W].) By (6.4), we have $T(t)=[\bar{\lambda}(t)]^{1/2}$, where $t=u(0)$, $\bar{\lambda}(t)$ is defined as in Lemma 6.1, and

$$T(t)=\frac{1}{\sqrt{2}}\int_0^t \frac{du}{\sqrt{F(\varepsilon,t)-F(\varepsilon,u)}}, \qquad T'(t)=\frac{1}{(2)^{3/2}\,t}\int_0^t \frac{[\theta(\varepsilon,t)-\theta(\varepsilon,u)]\,du}{[F(\varepsilon,t)-F(\varepsilon,u)]^{3/2}}, \tag{6.8}$$

where $F(\varepsilon, u)=\int_0^u f(\varepsilon, x)\,dx$. Since $T'(t)=(1/2)\,\bar{\lambda}^{-1/2}(t)\,\bar{\lambda}'(t)$, then the result follows from (6.8). ∎

6.3. *An Equation with S-Shaped Curve*

We consider (6.4) and $f(\varepsilon, u)=u+u^3-u^4-\varepsilon u^2$. We will show that there are exactly three degenerate solutions on Σ_ε for certain ε using Theorem 5.1 and the results in [OS2]. When $\varepsilon=0$, $f(0,u)=u+u^3-u^4$ satisfies $f_{uu}\geqslant 0$ in $[0,\alpha]$ and $f_{uu}\leqslant 0$ in $[\alpha,\infty)$. By Lemma 6.2, at any degenerate solution (λ_0, u_0), the solution of linearized equation $w_0>0$. Thus by Theorem 3.13 in [OS2], for any degenerate solution $(\bar{\lambda}(t), t)\in\Sigma_0^+$, $t>0$, $\bar{\lambda}''(t)>0$. So there is at most one turning point on each connected component of Σ_0^+. On the other hand, $f_u(0,0)=1>0$, then $(\lambda_1, 0)$ is a point where a bifurcation from the trivial solutions occurs, and a curve $\Sigma_0=\{(\bar{\lambda}(t), t)\}$ of solutions of (6.4) bifurcates from $(\lambda_1, 0)$ with $\bar{\lambda}'(0)=0$ by (4.1) and $f_{uu}(0,0)=0$. And by (4.2),

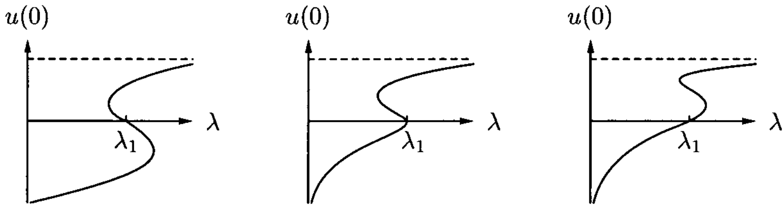
$$\bar{\lambda}''(0)=-\frac{\lambda_1 f_{uuu}(0,0)\int_0^1 w_0^4\,dx}{3f_u(0,0)\int_0^1 w_0^2\,dx}<0,$$

where w_0 is the eigenfunction corresponding to λ_1. So Σ_0^+ bifurcates to the left of $(\lambda_1, 0)$, and Σ_0^- also bifurcates to the left of $(\lambda_1, 0)$. Since $f(0,u)\leqslant ku$ for some $k>0$ and all $u\geqslant 0$, then there is no positive solution for $\lambda>0$

small by Lemma 6.17 of [OS2], and for any positive solution u, $\|u\|_\infty < M_0$ where M_0 is the positive zero of $f(0, u)$. Therefore, there is a unique turning point $(\bar{\lambda}(t_0), t_0)$ such that Σ_0^+ starts from $(\lambda_1, 0)$, continues to the left up to $(\bar{\lambda}(t_0), t_0)$, then Σ_0^+ bends back and continues right to $(\bar{\lambda}, t) = (\infty, M_0)$. There is no solution with $u(0) > M_0$ by the maximal principle. Σ_0^+ is an exact $\subset$-shaped curve, and $\int_0^1 f_{uu}(0, u(t_0))\, w(t_0)^3\, dx < 0$. We can study Σ_0^- similarly. In fact, if u is a negative solution, then $-u$ is a positive solution of $u_{xx} + \lambda[-f(\varepsilon, -u)] = 0$. For $g(0, u) = -f(0, -u) = u + u^3 + u^4$, $g_{uu} > 0$ for $u > 0$. Thus by Theorem 3.12 in [OS2], for any degenerate solution $(\bar{\lambda}(t), t) \in \Sigma_0^-$, $t < 0$, $\bar{\lambda}''(t) < 0$. So there is at most one turning point on each connected component of Σ_0^-. On the other hand, Σ_0^- bifurcates to the left of $(\lambda_1, 0)$, $\bar{\lambda}'(t) > 0$ for $t \in (-\delta_1, 0)$, $\delta_1 > 0$. And by Lemma 6.3, $\bar{\lambda}'(t) > 0$ for $t < -\delta_2$, for $\delta_2 > \delta_1$. Therefore, Σ_0^- is connected and there is no turning point on it. So Σ_0 is exactly S-shaped with the lower turning point at $(\lambda_1, 0)$. (See Fig. 5(B).)

Next we show that when $\varepsilon > 0$ is small, Σ_ε^+ is exact S-shaped. First, Σ_ε^+ still bifurcates from $(\lambda_1, 0)$, but the turning direction is changed since $\bar{\lambda}'(0) = \lambda_1 \varepsilon \int_0^1 w_0^3\, dx / \int_0^1 w_0^2\, dx > 0$. So Σ_ε^+ bifurcates to right from $(\lambda_1, 0)$. On the other hand, by Theorem 5.1, there exists $\varepsilon_1 > 0$ such that for $\varepsilon \in (0, \varepsilon_1)$, Σ_ε^+ has a degenerate solution $(\bar{\lambda}(t_\varepsilon), t_\varepsilon)$ which is a perturbation of $(\bar{\lambda}(t_0), t_0)$. For any small $\delta > 0$, $(\bar{\lambda}(t_\varepsilon), t_\varepsilon)$ is the only degenerate solution with $t \in (\delta, M_0 - \delta)$ by Theorem 2 of Dancer [Da], since there is only one turning point on Σ_0^+, and at $(\bar{\lambda}(t_0), u(t_0))$, $\int_0^1 f_{uu}(0, u(t_0))\, w^3(t_0)\, dx < 0$. There is no turning points on Σ_ε with $\|u\|_\infty > M_0 - \delta$ for $\varepsilon > 0$ small. In fact, let M_ε be the zero of $f(\varepsilon, \cdot)$ perturbed from M_0, then for $\theta(\varepsilon, u) = f(\varepsilon, u) - u f_u(\varepsilon, u) = u^2(3u^2 - 2u + \varepsilon)$, $\theta(\varepsilon, M_\varepsilon) > \theta(\varepsilon, u)$ for $\varepsilon \in [0, \varepsilon_1]$ and $u \in [0, M_\varepsilon)$. Thus by Lemma 6.3, $\bar{\lambda}'(t) > 0$ for $t \in [M_0 - \delta, M_\varepsilon)$ for $\delta > 0$ small enough.

So the portion of Σ_ε^+ above $\|u\|_\infty = \delta$ is clear, there is exactly one turning point $(\bar{\lambda}(t_\varepsilon), t_\varepsilon)$. Here $\delta > 0$ is a small constant independent of ε, and we assume that $\delta < 1/4$. Then it is elementary to check that $f_{uu}(\varepsilon, \cdot)$ changes sign exactly one in $[0, \delta]$ for $\varepsilon > 0$ small enough. Therefore for any turning point in the portion $t \in (0, \delta)$, by Theorem 3.13 in [OS2], $\bar{\lambda}''(t) < 0$ since $f_{uu}(\varepsilon, u) \leqslant 0 \in [0, \alpha_\varepsilon]$ and $f_{uu}(\varepsilon, u) \geqslant 0 \in [\alpha_\varepsilon, \delta]$. Thus there is at most one



(A) $-\varepsilon_1 < \varepsilon < 0$ (B) $\varepsilon = 0$ (C) $0 < \varepsilon < \varepsilon_1$

FIG. 5. Global diagram for $f(\varepsilon, u) = u + u^3 + u^4 - \varepsilon u^2$.

turning point in that portion of Σ_ε, and there is indeed one since $\bar{\lambda}'(t)>0$ near $t=0$ and $\bar{\lambda}'(t)<0$ at $t=\delta$. Hence Σ_ε^+ is exactly S-shaped. Similarly, we can show that Σ_ε^- is monotone increasing, $\bar{\lambda}'(t)>0$ for $t\in(-\infty,0]$. So for $\varepsilon\in(0,\varepsilon_1)$, Σ_ε is still exactly S-shaped with the lower turning point above $t=0$. $(\lambda_1,0)$ is still a degenerate solution, but not a turning point, since $\bar{\lambda}'(0)>0$. (See Fig. 5(C).) For $\varepsilon<0$, we can do a similar analysis. Σ_ε is still exactly S-shaped with the lower turning point below $t=0$, so Σ_ε^+ is exactly $\supset$-shaped, and Σ_ε^- is exactly $\subset$-shaped. (See Fig. 5(A).)

From the view of degenerate solutions, a transcritical bifurcation (with respect to ε) occurs at $\varepsilon=0$ and $(\lambda,u)=(\lambda_1,0)$. However Theorem 1.2 can not be applied to the bifurcation of the degenerate solutions here.

6.4. *Global Bifurcation*: *From Monotone to Reversed S-Shaped*

Here we consider (6.4) and $f(\varepsilon,u)=f_0(u+\varepsilon)$, where $f_0(u)=u^3-bu^2+cu$, $b,c>0$ and $3c>4b^2$. Note that if $3c>4b^2$, then $f_u(\varepsilon,u)>0$ for all $u\in\mathbf{R}$. Thus we study

$$u_{xx}+\lambda f_0(u+\varepsilon)=0,\qquad x\in(0,1),\qquad u_x(0)=u(1)=0. \tag{6.9}$$

We will classify Σ_ε for (6.9) and all $\varepsilon\in\mathbf{R}$. Our main result in this subsection is

PROPOSITION 6.4. *Let $f_0(u)=u^3-bu^2+cu$, where $b,c>0$ and $3c>4b^2$. Then there exists $\varepsilon_0<0$ such that*

(1) *For $\varepsilon<0$, there exists $t_1>0$ such that*

$$\Sigma_\varepsilon^+=\{(\bar{\lambda}(t),t)\colon t\in I_+\equiv[t_1,\infty)\},\qquad \textit{and}$$
$$\Sigma_\varepsilon^-=\{(\bar{\lambda}(t),t)\colon t\in I_-\equiv(-\infty,0)\}.$$

Σ_ε^- is exactly $\supset$-shaped, and has exactly one turning point at $t=t_2<0$, $\lim_{t\to 0^-}\bar{\lambda}(t)=0$. For Σ_ε^+, if $\varepsilon\in(-\infty,\varepsilon_0)$, Σ_ε^+ is monotone decreasing, $\bar{\lambda}'(t)<0$ for $t\in I_+$; if $\varepsilon=\varepsilon_0$, Σ_ε^+ has a cusp type degenerate solution at $t=t_3>t_1$, i.e. $\bar{\lambda}'(t_3)=\bar{\lambda}''(t_3)=0$ and $\bar{\lambda}'''(t_3)<0$, and $\bar{\lambda}'(t)<0$ for $t\in I_+\backslash\{t_3\}$; if $\varepsilon\in(\varepsilon_0,0)$, Σ_ε^+ is reversed S-shaped, there are exactly two turning points at $t=t_4$ and $t=t_5$ on Σ_ε^+. (See Figs. 6(A)–6(C).)

(2) *For $\varepsilon=0$,*

$$\Sigma_\varepsilon^+=\{(\bar{\lambda}(t),t)\colon t\in I_+\equiv(0,\infty)\},\quad \textit{and}\quad \Sigma_\varepsilon^-=\{(\bar{\lambda}(t),t)\colon t\in I_-\equiv(-\infty,0)\}.$$

Σ_ε^- is monotone increasing, $\bar{\lambda}'(t)>0$ for $t\in I_-$. Σ_ε^+ is exactly $\supset$-shaped and has exactly one turning point at $t=t_5>0$. $\lim_{t\to 0}\bar{\lambda}(t)=\lambda_0=\lambda_1/c$; (See Fig. 6(D).)

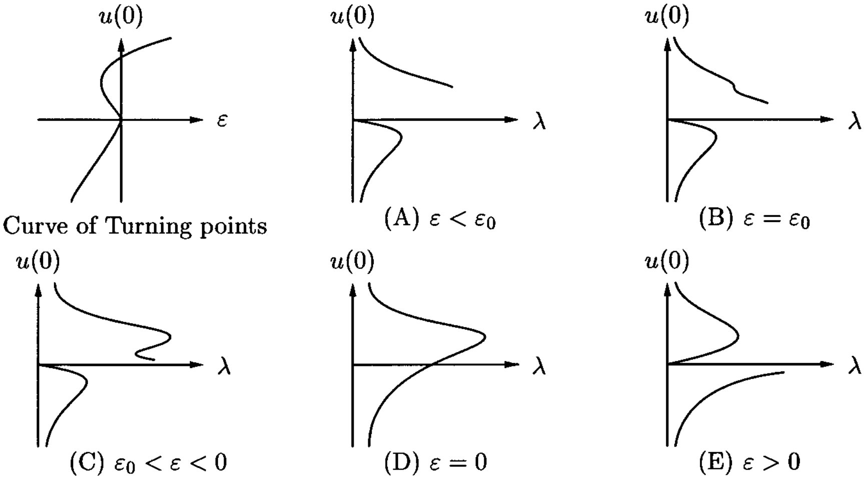


FIG. 6. Global bifurcation for (6.9).

(3) *For* $\varepsilon > 0$, *there exists* $t_6 > 0$ *such that*

$$\Sigma_\varepsilon^+ = \{(\bar{\lambda}(t), t) \colon t \in I_+ \equiv (0, \infty)\}, \quad \textit{and} \quad \Sigma_\varepsilon^- = \{(\bar{\lambda}(t), t) \colon t \in I_- \equiv (-\infty, t_6)\}.$$

Σ_ε^- *is monotone increasing*, $\bar{\lambda}'(t) > 0$ *for* $t \in I_-$. Σ_ε^+ *is exactly* $\supset$*-shaped and has exactly one turning point at* $t = t_5 > 0$. $\lim_{t \to 0^+} \bar{\lambda}(t) = 0$; (*See Fig.* 6(E).)

(4) *For all* ε, $\lim_{t \to \pm\infty} \bar{\lambda}(t) = 0$.

Proof of Proposition 6.4, *Part* (1). Our proof will be in two parts. In the first part, we prove the results for Σ_ε^-, $\varepsilon \in \mathbf{R}$ and Σ_ε^+, $\varepsilon \in [0, \infty)$. Note in this part, all solution curves have at most one turning point, so the results basically follows from [OS2]. In the second part, we will complete the proof for Σ_ε^+, $\varepsilon \in (-\infty, 0)$.

First we consider $\varepsilon = 0$. For $f(0, u) = u^3 - bu^2 + cu$, $f(0) = 0$ and $f_u(0, 0) = c > 0$, thus $\lambda_0 = \lambda_1 / c$ is a point where a bifurcation from the trivial solutions occurs. (See Theorem 3.1 in [OS2].) At $(\lambda, u) = (\lambda_0, 0)$, $\Sigma_0^+ = \{(\bar{\lambda}(t), t)\}$ satisfies $\bar{\lambda}'(0) > 0$, so Σ_0^+ bifurcates to the right of λ_0. By Theorem 3.13 of [OS2], since $f(0, u)$ satisfies $f_{uu} \leqslant 0$ for $u \in [0, \alpha]$ and $f_{uu} \geqslant 0$ for $u \in [\alpha, \infty)$, then any turning point on Σ_0^+ satisfies $\bar{\lambda}''(t) < 0$, which implies there is at most one turning point on Σ_0^+. On the other hand, by Mountain Pass Theorem, we can show there is a solution for any $\lambda \in (0, \delta)$ for some $\delta > 0$. So Σ_0^+ continues to right until it reaches its only turning point $(\bar{\lambda}(t_5), t_5)$, then it bends back to left and continues to $(\lambda, t) = (0, \infty)$. We can study Σ_0^- similarly. Recall that, if u is a negative solution, then $-u$ is a positive solution of $u_{xx} + \lambda[-f_0(-u)] = 0$. For $g(u) = -f_0(-u) = u^3 + bu^2 + cu$, we have $g_{uu} > 0$ for $u > 0$. $(\lambda, t) = (\lambda_0, 0)$ is a point where a bifurcation from the trivial solutions occurs, and Σ_0^- connects with Σ_0^+ at $(\lambda_0, 0)$. So a curve of negative solutions bifurcates to the left of $(\lambda_0, 0)$, and there is no turning points on Σ_0^- by Theorem 6.5 of [OS2].

When $b/3 > \varepsilon > 0$, $f(\varepsilon, 0) > 0$, $f_{uu} < 0$ for $u \in (0, b/3 - \varepsilon)$, $f_{uu} > 0$ for $u > b/3 - \varepsilon$ and $\theta(\varepsilon, u) = f(\varepsilon, u) - uf_u(\varepsilon, u)$ changes sign exactly once in $(0, \infty)$, so by Theorem 6.21 of [OS2], Σ_ε^+ is exactly $\supset$-shaped. For $g(\varepsilon, u) = -f(\varepsilon, -u)$, $g(\varepsilon, 0) < 0$, $g_{uu} > 0$ for $u > 0$, then Σ_ε^- is monotone increasing, and the curve is "broken" at $t = t_6$ by Theorem 6.11 of [OS2]. When $\varepsilon \geqslant b/3$, $f(\varepsilon, 0) > 0$, $f_{uu} > 0$ for $u > 0$ and $\theta(\varepsilon, u)$ changes sign exactly once, so by Theorem 6.21 of [OS2], Σ_ε^+ is exactly $\supset$-shaped. For $g(\varepsilon, u) = -f(\varepsilon, -u)$, $g(\varepsilon, 0) < 0$, $g_{uu} < 0$ for $u \in (0, \varepsilon - b/3)$, $g_{uu} > 0$ for $u > \varepsilon - b/3$, but $\theta(\varepsilon, u) < 0$ for all $u > 0$, thus Σ_ε^- is monotone increasing by Theorem 6.11 of [OS2]. This completes the proof for $\varepsilon > 0$.

When $\varepsilon < 0$, $g(\varepsilon, u) = -f(\varepsilon, -u)$ satisfies $g(\varepsilon, 0) > 0$, $g_{uu} > 0$ for $u > 0$ and $\theta_1(\varepsilon, u) = g(\varepsilon, u) - ug_u(\varepsilon, u)$ changes sign exactly once in $(0, \infty)$, so by Theorem 6.21 of [OS2], Σ_ε^+ is exactly $\supset$-shaped. ∎

We prove the result for Σ_ε^+ and $\varepsilon < 0$ in several steps. First we analyze the bifurcation of solution curves near $\varepsilon = 0$. When $\varepsilon = 0$, we have two degenerate solutions on Σ_0: $(\lambda_0, 0)$ and $(\bar{\lambda}(t_5), t_5)$. At $t = t_5$, $\bar{\lambda}''(t_5) < 0$, and $\int_0^1 f_{uu}(0, u(t_5))\, w^3(t_5)\, dx > 0$, where $w(t_5)$ is the solution of linearized equation. When ε is small, $(\bar{\lambda}(t_5), t_5)$ persists under the perturbation, and for $\varepsilon < 0$, $(\lambda_0, 0)$ bifurcates to two degenerate solutions, one is positive, and the other is negative. In fact, at $(\bar{\lambda}(t_5), t_5)$, (F1) and (F4) are satisfied, and so is (F2), since $\int_0^1 f(0, u(t_5))\, w(t_5)\, dx = (2\bar{\lambda}(t_5))^{-1}\, (u(t_5))_x\, (1)(w(t_5))_x\, (1) \neq 0$. So Theorem 5.1 can be applied. On the other hand, same as Section 6.1, the degenerate solutions near $(\varepsilon, \lambda, u, w) = (0, \lambda_0, 0, w_0)$ forms a curve with $\varepsilon''(0) < 0$. So there are two degenerate solutions near $(\lambda_0, 0)$ for $\varepsilon < 0$, the two degenerate solutions are positive and negative respectively. In particular, there are at least two turning points $(\bar{\lambda}(t_5), t_5)$ and $(\bar{\lambda}(t_4), t_4)$ (which is near $(\lambda_0, 0)$) on Σ_ε^+ when $\varepsilon \in (\varepsilon_1, 0)$ for some $\varepsilon_1 < 0$.

To show there is no other turning points when $\varepsilon \in (\varepsilon_1, 0)$, we prove

Lemma 6.5. *Let $\theta(\varepsilon, u) = f(\varepsilon, u) - uf_u(\varepsilon, u)$. Define*

$$\varepsilon_2 = -\frac{b^3}{27c - 9b^2}.$$

Then when $\varepsilon < \varepsilon_2$, $\theta(\varepsilon, u) < 0$ for $u \in [0, \infty)$. When $\varepsilon \in [\varepsilon_2, 0]$, there exists $M > 0$ (which does not depends on ε) such that for any $u > M$, $\theta(\varepsilon, u) < \theta(\varepsilon, v)$ for $v \in [0, u]$.

Proof. Since $\theta_u(\varepsilon, u) = -uf_{uu}(\varepsilon, u)$, then the only maximum of $\theta(\varepsilon, \cdot)$ is achieved at $u = b/3 - \varepsilon$. When $\varepsilon < \varepsilon_2$, it is easy to calculate that $\theta(\varepsilon, b/3 - \varepsilon) < 0$, thus $\theta(\varepsilon, u) < 0$ for $u \in [0, \infty)$. For $\varepsilon \in [\varepsilon_2, 0]$, we notice that $\theta(\varepsilon, u) \to -\infty$ uniformly for $\varepsilon \in [\varepsilon_2, 0]$, so such an M exists. ∎

By Lemma 6.5, we have

LEMMA 6.6. (1) *When $\varepsilon < \varepsilon_2$, Σ_ε^+ is monotone;*

(2) *When $\varepsilon \in (\varepsilon_1, 0)$ (maybe choose $|\varepsilon_1|$ even smaller), Σ_ε^+ is exactly reversed S-shaped;*

(3) *When $\varepsilon \in [\varepsilon_2, 0]$, $\bar{\lambda}'(t) < 0$ for $t > M$.*

Proof. When $\varepsilon < \varepsilon_2$, $\bar{\lambda}'(t) < 0$ for any $t > t_1 > 0$ by Lemma 6.3, so Σ_ε^+ is monotone decreasing. And also by Lemma 6.3, and Lemma 6.5, $\bar{\lambda}'(t) < 0$ for $t > M$ when $\varepsilon \in [\varepsilon_2, 0]$. When $\varepsilon \in (\varepsilon_1, 0)$, there are two turning points $(\bar{\lambda}(t_4), t_4)$ and $(\bar{\lambda}(t_5), t_5)$ (which depend on ε). If there is another degenerate solution $(\varepsilon^n, \lambda^n, t^n)$ for $\varepsilon^n \to 0^-$, then $t_n \in (0, M)$, so we can assume that $(\varepsilon^n, \lambda^n, t^n) \to (0, \lambda_*, t_*)$ as $\varepsilon^n \to 0^-$. Here λ^n are also bounded since $\lambda^n = \bar{\lambda}(t^n)$ and $\bar{\lambda}(\cdot)$ is uniformly bounded for $\varepsilon \in [\varepsilon_2, 0]$ and $t \in [t_1, M]$. And $(0, \lambda_*, t_*)$ is a degenerate solution for $\varepsilon = 0$, so $(\varepsilon^n, \lambda^n, t^n)$ must be coincident with one of $(\bar{\lambda}(t_4), t_4)$ and $(\bar{\lambda}(t_5), t_5)$ since they are the only degenerate solutions perturbed from the degenerate solutions when $\varepsilon = 0$. That is a contradiction. So (6.9) has exactly two degenerate solutions when $\varepsilon \in (\varepsilon_1, 0)$. On the other hand, Σ_ε^+ is connected by using the arguments in the proof of Theorem 6.16 and Proposition 7.2 of [OS2]. Since $(\bar{\lambda}(t_4), t_4)$ and $(\bar{\lambda}(t_5), t_5)$ are the perturbation of degenerate solutions when $\varepsilon = 0$, then $\bar{\lambda}''(t_4) > 0$ and $\bar{\lambda}''(t_5) < 0$, and consequently Σ_ε^+ is exactly reversed S-shaped. ∎

From Lemma 6.6, (6.9) has no turning points on Σ_ε^+ when $\varepsilon < \varepsilon_2$, and there are two turning points on Σ_ε^+ when $\varepsilon \in [\varepsilon_1, 0)$. So to complete the proof, the key is to prove there is only one point $\varepsilon \in (\varepsilon_2, \varepsilon_1)$ such that the number of degenerate solutions changes. We will apply Theorem 5.2 in that part of proof, and the key estimate is

LEMMA 6.7. *Let $(\varepsilon, \lambda, u, w)$ be a degenerate solution of* (6.9), $\lambda > 0$, $u > 0$ *and* $\int_0^1 f(\varepsilon, u)\, w^3\, dx = 0$. *Then*

$$D = \det \begin{bmatrix} \lambda \int_\Omega f_\varepsilon w_0 \, dx & \int_\Omega f w_0 \, dx \\ \lambda \int_\Omega f_\varepsilon \theta \, dx + \lambda_0 \int_\Omega f_{\varepsilon u} w_0^2 \, dx & \int_\Omega f\theta \, dx + \int_\Omega f_u w_0^2 \, dx \end{bmatrix} > 0, \tag{6.10}$$

and

$$3\lambda \int_0^1 f_{uu}(\varepsilon, u(x))\, w^2(x)\, \theta(x)\, dx + \lambda \int_0^1 f_{uuu}(\varepsilon, u(x))\, w^4(x)\, dx > 0. \qquad (6.11)$$

Proof. We first prove (6.11). Since θ is the solution of (5.11), then

$$\lambda \int_0^1 f_{uu}(\varepsilon, u(x))\, w^2(x)\, \theta(x)\, dx = \int_0^1 \theta_x^2(x)\, dx - \lambda \int_0^1 f_u(\varepsilon, u(x))\, \theta^2(x)\, dx.$$

On the other hand, $\int_0^1 \theta w\, dx = 0$, and w is the eigenfunction corresponding to the first eigenvalue $\lambda_1(u)$ of operator $L\psi = \psi_{xx} + \lambda f_u(\varepsilon, u)\psi$, thus by the variational characterization of eigenvalues, we have

$$\frac{\int_0^1 \theta_x^2(x)\, dx - \lambda \int_0^1 f_u(\varepsilon, u(x))\, \theta^2(x)\, dx}{\int_0^1 \theta^2(x)\, dx} \geqslant \lambda_2(u) > \lambda_1(u) = 0.$$

Hence (6.11) is proved since $f_{uuu} \equiv 6 > 0$.

For the estimate of D, we notice that

$$f_\varepsilon(\varepsilon, u) = \frac{\partial f_0(\varepsilon + u)}{\partial \varepsilon} = D_u f_0(\varepsilon + u) = f_u(\varepsilon, u).$$

Thus

$$\int_0^1 f_\varepsilon w\, dx = \int_0^1 f_u w\, dx = -\frac{1}{\lambda} w_x(1) \qquad (6.12)$$

by (6.6). Similarly,

$$\int_0^1 f_\varepsilon \theta\, dx + \int_0^1 f_{\varepsilon u} w^2\, dx = \int_0^1 f_u \theta\, dx + \int_0^1 f_{uu} w^2\, dx = -\frac{1}{\lambda} \theta_x(1) \qquad (6.13)$$

by (5.11). On the other hand, by Lemma 6.2,

$$\int_0^1 f w\, dx = \frac{1}{2\lambda} w_x(1)\, u_x(1). \qquad (6.14)$$

Finally, we estimate $\int_0^1 f\theta\, dx + \int_0^1 f_u w^2\, dx$. Here we use the equations

$$u_{xx} + \lambda f(\varepsilon, u) = 0, \qquad u_x(0) = u(1) = 0, \qquad (6.15)$$

$$\theta_{xx} + \lambda f_u(\varepsilon, u)\theta + \lambda f_{uu}(\varepsilon, u) w^2 = 0, \qquad \theta_x(0) = \theta(1) = 0. \qquad (6.16)$$

Multiplying (6.15) by $x\theta_x$, we have

$$\int_0^1 u_{xx} x\theta_x \, dx + \lambda \int_0^1 f x\theta_x \, dx = 0. \tag{6.17}$$

Similarly, multiplying (6.16) by xu_x, we have

$$\int_0^1 \theta_{xx} xu_x \, dx + \lambda \int_0^1 xu_x f_u \theta \, dx + \lambda \int_0^1 xu_x w^2 f_{uu} \, dx = 0. \tag{6.18}$$

Using integral by parts, we obtain

$$\int_0^1 u_{xx} x\theta_x \, dx = u_x(1)\, \theta_x(1) - \int_0^1 u_x \theta_x \, dx - \int_0^1 u_x x\theta_{xx} \, dx. \tag{6.19}$$

Combining (6.17), (6.18), and (6.19), we obtain

$$\begin{aligned} &\int_0^1 u_x \theta_x \, dx - u_x(1)\, \theta_x(1) \\ &\qquad = \lambda \int_0^1 f x\theta_x \, dx + \lambda \int_0^1 xu_x f_u \theta \, dx + \lambda \int_0^1 xu_x w^2 f_{uu} \, dx. \end{aligned} \tag{6.20}$$

On the other hand, using integration by parts, we have

$$\int_0^1 f x\theta_x \, dx = -\int_0^1 f\theta \, dx - \int_0^1 x f_u \theta u_x \, dx, \tag{6.21}$$

and

$$\int_0^1 xu_x w^2 f_{uu} \, dx = -\int_0^1 f_u w^2 \, dx - 2 \int_0^1 xww_x f_u \, dx. \tag{6.22}$$

Multiplying (6.16) by θ, we get

$$\int_0^1 u_x \theta_x \, dx = \lambda \int_0^1 f\theta \, dx. \tag{6.23}$$

Combining (6.20), (6.21), (6.22), and (6.23), we obtain

$$2\lambda \int_0^1 f\theta \, dx + 2\lambda \int_0^1 f_u w^2 \, dx = u_x(1)\, \theta_x(1) - 2\lambda \int_0^1 xww_x f_u \, dx + \lambda \int_0^1 f_u w^2 \, dx. \tag{6.24}$$

Therefore by (6.12), (6.13), (6.14), and (6.20), we get

$$D = -w_x(1) \left[-\int_0^1 xww_x f_u \, dx + \tfrac{1}{2} \int_0^1 f_u w^2 \, dx \right] > 0. \tag{6.25}$$

Here, note that w satisfies $w_{xx} = -\lambda f_u(\varepsilon, u) w < 0$ since $f_u(\varepsilon, u) > 0$ from $3c > 4b^2$ and $w > 0$, thus $w_x(x) < 0$ for $x \in (0, 1]$ since $w_x(0) = 0$. ▮

COROLLARY 6.8. *Let* $(\varepsilon_0, \lambda_0, u_0, w_0)$ *be a degenerate solution of* (6.9) *with* $\lambda > 0$, $u_0 > 0$ *and* $\int_0^1 f(\varepsilon_0, u_0) w_0^3 \, dx = 0$. *Then there exists* $\delta > 0$ *such that for* $\varepsilon \in (\varepsilon_0 - \delta, \varepsilon_0 + \delta)$, *all the degenerate solutions of* (6.9) *near* $(\varepsilon_0, \lambda_0, u_0, w_0)$ *is of form* $\{T_\varepsilon = (\varepsilon(s), \lambda(s), u(s), w(s)) : s \in (-\delta, \delta)\}$, *where* $\varepsilon(0) = \varepsilon_0$, $\lambda(0) = \lambda_0$, $\varepsilon'(0) = \lambda'(0) = 0$ *and* $\varepsilon''(0) > 0$, $\lambda''(0) < 0$.

Proof of Proposition 6.4, *Part* (2). By Lemma 6.3, any degenerate solution $T = (\varepsilon, \lambda, u, w)$ satisfies (F1). But otherwise there are three possibilities: (a) T satisfies (F2) and (F4), (T is a fold;) (b) T satisfies (F2) and (F4′) (T is a cusp;) (c) T satisfies (F2′). We claim that, if T is type c, then $u \equiv 0$ and $\varepsilon = 0$. In fact, by Lemma 6.3, $\langle l, F_\lambda \rangle = \int_0^1 f(\varepsilon, u) w \, dx = (2\lambda)^{-1} u_x(1) w_x(1)$. So if T is type c, then $u_x(1) w_x(1) = 0$. If $w_x(1) = 0$, then $w(x) \equiv 0$ for $x \in [0, 1]$ since w satisfies a second order linear equation and $w(1) = w_x(1) = 0$, which contradicts with w being a nontrivial solution. So $u_x(1) = 0$. If $u_x \not\equiv 0$, then by Lemma 6.1, $u_x(x) < 0$ for $x \in (0, 1)$ and $u_x(0) = u_x(1) = 0$. Since w and u_x both satisfy the equation $\phi'' + \lambda f_u(\varepsilon, u)\phi = 0$, then by Sturm comparison lemma, w must have a zero in $(0, 1)$, which contradicts with $w > 0$ from Lemma 6.3. Thus $u_x \equiv 0$, and $u \equiv 0$ since $u(1) = 0$. On the other hand, $f(\varepsilon, 0) = 0$ if and only if $\varepsilon = 0$, hence $\varepsilon = 0$. Therefore, T satisfies $w_{xx} + \lambda f_u(0, 0) w = 0$, λc must be an eigenvalue of $\phi_{xx} + \lambda\phi = 0$, $\phi_x(0) = \phi(1) = 0$. In particular, if $u > 0$, then T is either type a or type b.

Define Γ be the set of degenerate solutions $(\varepsilon, \lambda, u, w)$ with $\varepsilon < 0$ and $u(x) > 0$ for $x \in (0, 1)$. Let $T \in \Gamma$. By Lemma 6.2, $u_x(1) < 0$. If T is type a, then we can apply Theorem 5.1 near T, and all degenerate solutions near T are on a curve which can be parameterized by ε. If T is type b, by Corollary 6.8, the degenerate solutions near T forms a curve $T(t) = (\varepsilon(t), \lambda(t), u(t), w(t))$, $t \in (-\eta, \eta)$, with $T(0) = T$ and $\varepsilon''(0) > 0$. In either case, for a degenerate solution T_1 near T, $T_1 \in \Gamma$ since $u_x(1) < 0$ at T.

Let Γ_1 be a connected component of Γ. From the last paragraph, there is at least one T on Γ_1 which is type a, thus near T, Γ_1 is a curve. We continue this curve Γ_1 to the left (ε decreases). Since there is no degenerate solution for $\varepsilon < \varepsilon_2$, then before ε reaches ε_2, Γ_1 either blows up ($|\varepsilon| + |\lambda| + \|u\| + \|w\| \to \infty$), or Γ_1 reaches a boundary point, or Γ_1 reaches a point where it can not be continued to further left, that is, a type b degenerate solution. However, blowing up of Σ_1 is impossible, since $\|u\|_\infty \leqslant M$, $\|w\| \leqslant 1$, $|\varepsilon| \leqslant |\varepsilon_2|$ and $|\lambda|$ is bounded because

$$\lambda = \frac{\int_0^1 u_x^2 \, dx}{\int_0^1 f(u) u \, dx} \leqslant \frac{\lambda_1 \int_0^1 u^2 \, dx}{C_1 \int_0^1 u^2 \, dx} = \frac{\lambda_1}{C_1},$$

where $C_1 = \min_{u \geqslant 0} f(u)/u$. If Σ_1 reaches a boundary point $T_1 = (\varepsilon, \lambda, u, w)$, then $\varepsilon < 0$, and T_1 is a degenerate solution which is not type c, so it must be type a or b, then it is not a boundary point. Therefore, Σ_1 reaches a type b degenerate solution T_0 at $\varepsilon = \varepsilon_0$, then bends back at T_0, continue to the right (ε increases). There is no any other type b degenerate solutions on Γ_1, since for any type b degenerate solution, the curve of degenerate solution bends to the right. Hence Γ_1 has two branches for $\varepsilon > \varepsilon_0$, and both branches can be extended to the right until they approach the boundary of Γ or blow up. However, we have shown that Γ_1 can not blow up and it can only reach the boundary at $\varepsilon = 0$. In particular, for any $\varepsilon \in (\varepsilon_0, 0)$, there are exactly two points on Γ_1, and Γ_1 is an exactly $\subset$-shaped curve. Since for $\varepsilon \in (\varepsilon_1, 0)$, (6.9) has exactly two degenerate solutions on Σ_0^+, then Γ has one and only one connected component, which is exactly $\subset$-shaped.

Therefore, when $\varepsilon \in (\varepsilon_0, 0)$, Σ_ε^+ is exactly reversed S-shaped; when $\varepsilon < \varepsilon_0$, Σ_ε^+ is monotone and when $\varepsilon = \varepsilon_0$, there is one and only one cusp type degenerate solution. ∎

The result in Proposition 6.4 can be generalized to a more general class of functions. In fact, it is easy to verify that a similar result holds if $f(\varepsilon, u)$ satisfies the following conditions:

(1) $f(\varepsilon, u) = f(\varepsilon + u)$;

(2) $f \in C^3(\mathbf{R})$, $f(0) = 0$, $f'(0) > 0$, $f'(u) \geqslant 0$ for $u \in \mathbf{R}$;

(3) $f'''(u) \geqslant 0$ $u \in \mathbf{R}$;

(4) There exists $\alpha > 0$ such that $f''(u) < 0$ in $(-\infty, \alpha)$ and $f''(u) > 0$ in (α, ∞);

(5) There exists $M > 0$ such that for $|u| \geqslant M$, and $\varepsilon \in [0, \alpha]$, $\theta_u(\varepsilon, u) < 0$, where $\theta(\varepsilon, u) = f(\varepsilon, u) - u f_u(\varepsilon, u)$.

6.5. *Perturbed Gelfand Equation: Conjecture and Perspective*

Finally, we consider the Perturbed Gelfand equation

$$\Delta u + \lambda \exp[-1/(u+\varepsilon)] = 0, \qquad x \in B^n, \quad v(x) = 0, \quad x \in \partial B^n, \tag{6.26}$$

where B^n is the unit ball in n-dimensional space. The original perturbed Gelfand equation is

$$\Delta v + \mu \exp[v/(\varepsilon v + 1)] = 0, \qquad x \in \Omega, \quad v(x) = 0, \quad x \in \partial\Omega. \tag{6.27}$$

Equation (6.26) is obtained by a change of variables: $u = \varepsilon^2 v$ and $\lambda = \varepsilon^2 \exp(1/\varepsilon)\mu$, which was introduced by Du and Lou [DL2]. Here we consider (6.26) since it is in the form of equation in (6.9) with $f_0(u) = \exp(-1/u)$.

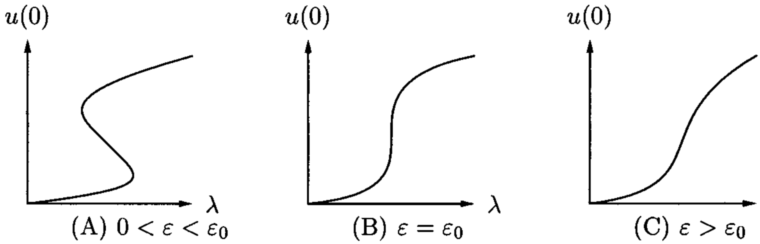


FIG. 7. Conjecture for perturbed Gelfand equation, $n = 1, 2$.

Let Σ_ε be the solution set of (6.26). Then any solution of (6.26) is positive and radially symmetric. Thus Σ_ε can be represented in a form $(\lambda(t), t)$ where $t = u(0) \in (0, \infty)$. It has been a long-time conjecture that when $n = 1$ or 2, there exists $\varepsilon_0 > 0$ such that when $\varepsilon \in (0, \varepsilon_0)$, Σ_ε is exactly S-shaped; when $\varepsilon \in (\varepsilon_0, \infty)$, Σ_ε is monotone increasing; and when $\varepsilon = \varepsilon_0$, there is a cusp type degenerate solution. (See Fig. 7.)

It is easy to show that when $\varepsilon \geqslant 1/4$, Σ_ε is monotone increasing. The proof of exact S-shaped curve for small $\varepsilon > 0$ has been studied by several authors. Dancer [Da] first proved (6.26) has exactly three solutions for $\lambda \in (\lambda_0, \lambda_1(\varepsilon))$, $\varepsilon \in (0, \varepsilon_1)$ and $n = 1, 2$. Hastings and McLeod [HM] proved Σ_ε is exactly S-shaped for $n = 1$, $\varepsilon \in (0, \varepsilon_1)$ using quadratures. Recently, Wang showed that for $n = 1$ and $\varepsilon \in (0, 1/4.4967]$, Σ_ε is exactly S-shaped using a different quadrature method, and the upper bound was improved to 1/4.35 by Korman and Li [KL] using the bifurcation method (their proof also uses a little quadrature method, but it can be avoided, see [KS].) Du and Lou [DL2] proved Σ_ε is exactly S-shaped for $n = 2$ and $\varepsilon \in (0, \varepsilon_1)$ for some small $\varepsilon_1 > 0$ using a combination of bifurcation method and a perturbation argument.

Using Theorem 5.1 and previous results in [OS1], we can also obtain the same result as [DL2] in a more direct way. The proof is similar to those in the previous examples, so we omit it. To conclude our paper, we point out that, our approach in this paper implies, to completely resolve the conjecture for all $\varepsilon > 0$, we only need to prove the following estimate: for a degenerate solution $(\varepsilon, \lambda, u, w)$ of (6.26),

$$3 \int_{B^n} f_{uu}(\varepsilon, u(x))\, w^2(x)\, \theta(x)\, dx + \int_{B^n} f_{uuu}(\varepsilon, u(x))\, w^4(x)\, dx < 0. \qquad (6.28)$$

In fact, if (6.28) holds for any degenerate solution of (6.26), then our proof of Proposition 6.4 can follow through, and eventually solve the conjecture on perturbed Gelfand equation for $n = 1, 2$. However, the estimate of the integral in (6.28) seems to be quite challenging at this moment.

7. APPENDIX

LEMMA 7.1. *Let X be a Banach space, and $w \in X$ with $\|w\| = 1$. Then there exists a linear continuous functional $f \in X^*$ and a closed subspace X_0 of X with codimension one such that*

(1) $N(f) = X_0$, $f(w) = 1$ *and* $\|f\| = d^{-1} > 0$;

(2) $d(w, X_0) = \inf\{\|w - x\| : x \in X_0\} = d > 0$, $d \in (0, 1)$.

Proof. Since $L(w) = span\{w\}$ is one dimensional, then there exists a continuous projection $P: X \to L(w)$ such that $N(P) = X_0$ is a closed subspace of X with codimension 1. Since X_0 is closed, and $P(w) \neq 0$, then $d = d(w, X_0) > 0$. Since $0 \in X_0$, then $d < 1$. The existence of f follows from the Hahn–Banach Theorem, see corollary of Mazur Theorem in [Y, p. 109]. ∎

ACKNOWLEDGMENTS

I thank Professor Tiancheng Ouyang for his suggestions of considering the pair of the original equation and the linearized equation, and many discussions while this work was prepared. I also thank Professor Yihong Du, Yuan Lou, and Chongchun Zeng for their helpful discussions on the subject.

REFERENCES

[BIS] K. J. Brown, M. M. A. Ibrahim, and R. Shivaji, *S*-shaped bifurcation curves, *Nonlinear Anal.* **5**, No. 5 (1981), 475–486.

[CH] Shui Nee Chow and J. K. Hale, "Methods of Bifurcation Theory," Springer-Verlag, New York/Berlin, 1982.

[CT] P. T. Church and J. G. Timourian, Global structure for nonlinear operators in differential and integral equations. I. Folds; II. Cusps: Topological nonlinear analysis, *in* "Progr. Nonlinear Differential Equations Appl.," Vol. 27, pp. 109–160, 161–245, Birkhaüser, Boston, 1997.

[CR1] M. G. Crandall and P. H. Rabinowitz, Bifurcation from simple eigenvalues, *J. Funct. Anal.* **8** (1971), 321–340.

[CR2] M. G. Crandall and P. H. Rabinowitz, Bifurcation, perturbation of simple eigenvalues and linearized stability, *Arch. Rational Mech. Anal.* **52** (1973), 161–180.

[Da] E. N. Dancer, On the structure of solutions of an equation in catalysis theory when a parameter is large, *J. Differential Equations* **37**, No. 3 (1980), 404–437.

[Du] Yihong Du, Exact multiplicity and S-shaped bifurcation curve for some semilinear elliptic problems from combustion theory, preprint, 1998.

[DL1] Yihong Du and Yuan Lou, S-shaped global bifurcation curve and Hopf bifurcation of positive solutions to a predator-prey model, *J. Differential Equations* **144**, No. 2 (1998), 390–440.

[DL2] Yihong Du and Yuan Lou, Proof of a conjecture for the perturbed Gelfand equation from combustion theory, preprint, 1997.

[HM] S. P. Hastings and J. B. McLeod, The number of solutions to an equation from catalysis, *Proc. Roy. Soc. Edinburgh Sect. A* **101**, Nos. 1–2 (1985), 15–30.

[KL] P. Korman and Yi Li, On the exactness of an S-shaped bifurcation curve, *Proc. Amer. Math. Soc.* **127**, No. 4 (1999), 1011–1020.

[KS] P. Korman and Junping Shi, Instability and exact multiplicity of solutions of semilinear equations, submitted for publication.

[L] P.-L. Lions, On the existence of positive solutions of semilinear elliptic equations, *SIAM Rev.* **24**, No. 4 (1982), 441–467.

[OS1] Tiancheng Ouyang and Junping Shi, Exact multiplicity of positive solutions for a class of semilinear problem, *J. Differential Equations* **146**, No. 1 (1998), 121–156.

[OS2] Tiancheng Ouyang and Junping Shi, Exact multiplicity of positive solutions for a class of semilinear problem, II, *J. Differential Equations* **158**, No. 1 (1999), 94–151.

[S] Junping Shi, Ph.D. Dissertation, Brigham Young University, 1998.

[SW] J. Smoller and A. Wasserman, Global bifurcation of steady-state solutions, *J. Differential Equations* **39**, No. 2 (1981), 269–290.

[W] Shin Hwa Wang, On *S*-shaped bifurcation curves, *Nonlinear Anal.* **22**, No. 12 (1994), 1475–1485.

[WL] Shin Hwa Wang and Fu Ping Lee, Bifurcation of an equation from catalysis theory, *Nonlinear Anal.* **23**, No. 9 (1994), 1167–1187.

[Y] Kosaku Yosida, "Functional Analysis," 5th ed., Grundlehren der Mathematischen Wissenschaften, Vol. 123, Springer-Verlag, Berlin/New York, 1978.