

# Exact Multiplicity of Positive Solutions for a Class of Semilinear Problems

Tiancheng Ouyang and Junping Shi

*Department of Mathematics, Brigham Young University, Provo, Utah 84602*

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We establish an exact multiplicity result of positive solutions of

$$\Delta u + \lambda f(u) = 0 \quad \text{in } B^n, \quad u = 0 \quad \text{on } \partial B^n,$$

where  $B^n$  is the unit ball,  $f$  satisfies  $f''$  changes sign only once and asymptotically sublinear or linear. The nonlinearities which we are concerned with here include  $f(u) = u(u-b)(c-u)$  for  $0 < 2b < c$  and  $f(u) = u^p - u^q$  for  $1 < p < q$ . Precise global bifurcation diagrams are obtained. © 1998 Academic Press

## INTRODUCTION

We study global structure of all positive solutions of the Dirichlet problem for the semilinear elliptic equation

$$\begin{cases} \Delta u + \lambda f(u) = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where  $\Omega$  is a bounded domain in  $\mathbf{R}^n$ , and  $\lambda$  is a positive parameter. The problem (1.1) arises in many different physical situations, for instance, in the theory of thermal ignition of gases, in the quantum field theory, and in the population dynamics. (See [18] and the references therein.) Also, the solutions of (1.1) can be treated as steady-states of nonlinear evolution problem:

$$\begin{cases} u_t = \Delta u + \lambda f(u) & \text{in } \Omega \times [0, \infty), \\ u(x, t) = 0 & \text{on } \partial\Omega \times [0, \infty), \\ u(x, 0) = u_0(x) & \text{on } \Omega. \end{cases}$$

The problem (1.1) has been extensively studied by many mathematicians. Many existence and multiplicity results have been obtained. For instance, see [18, 26] and the references therein.

In this paper, we investigate the exact multiplicity and the global bifurcation diagrams of (1.1) with a class of nonlinearities  $f$ . The results obtained in this paper give the global structure of solution set, the dependence between solution curve and nonlinearity  $f(u)$ , and the asymptotic behavior of solutions.

Notice that the number of solutions of (1.1) depends on the nonlinearity  $f$ , the range of parameter  $\lambda$ , and the symmetry or the shape of the domain  $\Omega$ . In this paper, we consider problem (1.1) with domain  $\Omega$  being the unit ball in  $\mathbf{R}^n$ . The nonlinearity of  $f$  in (1.1) is either of class (C) or  $(C_0)$ , defined precisely in Section 1. Some prototype examples in class (C) or  $(C_0)$  are:

- (1)  $f(u) = u(u - b)(c - u)$ , where  $0 < 2b < c$ ;
- (2)  $f(u) = u^p - u^q$ , where  $0 < p < q$ ;
- (3)  $f(u) = u(u - b)/(1 + au^p)$ , where  $a > 0$ ,  $b \geq 0$ ,  $1 < p \leq 2$ .

If  $f$  is of class (C) or  $(C_0)$ , we prove that all solutions  $(\lambda, u)$  of (1.1) lie on a single smooth curve in  $\mathbf{R} \times C^2(\Omega)$ , and there is at most one turning point on this curve. Hence we have the exact multiplicity of solutions of (1.1) with  $f$  being of class (C). Moreover, we completely classify the global bifurcation diagram of (1.1) according to the different asymptotic behaviors of  $f$  in class (C) or  $(C_0)$ , see the figures in Section 1. For instance, when  $f(u) = u(u - b)(c - u)$ ,  $0 < 2b < c$ , we prove that there exists a  $\lambda_* > 0$ , such that (1.1) has exactly 0, 1 or 2 positive solutions, depending on  $\lambda < \lambda_*$ ,  $\lambda = \lambda_*$  or  $\lambda > \lambda_*$ , respectively. Moreover, all of the positive solutions consists of a one parameter  $\subset$ -shaped curve.

In [18], Lions used many “bifurcation diagrams” to describe the solution set of (1.1) with a variety of nonlinearities  $f$ . Those “bifurcation diagrams” were “minimal” in some sense and were formally discussed. In this paper, we establish rigorously the exact multiplicity and the precise bifurcation diagrams for nonlinearities of class (C) and  $(C_0)$  with ball domain.

The one-dimensional version of (1.1) with cubic nonlinearity was studied by Smoller and Wasserman [28] by using phase-plane analysis. In [12], Korman, Li, and Ouyang developed a new global bifurcation approach for more general nonlinearities  $f$ . A two-dimensional version of (1.1) with generalizing cubic nonlinearity was studied by Korman, Li, and Ouyang in [13]. In this paper, we follow the bifurcation approach in [12, 13]. Our results classify the different global behaviors of (1.1) with  $f$  of class (C) and  $(C_0)$ . Moreover, we improve the results in [13] of two-dimensional case by weakening the assumption (2.7–2.8) there. The results here appears to be the first exact multiplicity result for nonlinearity  $f$  of class (C) with dimension  $n \geq 3$ .

The behavior of the solution curve when  $\lambda$  is near infinity for a class of  $f$  has been studied by many mathematicians. Gardner and Peletier [8] proved the exact multiplicity results when  $\lambda$  near infinity for a class of nonlinearities  $f$  which cover the cubic case  $f(u) = u(u - b)(c - u)$  with  $0 < 2b < c$

and  $c$  close to  $2b$  with ball domain. Recently, Dancer [6, 7] proved the exact multiplicity results when  $\lambda$  near infinity for a class of symmetric domains in  $\mathbf{R}^n$  and more general nonlinearities which cover the cubic case  $f(u) = u(u-b)(c-u)$ ,  $0 < 2b < c$ . Both results concerned only the asymptotic behavior of the bifurcation diagram. Our results complete the description of the global bifurcation diagram for all  $\lambda$  by analyzing the local behavior near turning points. Our method can also be applied to more nonlinearities.

The study of “turning points” on the solution curves is crucial to our analysis. At a turning point  $(\lambda, u)$  of (1.1), the linearized equation

$$\begin{cases} \Delta w + \lambda f'(u) w = 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega \end{cases} \quad (1.2)$$

has a nontrivial solution  $w$ . We show that, at such point, a bifurcation argument can be applied. Moreover, if  $w$  does not change sign in  $B^n$ , i.e., 0 is the first eigenvalue of (1.2), then the solution curve “turns to the same direction” at *any* turning point for the given  $f$ . This implies there is only one turning point. The original idea of this argument is from Korman, Li, and Ouyang [12, 13].

The study of positivity of nontrivial solution  $w$  of the linearized equation (1.2) at turning point in high dimensional ball is also very important. We use a similar argument as in Kwong and Zhang [17] and a Sturm Comparison Lemma to prove that  $w > 0$  if  $f$  is of class (C) or  $(C)_0$ .

The global bifurcation diagrams which we obtain also give information of the existence and uniqueness of the ground state solution of  $\Delta u + f(u) = 0$ . The existence of ground state solution was proved in [2] and [3], and the uniqueness of ground state with the same  $f$  in [8] was first proved in Peletier and Serrin [23], (see also [20, 24]). Our results in this paper can be used to prove the uniqueness of ground state for a more general nonlinearity. We shall discuss this issue in our forthcoming paper [22].

Our results have applications to reaction–diffusion equations (see Theorem 5.4).

For the nonlinearity  $f$  not belonging to class of (C) or  $(C)_0$ , there are also many results on the bifurcation diagrams. (See, for example, [1, 4, 11, 16, 17, 19, 21, 27].)

We organize our paper as follows. We state our main results in Section 1. In Section 2, we prove, under the assumption  $w$  does not change sign, that the “turning direction” can be determined. Then, in Section 3, we show  $w$  does not change sign for  $f$  of class either (C) or  $(C)_0$ . We give the proof of our main theorems, Theorems 1.1, 1.2, and 1.3 in Section 4. In Section 5, we discuss some examples which fit our assumptions, and give an application of our result to a reaction–diffusion equation.

## 1. MAIN RESULTS

A function  $f$  is of class (C) if it satisfies

(C1)  $f \in C^2(\overline{\mathbf{R}^+}, \mathbf{R})$ ;  $f(0) = f(b) = 0$  for some  $b > 0$ ;  $f(u) < 0$  for  $0 < u < b$ ,

(C2A)  $f(u) > 0$  for  $b < u < c$  and  $f(u) < 0$  for  $u > c$ ;  $\int_0^c f(u) du > 0$ , or

(C2B)  $f(u) > 0$  for  $u > b$ ,

(C3)  $f''(u)$  changes sign only once in  $(0, \infty)$ , and there exists  $\alpha \in (0, \infty)$  such that

$$f''(u) \geq 0 \quad \text{in } (0, \alpha);$$

$$f''(u) \leq 0 \quad \text{in } (\alpha, \infty),$$

(C4) Let  $\theta$  is the unique positive number such that  $F(\theta) = \int_0^\theta f(s) ds = 0$ ,  $\rho = \alpha - (f(\alpha)/f'(\alpha))$ , and  $K(u) = uf'(u)/f(u)$ . If  $\rho > \theta$ , then

$$K(u) \geq K(\theta) \quad \text{for } u \in [b, \theta];$$

$$K(u) \text{ is nonincreasing in } [\theta, \rho];$$

$$K(u) \leq K(\rho) \quad \text{for } u \in [\rho, \alpha].$$

We will say a function  $f$  is of class (C-A) (or (C-B)) if  $f$  satisfies (C1), (C2-A) (respectively (C2-B)), (C3), (C4), and a function  $f$  is of class (C), if  $f$  is either of class (C-A) or (C-B).

*Remark.*  $K(u)$  could be understood as a measurement of growth rate of  $f(u)$  (for example, if  $f(u) = u^p$ , then  $K(u) = p$ ). The restriction on  $K(u)$  was also used by [21] to obtain some uniqueness results for (1.1). An example shows that this technical growth condition is necessary (see [22]). On the other hand, we notice that our growth condition is weaker than that of [23] and [8], and their conditions are equivalent to  $\rho \leq \theta$  in our context.

We state some well-known results which will be used frequently in the follows.

**PROPOSITION 1.0.** *Let domain  $\Omega$  be the unit ball in  $\mathbf{R}^n$ .*

(1) *If  $f$  is locally lipschitz, then all positive solutions of (1.1) are radially symmetric and satisfy*

$$\begin{cases} u'' + \frac{n-1}{r} u' + \lambda f(u) = 0 & r \in (0, 1), \\ u'(0) = u(1) = 0. \end{cases} \quad (1.3)$$

(2) If  $u$  is a positive solution of (1.1),  $f(0) \geq 0$ , and  $w$  is a solution of linearized problem (1.2), then  $w$  is also radially symmetric and satisfy

$$\begin{cases} w'' + \frac{n-1}{r} w' + \lambda f'(u)w = 0 & r \in (0, 1), \\ w'(0) = w(1) = 0. \end{cases} \tag{1.4}$$

(3) For any  $d > 0$ , there is at most one  $\lambda_d > 0$  such that (1.3) has a positive solution  $u$  with  $\lambda = \lambda_d$  and  $u(0) = d$ .

(4) Let  $T = \{d > 0 \mid (1.3) \text{ has a solution with } u(0) = d\}$ , then  $T$  is open;  $\lambda(d) = \lambda_d$  (defined in (3)) is a well-defined continuous function from  $T$  to  $\mathbf{R}^+$ .

(1) and (2) are the results in [9] and [19]. (3) and (4) can be proved easily by a rescaling argument and the uniqueness of corresponding ODE.

By Proposition 1.0, we could use  $\mathbf{R}^+ \times \mathbf{R}^+ = \{(\lambda, d) \mid \lambda > 0, d > 0\}$  as the phase space of the bifurcation diagram. We define the *bifurcation diagram* of (1.1) as  $D = \{(\lambda(d), d) \mid d \in T\}$ , then each component of  $D$  is homeomorphic to an open interval, and it is a parametrized curve with  $d$  as parameter. If  $(\lambda, u)$  is a solution of (1.1) such that the linearized equation (1.2) has a nontrivial solution, then we call  $(\lambda, u)$  a *critical solution*. Sometimes we also call  $(\lambda, u)$  a *turning point* in the bifurcation diagram.

Our main results can be stated as follows:

**THEOREM 1.1.** *Assume  $f$  is of class (C-A). Then there exists  $\lambda^* > 0$ , such that (1.1) has exactly*

*no solution when  $\lambda < \lambda^*$ ;*

*one solution when  $\lambda = \lambda^*$ ;*

*two solutions when  $\lambda > \lambda^*$ .*

*Moreover, all positive solutions of (1.1) lie on a single smooth solution curve in space  $\mathbf{R} \times C^2(B^n)$ , which consists of two branches  $u_1(x, \lambda) < u_2(x, \lambda)$  for  $\lambda > \lambda^*$ . Furthermore,  $\lim_{\lambda \rightarrow \infty} u_2(x, \lambda) = c$  if  $|x| < 1$ ;  $\lim_{\lambda \rightarrow \infty} u_1(x, \lambda) = 0$  if  $|x| \neq 0$ ,  $\lim_{\lambda \rightarrow \infty} u_1(0, \lambda) = a$  where  $a \in (b, c)$ . If we rescale  $u_1$  by letting  $v_\lambda(x) = u_1(\lambda^{-1/2}x)$ , then  $\lim_{\lambda \rightarrow \infty} v_\lambda(x) = u_0(x)$ , where  $u_0(x)$  is the unique ground state solution of  $\Delta u + f(u) = 0$ .*

Figure 1 shows the graph of  $f$  of class (C-A) and the corresponding bifurcation diagram.

Let  $\lambda_1$  be the first eigenvalue of  $-\Delta$  in  $B^n$ , and  $\lambda_* = \lambda_1/f'(\infty)$ , when  $f'(\infty) = \lim_{u \rightarrow \infty} (f(u)/u) \geq 0$ .

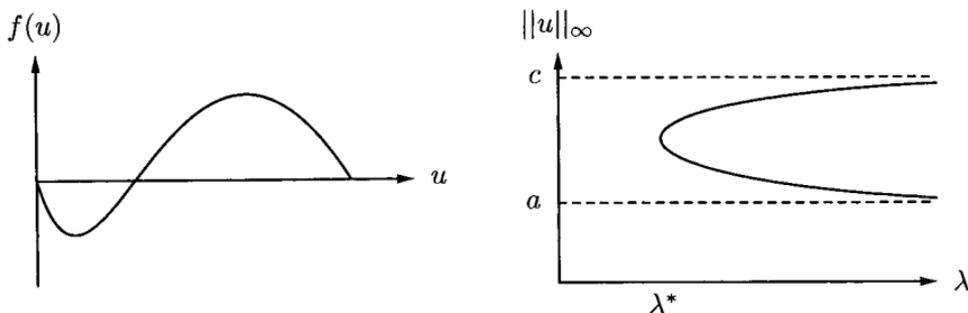


FIG. 1. (Left)  $f(u) = u(u-b)(c-u)$  for  $0 < 2b < c$ . (Right) Bifurcation diagram.

THEOREM 1.2. Assume  $f$  is of class (C-B).

(1) If  $\liminf_{u \rightarrow \infty} K(u) \geq 1$ , then (1.1) has

*no solution when  $\lambda \leq \lambda_*$ ;*

*one solution when  $\lambda > \lambda_*$ .*

(2) If there is a  $\beta > \alpha$  such that  $K(\beta) = 1$ , and  $f'(\infty) > 0$ , then there exist  $0 < \lambda^* < \lambda_*$  such that (1.1) has exactly

*no solution when  $\lambda < \lambda^*$ ;*

*one solution when  $\lambda = \lambda^*$ ;*

*two solutions when  $\lambda_* > \lambda > \lambda^*$ ;*

*one solution when  $\lambda \geq \lambda_*$ .*

(3) If there is a  $\beta > \alpha$  such that  $K(\beta) = 1$ , and  $f'(\infty) = 0$ , then there exists  $\lambda^* > 0$ , such that (1.1) has exactly

*no solution when  $\lambda < \lambda^*$ ;*

*one solution when  $\lambda = \lambda^*$ ;*

*two solutions when  $\lambda > \lambda^*$ .*

Figures 2, 3, and 4 show the graphs of  $f$  of class (C-B) and the corresponding bifurcation diagrams.

*Remark.* For  $f$  being of class (CA), the positive part of  $f$  is bounded; for being of class (CB),  $f$  is eventually positive.  $f'(\infty)$  can not be  $\infty$  since  $f''(u) \leq 0$  for  $u$  large. (1) and (2) in Theorem 1.2 are the cases that  $f$  is asymptotically linear, while the existence of turning point depends on whether  $f'u - f$  changes sign, and (3) is the case that  $f$  is asymptotically sublinear.

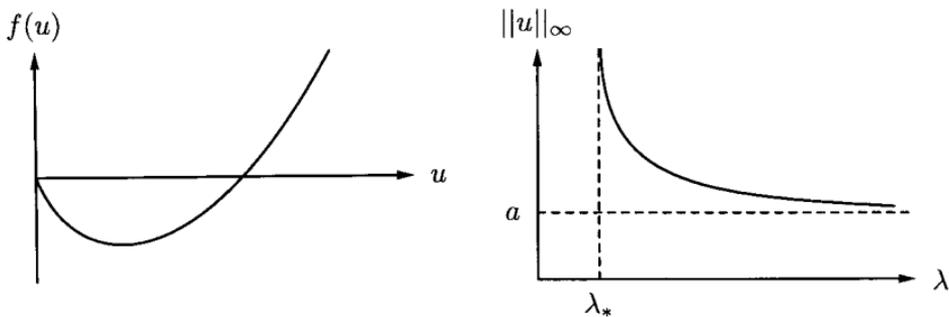


FIG. 2. (Left)  $f(u)$  in (C-B) and  $K(u) \geq 1$ . (Right) Bifurcation diagram.

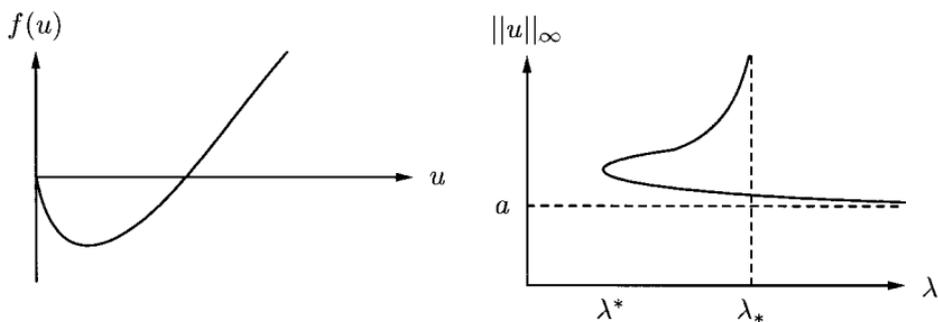


FIG. 3. (Left)  $f(u)$  in (C-B) and  $f'(\infty) > 0$ . (Right) Bifurcation diagram.

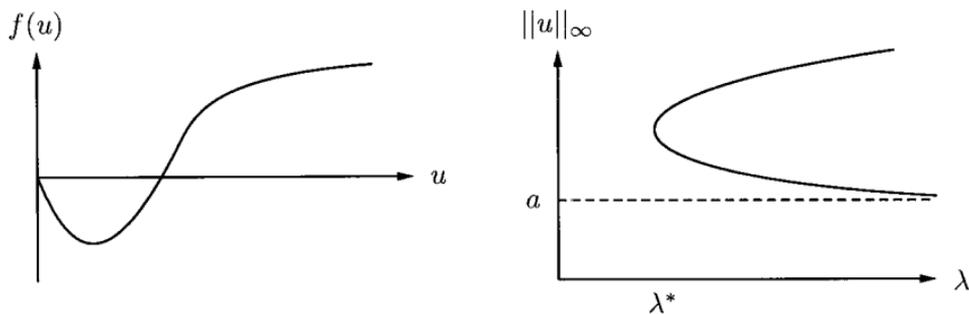


FIG. 4. (Left)  $f(u)$  in (C-B) and  $f'(\infty) = 0$ . (Right) Bifurcation diagram.

Now we turn to the degenerate case that the negative part of  $f$  shrinks to zero and  $f'(0) = 0$ . More specifically, we will consider  $f$  satisfying

$$(C1)_0 \quad f \in C^2(\overline{\mathbf{R}^+}, \mathbf{R}); f(0) = 0; f'(0) = 0,$$

$$(C2A)_0 \quad f(u) > 0 \text{ for } 0 < u < c \text{ and } f(u) < 0 \text{ for } u > c,$$

or

$$(C2B)_0 \quad f(u) > 0 \text{ for } u > 0,$$

(C3)<sub>0</sub>  $f''(u)$  changes sign only once in  $(0, \infty)$ , and there exists  $\alpha \in (0, \infty)$  such that

$$f''(u) \geq 0 \quad \text{in } (0, \alpha),$$

$$f''(u) \leq 0 \quad \text{in } (\alpha, \infty),$$

and also the growth condition:

$$(C4)_0 \quad \text{Define } \rho = \alpha - (f(\alpha)/f'(\alpha)), \text{ and } K(u) = (uf'(u)/f(u)). \text{ And}$$

$$\lim_{u \rightarrow 0^+} K(u) > 1;$$

$$K(u) \text{ is nonincreasing in } [0, \rho];$$

$$K(u) \leq K(\rho) \quad \text{for } u \in [\rho, \alpha].$$

Similar to the previous cases, we will say that functions satisfying the above conditions belong to one class from  $(C-A)_0$ ,  $(C-B)_0$ , and  $(C)_0$ .

**THEOREM 1.3.** *For (C) replaced by  $(C)_0$ , the statements in Theorems 1.1 and 1.2 remain true.*

## 2. A BIFURCATION THEOREM

First we state a bifurcation theorem of Crandall and Rabinowitz.

**THEOREM 2.1** (Crandall–Rabinowitz [5]). *Let  $X$  and  $Y$  be Banach spaces. Let  $(\bar{\lambda}, \bar{x}) \in \mathbf{R} \times X$  and let  $F$  be a continuously differentiable mapping of an open neighborhood of  $(\bar{\lambda}, \bar{x})$  into  $Y$ . Let the null-space  $N(F_x(\bar{\lambda}, \bar{x})) = \text{span}\{x_0\}$  be one-dimensional and  $\text{codim } R(F_x(\bar{\lambda}, \bar{x})) = 1$ . And  $F_\lambda(\bar{\lambda}, \bar{x}) \notin R(F_x(\bar{\lambda}, \bar{x}))$ . If  $Z$  is a complement of  $\text{span}\{x_0\}$  in  $X$ , then the solutions of  $F(\lambda, x) = F(\bar{\lambda}, \bar{x})$  near  $(\bar{\lambda}, \bar{x})$  form a curve  $(\lambda(s), x(s)) = (\bar{\lambda} + \tau(s), \bar{x} + sx_0 + z(s))$ , where  $s \rightarrow (\tau(s), z(s)) \in \mathbf{R} \times Z$  is a continuously differentiable function near  $s = 0$  and  $\tau(0) = \tau'(0) = 0$ ,  $z(0) = z'(0) = 0$ .*

In this section we will assume  $f$  only satisfies

$$(A1) \quad f \in C^2(\overline{\mathbf{R}^+}, \mathbf{R}), f(0) = 0,$$

(A2)  $f''(u) \not\equiv 0$ , and there exists  $\alpha \in (0, \infty)$  such that

$$f''(u) \geq 0 \ (\leq 0) \quad \text{on} \quad [0, \alpha];$$

$$f''(u) \leq 0 \ (\geq 0) \quad \text{on} \quad [\alpha, \infty).$$

It is easy to see that (C) and  $(C)_0$  are special cases of (A). For this more general nonlinearity, we will prove the following theorem which is a strengthened form of the Crandall–Rabinowitz theorem in our context.

**THEOREM 2.2.** *Suppose  $f$  satisfies (A1), (A2),  $\Omega = B^n$ , and  $(\lambda_0, u_0)$  is a critical solution of (1.1),  $u_0 \not\equiv 0$ , with  $w$  being the corresponding solution of linearized problem (1.2). And we assume  $w > 0$  in  $B^n$ . Then*

(1) *All the solutions of (1.1) near  $(\lambda_0, u_0)$  have form of  $(\lambda_0 + \tau(s), u_0 + sw + z(s))$ , with  $\tau(0) = \tau'(0) = 0$ ,  $z(0) = z'(0) = 0$ .*

(2) *The solution curve is  $C^2$  near  $(\lambda_0, u_0)$ , and  $\tau''(0) \neq 0$ ,  $\tau''(0) > 0$  if  $f''(u) \geq 0$  near  $u = 0$  and  $\tau''(0) < 0$  if  $f''(u) \leq 0$  near  $u = 0$ .*

Part 1 of the theorem is just the consequence of Theorem 2.1. We are more interested in Part 2, which states that the turning directions are same for *all* the critical solutions with  $f$  satisfying (A). In fact, that implies that there is only one turning point on each component of solution curve, as we show below.

To prove Theorem 2.2, first we begin with some elementary lemmas.

**LEMMA 2.3.** *Let  $u$  and  $w$  be the solutions of (1.1) and (1.2) respectively,  $w(x) > 0$  for all  $x \in \Omega$ , and  $\Omega$  be a star-shaped domain with respect to origin in  $\mathbf{R}^n$ , then*

$$\int_{\Omega} f(u)w \, dx = \frac{1}{2\lambda} \int_{\partial\Omega} |\nabla u| \cdot |\nabla w| (x \cdot \nu) \, ds.$$

*If in addition  $f(0) \geq 0$ , then  $\int_{\Omega} f(u)w \, dx > 0$ .*

*Proof.* We will prove a more general integral equality which is very similar to Pohozaev's identity. Let  $u$  and  $w$  be the solutions of (1.1) and (1.2) respectively, and  $\Omega$  be a smooth domain in  $\mathbf{R}^n$ , then

$$\int_{\Omega} f(u)w \, dx = \frac{1}{2\lambda} \int_{\partial\Omega} [(x \cdot \nabla w)(\nabla u \cdot \nu) + (x \cdot \nabla u)(\nabla w \cdot \nu) - (\nabla u \cdot \nabla w)(x \cdot \nu)] \, ds.$$

The conclusion of Lemma 2.3 is a corollary of above equality.

Let  $\nu$  be the outward normal derivative of  $\partial\Omega$ . Multiply (1.1) by  $x \cdot \nabla w$ , multiply (1.2) by  $x \cdot \nabla u$  and integrate over  $\Omega$ ,

$$A_1 \equiv \int_{\Omega} \Delta u(x \cdot \nabla w) \, dx = -\lambda \int_{\Omega} f(u)(x \cdot \nabla w) \, dx \equiv -A_2,$$

$$B_1 \equiv \int_{\Omega} \Delta w(x \cdot \nabla u) \, dx = -\lambda \int_{\Omega} f'(u) w(x \cdot \nabla u) \, dx \equiv -B_2.$$

We first calculate  $A_1$ .

$$\begin{aligned} A_1 &= \int_{\partial\Omega} (x \cdot \nabla w)(\nabla u \cdot \nu) \, ds - \int_{\Omega} \nabla u \cdot \nabla(x \cdot \nabla w) \, dx \\ &= \int_{\partial\Omega} (x \cdot \nabla w)(\nabla u \cdot \nu) \, ds - \int_{\Omega} \nabla u \cdot \nabla w \, dx - \int_{\Omega} \sum_{i,j=1}^n u_i x_j w_{ij} \, dx. \end{aligned} \quad (2.1)$$

By a similar calculation

$$B_1 = \int_{\partial\Omega} (x \cdot \nabla u)(\nabla w \cdot \nu) \, ds - \int_{\Omega} \nabla u \cdot \nabla w \, dx - \int_{\Omega} \sum_{i,j=1}^n w_i x_j u_{ij} \, dx. \quad (2.2)$$

And

$$\begin{aligned} \int_{\Omega} \sum_{i,j=1}^n u_i x_j w_{ij} \, dx &= \int_{\partial\Omega} \sum_{i,j=1}^n u_i x_j w_i \nu_j \, ds - \int_{\Omega} \sum_{i,j=1}^n w_i (u_i x_j)_j \, dx \\ &= \int_{\partial\Omega} (\nabla u \cdot \nabla w)(x \cdot \nu) \, ds - n \int_{\Omega} \nabla u \cdot \nabla w \, dx - \int_{\Omega} \sum_{i,j=1}^n w_i x_j u_{ij}. \end{aligned}$$

So we have

$$\int_{\Omega} \sum_{i,j=1}^n (u_i x_j w_{ij} + w_i x_j u_{ij}) \, dx = \int_{\partial\Omega} (\nabla u \cdot \nabla w)(x \cdot \nu) \, ds - n \int_{\Omega} \nabla u \cdot \nabla w \, dx. \quad (2.3)$$

On the other hand,

$$\begin{aligned}
 B_2 &= \lambda \int_{\Omega} w(x \cdot \nabla f(u)) \, dx = \lambda \int_{\Omega} \sum_{i=1}^n w x_i [f(u)]_i \, dx \\
 &= \lambda \int_{\partial\Omega} \sum_{i=1}^n w x_i f(u) \nu_i \, ds - \lambda \int_{\Omega} \sum_{i=1}^n f(u) (w x_i)_i \, dx \\
 &= -n\lambda \int_{\Omega} f(u) w \, dx - \lambda \int_{\Omega} f(u) (x \cdot \nabla w) \, dx \\
 &= -n\lambda \int_{\Omega} f(u) w \, dx - A_2.
 \end{aligned}$$

Therefore, by (2.1), (2.2), and (2.3)

$$\begin{aligned}
 \int_{\Omega} f(u) w \, dx &= \frac{1}{n\lambda} (-A_2 - B_2) = \frac{1}{n\lambda} (A_1 + B_1) \\
 &= \frac{n-2}{n\lambda} \int_{\Omega} \nabla u \cdot \nabla w \, dx + \frac{1}{n\lambda} \int_{\partial\Omega} [(x \cdot \nabla w)(\nabla u \cdot \nu) \\
 &\quad + (x \cdot \nabla u)(\nabla w \cdot \nu) - (\nabla u \cdot \nabla w)(x \cdot \nu)] \, ds.
 \end{aligned}$$

Finally, multiply (1.1) by  $w$  and integrate

$$\int_{\Omega} \nabla u \cdot \nabla w \, dx = \lambda \int_{\Omega} f(u) w \, dx.$$

So

$$\begin{aligned}
 \int_{\Omega} f(u) w \, dx &= \frac{1}{2\lambda} \int_{\partial\Omega} [(x \cdot \nabla w)(\nabla u \cdot \nu) + (x \cdot \nabla u)(\nabla w \cdot \nu) \\
 &\quad - (\nabla u \cdot \nabla w)(x \cdot \nu)] \, ds.
 \end{aligned}$$

If  $w > 0$ , and  $u > 0$ , then

$$\nabla u|_{\partial\Omega} = -|\nabla u| \cdot \nu, \quad \nabla w|_{\partial\Omega} = -|\nabla w| \cdot \nu$$

Then

$$\int_{\Omega} f(u) w \, dx = \frac{1}{2\lambda} \int_{\partial\Omega} |\nabla u| \cdot |\nabla w| (x \cdot \nu) \, ds.$$

If in addition,  $\Omega$  is star-shaped with respect to the origin,  $x \cdot \nu \geq 0$  for  $x \in \partial\Omega$ , and  $f(0) \geq 0$  implies  $|\nabla u| > 0$  for any  $x \in \partial\Omega$  by Hopf's Lemma, then

$$\int_{\Omega} f(u) w \, dx > 0. \quad \blacksquare$$

LEMMA 2.4. *Let  $u$  and  $w$  be the solutions of (1.1) and (1.2) respectively, then*

$$\int_{\Omega} f''(u) |\nabla u|^2 w \, dx = -f(0) \int_{\partial\Omega} \nabla w \cdot \nu \, ds.$$

*Proof.* Differentiate (1.1) and (1.2)

$$\Delta(\nabla u) + \lambda f'(u) \nabla u = 0, \quad (2.4)$$

$$\Delta(\nabla w) + \lambda f'(u) \nabla w + \lambda f''(u) w \nabla u = 0. \quad (2.5)$$

Innerproduct (2.4) with  $\nabla w$ , (2.5) with  $\nabla u$ , subtract and integrate

$$\begin{aligned} \lambda \int_{\Omega} f''(u) |\nabla u|^2 w \, dx &= \int_{\Omega} [\nabla(\Delta u) \cdot \nabla w - \nabla(\Delta w) \cdot \nabla u] \, dx \\ &= \int_{\partial\Omega} [\Delta u (\nabla w \cdot \nu) - \Delta w (\nabla u \cdot \nu)] \, ds \\ &= -\lambda \int_{\partial\Omega} f(u) (\nabla w \cdot \nu) \, ds + \lambda \int_{\partial\Omega} f'(u) w (\nabla u \cdot \nu) \, ds \\ &= -\lambda f(0) \int_{\partial\Omega} \nabla w \cdot \nu \, ds. \quad \blacksquare \end{aligned}$$

*Proof of Theorem 2.2.* Let  $F: \mathbf{R}^+ \times X \rightarrow Y$  be given by

$$F(\lambda, u) = \Delta u + \lambda f(u),$$

where  $u \in X = \{u \in C^{2,\alpha}(\overline{B^n}) \mid |u|_{\partial B^n} = 0\}$  and  $Y = C^\alpha(\overline{B^n})$ . And it is easy to show that

$$\langle F_u(\lambda, u), w \rangle = \Delta w + \lambda f'(u) w.$$

The null space of  $F_u(\lambda_0, u_0)$  is one-dimensional, and  $\text{codim } R(F_u(\lambda_0, u_0)) = 1$  since  $F_u(\lambda_0, u_0)$  is a Fredholm operator of index 0. Finally, if  $F_\lambda(\lambda_0, u_0) \notin R(F_u(\lambda_0, u_0))$  is not true, one can find  $z \in X$  satisfying

$$\begin{cases} \Delta z + \lambda_0 f'(u_0) z = f(u_0) & \text{in } B^n, \\ z = 0 & \text{on } \partial B^n. \end{cases}$$

since  $w$  satisfies

$$\begin{cases} \Delta w + \lambda_0 f'(u_0) w = 0 & \text{in } B^n, \\ w = 0 & \text{on } \partial B^n. \end{cases}$$

we have

$$0 = \int_{B^n} (w \cdot \Delta z - z \cdot \Delta w) = \int_{B^n} f(u_0) w \, dx,$$

which contradicts Lemma 2.3. Now applying Theorem 2.1, we can conclude that  $(\lambda_0, u_0)$  is a bifurcation point, near  $(\lambda_0, u_0)$  the solutions of (2.1) form a curve  $(\lambda_0 + \tau(s), u_0 + sw + z(s))$  with  $s$  near  $s = 0$ , and  $\tau(0) = \tau'(0) = 0, z(0) = z'(0) = 0$ . Since we assume  $f$  is  $C^2$ , then by the implicit function theorem, the solution curve near  $(\lambda_0, u_0)$  is also  $C^2$ .

For the sign of  $\tau''(0)$ , we first claim that

$$\tau''(0) = \frac{-\lambda_0 \int_{\Omega} f''(u_0) w^3 \, dx}{\int_{\Omega} f(u_0) w \, dx} = \frac{-\lambda_0 \int_0^1 r^{n-1} f''(u_0) w^3 \, dr}{\int_0^1 r^{n-1} f(u_0) w \, dr}. \tag{2.6}$$

*Proof of Claim.* Let  $(\lambda, u) = (\lambda_0 + \tau(s), u_0 + sw + z(s))$ , differentiate (1.1) twice, and evaluate at  $s = 0$ .

$$\begin{aligned} \Delta u_{ss} + \lambda f'(u) u_{ss} + 2\lambda' f'(u) u_s + \lambda f''(u) u_s^2 + \lambda'' f(u) &= 0, \\ \Delta u_{ss} + \lambda_0 f'(u) u_{ss} + \lambda_0 f''(u) w^2 + \lambda''(0) f(u) &= 0. \end{aligned} \tag{2.7}$$

Multiply (2.7) by  $w$ , (2.2) by  $u_{ss}$ , subtracting and integrating, we obtain (2.6).

From Lemma 2.3,  $\int_0^1 r^{n-1} f(u_0) w \, dr > 0$ . From Lemma 2.4,  $\int_0^1 r^{n-1} f''(u_0) u_r^2 w \, dr = 0$ , since  $f(0) = 0$ . That means  $f''(u(r))$  must change sign, since  $w > 0$ . Next we will compare  $\int_0^1 r^{n-1} f''(u_0) u_r^2 w \, dr$  and  $\int_0^1 r^{n-1} f''(u_0) w^3 \, dr$ . If  $f''(u) \geq 0$  near  $u = 0$ , then there exists  $r_0 \in (0, 1)$  such that

$$\begin{aligned} f''(u_0(r)) &\leq 0 & \text{in } [0, r_0], \\ f''(u_0(r)) &\geq 0 & \text{in } [r_0, 1]. \end{aligned}$$

Next we claim: There exists  $k > 0$ , such that

$$\begin{aligned} kw &\geq -u_r & \text{in } [0, r_0], \\ kw &\leq -u_r & \text{in } [r_0, 1]. \end{aligned}$$

*Proof of Claim.* Let  $v = w + u_r$ . Then  $v(0) = w(0) > 0, v(1) = u_r(1) < 0$ . So there is at least one point  $t_0 \in (0, 1)$  such that  $v(t_0) = 0$ . We prove there is only one point such that  $v = 0$ . Since  $u_r$  satisfies

$$u_r'' + \frac{n-1}{r} u_r' - \frac{n-1}{r^2} u_r + \lambda f'(u) u_r = 0, \tag{2.8}$$

then by (1.4) and (2.8),  $v$  satisfies

$$v'' + \frac{n-1}{r} v' + \lambda f'(u)v = \frac{n-1}{r^2} u_r. \quad (2.9)$$

Multiply (1.4) by  $r^{n-1}v$ , (2.9) by  $r^{n-1}w$  and subtract, we have

$$[r^{n-1}(w'v - v'w)]' = -(n-1)r^{n-3}wu_r. \quad (2.10)$$

Suppose  $v$  has more than one zeros in  $(0, 1)$ . Let  $t_1 < t_2$  be the last two zeros of  $v$ , then  $v'(t_1) \geq 0$ ,  $v'(t_2) \leq 0$ . Integrate (2.10) over  $(t_1, t_2)$ ,

$$-t_2^{n-1}v'(t_2)w(t_2) + t_1^{n-1}v'(t_1)w(t_1) = -(n-1) \int_{t_1}^{t_2} r^{n-3}wu_r dt,$$

where the left-hand side  $\leq 0$ , and right-hand side  $> 0$ , that is a contradiction. So  $v$  has only one zero in  $(0, 1)$ . In other words, there is only one  $t_0 \in (0, 1)$  such that  $-u_r(t_0) = w(t_0)$ . Since  $w$  satisfies a linear differential equation, by varying the coefficient  $k$ , we can get the claim.

From the claim, we have

$$k^2 \int_0^1 r^{n-1}f''(u_0)w^3 dr < \int_0^1 r^{n-1}f''(u_0)u_r^2 w dr = 0,$$

then  $\tau''(0) > 0$ .

If  $f''(u) \leq 0$  near 0, then proof is similar, and  $\tau''(0) < 0$ . ■

*Remark.* From the proof of Theorem 2.2, we can see that if  $f(0) \neq 0$ , in some cases, the conclusions of Theorem 2.2 are still true. In fact, if we replace (A1), (A2) by

$$(A1') \quad f \in C^2(\overline{\mathbf{R}^+}, \mathbf{R}), f(0) < 0,$$

$$(A2') \quad f''(u) \neq 0, \text{ and there exists } \alpha \in (0, \infty) \text{ such that}$$

$$f''(u) \geq 0 \quad \text{on } [0, \alpha];$$

$$f''(u) \leq 0 \quad \text{on } [\alpha, \infty).$$

then  $\tau''(0) > 0$ ; or we can replace (A1), (A2) by

$$(A1'') \quad f \in C^2(\overline{\mathbf{R}^+}, \mathbf{R}), f(0) > 0,$$

$$(A2'') \quad f''(u) \neq 0, \text{ and there exists } \alpha \in (0, \infty) \text{ such that}$$

$$f''(u) \leq 0 \quad \text{on } [0, \alpha];$$

$$f''(u) \geq 0 \quad \text{on } [\alpha, \infty).$$

then  $\tau''(0) < 0$ . We will discuss the applications of this remark and further applications of Theorem 2.2 in [22].

### 3. THE POSITIVITY OF $w$

In this section, we will prove the solution  $w$  of linearized equation (1.4) does not change sign in  $(0, 1)$ . Since  $w$  is a solution of linear equation, then  $w$  must be either positive or negative in  $(0, 1)$ . So without loss of generality, we always assume that  $w(0) > 0$  and prove  $w > 0$  in  $(0, 1)$ . We will first prove  $w > 0$  for  $f$  of class (C-A) in detail, then sketch the proof of  $w > 0$  for  $f$  of class (C-B) and  $(C)_0$ .

First we state a Sturm Comparison Theorem which will be used frequently in this section.

**LEMMA 3.1.** *Let  $Lu(t) = [(p(t)u'(t))' + q(t)u(t)]$ , where  $p(t)$  and  $q(t)$  are continuous in  $[a, b]$  and  $p(t) \geq 0$ ,  $t \in [a, b]$ . Suppose  $Lw(t) = 0$ ,  $Lv(t) \leq 0$  ( $\neq 0$ ),  $v(t) \geq 0$ . Then  $w$  has at most one zero in  $[a, b]$ . If in addition,  $w'(a) = 0$  or  $p(a) = 0$ , then  $w$  does not have any zero in  $[a, b]$ .*

*Proof.* Assume  $w$  has two consecutive zeros  $a \leq t_1 < t_2 \leq b$ , and  $w(t) > 0$  in  $(t_1, t_2)$ . We have

$$[p(wv' - w'v)]' = wLv - vLw = wLv. \quad (3.1)$$

Integrate (3.1) over  $[t_1, t_2]$

$$p(wv' - w'v)|_{t_1}^{t_2} = \int_{t_1}^{t_2} wLv dt.$$

The left-hand side  $= -p(t_2)w'(t_2)v(t_2) + p(t_1)w'(t_1)v(t_1) \geq 0$ , while the right-hand side  $< 0$ , which is a contradiction. So  $w$  has at most one zero in  $[a, b]$ .

If in addition  $w'(a) = 0$  or  $p(a) = 0$ , we assume  $t_1$  is the first zero of  $w$ , and  $w > 0$  in  $(a, t_1)$ . Then integrate (3.1) over  $[a, t_1]$ , we will have a similar contradiction. Thus  $w$  does not have any zero in this case. ■

Recall that  $\alpha$  is the point where  $f''$  changes sign,  $\rho = \alpha - f(\alpha)/f'(\alpha)$  and  $\theta$  is the unique point such that  $F(\theta) = \int_0^\theta f(u) du = 0$ . First, to clarify the definition of  $\rho$ , we have

LEMMA 3.2. Assume  $f$  is either of class (C) with  $\alpha > b$ , or of class (C)<sub>0</sub>, then, for any  $u > \rho$ , we have

$$f(u) \geq (u - \rho) f'(u).$$

And  $\rho$  is the smallest point satisfying above property.

*Proof.* Let  $f$  be either of class (C) with  $\alpha > b$ , or of class (C)<sub>0</sub>, then  $f(\alpha) > 0$  and  $f'(\alpha) > 0$ . So  $\rho < \alpha$ . Let  $g(u) = f(u) - (u - \rho) f'(u)$ , then  $g'(u) = -(u - \rho) f''(u) \leq 0$  in  $[\rho, \alpha]$ , and  $g' \geq 0$  in  $[\alpha, \infty)$ . Therefore  $g$  achieves the unique minimum in  $[\rho, \infty)$  at  $u = \alpha$ . By the definition of  $\rho$ ,  $g(\alpha) = 0$ . So  $g(u) \geq g(\alpha) = 0$  for all  $u \geq \rho$ . For any  $\rho' < \rho$ ,  $f(\alpha) < (\alpha - \rho') f'(\alpha)$ . So  $\rho$  is the smallest point which has such property. ■

Now we are ready to prove that  $w > 0$  in the case that  $f$  is of class (C) and  $\theta \geq \rho$ .

PROPOSITION 3.3. Assume that  $f$  satisfies (C1), (C2-A)(or (C2-B)), and (C3), and  $\theta \geq \rho$ ,  $w(0) > 0$ , then  $w(r) > 0$  for  $r \in [0, 1)$ .

*Proof.* For  $f$  satisfying (C1), (C2-A)(or (C2-B)), and (C3), we have two different cases. One case is  $\alpha > b$ , then  $f(\alpha) > 0$ ,  $f'(\alpha) > 0$ , and  $b < \rho < \alpha$ . The other case is  $0 < \alpha \leq b$ , then  $f(\alpha) \leq 0$ ,  $f'(\alpha) > 0$ , and  $0 < \alpha \leq \rho \leq b$ .

*Case 1.*  $\alpha \leq b$ . In this case,  $f(\alpha) \leq 0$ ,  $f'(\alpha) > 0$ , and  $0 < \alpha \leq \rho \leq b$ . Since  $u(0) > \theta$  for any solution  $u$  of (2.1) and  $u_r < 0$ , we can assume that there exist  $0 < r_3 < r_4 < 1$  such that  $u(r_3) = \theta$  and  $u(r_4) = b$ . First  $w(r) > 0$  in  $[r_3, 1)$  by Lemmas 15 and 16 in [17]. In  $[0, r_3]$ , we consider  $v(r) = f(u(r))$ . Let  $p(r) = r^{n-1}$ ,  $q(r) = \lambda f'(u(r)) r^{n-1}$ , then

$$Lv(r) \equiv (pv')' + qv = \lambda r^{n-1} f''(u(r)) u_r^2.$$

In  $[0, r_3]$ ,  $v(r) = f(u(r)) > 0$  and  $Lv(r) \leq 0$  since  $\alpha \leq b < \theta$ . By Lemma 3.1,  $w > 0$  in  $[0, r_3]$ . So  $w > 0$  in  $(0, 1)$ .

*Case 2.*  $\alpha > b$ . In this case, we have  $f(\alpha) > 0$ ,  $f'(\alpha) > 0$ , and  $b < \rho < \alpha$ , so we can assume that there exist  $0 < r_3 \leq r_2 < 1$  such that  $u(r_3) = \theta$  and  $u(r_2) = \rho$ . First  $w(r) > 0$  in  $[r_3, 1)$  by Lemmas 15 and 16 in [17]. In  $[0, r_2]$ , suppose  $p(r) = r^{n-1}$ ,  $q(r) = \lambda f'(u(r)) r^{n-1}$ ,  $v(r) = u(r) - \rho$ , then it is easy to show that

$$Lv(r) \equiv (pv')' + qv = -\lambda r^{n-1} [f(u(r)) - f'(u(r))(u(r) - \rho)].$$

By Lemma 3.2,  $Lv(r) \leq 0$  on  $[0, r_2]$ , and  $v(r) > 0$  in  $(0, r_2)$ , and also  $p(0) = 0$ ,  $w'(0) = 0$ , then by Lemma 3.1,  $w$  does not have any zero in  $(0, r_2)$ . ■

The case that  $\theta < \rho$  is much more difficult to prove. We need a more careful analysis. The behavior of  $K(u) = uf'(u)/f(u)$  will be important in this case. We define

$$Q(u) = uf(u) - 2F(u),$$

$$p(u) = Q'(u) = f'(u)u - f(u).$$

Then we have following basic properties of  $K(u)$ ,  $Q(u)$ , and  $p(u)$ .

LEMMA 3.4. *Assume  $f$  is either of class (C) or  $(C)_0$ . Then one of the following holds:*

1. *There exists  $\beta > \alpha$ , such that  $K(u) > 1$ ,  $p(u) > 0$  in  $(0, \beta)$ , and  $K(u) \leq 1$ ,  $p(u) \leq 0$  in  $(\beta, \infty)$ ; and if  $p(u) \not\equiv 0$  in  $(\beta, \infty)$ ,  $Q(u) < 0$  for  $u$  large and  $\lim_{u \rightarrow \infty} Q(u) = -\infty$ .*
2.  *$K(u) > 1$ ,  $p(u) > 0$  for all  $u \in (0, \infty)$  and  $Q(u) > 0$  in  $(0, \infty)$ .*

*Proof.*  $p(0) = Q(0) = 0$ ,  $p'(u) = uf''(u)$ , so  $p$  increases in  $(0, \alpha)$  and decreases in  $(\alpha, \infty)$ . So if  $\beta$  is the first zero of  $p$ , then  $p(u) > 0$ ,  $K(u) > 1$  in  $(0, \beta)$ , and  $p(u) \leq 0$ ,  $K(u) \leq 1$  in  $(\beta, \infty)$ . If  $p(u) \not\equiv 0$  in  $(\beta, \infty)$ , there exists  $u_1 > \beta$  such that  $p(u) \leq p(u_1) < 0$  for  $u > u_1$ . So  $Q(u) = Q(u_1) + \int_{u_1}^u p(t) dt \rightarrow -\infty$  for  $u$  large. If  $p(u)$  has no zero in  $(0, \infty)$ , then  $p(u) > 0$ ,  $K(u) > 1$  for all  $(0, \infty)$ , therefore  $Q(u) = \int_0^u p(t) dt > 0$  for all  $u > 0$ . ■

In the second case of Lemma 3.4, we could define  $\beta = \infty$ . Then if  $f$  is of class (C), and  $\theta < \rho$ , we have the relation

$$b < \theta < \rho < \alpha < \beta.$$

Let  $u(r)$  be a solution of (2.1),  $u(0) > \beta$ , we define  $0 < r_1 < r_2 < r_3 < r_4 < 1$ , satisfying  $u(r_1) = \beta$ ,  $u(r_2) = \rho$ ,  $u(r_3) = \theta$ ,  $u(r_4) = b$ . (If  $u(0) \leq \beta$ , we only define  $r_i$  when they exist, that will only make our proof easier.)

LEMMA 3.5. *Assume  $f$  is either of class (C) with  $\alpha > b$ , or of class  $(C)_0$ , then  $K(u)$  is decreasing on  $(\alpha, \beta)$ .*

*Proof.* In this case,  $f > 0$  and  $f' > 0$  in  $(\alpha, \beta)$ .  $f \in C^2$  implies that

$$K'(u) = -\frac{(f'(u))^2 u - f'(u) f(u) - f''(u) f(u) u}{f^2(u)} \equiv -\frac{N(u)}{f^2(u)}.$$

In  $(\alpha, \beta)$ ,  $p(u) = f'(u)u - f(u) > 0$  by Lemma 3.4,  $f''(u) \leq 0$ , and  $f, f'$  are both positive, then  $N(u) > 0$ . So  $K'(u) < 0$ . ■

At last, we need a Pohozaev's identity. Let  $u(r)$  be a solution of (2.1), and we define

$$H(r) = \frac{1}{2}[ru_r^2(r) + (n-2)u_r(r)u(r)] + \lambda rF(u(r)) \quad (3.2)$$

where  $F(u) = \int_0^u f(s) ds$ . By a direct calculation, for  $0 \leq r_1 < r_2 \leq 1$ , we have

$$r_2^{n-1}H(r_2) - r_1^{n-1}H(r_1) = \int_{r_1}^{r_2} \lambda r^{n-1} \left[ nF(u(r)) - \frac{n-2}{2} f(u(r))u(r) \right] dr. \quad (3.3)$$

We have following property of  $H(r)$ .

**LEMMA 3.6.** *Assume  $f$  is of class (C-A),  $\theta < \rho$ , and  $u$  is a solution of (2.1), then  $H(r) \geq 0$  in  $[r_2, 1]$ . (If  $u(0) \leq \rho$ , we assume  $r_2 = 0$ .)*

*Proof.* First we assume  $n > 2$ . We will show that  $J(r) = r^{n-1}H(r) \geq 0$  in  $[r_2, 1]$ . In fact,  $J(1) = H(1) = \frac{1}{2}u_r^2(1) \geq 0$ , and  $J'(r) = \lambda r^{n-1}[nF(u(r)) - (n-2)/2 f(u(r))u(r)] \equiv \lambda r^{n-1}G(u(r))$ , where  $G(u) = nF(u) - (n-2)/2 f(u)u$ . We study the property of  $G(u)$ . The function  $G(u)$  satisfies

$$G'(u) = \frac{n+2}{2}f - \frac{n-2}{2}f'u = \frac{n-2}{2}(f - f'u) + 2f, \quad (3.5)$$

$$G''(u) = 2f' - \frac{n-2}{2}f''u. \quad (3.6)$$

Let  $\gamma$  satisfy  $f(\gamma) = \min_{0 \leq u \leq b} f(u)$ . Since  $G(0) = 0$ ,  $G'(0) = 0$ ,  $G''(u) < 0$  in  $[0, \gamma]$ , then  $G'(u) < 0$  in  $[0, \gamma]$ . And  $f' > 0$  in  $(\gamma, b)$ , so  $G' = (n+2)/2 f - (n-2)/2 f'u < 0$  in  $[\gamma, b]$ . Therefore  $G(u)$  decreases in  $[0, b]$ , and  $G(b) < 0$ . On  $[b, \theta]$ ,  $f > 0$ ,  $F < 0$ , so  $G(u) < 0$ . Therefore  $G(u) < 0$  in  $(0, \theta)$ , that implies  $J(r) = J(1) - \int_r^1 \lambda r^{n-1}G(u(s)) ds > 0$  in  $[r_3, 1]$ . To prove  $J(r) \geq 0$  on  $[r_2, r_3]$ , we discuss following two cases.

*Case A.*  $K(\rho) \geq (n+2)/(n-2)$ .

By (C4),  $K(u) \geq K(\rho) \geq (n+2)/(n-2)$ , for all  $u \in (b, \rho)$ , then  $G'(u) < 0$  for all  $u \in (0, \rho)$ . So  $G(u) < 0$  in  $(b, \rho)$  and  $J(r) \geq 0$  on  $[r_2, 1]$ .

*Case B.*  $K(\rho) < (n+2)/(n-2)$ .

Since  $G' = 0$  is equivalent to  $K(u) = (f'(u)u/f(u)) = (n+2)/(n-2)$ , and by (C4)  $K(u)$  is non-increasing in  $(\theta, \rho)$ , so there is at most one  $d \in (\theta, \rho)$  such that  $K(u) = (n+2)/(n-2)$ . Also by (C4),  $K(u) \leq K(\rho) < (n+2)/(n-2)$  for  $u \in (\rho, \alpha)$ . So  $G'(u) > 0$  in  $(\rho, \alpha)$ . For  $u \in (\alpha, \beta)$ , by Lemma 3.5,  $K(u) < (n+2)/(n-2)$ , so  $G'(u) > 0$  in  $(\alpha, \beta)$ .

For  $u > \beta$ , there is  $\eta > \beta$  such that  $f(\eta) = \max_{\beta \leq u \leq c} f(u)$ . For  $u \in (\beta, \eta)$ ,  $G'' > 0$ , and  $G'(\beta) > 0$ , so  $G' > 0$ ; and for  $u \in (\eta, c)$ ,  $f' < 0$ ,  $f > 0$ , so  $G' = (n+2)/2 f - (n-2)/2 f' u > 0$ .

Hence,  $G' > 0$  for  $u > \beta$ . Since  $G(c) > 0$  and  $G(u) < 0$  in  $(0, \theta)$ , so there exists a unique point  $p > d$  such that  $G(p) = 0$ .

*Claim.* For any solution  $u$  of (1.1),  $u(0) > p$ .

Suppose not, then  $u(0) \leq p$ , and  $u_r < 0$ , then for  $r \in [0, 1]$ ,  $0 \leq u(r) \leq p$ . So  $J'(r) = \lambda r^{n-1} G(u(r)) < 0$  for  $r \in (0, 1)$ . But  $J(1) = H(1) = \lambda \int_0^1 r^{n-1} G(u(r)) dr = \frac{1}{2} u_r^2(1) \geq 0$ . So  $u(0) > p$ .

Now since  $u(0) > p$ ,  $u(1) = 0$ , and  $u_r < 0$ , then there exists  $r_0 \in (0, 1)$  such that  $u(r_0) = p$ . For  $r \in [0, r_0]$ ,  $G(u(r)) > 0$ , so  $J' > 0$ , and  $J(r) \geq 0$ . For  $r \in [r_0, 1]$ ,  $G(u(r)) \leq 0$ ,  $H(1) \geq 0$ , so  $H(r) > 0$  in  $(r_0, 1)$ . Then  $H(r) \geq 0$  in  $[0, 1]$ . In particular,  $H(r) \geq 0$  in  $[r_2, 1]$ .

For the case  $n = 2$ , the proof is similar. ■

Now we are ready to prove  $w > 0$  for  $f$  in class (C-A) and  $\theta < \rho$ .

**PROPOSITION 3.7.** *If  $u$  and  $w$  are solutions of (2.1) and (2.2) respectively, and  $f$  is of class (C-A) with  $\theta < \rho$ ,  $w(0) > 0$ , then  $w(r) > 0$  in  $(0, 1)$ .*

*Proof.* First we assume  $r_i, i = 1, 2, 3, 4$  all exist, if  $r_i$  does not exist, it can only make our proof easier, as shown below.

*Step 1.*  $w > 0$  on  $[0, r_2]$ .

Let  $p(r) = r^{n-1}$ ,  $q(r) = \lambda f'(u(r)) r^{n-1}$ ,  $v(r) = u(r) - \rho$ , then it is easy to show that

$$Lv(r) \equiv (pv')' + qv = -\lambda r^{n-1} [f(u(r)) - f'(u(r))(u(r) - \rho)].$$

By Lemma 3.2,  $Lv(r) \leq 0$  on  $[0, r_2]$ , and  $v(r) > 0$  in  $(0, r_2)$ , and also  $p(0) = 0, w'(0) = 0$ , then by Lemma 3.1,  $w$  cannot have any zero in  $(0, r_2)$ .

*Step 2.*  $w \neq 0$  on  $[r_3, 1)$ .

This is the same as that of  $\alpha < b$ , by Lemmas 15 and 16 in [17].

*Step 3.*  $w \neq 0$  in  $(r_2, r_3)$ .

This case is more delicate. To apply Lemma 3.1, we will use another test function  $v(r) = ru_r(r) + \mu u(r)$ , where  $\mu > 0$  is a constant to be specified later. It is easy to calculate that  $Lv(r) = \lambda g(u(r)) r^{n-1}$ , where  $g(u) = \mu [f'(u) u - f(u)] - 2f(u)$ . Define

$$h(r) = -\frac{ru_r(r)}{u(r)} \quad \text{in } (0, 1),$$

$$\mu(r) = \frac{2f(u(r))}{f'(u(r))u(r) - f(u(r))} \quad \text{in } (r_1, 1).$$

Then

$$\begin{aligned} h'(r) &= \frac{(n-2)uu_r + ru_r^2 + \lambda fru}{u^2} \\ &= \frac{2H(r) - 2\lambda rF + \lambda fru}{u^2} = \frac{2H(r)}{u^2} + \lambda r \frac{fu - 2F}{u^2}. \end{aligned}$$

Here, in the second equality, we use Pohozaev's identity. Let  $Q(u) = uf(u) - 2F(u)$ , then  $Q'(u) = uf'(u) - f(u)$ . We know  $Q'(u) \geq 0$  in  $(0, \beta)$  and  $Q(0) = 0$ , so  $Q(u(r)) > 0$  in  $(r_1, 1)$ . And by Lemma 3.6,  $H(r) \geq 0$  in  $[r_2, 1]$ . So  $h'(r) > 0$  in  $(r_2, 1)$ .

On the other hand,

$$\mu'(r) = \frac{2[(f')^2 u - ff' - ff''u]}{(f'u - f)^2} u_r = \frac{2N(u(r))}{(f'u - f)^2} u_r.$$

Since  $N(u) \geq 0$  in  $(\theta, \rho)$  by (C4) and  $u_r < 0$ , then  $\mu'(r) \leq 0$  in  $(r_2, r_3)$ . And  $\lim_{r \rightarrow r_1^+} \mu(r) = +\infty$ ,  $\mu(r_4) = 0$ ,  $\mu(r) \geq \mu(r_2)$  for  $r \in (r_1, r_2)$  by (C4) and Lemma 3.5,  $\mu(r) \leq \mu(r_3)$  for  $r \in (r_3, r_4)$  by (C4). Also  $h(r) > 0$  in  $(r_1, r_3)$ , and  $h'(r) > 0$ ,  $\mu'(r) \leq 0$  in  $(r_2, r_3)$ , there will be three cases.

*Case A.* There exists a unique  $r_0 \in (r_2, r_3)$ , such that  $h(r_0) = \mu(r_0) = \mu_0$ .

We fix  $v = ru_r + \mu_0 u$ . Then by the definition of  $h$  and  $\mu$ , it is easy to show that  $v(r) > 0$  in  $(r_2, r_0)$  and  $v(r) < 0$  in  $(r_0, 1]$ , while  $g(r) < 0$  in  $(0, r_0)$  and  $g(r) > 0$  in  $(r_0, 1]$ . Let  $p, q$  be same as step 1, then on  $[r_0, 1]$ ,  $v(r) \leq 0$ ,  $Lv(r) \geq 0$ ,  $w(1) = 0$ , so by Lemma 3.1  $w \neq 0$  in  $[r_0, 1)$ . Assume  $w$  has a zero in  $(r_2, r_0)$ , and assume  $t_1 > r_2$  is the first zero of  $w$  and  $w > 0$  in  $[0, t_1)$ . Then we have

$$r^{n-1}(wv' - w'v)|_0^{t_1} = \int_0^{t_1} wLv \, dr = \lambda \int_0^{t_1} wgr^{n-1} \, dr,$$

where the left-hand side  $= -t_1^{n-1}w'(t_1)v(t_1) > 0$ , while the right-hand side  $< 0$ , which makes a contradiction. So  $w \neq 0$  in  $(r_2, r_0)$ .

*Case B.* For all  $r \in (r_2, r_3)$ ,  $\mu(r) \geq h(r)$ .

We choose  $\mu_0 = \mu(r_3)$ , and  $v = ru_r + \mu_0 u$ . Then  $v(r) \geq 0$  in  $(r_2, r_3)$ , and  $g(r) < 0$  in  $(0, r_3)$ . So by a similar argument in Case A, we can show that  $w \neq 0$  in  $(r_2, r_3)$ .

*Case C.* For all  $r \in (r_2, r_3)$ ,  $\mu(r) \leq h(r)$ .

We choose  $\mu_0 = h(r_2)$ . And other argument is similar to that of Case A. ■

Next we consider the case that  $f$  is of class (C-B). This case is essentially same as the case that  $f$  is of class (C-A). With slightly modification to the proof of Proposition 3.7, we can prove

**PROPOSITION 3.8.** *If  $u$  and  $w$  are solutions of (2.1) and (2.2) respectively, with  $f$  of class (C-B),  $w(0) > 0$ , then  $w(r) > 0$  in  $(0, 1)$ .*

Here we make a remark on the *existence* of solutions  $w$  of (1.2) or equivalently (2.2). Remember we assume that in Proposition 3.7 and 3.8. In fact, we have following result contradicting previous theorems.

**LEMMA 3.9.** *If  $u$  and  $w$  are solutions of (1.1) and (1.2) respectively, and  $p(u) = f'(u)u - f(u)$  does not change sign in  $\Omega$ , then  $w$  cannot have same sign in  $\Omega$ .*

*Proof.* Multiply (1.1) by  $w$ , (1.2) by  $u$ , subtract and integrate, we have

$$0 = \int_{\Omega} (\Delta u \cdot w - \Delta w \cdot u) dx = \lambda \int_{\Omega} (f'(u)u - f(u))w dx = \lambda \int_{\Omega} p(u)w dx.$$

So if  $p(u)$  does not change sign,  $w$  must change sign in  $\Omega$ . ■

**COROLLARY 3.10.** (1) *If  $f$  is of class (C-A), and  $u$  is a critical solution of (2.1), then  $u(0) > \beta$ .*

(2) *If  $f$  is of class (C-B), then (2.1) has a critical solution only if there is a  $\beta > \alpha$  such that  $K(\beta) = 1$  (or equivalently  $p(\beta) = 0$ ). And if  $u$  is a critical solution, then  $u(0) > \beta$ .*

*Proof.* This is a direct consequence of Propositions 3.3, 3.7, 3.8, and Lemma 3.9. ■

To conclude this section, we prove  $w > 0$  for  $f$  of class  $(C)_0$ , the degenerate case of (C). The proof is similar to that of Proposition 3.7. First Lemmas 3.2–3.5 are still true if  $\beta < \infty$ . If  $\beta = \infty$ , we can get similar results. We only need to modify the proof of Lemma 3.6, and we get following version.

LEMMA 3.11. *If  $u(r)$  is a solution of (2.1), with  $f$  of class  $(C)_0$ , then  $H(r) \geq 0$  in  $[r_2, 1]$ .*

*Proof.* Suppose  $J, G$  are same as Lemma 3.6.

*Case A.*  $\lim_{u \rightarrow 0} K(u) \geq K(\rho) > (n+2)/(n-2)$ . Follow in the same way as Case A in proof of Lemma 3.6,  $J(r) \geq 0$  in  $[r_2, 1]$ .

*Case B.*  $\lim_{u \rightarrow 0} K(u) > (n+2)/(n-2) \geq K(\rho)$ . Follow the same way as Case B in proof of Lemma 3.6, there is a unique  $d \in (0, \rho]$  such that  $G(d) = \min_u G(u) < 0$ , and  $G'(u) \geq 0$  for  $u \geq d$ . Then  $H(r) \geq 0$  for all  $r \in [0, 1]$ .

*Case C.*  $\lim_{u \rightarrow 0} K(u) \leq (n+2)/(n-2)$ . By  $(C4)_0$ ,  $K(u) \leq (n+2)/(n-2)$  for  $u \in (0, \alpha]$ . And by Lemma 3.5,  $K(u) \leq (n+2)/(n-2)$  for  $u \in (\alpha, \beta]$ . Then  $G'(u) > 0$  in  $(0, \beta]$ . By the same way as that in Lemma 3.6, we can show  $G'(u) > 0$  for  $u > \beta$ . So  $G(u) > 0$  for all  $u > 0$ , that implies  $H(r) \geq 0$  for all  $r \in [0, 1]$ , since  $J(0) = 0$ . ■

Now we have no difficulty to prove the counterparts of Proposition 3.7 and Corollary 3.10 under  $(C)_0$ .

THEOREM 3.12. *If  $u$  and  $w$  are solutions of (2.1) and (2.2) respectively, with  $f$  of class  $(C)_0$ ,  $w(0) > 0$ , then  $w(r) > 0$  in  $(0, 1)$ .*

COROLLARY 3.13. (1) *If  $f$  is of class  $(C-A)_0$ , and  $u$  is a critical solution of (2.1), then  $u(0) > \beta$ .*

(2) *If  $f$  is of class  $(C-B)_0$ , then (2.1) has a critical solution only if there is a  $\beta > \alpha$  such that  $K(\beta) = 1$  (or equivalently  $p(\beta) = 0$ ). And if  $u$  is a critical solution, then  $u(0) > \beta$ .*

#### 4. PROOF OF MAIN THEOREMS

*Proof of Theorem 1.1.* By Theorem A.2, there exists a  $\lambda^* > 0$  which is the infimum of all  $\lambda > 0$  such that (1.1) has a solution, and for  $\lambda \geq \lambda^*$ , there exists a maximum positive solution  $u_1$  of (1.1) which is less than  $c$  pointwise. Moreover, for  $\lambda > \lambda^*$ , there exists a second solution  $u_2$  of (1.1) which satisfies  $0 < u_2 < u_1$ . Note, by maximum principle, there is no positive solution  $u$  with  $u(0) \geq c$ .

At  $\lambda = \lambda^*$ , the solution  $u^*$  of (1.1) must be a critical solution, otherwise by implicit function theorem, there would be solutions for  $\lambda < \lambda^*$ . And by Propositions 3.3 and 3.7, the solution  $w$  of linearized problem (1.2) does not change sign in  $B^n$ . Therefore, Theorem 2.2 is applicable and all solutions of (1.1) near  $(\lambda^*, u^*)$  have the form  $(\lambda^* + \tau(s), u^* + sw + z(s))$ , with  $\tau(0) = \tau'(0) = 0$ ,  $z(0) = z'(0) = 0$ , and  $\tau''(0) > 0$  since  $f''(u) \geq 0$  near  $u = 0$ . So

the solution curve “turns right” at  $(\lambda^*, u^*)$ . We call the branch of solutions with larger solutions “upper branch,” and the other “lower branch,” then the critical solution corresponds to a turning point on the solution curve.

We claim there is no any other turning point on either upper branch or lower branch. Suppose there is, for example, a turning point on the upper branch. Let  $(\lambda^{**}, u^{**})$  be the first critical solution on upper branch when we continue the solution curve rightward in  $\lambda$  from  $\lambda^*$ . Then Proposition 3.7 show that the corresponding  $w$  does not change sign, and Theorem 2.2 is applicable, then  $\tau''(0) > 0$  at  $(\lambda^{**}, u^{**})$ , but that is impossible, since we continue the solution curve from left to right, there always exists solution for  $\lambda$  near  $\lambda^{**}$  and  $\lambda < \lambda^{**}$ . So  $(\lambda^*, u^*)$  is the unique critical solution on this component of solution curve.

Thus both upper and lower branch can be continued for  $\lambda > \lambda^*$  without turning points. Now we prove that the upper branch solutions are increasing with respect to  $\lambda$ , i.e. for  $\lambda_1 > \lambda_2$ ,  $u(r, \lambda_1) > u(r, \lambda_2)$  for  $r \in (0, 1)$ , where  $u(\cdot, \lambda)$  is the upper branch solution.

We denote by  $u_\lambda(r, \lambda)$  the derivative of  $u(r, \lambda)$  with respect to  $\lambda$ , then we show that  $u_\lambda(r, \lambda) > 0$  for  $\lambda > \lambda^*$  and  $r \in (0, 1)$ . Since  $w(r) > 0$ , and  $w'(1) < 0$  by the uniqueness of solution of ordinary differential equation, then for  $\lambda > \lambda^*$  and close to  $\lambda^*$ , it is also true. Now let  $\lambda_1$  be the supremum of  $\lambda > \lambda^*$  such that  $u_\lambda(r, \lambda) > 0$  holds. There are two cases possible.

*Case A.*  $u_\lambda(r, \lambda_1) \geq 0$  for all  $r \in (0, 1)$  and  $u_\lambda(r_1, \lambda_1) = 0$  for some  $r_1 \in (0, 1)$ . Then  $r_1$  is a minimum point of  $u_\lambda(r, \lambda_1)$ ,  $u'_\lambda(r_1, \lambda_1) = 0$ . Notice that  $u_\lambda$  satisfies

$$\begin{cases} u''_\lambda + \frac{n-1}{r} u'_\lambda + \lambda f'(u) u_\lambda + f(u) = 0 & r \in (0, 1), \\ u'_\lambda(0) = u_\lambda(1) = 0. \end{cases} \tag{4.1}$$

On the other hand,  $u_r$  satisfies

$$\begin{cases} u''_r + \frac{n-1}{r} u'_r + \lambda f'(u) u_r - \frac{n-1}{r^2} u_r = 0 & r \in (0, 1), \\ u_r(0) = 0. \end{cases} \tag{4.2}$$

Multiply (4.1) by  $r^{n-1}u_r$ , (4.2) by  $r^{n-1}u_\lambda$ , subtract and integrate, we have

$$r^{n-1}(u'_\lambda u_r - u_\lambda u'_r)|_0^{r_1} = \int_0^{r_1} (-(n-1)r^{n-3}u_\lambda u_r - r^{n-1}f(u)u_r) dr. \tag{4.3}$$

The left-hand side is 0, but the right-hand side is positive, so it is a contradiction.

*Case B.*  $u_\lambda(1, \lambda_1) = 0$ . Use (4.3) with  $r_1 = 1$ , we still get a contradiction. So the claim is true.

So for all  $\lambda \geq \lambda^*$ ,  $u_\lambda(r, \lambda) > 0$  holds. Hence the upper branch solutions are increasing and bounded by  $c$ . Let  $v(r) = \lim_{\lambda \rightarrow \infty} u(r, \lambda)$ . Then  $v$  is a monotone decreasing function in  $(0, 1)$  and  $0 \leq v(r) \leq c$ . We claim  $v(r) = c$  if  $r < 1$  and  $v(1) = 0$ .

Let  $\phi_\lambda(r) = u(r/\sqrt{\lambda}, \lambda)$ . Then  $\phi_\lambda$  satisfies

$$\begin{cases} u'' + \frac{n-1}{r} u' + f(u) = 0 & r \in (0, \sqrt{\lambda}), \\ u'(0) = u(\sqrt{\lambda}) = 0. \end{cases} \quad (4.4)$$

Multiply (4.4) by  $\phi'_\lambda$ , and integrate over  $[0, r]$ , we have

$$\frac{1}{2} [\phi'_\lambda(r)]^2 + (n-1) \int_0^r \frac{[\phi'_\lambda(s)]^2}{s} ds = F(\phi_\lambda(0)) - F(\phi_\lambda(r)).$$

Therefore  $\int_0^\infty ([\phi'_\lambda(s)]^2/s) ds < \infty$ , and  $\lim_{r \rightarrow \infty} \phi'_\lambda(r) = 0$ ,  $\lim_{r \rightarrow \infty} \phi''_\lambda(r) = 0$ .

Similarly, by

$$\phi''_\lambda + \frac{n-1}{r} \phi''_\lambda + f'(\phi_\lambda) \phi'_\lambda - \frac{n-1}{r^2} \phi'_\lambda = 0,$$

$\lim_{r \rightarrow \infty} \phi'''_\lambda(r) = 0$ . In particular,  $\phi'''_\lambda$ ,  $\phi''_\lambda$ ,  $\phi'_\lambda$  are all uniformly bounded in  $[0, \infty)$  with respect to  $\lambda$ . Therefore, by Ascoli–Arzela Theorem, there exists a  $\phi_\infty \in C^2(\mathbf{R}^n)$  such that  $\phi_\lambda \rightarrow \phi_\infty$  ( $\lambda \rightarrow \infty$ ) in  $C^2(B_r)$  for any  $r > 0$ , where  $B_r$  is the ball centered at origin and with radius  $r$  in  $\mathbf{R}^n$ . Obviously,  $\phi_\infty$  is radially symmetric and satisfies

$$\begin{cases} \phi''_\infty + \frac{n-1}{r} \phi'_\infty + f(\phi_\infty) = 0 & r \in (0, \infty) \\ \phi'_\infty(0) = 0, \\ \phi_\infty(0) = \lim_{\lambda \rightarrow \infty} \phi_\lambda(0). \end{cases}$$

We claim that  $\phi_\infty(0) = c$ . Suppose not, then  $\theta < \phi_\infty(0) < c$  and  $\phi'_\infty(r) < 0$  for all  $r > 0$ . Since  $\lim_{r \rightarrow \infty} \phi'_\infty(r) = 0$ ,  $\lim_{r \rightarrow \infty} \phi''_\infty(r) = 0$ , by the equation,  $\lim_{r \rightarrow \infty} f(\phi_\infty(r)) = 0$ . Hence,  $\lim_{r \rightarrow \infty} \phi_\infty(r)$  is either  $b$  or  $0$ . First we assume that  $\lim_{r \rightarrow \infty} \phi_\infty(r) = b$ . We rewrite Eq. (4.4) as

$$(r^{n-1} \phi'_\infty)' + r^{n-1} f(\phi_\infty) = 0.$$

Let  $p(r) = r^{n-1} \phi'_\infty(r)$  and  $\psi(r) = p'(r)/p(r)$ , then we can check that  $\psi'(r) = -\psi^2 - f'(\phi_\infty(r)) + (n-1/r)\psi$ . So if  $r > 1$ ,  $\psi'(r) < -\psi^2 + (n-1)\psi$ , that implies  $\psi(r) < n-1$ . Since  $\phi_\infty(r) > b$  for all  $r > 0$ ,  $f'(\phi_\infty(r)) > k > 0$  for  $r$

large enough. Hence  $\psi'(r) \leq -\psi^2 - f'(\phi_\infty(r)) + ((n-1)^2/r) \leq -k/2$  for  $r$  large enough. So there exists a  $r_0 > 0$  such that  $\psi(r_0) = 0$ , which implies  $f(\phi_\infty(r_0)) = 0$ . That is a contradiction. Therefore  $\lim_{r \rightarrow \infty} \phi_\infty(r) = 0$ . And  $\phi_\infty$  is a ground state solution of  $\Delta u + f(u) = 0$ . By [23], we know  $|\phi_\infty(r)| \leq Ce^{-kr}$  for all  $r > 0$  and constants  $k, C > 0$ . That implies when  $\lambda$  large enough,  $\phi_\lambda(r) \leq Ce^{-kr}$  for  $0 < r \leq C_1 \sqrt{\lambda}$  and  $u(r, \lambda) \leq Ce^{-(kr/\sqrt{\lambda})}$  for  $0 < r \leq C_1$ , where  $C_1$  is any constant less than 1. But on the other hand if we fix  $r_0$  in  $(0, 1)$ ,  $u(r_0, \lambda)$  is increasing with respect to  $\lambda > 0$ , then  $u(r_0, \lambda) \geq \delta$  for some constant  $\delta$ , that is a contradiction. Therefore  $\phi_\infty(0) = c$  and  $\phi_\infty \equiv c$ .

So we have proved for any upper branch solution  $u(r, \lambda)$ , that  $\lim_{\lambda \rightarrow \infty} u(0, \lambda) = c$ . In fact, for any  $r \in (0, 1)$ , if  $\lim_{\lambda \rightarrow \infty} u(r, \lambda) < c$ , we can apply our above argument in the interval  $[r, 1]$  to get a contradiction. So

$$v(r) = \lim_{\lambda \rightarrow \infty} u(r, \lambda) = c \quad r \in [0, 1),$$

and  $v(1) = 0$ .

On lower branch, we denote  $L = \lim_{\lambda \rightarrow \infty} u(0)$ . In Lemma 3.6, we showed that  $L \geq p$ , where  $p$  is the first zero of  $G(u) = nF(u) - (n-2)/2 f(u)$ . Similar to upper branch, we can show that the lower branch solutions are decreasing with respect to  $\lambda$  (detail can be found in [12]). Then we can repeat our above argument to show that, after a rescaling, the lower branch solution converges to a ground state solution.

Finally, we rule out the possibility of another component of solution curve. Since our above argument works for any component of solution curve, so if there are at least two components  $\gamma_1, \gamma_2$ , the upper branches of both components will converge to limits  $v_1, v_2$ . And  $v_1(0) = v_2(0) = c$ . But, on the other hand, by Proposition 1.0, in the bifurcation diagram  $\{(\lambda(d), d) | d \in T\}$ , one component must be entirely above the other one. So  $v_1(0) > v_2(0)$  or  $v_1(0) < v_2(0)$ . That reaches a contradiction. The other statements in Theorem 1.1 can be proved the same way as [7] and [12, 13]. ■

Following lemma concerns the asymptotic behavior of  $f$  when  $f$  is either of class (C-B) or (C-B)<sub>0</sub>.

LEMMA 4.1. *Let  $f$  be either of class (C-B) or (C-B)<sub>0</sub>, then*

- (1)  $f(u) \leq a_1 u$ , with  $a_1 = f'(\alpha)$ ;
- (2)  $f'(\alpha) + \int_\alpha^\infty f''(u) du \geq 0$ ;
- (3) If  $f'(\alpha) + \int_\alpha^\infty f''(u) du = 0$ , then  $\lim_{u \rightarrow \infty} (f(u)/u) = 0$ ;
- (4) If  $f'(\alpha) + \int_\alpha^\infty f''(u) du > 0$ , then  $f(u) \geq a_2 u + b_2$ , with  $a_2 = f'(\alpha) + \int_\alpha^\infty f''(u) du > 0$ . And  $\lim_{u \rightarrow \infty} (f(u)/u) = a_2$ .

*Proof.* (1)  $(f(u) - f'(\alpha)u)' = f'(u) - f'(\alpha) \leq 0$  for all  $u > 0$ , and  $f(0) = 0$ . So  $f(u) \leq f'(\alpha)u$  for all  $u > 0$ .

(2) If  $f'(\alpha) + \int_{\alpha}^{\infty} f''(u) du < 0$ , then there exists  $u_0 > \alpha$  such that  $f'(u_0) < 0$ , and  $f'(u) \leq f'(u_0)$  for all  $u > u_0$ , then for some  $u_1 > \alpha$ ,  $f(u) < 0$ . That is not the case for  $f$  of class (C-B) or  $(C-B)_0$ .

(3) For any  $\varepsilon > 0$ , there exists  $u_0 > \alpha$  such that for  $u > u_0$ ,

$$f'(u) = f'(\alpha) + \int_{\alpha}^u f''(t) dt \leq f'(\alpha) + \int_{\alpha}^{u_0} f''(t) dt = f'(u_0) < \varepsilon$$

Then

$$\begin{aligned} 0 < \frac{f(u)}{u} &= \frac{f(u) - f(u_0)}{u} + \frac{f(u_0)}{u} = \frac{\int_{u_0}^u f'(t) dt}{u} + \frac{f(u_0)}{u} \\ &< \frac{\varepsilon(u - u_0)}{u} + \frac{f(u_0)}{u} < 2\varepsilon \end{aligned}$$

for  $u$  large enough. So  $\lim_{u \rightarrow \infty} (f(u)/u) = 0$ .

(4)  $f(u) = f(\alpha) + \int_{\alpha}^u f'(t) dt \geq f(\alpha) + (u - \alpha)a_2 = a_2u + f(\alpha) - a_2\alpha$  for  $u > \alpha$ . Choose  $b_2$  even larger, we have  $f(u) \geq a_2u + b_2$  for all  $u > 0$ . On the other hand,  $f'' \leq 0$  for  $u > \alpha$  implies  $f(u)/u$  is decreasing for  $u > \alpha$ . So  $\lim_{u \rightarrow \infty} (f(u)/u)$  exists and equal to  $a_2$ . ■

**LEMMA 4.2.** *If  $f$  is either of class (C) or  $(C)_0$ , and  $0 < \lambda < \lambda_1/a_1$ , where  $a_1 = f'(\alpha)$ , then (1.1) has no solution.*

*Proof.* By Lemma 4.1 (1),  $f(u) \leq a_1u$  for  $u > 0$ . If  $u$  is a solution of (1.1), then

$$\lambda_1 \int_{\Omega} u^2 dx \leq \int_{\Omega} |\nabla u|^2 dx = \lambda \int_{\Omega} uf(u) dx \leq \lambda a_1 \int_{\Omega} u^2 dx$$

So  $\lambda \geq \lambda_1/a_1$ . ■

*Proof of Theorem 1.2 (1).* Let  $\lambda_* = \lambda_1/f'(\infty) = \lambda_1/a_2$ . Since  $\beta = \infty$ , then  $(f(u)/u)' = (uf'(u) - f(u))/u^2 > 0$ , and by Lemma 4.1 (1),  $f(u)/u \leq a_1$ , then  $0 < a_2 = \lim_{u \rightarrow \infty} (f(u)/u) \leq a_1$ . By Theorem A.4, there exists a component  $D_1$  of  $D$  bifurcating from  $(\lambda_*, \infty)$ . By Corollary 3.10, (1.1) has no critical solution, and by Lemma 4.2, (1.1) has no solution for  $\lambda < \lambda_1/a_1$ , hence  $D_1$  must be a curve  $(\lambda(d), d)$  for  $d$  in some  $(a, \infty)$  such that  $\lambda(d)$  is decreasing and  $\lim_{d \rightarrow \infty} \lambda(d) = \lambda_*$ ,  $\lim_{d \rightarrow a^+} \lambda(d) = \infty$ , since  $D_1$  is bounded by lines  $\lambda = \lambda_1/a_1$  and  $d = b$ .

Next we rule out the possibility of another component  $D_2$  of  $D$ . Suppose there exists  $D_2$ , then  $D_2$  must be entirely below  $D_1$ , bounded from left by  $\lambda_1/a_1$  and bounded from below by  $d=b$ . That forces  $D_2$  must have a turning point. But by Corollary 3.10, (1.1) has no critical solution. That is a contradiction. Therefore,  $D_1$  is the unique component of  $D$ . ■

*Proof of Theorem 1.2 (3).* Since  $\lim_{u \rightarrow \infty} (f(u)/u) = 0$ , then the assumptions in Theorem A.2 are satisfied. So we can use the same arguments in the proof of Theorem 1.1 to get one component  $D_1$  which has exactly one turning point. To prove that there is only one component, we can assume there is another component  $D_2$ . We can conclude that for  $\lambda \rightarrow \infty$ , (1.1) has at least three solutions. One of  $D_1$  and  $D_2$  must be bounded, say  $D_2$  is bounded such that  $a < \|u\|_{L^\infty} < a'$  for all solutions on  $D_2$ . Then we can truncate  $f$  to make  $f$  bounded for  $u > 2a'$ ,  $f$  remains unchanged for  $u \leq 2a'$ , and truncated  $f$  is of class (C-A). By Theorem 1.1, the new problem has only two solutions for  $\lambda$  large, but that is a contradiction. Therefore,  $D_1$  is the unique component of  $D$ . ■

Proof of Theorem 1.2 (2) is a little harder. In the following discussion, we will use a theorem of bifurcation from infinity by Rabinowitz and the variational method to study the solutions of (1.1). Let  $X = H_0^1(\Omega)$ ,  $\|u\|^2 = \int_\Omega |\nabla u|^2 dx$ , define  $f(u) = 0$  for  $u < 0$ , and define the energy functional

$$I(\lambda, u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega F(u) dx.$$

It is well-known that  $u(\lambda, \cdot)$  is a classical solution of (1.1) if and only if  $u$  is a critical point of  $I(\lambda, u)$  provided  $f$  is smooth. We still denote  $\lambda_* = \lambda_1/a_2$ . The following results are elementary.

LEMMA 4.3. *If  $f$  is either of class (C-B) or (C-B)<sub>0</sub> and  $f'(\infty) > 0$ , then*

- (1)  $0$  is a local minimum of  $I$  for all  $\lambda > 0$ ;
- (2)  $I(\lambda, u) \geq 0$  for all  $u \in X$  and  $0 < \lambda < \lambda_1/a_1$ ;
- (3)  $I(\lambda, u)$  is bounded from below if  $0 < \lambda < \lambda_*$ , and the global minimum of  $I$  exists;
- (4)  $I(\lambda, u)$  is not bounded from below if  $\lambda > \lambda_*$ .

*Proof.* (1) Since  $f'(0) \leq 0$  for  $f$  of class (C) or (C)<sub>0</sub>, so by an argument in [26], we know  $0$  is local minimum of  $I$ .

- (2) By Lemma 4.1

$$\frac{1}{2}a_2u^2 + b_2u \leq F(u) \leq \frac{1}{2}a_1u^2.$$

Hence

$$I(\lambda, u) \geq \frac{1}{2}\lambda_1 \int_{\Omega} u^2 dx - \lambda \cdot \frac{1}{2}a_1 \int_{\Omega} u^2 dx \geq 0.$$

(3) Since  $\lim_{u \rightarrow \infty} (f(u)/u) = a_2$ , then for any  $\varepsilon > 0$ , there exists  $M > 0$  such that  $f(u) \leq (a_2 + \varepsilon)u$  for  $u > M$ , and  $F(u) \leq \frac{1}{2}(a_2 + \varepsilon)u^2$ .

$$\begin{aligned} I(\lambda, u) &\geq \frac{1}{2}\lambda_1 \int_{\Omega} u^2 dx - \lambda \int_{\{u > M\}} F(u) dx - \lambda \int_{\{u \leq M\}} F(u) dx \\ &\geq \frac{1}{2}\lambda_1 \int_{\Omega} u^2 dx - \lambda \cdot (a_2 + \varepsilon) \int_{\{u > M\}} u^2 dx - \lambda \max_{u \leq M} F(u) \cdot m(\Omega) \\ &\geq \frac{1}{2}[\lambda_1 - (a_2 + \varepsilon)\lambda] \int_{\Omega} u^2 dx - \lambda \max_{u \leq M} F(u) \cdot m(\Omega), \end{aligned}$$

where  $m(\cdot)$  is the Lebesgue measure. Choose  $\varepsilon > 0$  small enough that  $\lambda_1 - (a_2 + \varepsilon)\lambda > 0$ , then  $I(\lambda, u)$  is bounded from below and coercive. It is also clear that  $I$  is weakly lower semicontinuous, and  $I(u) \rightarrow \infty$  if  $\|u\| \rightarrow \infty$ . Therefore, the global minimum of  $I$  exists (but possibly 0).

(4) Let  $\phi_1$  be the first eigenfunction of  $\Omega$ , i.e.

$$\begin{cases} \Delta v + \lambda_1 v = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Then

$$\begin{aligned} I(\lambda, k\phi_1) &= \frac{1}{2}\lambda_1 k^2 \int_{\Omega} \phi_1^2 dx - \lambda \int_{\Omega} F(k\phi_1) dx \\ &\leq \frac{1}{2}\lambda_1 k^2 \int_{\Omega} \phi_1^2 dx - \frac{1}{2}a_2 \lambda k^2 \int_{\Omega} \phi_1^2 dx - b_2 \lambda k \int_{\Omega} \phi_1 dx \\ &\leq \frac{1}{2}(\lambda_1 - a_2 \lambda) k^2 \int_{\Omega} \phi_1^2 dx - b_2 \lambda k \int_{\Omega} \phi_1 dx. \end{aligned}$$

Therefore  $k \rightarrow \infty$ ,  $I(\lambda, k\phi_1) \rightarrow -\infty$ . ■

LEMMA 4.4. *If*

$$\lim_{d \rightarrow \infty} \lambda(d) = \lambda_* \quad \text{and} \quad \lim_{u \rightarrow \infty} Q(u) = \lim_{u \rightarrow \infty} (uf(u) - 2F(u)) = -\infty.$$

Then  $I(\lambda(d), u(\lambda(d), d)) < 0$  for  $d$  large enough, where  $u(\lambda(d), d)$  is a solution of

$$\begin{cases} u'' + \frac{n-1}{r} u' + \lambda(d) f(u) = 0 & r \in (0, 1) \\ u(0) = d, \quad u'(0) = u(1) = 0. \end{cases} \tag{4.5}$$

*Proof.* For any  $M > 0$ , there exists  $d_1 > 0$  such that for  $d > d_1$ ,  $Q(u) = uf(u) - 2F(u) < -M$ . Now consider

$$\begin{cases} (r^{n-1}v')' + \lambda r^{n-1}v = 0 & r \in (0, 1) \\ v(0) = d, \quad v'(0) = 0. \end{cases} \tag{4.6}$$

If in addition,  $v(1) = 0$ , we know when  $\lambda = \lambda_1$ , (4.6) has a positive solution  $v(r, \lambda_1, d)$  and  $v' < 0$ . And for any  $\varepsilon > 0$ , there exists  $d_2 > 0$ , such that for  $d > d_2$ ,  $v(\bar{r}, \lambda_1, d) = d_1$ , and  $\bar{r} \in (1 - \varepsilon, 1)$ . By taking  $d_2$  even larger, we can assume that for some  $\varepsilon_1 > 0$  and for all  $\lambda \in (\lambda_1 - \varepsilon_1, \lambda_1 + \varepsilon_1)$  and  $d > d_2$ ,  $v(\bar{r}, \lambda, d) = d_1$ , and  $\bar{r} \in (1 - \varepsilon, 1)$ .

For the same  $\varepsilon_1 > 0$ , there exists  $d_3 > 0$ , such that for  $u > d_3$ , we have

$$(\lambda_1 - \varepsilon_1) u \leq \lambda(d) f(u) \leq (\lambda_1 + \varepsilon_1) u. \tag{4.7}$$

Now we suppose  $d > \max(d_1, d_2, d_3)$ , rewrite (4.5) as

$$\begin{cases} (r^{n-1}u')' + \lambda(d) r^{n-1}f(u) = 0 & r \in (0, 1), \\ u(0) = d, \quad u'(0) = u(1) = 0, \end{cases}$$

and suppose  $u(\bar{r}_1, \lambda(d), d) = d_1$ . We also assume  $v(\bar{r}_2, \lambda_1 + \varepsilon_1, d) = d_1$ ,  $v(\bar{r}_3, \lambda_1 - \varepsilon_1, d) = d_1$ . By (4.6) and Sturm's comparison theorem, we have  $\bar{r}_3 \leq \bar{r}_1 \leq \bar{r}_2$ . So  $\bar{r}_1 \in (1 - \varepsilon, 1)$ . Since  $M$  and  $\varepsilon$  are arbitrary, we can conclude that  $\int_{\Omega} (uf(u) - 2F(u)) dx < 0$  for  $u = u(\lambda(d), d)$  and  $d$  large enough. But for the solution  $u$  of (1.1),  $u$  satisfies  $\int_{\Omega} |\nabla u|^2 dx - \lambda \int_{\Omega} uf(u) dx = 0$ , therefore

$$I(\lambda, u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \lambda \int_{\Omega} F(u) dx = \frac{1}{2} \lambda \int_{\Omega} (uf(u) - 2F(u)) dx < 0$$

for  $d$  large enough. ■

*Proof of Theorem 1.2 (2).* By Theorem A.4, there is a component of solution curve  $D_1$  bifurcating from  $(\lambda_*, \infty)$ . Using the same argument in the proof of Theorem 1.2 (1), we can show that  $D_1$  is the unique component of  $D$ , and the solution curve  $D_1 = (\lambda(d), d)$  satisfies  $\lim_{d \rightarrow a} \lambda(d) = \infty$ , and  $\lim_{d \rightarrow \infty} \lambda(d) = \lambda_*$ . By the same argument in the proof of Theorem 1.1, we can show that there is at most one turning point on  $D_1$ . We claim there is a turning point on  $D_1$ .

If not, then  $\lambda(d)$  is decreasing for  $d \in (a, \infty)$ . And for  $\lambda \in (\lambda_*, \infty)$ , (1.1) has exactly one positive solution. Then we show that  $I(\lambda, u) > 0$  for all the solutions  $u$  on  $D_1$ . First we notice if  $f$  is of class (C-A), then for all lower branch solution  $u_2$ , we have  $I(\lambda, u_2) > 0$ . In fact, if  $I(\lambda, u_2) \leq 0$ , and 0 is a local minimizer of  $I(\lambda, u)$ , then we can use the Mountain Pass Lemma to show the existence of another solution  $u_3$  such that  $I(\lambda, u_3) > 0$ . By maximum principle,  $u_3$  is a positive solution of (1.1). This  $u_3$  does not equal to  $u_2$ , and also does not equal to upper branch solution  $u_1$ , because  $u_1$  must be a local minimizer of  $I$ , which contradicts Theorem 1.1. For  $f$  in Theorem 1.2 (2), and for any  $\lambda < \lambda_*$ , if  $u(0, \lambda) = d_1$ , we can truncate  $f$  for  $u > d_1 + \varepsilon$  to make  $f$  bounded, and unchanged for  $u \leq d_1 + \varepsilon$ . Here  $\varepsilon > 0$ . We can choose the suitable truncation such that (C-A) is satisfied by truncated  $f$ . Remember that the solution curve of  $d < d_1 + \varepsilon$  is unchanged after truncation. So for the truncated  $f$ , the conclusion of Theorem 1.1 is true. Therefore  $I(\lambda, u) > 0$  for all the solutions  $u$  on  $D_1$ .

But on the other hand, if  $f$  satisfies the condition in Theorem 1.2 (2), by Lemma 3.3,  $\lim_{u \rightarrow \infty} Q(u) = -\infty$ . So by Lemma 4.4,  $I(u) < 0$  for  $d$  large enough. That is a contradiction. Hence the conclusion in Theorem 1.2 (2) is proved. ■

*Remark.* From Lemma 4.3 and the proof of Theorem 1.2, we can have a complete variational picture for  $f$  of class (C-B). If the graph of  $f$  is star-shaped with respect to origin, by Theorem 1.2 (1), there is a unique solution  $u$  of (1.1) with  $\lambda > \lambda_*$ .  $u$  is a saddle-point of energy functional with  $I(u) > 0$  and the Morse index of  $u$  is  $+1$ , so the solution is mountain-pass type. (But the classical mountain pass theorem cannot be applied here, since Palais–Smale condition is not satisfied.) When the graph of  $f$  is no longer star-shaped, but bounded by a ray starting from origin, then the branch of mountain-pass type solution will cross the line  $\lambda = \lambda_*$ , and reach  $\lambda^*$ , the infimum of all  $\lambda$  such that (1.1) has a solution. At  $\lambda^*$ , the mountain-pass solution changes to a local minimizer on the upper branch, and the Morse index of  $u$  changes to 0. When  $\lambda$  increases, the local minimizer will become the global minimizer. For  $f$  being asymptotic linear, by Theorem 1.2 (2), the global minimizer will blow up for finite  $\lambda = \lambda^*$ ; And for  $f$  being sublinear, by Theorem 1.2 (3), the global minimizer will keep existence for all  $\lambda > \lambda^*$ .

Finally, we sketch a proof for Theorem 1.3, when  $f$  is of class  $(C)_0$  instead of (C). From the above proof, we need following ingredients to prove the theorem:

(1)  $w > 0$ ;

(2) Existence of solutions (more specifically, existence of two positive solution for sublinear case and bifurcation from infinity result for asymptotic linear case);

- (3) Lemma 4.4;
- (4) Solution curve has only one component.

That  $w > 0$  has been proved in Section 3 and Lemma 4.4 is also available; the existence of two solutions for the sublinear case can be provided by Theorem A.3 and a similar argument (minimization and mountain pass theorem); the result of bifurcation from infinity is also true. So we only need to prove that solution curve has only one component. Since we can have a same truncation argument as in the proof of Theorem 1.2, then (4) can also be proved similarly.

*Remark.* Our argument in this section can be further exploited to prove the uniqueness of ground state solution. In fact, if  $f$  is of class (C), our result in Section 3 implies that if  $u$  is a ground state solution, then the solution of corresponding linearized equation  $w$  changes sign exactly once, so we can follow [17] Lemma 10 to show that a lower branch of solution (in ball) will bifurcate from that ground state. Therefore there is only one branch of solutions (in ball), which indicates that there is only one ground state. For  $f$  of class  $(C)_0$ , a similar result can be obtained. We will discuss the problem of ground state in detail in our forthcoming paper [22].

## 5. EXAMPLES AND APPLICATION

Our first example is cubic function  $f_0(u) = -u(u-b)(u-c)$ ,  $c > 2b > 0$ . The equation (1.1) with nonlinearity  $f_0$  arises in population models.

**PROPOSITION 5.1.**  $f_0$  is of class (C-A).

*Proof.* (C1) and (C2-A) are clear. The second derivative of all cubic functions changes sign exactly once. For this case,  $\alpha = (b+c)/3 \in (b, c)$ . So (C3) is true. For (C4), notice  $K'(u) = -N(u)/f^2(u)$ , where  $N(u) = u[f'(u)]^2 - f(u)f'(u) - uf(u)f''(u) = u^2[(b+c)u^2 - 4bcu + bc(b+c)] > 0$  for all  $u > 0$ . So  $K(u)$  is decreasing for all  $u > 0$ . Therefore (C4) is also satisfied. ■

Hence Theorem 1.1 can be applied to  $f_0(u) = -u(u-b)(u-c)$ . The examples of  $f$  of class (C-B) can be derived from  $f_0(u)$ . We calculate that for  $f_0(u)$ ,  $\beta = (b+c)/2$ , the maximum point  $\gamma = [(b+c) + (b^2 - bc + c^2)^{1/2}]/3 > \beta$ . Let  $u_1, u_2$  satisfies  $0 < b < u_1 < \beta < u_2 < \gamma$ , and define

$$\begin{aligned}
 f_1(u) &= \begin{cases} f_0(u) & \text{for } 0 \leq u \leq u_1, \\ f_0(u_1) + f'_0(u_1)(u - u_1) & \text{for } u \geq u_1. \end{cases} \\
 f_2(u) &= \begin{cases} f_0(u) & \text{for } 0 \leq u \leq u_2, \\ f_0(u_2) + f'_0(u_2)(u - u_2) & \text{for } u \geq u_2. \end{cases} \\
 f_3(u) &= \begin{cases} f_0(u) & \text{for } 0 \leq u \leq u_2, \\ f_0(u_2) + f'_0(u_2) \ln(u - u_2 + 1) & \text{for } u \geq u_2. \end{cases}
 \end{aligned}$$

Then  $f_i$  ( $i = 1, 2, 3$ ) satisfies (C1), (C2-B), (C4) and  $f_i \in C^1$ . Though  $f_i$  is not  $C^2$  at  $u = u_i$  ( $i = 1, 2$ ), we can use a limiting process to show the arguments in Section 2 are still valid for  $f_i$ , and remember that all the other proofs only require  $f$  to be  $C^1$ . Hence Theorem 1.2 (i) can be applied to  $f_i$  ( $i = 1, 2, 3$ ).

A more general example of nonlinearity that is of class (C-A) is  $f_4(u) = -u + su^q - ru^p$  for some  $s, r > 0$  and  $p > q > 1$ . By a tedious but elementary calculation, we can show that (C-A) is satisfied if following conditions are true:

$$\begin{aligned}
 (1) \quad & s \left[ \frac{s(q-1)}{r(p-1)} \right]^{(q-1)/(p-q)} - r \left[ \frac{s(q-1)}{r(p-1)} \right]^{(p-1)/(p-q)} > 1, \\
 (2) \quad & \frac{s}{q+1} \left[ \frac{s(q-1)(p+1)}{r(p-1)(q+1)} \right]^{(q-1)/(p-q)} \\
 & - \frac{r}{p+1} \left[ s \frac{(q-1)(p+1)}{r(p-1)(q+1)} \right]^{(p-1)/(p-q)} > \frac{1}{2}, \\
 (3) \quad & r(p-1)^2 \left[ \frac{p-1}{s(p-q)} \right]^{(p-q)/(q-1)} \\
 & - sr(p-q)^2 \left[ \frac{p-1}{s(p-q)} \right]^{(p-1)/(q-1)} < s(q-1)^2.
 \end{aligned}$$

In fact, (1) implies  $f_4$  has two positive zeros; (2) implies  $\int_0^c f_4(u) du > 0$ ; (3) is a sufficient condition of  $K(u)$  is decreasing for all  $u > 0$ . Other conditions in (C-A) are easy to check. So we can also apply Theorem 1.1 to  $f_4$ .

Now we turns to the examples of (C)<sub>0</sub>. First we show that condition  $\lim_{u \rightarrow 0^+} K(u) > 1$  in (C4)<sub>0</sub> is not restrictive. In fact, we have following proposition.

**PROPOSITION 5.2.** *Let  $f \in C^2(\overline{\mathbf{R}}_+, \mathbf{R})$ ,  $f(0) = 0$ ,  $f' > 0$ ,  $f'' \geq 0$  near 0. Then*

- (1)  $\lim_{u \rightarrow 0} K(u) \geq 1$ ;
- (2) *If  $f'(0) \neq 0$ ,  $\lim_{u \rightarrow 0} K(u) = 1$ ;*

- (3) If  $f'(0) = 0, f''(0) \neq 0$ , then  $\lim_{u \rightarrow 0} K(u) > 1$ ;
- (4) If  $f(u) = u^p + \sum a_n u^{p_n}, p_n > p \geq 1$ , then  $\lim_{u \rightarrow 0} K(u) > 1$ .

The proof is elementary, basically just using L'Hospital's rule. The model example is  $g_0(u) = u^q - ru^p$  for  $r > 0$  and  $p > q > 1$ . For this nonlinearity  $g_0(u)$ , it is  $C^2$  at  $u = 0$  only if  $q \geq 2$ , but for  $1 < q < 2$ , we can still use a limiting process to get same results.

**PROPOSITION 5.3.**  $g_0$  is of class  $(C-A)_0$  for all  $r > 0$ .

*Proof.*  $(C1)_0$  and  $(C2-A)_0$  are easy to check.  $(C3)_0$  is also true and  $\alpha = [q(q-1)/rp(p-1)]^{1/(p-q)}$ . Finally, Proposition 5.2 (4) implies  $\lim_{u \rightarrow 0} K(u) = q > 1$ , and  $K'(u) = -(r(p-q)^2 u^{p+q-1}/g_0^2(u)) < 0$  for  $u > 0$ . Therefore,  $(C4)_0$  is satisfied.

By the same modification to  $f_0$ , we can also modify  $g_0$  to construct the examples that is of  $(C-B)_0$ .

From the remark at the end of Section 4, we know if  $n = 2$ , then for all  $p > q > 1$ , the lower branch will approach  $d = 0$ . If  $n \geq 3$ , then when  $q \leq (n+2)/(n-2)$ , the lower branch will approach  $d = 0$ . In both cases there is no ground state solution. But when  $q > (n+2)/(n-2)$ , the lower branch will stop at some  $d_0 > 0$ , and there is a family of ground state solutions.

Finally, we apply our result to a high space dimension reaction-diffusion equation.

$$\begin{cases} u_t = \Delta u + \lambda u(u-b)(c-u) & \text{in } B^n, \\ u = 0 & \partial B^n. \end{cases} \tag{5.1}$$

It is known (see [10] and the references therein) that (5.1) has a connected global attractor  $A$  in  $H_0^1(B^n)$ . If each equilibrium of (5.1) (solution of (1.1)) is hyperbolic (same meaning as noncritical), then the attractor is consisted of all the equilibria and their unstable manifolds.

Applying our result, we can have stronger conclusion that is same as one dimension case.

**THEOREM 5.4.** For problem (5.1)

- (1) For  $\lambda < \lambda^*$ ,  $0$  is global asymptotically stable.
- (2) For  $\lambda > \lambda^*$ , (5.1) has a global attractor, that consists of three equilibria (include  $0$ ) and their unstable manifolds. Moreover,  $0$  and the maximal equilibrium are asymptotically stable, the other equilibrium is unstable with one dimensional unstable manifold.

(3) For  $\lambda = \lambda^*$ , (3.1) has a global attractor, that consists of two equilibria (including 0) and their center manifolds. Moreover, 0 is asymptotically stable, the other equilibrium is unstable with one dimensional center manifold.

This result for one dimension space was known (see [10]). Our result for higher-dimension ball domain is new, but the proof is the same as the one-dimension case, as long as the information of equilibria is known. It is easy to see this result is also true for all  $f$  of class (C-A).

## APPENDIX: EXISTENCE RESULTS FOR (1.1).

There are numerous results concerning the existence of solutions of (1.1), and we just quote a few that are useful to prove our main theorems. For convenience, we make some necessary modification to the original theorems according to our context. For the following theorems, we assume  $f: [0, \infty) \rightarrow \mathbf{R}$  is locally lipschitz,  $f(0) = 0$ .

**THEOREM A.1** ([18] Theorem 1.1). *If*

$$\liminf_{u \rightarrow \infty} \frac{\lambda f(u)}{u} > \lambda_1,$$

$$\limsup_{u \rightarrow 0^+} \frac{\lambda f(u)}{u} < \lambda_1$$

$$\lim_{u \rightarrow \infty} \frac{\lambda f(u)}{u^l} = 0 \quad \text{with } l = \frac{n+2}{n-2} \quad \text{if } n \geq 3, l < \infty \quad \text{if } n = 1, 2.$$

and  $\Omega$  is convex, then (1.1) has a positive solution  $u$ .

**THEOREM A.2** ([18] Theorem 1.5). *Let*

$$\limsup_{u \rightarrow 0^+} \frac{f(u)}{u} \leq 0,$$

$$\alpha = \inf\{u > 0, f(u) > 0\} \quad \text{exists and } \alpha > 0,$$

$$\lim_{u \rightarrow \infty} \frac{f(u)}{u} = 0 \quad (\text{resp. } f(\beta) = 0 \text{ for some } \beta > \alpha),$$

$$\exists \xi > 0, \quad (\text{resp. } \beta > \xi > 0), \quad F(\xi) > 0, \quad \text{where } F(u) = \int_0^u f(s) ds.$$

Let  $\lambda^*$  be the infimum of all  $\lambda > 0$  such that there exists a solution of (1.1) (resp. less than  $\beta$ ), then  $\lambda^*$  is finite and positive and we have, if  $\Omega$  is star-shaped,

(1) For  $\lambda \geq \lambda^*$ , there exists a maximum positive solution  $u_1$  of (1.1) (resp. maximum among all positive solutions of (1.1) less than  $\beta$ ).

(2) For  $\lambda > \lambda^*$ , there exists a second solution  $u_2$  of (1.1) which satisfies  $0 < u_2 < u_1$ .

THEOREM A.3 ([26], Theorem 2.32). *Let*

$$f(\xi) > 0 \quad \text{for } \xi \in (0, r), \quad f(r) = 0.$$

*Then there exists a  $\underline{\lambda} > 0$  such that for all  $\lambda > \underline{\lambda}$ , (1.1) has at least two classical solutions which are positive in  $\Omega$ , and  $0 < \|u\|_\infty < r$ .*

THEOREM A.4 ([27], Theorem 2.28). *Let*

$$\lim_{u \rightarrow \infty} \frac{f(u)}{u} = a_2 > 0.$$

*Let  $E = \{u \in C^1(\Omega) \mid u = 0 \text{ on } \partial\Omega\}$ ,  $T = \{(\lambda, u) \in \mathbf{R} \times E \mid (\lambda, u) \text{ is a solution of (1.1)}\}$ . Then there exists a component  $D_1$  of  $T$  which contains  $(\mu_1, \infty)$ , where  $\mu_1 = \lambda_1/a_2$ ,  $\lambda_1$  is the first eigenvalue of  $-\Delta$  on  $\Omega$ . Moreover let  $A =$  projection of  $D_1$  on  $\mathbf{R}$ , we have either*

1.  *$A$  is bounded.  $D_1$  contains  $(\hat{\mu}, \infty)$  or  $(\hat{\mu}, 0)$  for  $\hat{\mu}a_2 \in \sigma(-\Delta)$ , where  $\sigma(-\Delta)$  is the spectrum of  $-\Delta$ ; or*
2.  *$A$  is unbounded.*

In Theorem A.4, we say “a component  $D_1$  of  $T$  contains  $(\mu_1, \infty)$ ” means that there exist solution  $(\lambda_n, u_n)$  of (1.1) such that  $\lambda_n \rightarrow \mu_1$  and  $\|u_n\| \rightarrow \infty$ .

## REFERENCES

1. I. Ali, A. Castro, and R. Shivaji, Uniqueness and stability of nonnegative solutions for semipositone problems in a ball, *Proc. Amer. Math. Soc.* **117** (1993), 775–781.
2. H. Berestycki and P. L. Lions, Nonlinear scalar field equations 1, 2, *Arch. Rational Mech. Anal.* **82** (1983), 313–375.
3. H. Berestycki, P. L. Lions, and L. A. Peletier, An ode approach to the existence of positive solutions for semilinear problem in  $\mathbf{R}^n$ , *Indiana Univ. Math. J.* **30** (1981), 141–167.
4. A. Castro and S. Gadam, Uniqueness of stable and unstable positive solutions for semipositone problems, *Nonlinear Anal.* **22** (1994), 425–429.
5. M. G. Crandall and P. H. Rabinowitz, Bifurcation, perturbation of simple eigenvalues and linearized stability, *Arch. Rational Mech. Anal.* **52** (1973), 161–180.
6. E. N. Dancer, On positive solutions of some singular perturbed problems where the nonlinearity changes sign, *Topological Methods Nonlinear Anal.* **5** (1995), 141–175.
7. E. N. Dancer, A note on asymptotic uniqueness for some nonlinearities which change sign, preprint, 1995.

8. R. Gardner and L. A. Peletier, The set of positive solutions of semilinear equations in large balls, *Proc. Royal. Soc. Edinburgh* **104A** (1986), 53–72.
9. B. Gidas, W. Ni, and L. Nirenberg, Symmetry and related properties via the maximum principle, *Comm. Math. Phys.* **68** (1979), 209–243.
10. J. Hale, Asymptotic behavior of dissipative systems, Amer. Math. Soc., *Math. Surveys* **25**, 1988.
11. M. Holzmann and H. Kielhöfer, Uniqueness of global positive solution branches of nonlinear elliptic problem, *Math. Ann.* **300** (1994), 221–241.
12. P. Korman, Y. Li, and T. Ouyang, Exact multiplicity results for boundary-value problems with nonlinearities generalizing cubic, *Proc. Roy. Soc. Edinburgh Sect. A* **126** (1996), 599–616.
13. P. Korman, Y. Li, and T. Ouyang, An exact multiplicity result for a class of semilinear equations, *Comm. Partial Differential Equations* **22** (1997), 661–684.
14. P. Korman and T. Ouyang, Exact multiplicity results for two classes of boundary-value problems, *Differential Integral Equations* **6** (1993), 1507–1517.
15. P. Korman and T. Ouyang, Multiplicity results for two classes of boundary-value problems, *SIAM J. Math. Analysis* **26** (1995), 180–189.
16. M. Kwong, Uniqueness of positive solutions of  $\Delta u - u + u^p = 0$  in  $\mathbf{R}^n$ , *Arch. Rational Mech. Anal.* **105** (1989), 243–266.
17. M. Kwong and L. Zhang, Uniqueness of the positive solution of  $\Delta u + f(u) = 0$  in an annulus, *Differential Integral Equations* **4** (1991), 583–599.
18. P. L. Lions, On the existence of positive solutions of semilinear elliptic equations, *SIAM Rev.* **24** (1982), 441–467.
19. C. Lin and W. Ni, A counterexample to the nodal domain conjecture and a related semilinear equation, *Proc. Amer. Math. Soc.* **102** (1988), 271–277.
20. K. McLeod and J. Serrin, Uniqueness of the positive radial solutions of  $\Delta u + f(u) = 0$  in  $\mathbf{R}^n$ , *Arch. Rational Mech. Anal.* **99** (1987), 115–145.
21. W. Ni and R. D. Nussbaum, Uniqueness and nonuniqueness for positive radial solutions of  $\Delta u + f(u, r) = 0$ , *Comm. Pure Appl. Math.* **38** (1985), 67–108.
22. T. Ouyang and J. Shi, Exact multiplicity of positive solutions for a class of semilinear problem: II, preprint, 1996.
23. L. A. Peletier and J. Serrin, Uniqueness of positive solutions of semilinear equations in  $\mathbf{R}^n$ , *Arch. Rational Mech. Anal.* **81** (1983), 181–197.
24. L. A. Peletier and J. Serrin, Uniqueness of solutions of semilinear equations in  $\mathbf{R}^n$ , *J. Differential Equations* **61** (1986), 380–397.
25. P. H. Rabinowitz, On bifurcation from infinity, *J. Differential Equations* **14** (1973), 462–475.
26. P. H. Rabinowitz, Minimax methods in critical point theory with applications to differential equations, Amer. Math. Soc. CBMS Reg. Conf. Ser. in Math., No. 65, 1986.
27. R. Schaaf and K. Schmitt, Asymptotic behavior of positive branches of elliptic problems with linear part at resonance, *Z. Angew. Math. Phys.* **43** (1992), 645–676.
28. J. Smoller and A. Wasserman, Global bifurcation of steady-state solutions, *J. Differential Equations* **39** (1981), 269–290.