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J. Differential Equations 263 (2017) 6537–6575

Journal of Differential Equations

www.elsevier.com/locate/jde

Hopf bifurcation in a reaction–diffusion equation with distributed delay and Dirichlet boundary condition ☆

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Received 4 May 2017; revised 11 July 2017
Available online 10 August 2017

Abstract

The stability and Hopf bifurcation of the positive steady state to a general scalar reaction–diffusion equation with distributed delay and Dirichlet boundary condition are investigated in this paper. The time delay follows a Gamma distribution function. Through analyzing the corresponding eigenvalue problems, we rigorously show that Hopf bifurcations will occur when the shape parameter $n \geq 1$, and the steady state is always stable when $n = 0$. By computing normal form on the center manifold, the direction of Hopf bifurcation and the stability of the periodic orbits can also be determined under a general setting. Our results show that the number of critical values of delay for Hopf bifurcation is finite and increasing in n, which is significantly different from the discrete delay case, and the first Hopf bifurcation value is decreasing in n. Examples from population biology and numerical simulations are used to illustrate the theoretical results.

MSC: 35R10; 92B05; 35B32; 35K57; 35B10; 92D40

Keywords: Hopf bifurcation; Distributed delay; Dirichlet boundary condition; Reaction–diffusion equation; Normal form

☆ Partially supported by a grant from China Scholarship Council, National Natural Science Foundations of China (No. 11571257), National Science Foundation grant DMS-1313243, and the Scientific Research Start-up Foundation of Hangzhou Normal University (No. 201603).

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http://dx.doi.org/10.1016/j.jde.2017.07.024

1. Introduction

Reaction–diffusion models have been used to describe the spatiotemporal distribution of density functions of substances from particles, chemicals, organisms, to plants and animals in modeling biological and ecological systems [4,30,31]. In 1952, Alan Turing [40] proposed that spatial patterns in embryonic morphogenesis were driven by diffusion-induced instability. Since then, researchers in chemistry and developmental biology have successfully applied Turing theory to explain and simulate the patterns arisen in Hydra growth [12,28], pigmentation patterning in fish [25], spatial patterns in Chlorite-Iodide-Malonic Acid-starch chemical reaction [26], regulation of Hox gene in the transition of fins to limbs during evolution [32], to name just a few.

On the other hand, real biochemical or ecological dynamics often depends on the historical information of systems so time delays could occur in various modeling mechanisms, and the presence of time delay may have profound impact on the dynamics of reaction–diffusion models [6,17,42]. The delay effect to a scalar reaction–diffusion population model has been considered in, for example, [3,5,18,36,38,43]. In general, a larger delay destabilizes the stable steady state of the system and an oscillatory pattern arises from a Hopf bifurcation. The stable steady state under Neumann boundary condition is usually a constant one, thus the Hopf bifurcation analysis is relatively easier [29,44]. For Dirichlet boundary problem, a positive steady state is always spatially non-homogeneous which makes such analysis difficult. Following the approach in [3], Su et al. [36] considered a general scalar diffusive equation with delayed growth rate per capita and Dirichlet boundary condition:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = d\dfrac{\partial^2 u(x,t)}{\partial x^2} + \lambda u(x,t) f(u(x,t-\tau)), & x \in (0,l),\ t>0, \\ u(0,t) = u(l,t) = 0, & t > 0, \\ u(x,t) = \eta(x,t), & x \in (0,l),\ t \in [-\tau, 0], \end{cases}$$

where $d > 0$ is the diffusion coefficient, $\tau > 0$ represents the time delay, $\lambda > 0$ is a scaling constant. The nonlinear function f is the growth rate per capita which can be chosen properly so that this equation can embrace different kinds of population dynamics. They proved that the non-homogeneous positive steady state loses its stability when τ increases and analyzed associated Hopf bifurcations. In [43], Yan and Li extended the results to a higher dimensional domain and also proved the stability of the bifurcating periodic orbits.

The dependence of the rate of change of current population on the population at a particular point of past time is usually a simplified assumption, and a more reasonable dependence would be on the whole historical information of the population. A distributed delay has been proposed to describe the population growth of some species, which can date back to the work of Volterra [41]. Here we propose a diffusive population model with general growth rate incorporating a distributed delay and Dirichlet boundary condition based on the previous work in [36]:

$$\begin{cases} u_t(x,t) = d\Delta u(x,t) + \lambda u(x,t) f(g * u(x,t-s)), & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,t) = u_0(x,t), & x \in \Omega,\ t \in (-\infty, 0], \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbf{R}^k$ $(k \geq 1)$ with smooth boundary, and $u_0 \in \mathcal{C} \triangleq C((-\infty,0], Y)$ with $Y = L^2(\Omega)$. Here, $d > 0$ represents the diffusion coefficient, $\lambda > 0$ is a growth rate coefficient and the growth rate per capita function f is assumed to be logistic type. Instead of using $f(u(t-\tau))$, we use $f(g*u)$ in (1.1) where $g*u(x,t-s)$ is a distributed delayed population density which has the following form,

$$g*u(x,t-s)=\int_{-\infty}^{t} g(\tau,t-s)u(x,s)ds=\int_{-\infty}^{0} g(\tau,-s)u(x,t+s)ds,$$

where the kernel function $g(\tau,t)$ is a probability distribution function satisfying $g(\tau,t)\geq 0$, and $\int_0^\infty g(\tau,t)dt=1$. Here we choose $g(\tau,t)$ to be the Gamma distribution (see [15,27]):

$$g_n(\tau,t)=\frac{t^n e^{-\frac{t}{\tau}}}{\tau^{n+1}\Gamma(n+1)}, \tag{1.2}$$

where $n\geq 0$ is the shape parameter of Gamma distribution, and the mean and variance of $g_n(\tau,\cdot)$ are given by $\mathbb{E}(g_n(\tau,\cdot))=(n+1)\tau$ and $\mathbb{V}ar(g_n(\tau,\cdot))=(n+1)\tau^2$ respectively. This includes the two well known distributions used in ecological studies: the weak kernel ($n=0$) and the strong kernel ($n=1$):

$$g_0(\tau,t)=\frac{1}{\tau}e^{-\frac{t}{\tau}},\quad g_1(\tau,t)=\frac{t}{\tau^2}e^{-\frac{t}{\tau}}.$$

In our analysis, we will take τ as the bifurcation parameter, which is associated with the average delay upon a scaling of the shape parameter. It is known that (1.1) has a unique positive steady state u_λ when $\lambda>\lambda_*$ (principal eigenvalue of $-d\Delta$ in $H_0^1(\Omega)$), and when $\tau=0$, the positive steady state u_λ is locally asymptotically stable [4]. Our main results are for the dynamical behavior of (1.1) when $\tau>0$ and λ is near λ_*, and our results can be summarized as follows:

1. for $n=0$, that is the weak kernel case, the positive steady state u_λ is locally asymptotically stable for any $\tau>0$;
2. for each positive integer n, there exists an increasing finite sequence $\{\tau_{n\lambda}^m\}_{m=0}^{m_n}$ where $m_n=[(n-1)/4]$ (the integer part of $(n-1)/4$), such that the positive steady state u_λ of Eq. (1.1) is locally asymptotically stable when $\tau\in(0,\tau_{n\lambda}^0)$ and is unstable when $\tau\in(\tau_{n\lambda}^0,\infty)$; moreover, the system (1.1) undergoes a Hopf bifurcation at $\tau=\tau_{n\lambda}^m$ near the steady state u_λ for each $0\leq m\leq m_n$;
3. the critical Hopf bifurcation values satisfy $q(m,n):=\lim\limits_{\lambda\to\lambda_*}(\lambda-\lambda_*)\tau_{n\lambda}^m=\dfrac{\tan(\eta_{n\lambda_*}^m)}{\cos^{n+1}(\eta_{n\lambda_*}^m)}$ with $\eta_{n\lambda_*}^m=\dfrac{(4m+1)\pi}{2(n+1)}$ for $0\leq m\leq m_n$. And $q(m,n)$ is strictly decreasing with respect to n, which means that the critical τ value is smaller for a larger n. In other words, it is easier (taking a smaller delay value) for the system (1.1) with a larger shape parameter n to lose its stability.

These results show that the shape parameter n significantly affects the dynamics of system (1.1). Note that the finiteness of bifurcation values $\tau_{n\lambda}^m$ is quite different from infinitely many bifurcation values in the case of discrete delay case considered in [36] which corresponds to a Dirac delta distribution function. Hence combining with this earlier results, we now know that the number of the critical values of Hopf bifurcation can be zero (weak kernel), a finite number (strong kernel or Gamma distribution with higher n), or infinitely many (Dirac delta distribution).

The spatiotemporal delay effect to the population models has been studied extensively in recent years. In fact, since the individuals in the population move freely, so the population density variation at a spatial location depends on the population in a neighborhood of the location, that is, on a spatial average weighted according to distance from the original position. Based on such assumption, Britton [2] proposed a model with spatiotemporal delay term:

$$u_t = \Delta u + u[1 + \alpha u - \beta u^2 - (1 + \alpha - \beta) g ** u], \ \ x \in \Omega, \ t > 0,$$

where

$$g ** u = \int_{-\infty}^{t} \int_{\Omega} g(x - y, t - s) u(y, s) dy ds.$$

Here the function $g(x,t)$ is a general spatiotemporal average, which is studied in [1,13,14,16] on an infinite spatial domain. Recently, Chen and Yu [7] considered a diffusive logistic model incorporating a class of spatiotemporal delay on a bounded domain with Dirichlet boundary condition, and found that the nonhomogeneous steady state is locally asymptotically stable and Hopf bifurcations do not occur. Zuo and Song [46] studied the bifurcation in a general scalar reaction–diffusion model with spatiotemporal delay under Neumann boundary condition. Guo [19] also considered Hopf bifurcation for a Dirichlet boundary value problem.

When the spatial variation of the averaging kernel is negligible, a reasonable simplification is that $g(x,t)$ is a purely temporal average, which means that $g(x,t) = \delta(x)\tilde{g}(t)$, that is the distributed delay case. In [15], Gourley and Ruan investigated a Nicholson's blowflies equation incorporating both distributed delay and diffusion under Neumann boundary condition, and they obtained the local and global stability of homogeneous steady state. Zuo and Song [45] studied the stability and bifurcation of positive constant steady state in a general scalar diffusive equation with distributed delay and Neumann boundary condition. For Dirichlet boundary problem with distributed delay, a two-species diffusive population model with distributed delay under Dirichlet boundary condition and its bifurcation problem were investigated in [23]. But the kernel function in [23] has support in $[\tau, \infty)$, which is quite different from the Gamma distribution considered here. Another idealized assumption is that $g(x,t)$ only depends on spatial information, that is, $g(x,t) = \bar{g}(x)\delta(t - \tau)$. Stability of positive steady state and associated Hopf bifurcations for such nonlocal spatial delay effect with Dirichlet boundary condition have been studied in for example [5,8,21]. And, in [24,39], the nonlocal delay effect in a scalar diffusive population equation under Neumann boundary condition is investigated. When $g(x,t) = \delta(x)\delta(t - \tau)$, then $g ** u$ becomes the discrete and local delay effect. The Dirichlet boundary value problem with discrete delay for diffusive logistic model has been considered in [3,36,38,43], and similar problem for Nicholson's blowflies and Dirichlet boundary condition equation [20,34,37] and the references therein.

This paper is organized as follows. In Section 2, we study the stability and Hopf bifurcation of the positive steady state through analyzing the corresponding eigenvalue problem. Then the normal form of Hopf bifurcation is calculated in Section 3 to determine the bifurcation direction and stability of the bifurcating periodic orbits. In Section 4, we apply our results to two reaction–diffusion population models: logistic model and food-limited model, and we perform some numerical simulations. Here, we want to introduce some notations in this paper. The Lebesgue space of square integrable functions defined on a bounded and smooth domain Ω is denoted by $L^2(\Omega)$ and we use H^k, H_0^k denote the real-valued Sobolev space based on $L^2(\Omega)$ space. Denote $X = H^2(\Omega) \cap H_0^1(\Omega)$ and $Y = L^2(\Omega)$. For a linear vector space Z, we define its complexification to be $Z_C \triangleq Z \oplus iZ = \{x_1 + ix_2 : x_1,\ x_2 \in Z\}$. The Banach space of continuous mappings from $s \in (-\infty, 0]$ into Y is denoted by $\mathcal{C} = C((-\infty, 0], Y)$ and the complex-valued Hilbert space Y_C has the inner product: $\langle u, v\rangle = \int_\Omega \bar{u}(x)^T v(x)dx$. And throughout the paper, we define λ_* as the principal eigenvalue of

$$\begin{cases} -d\Delta\phi(x) = \lambda\phi(x), & x \in \Omega, \\ \phi(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

where $\phi(x)$ is the corresponding eigenfunction of λ_* satisfying $\phi(x) > 0$ for all $x \in \Omega$ and $\int_\Omega \phi^2(x)dx = 1$.

2. Stability and Hopf bifurcation

In this section, we investigate the stability of the nonhomogeneous steady state of Eq. (1.1) which satisfies the following boundary problem:

$$\begin{cases} d\Delta u(x) + \lambda u(x) f(u(x)) = 0, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{2.1}$$

We always have the following assumptions for function f:

(H1) There exists a $\delta > 0$ such that f is a C^3 function on $[0, \delta]$,
(H2) $f(0) = 1$ and $f'(u) < 0$ for all $u \in [0, \delta]$.

The existence of the steady state of system (1.1) satisfying Eq. (2.1) has been proved in [36], so we will pay our attention to study its stability which can be determined by analyzing the corresponding eigenvalue problems. First we decompose the spaces X, Y as follows,

$$X = K \oplus X_1, \qquad Y = K \oplus Y_1,$$

where

$$K = \text{Span}\{\phi\},\ X_1 = \left\{ y \in X : \int_\Omega \phi(x)y(x)dx = 0 \right\},\ Y_1 = \left\{ y \in Y : \int_\Omega \phi(x)y(x)dx = 0 \right\}.$$

We directly give the result about the existence of steady state which is from Theorem 2.1 in [36].

Lemma 2.1. *There exist $\lambda^* > \lambda_*$ and a continuously differentiable mapping $\lambda \mapsto (\xi_\lambda, \alpha_\lambda)$ from $[\lambda_*, \lambda^*]$ to $X_1 \times \mathbb{R}^+$ such that Eq. (1.1) with f satisfying $(H2)$ has a positive steady state given by*

$$u_\lambda = \alpha_\lambda(\lambda - \lambda_*)\left[\phi + (\lambda - \lambda_*)\xi_\lambda\right], \quad \lambda \in [\lambda_*, \lambda^*]. \tag{2.2}$$

Moreover,

$$\alpha_{\lambda_*} = \frac{-\int_\Omega \phi^2(x)dx}{\lambda_* f'(0)\int_\Omega \phi^3(x)dx} = -\frac{1}{\lambda_* f'(0)\int_\Omega \phi^3(x)dx},$$

and $\xi_{\lambda_*} \in X_1$ is the unique solution of the equation

$$(d\Delta + \lambda_*)\,\xi + \left[1 + \lambda_*\alpha_{\lambda_*} f'(0)\phi(x)\right]\phi(x) = 0, \ \langle \phi, \xi\rangle = 0. \tag{2.3}$$

Let u_λ be the positive steady state obtained in Lemma 2.1, the discussion in the following sections is always based on the assumption that $\lambda \in (\lambda_*, \lambda^*]$ and $0 < \lambda^* - \lambda_* \ll 1$ unless otherwise specified. Here, we clarify that the following analysis is for a fixed shape parameter $n \in \mathbb{R}^+ \cup \{0\}$.

The linearization of Eq. (1.1) at u_λ is given by

$$\begin{cases} u_t(x,t) = d\Delta u(x,t) + \lambda f(u_\lambda)u(x,t) + \lambda u_\lambda f'(u_\lambda) g_n * u(x,t-s), & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{2.4}$$

where $g_n * u(x,t-s) = \int_{-\infty}^t g_n(\tau, t-s)u(x,s)ds$. We introduce the operator $A(\lambda) : \mathfrak{D}(A(\lambda)) \to Y_{\mathbb{C}}$ defined by

$$A(\lambda) := d\Delta + \lambda f(u_\lambda), \tag{2.5}$$

where $\mathfrak{D}(A(\lambda)) = X_{\mathbb{C}}$. By letting $\tilde{s} = s - t$ and dropping tilde on s, Eq. (2.4) can be rewritten as

$$\begin{cases} u_t(x,t) = A(\lambda)u(x,t) + \lambda u_\lambda f'(u_\lambda) \displaystyle\int_{-\infty}^{0} g_n(\tau, -s)u(x,t+s)ds, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{2.6}$$

From Chapter 3 of [42], the semigroup induced by the solutions of Eq. (2.6) has an infinitesimal generator $A_{n\tau}(\lambda)$ which is given by

$$A_{n\tau}(\lambda)\varphi_n = \dot{\varphi}_n, \tag{2.7}$$

and the domain of $A_{n\tau}(\lambda)$ is

$$\mathfrak{D}(A_{n\tau}(\lambda)) = \left\{ \varphi_n \in \mathcal{C}_{\mathbb{C}} \cap \mathcal{C}^1_{\mathbb{C}} : \dot{\varphi}_n(0) = A(\lambda)\varphi_n(0) + \lambda u_\lambda f'(u_\lambda) \int_{-\infty}^{0} g_n(\tau, -s)\varphi_n(s)ds \right\},$$

where $\mathcal{C}^1_{\mathbb{C}} = C^1((-\infty, 0], Y_{\mathbb{C}})$ and $\varphi_n(0) \in X_{\mathbb{C}}$. The spectrum of $A_{n\tau}(\lambda)$ is

$$\sigma(A_{n\tau}(\lambda)) = \{\mu \in \mathbb{C} : \Lambda(\lambda, \mu, \tau)\psi_n = 0, \text{ for some } \psi_n \in X_{\mathbb{C}} \backslash \{0\}\}, \tag{2.8}$$

where

$$\begin{aligned}\Lambda(\lambda, \mu, \tau) &= A(\lambda) + \lambda u_\lambda f'(u_\lambda) \int_{-\infty}^{0} g_n(\tau, -s) e^{\mu s} ds - \mu \\ &= A(\lambda) + \lambda u_\lambda f'(u_\lambda) \frac{1}{(1+\mu\tau)^{n+1}} - \mu.\end{aligned} \tag{2.9}$$

Note that (2.9) holds from the integral

$$\int_{-\infty}^{0} g_n(\tau, -s) e^{\mu s} ds = \frac{1}{\tau^{n+1}\Gamma(n+1)} \int_{-\infty}^{0} s^n e^{s/\tau} e^{\mu s} ds = \frac{1}{(1+\mu\tau)^{n+1}}. \tag{2.10}$$

The stability and associated Hopf bifurcation of the nonhomogeneous steady state u_λ of Eq. (1.1) are investigated in this section with the delay measure τ considered as the bifurcation parameter. In the following lemma, we show that, as $\tau \to 0$, the stability of steady state of Eq. (1.1) is determined by the limiting operator

$$A_{n0}(\lambda) = A(\lambda) + \lambda u_\lambda f'(u_\lambda). \tag{2.11}$$

Lemma 2.2. *For $\tau > 0$, we have the following results for the spectra of $A_{n\tau}$ and A_{n0}:*

(i) $\sigma(A_{n0}(\lambda)) \subseteq \mathbb{R}^-$;
(ii) $\lim_{\tau \to 0} \sigma_b(A_{n\tau}(\lambda)) = \sigma_b(A_{n0}(\lambda))$, *where* $\sigma_b(A_{n\tau}(\lambda)) := \sigma(A_{n\tau}(\lambda)) \cap \{\mu \in \mathbb{C} : \mathcal{R}e(\mu) > b\}$, $b = -\min\{1, \lambda_*\} + \varepsilon$ *with some small* $\varepsilon > 0$;
(iii) $\sup_{\mu \in \sigma(A_{n\tau}(\lambda))} \mathcal{R}e(\mu) < 0$ *holds for* $\tau > 0$ *sufficiently small.*

In particular, for $\lambda \in (\lambda_*, \lambda^*]$ and $\tau > 0$ sufficiently small, the positive steady state u_λ is locally asymptotically stable with respect to Eq. (1.1).

Proof. For (i), it is well-known that $A_{n0}(\lambda)$ is a self-joint linear operator, so that any spectral point of $A_{n0}(\lambda)$ is real-valued. Moreover, from the assumption $(H1)$ that $f'(u_\lambda) < 0$, the principal eigenvalue of $A_{n0}(\lambda)$ satisfies

$$\begin{aligned}\mu_1 &= \inf_{0 \neq y \in H_0^1(\Omega)} \frac{-d\int_\Omega |\nabla y|^2 + \lambda \int_\Omega f(u_\lambda) y^2 + \lambda \int_\Omega u_\lambda f'(u_\lambda) y^2}{\int_\Omega y^2} \\ &< \inf_{0 \neq y \in H_0^1(\Omega)} \frac{-d\int_\Omega |\nabla y|^2 + \lambda \int_\Omega f(u_\lambda) y^2}{\int_\Omega y^2} = 0,\end{aligned}$$

and the last equality holds because of the fact that $A(\lambda)u_\lambda = (d\Delta + \lambda f(u_\lambda))u_\lambda = 0$ and 0 is the principal eigenvalue of $A(\lambda)$ with u_λ being its eigenfunction. This proves (i).

To prove (ii), we follow the setting in [42]. Define $A_T y = d\Delta y$ for $y \in X_{\mathbb{C}}$. Then the closure of A_T generates an analytic compact semigroup $T(t)$ in $X_{\mathbb{C}}$, and it satisfies $|T(t)| \le Me^{-\lambda_* t}$ for $t \ge 0$ by [42, Theorem 1.1.5 and 3.1.4]. Then (2.6) can be rewritten as

$$v(t) = T(t)v(0) + \int_0^t T(t-s)\tilde{f}(v(0), v(s))ds, \quad t \ge 0,$$

with $\tilde{f} : \mathcal{C}_k \to Y_{\mathbb{C}}$ defined by

$$\tilde{f}(\tilde{\phi})(x) := \lambda f(u_\lambda)\tilde{\phi}(0) + \lambda u_\lambda f'(u_\lambda) \int_{-\infty}^{0} g_n(-s)\tilde{\phi}(x,s)ds, \; x \in \Omega, \tag{2.12}$$

where the space $\mathcal{C}_k$ is defined as (for $\gamma > 0$)

$$\mathcal{C}_k = \{\tilde{\phi} : (-\infty, 0] \to Y_{\mathbb{C}}, \tilde{\phi} \text{ is continuous and } \lim_{s \to -\infty} e^{\gamma s}|\tilde{\phi}(s)| = 0\}.$$

Here the norm in $\mathcal{C}_k$ is defined by $\|\tilde{\phi}\|_{\mathcal{C}_k} := \sup_{s \le 0} e^{\gamma s}|\tilde{\phi}(s)|$. For a fixed n, we choose $\gamma = 1$. When $0 < \tau < 1/2$, we have

$$\int_0^\infty e^{\gamma s}|g_n(s)|ds = \frac{1}{\tau^{n+1}\Gamma(n+1)} \int_0^\infty s^n e^{(\gamma - 1/\tau)s} ds = \frac{1}{(\gamma\tau - 1)^{n+1}} < 2^{n+1} < \infty. \tag{2.13}$$

Then, by (2.13) and the boundedness of $u_\lambda, \; f(u_\lambda), \; f'(u_\lambda)$, we have

$$\begin{aligned} |\tilde{f}(\tilde{\phi})| := \sup_{x \in \Omega} |\tilde{f}(\tilde{\phi})(x)| &\le |\lambda f(u_\lambda)| \|\tilde{\phi}\|_{\mathcal{C}_k} + |\lambda u_\lambda f'(u_\lambda)| \int_0^\infty e^{\gamma s}|g_n(s)|ds \|\tilde{\phi}\|_{\mathcal{C}_k} \\ &< \Big(|\lambda f(u_\lambda)| + 2^{n+1}|\lambda u_\lambda f'(u_\lambda)|\Big) \|\tilde{\phi}\|_{\mathcal{C}_k} < \tilde{M}\|\tilde{\phi}\|_{\mathcal{C}_k}, \end{aligned}$$

for some $\tilde{M} > 0$. Therefore $\tilde{f}$ is a bounded linear operator on $\mathcal{C}_k$. With $b = -\min\{1, \lambda_*\} + \varepsilon < 0$ and by using [42, Theorem 3.4.2 and 3.4.5], we know that the set $\sigma_b(A_{n\tau}(\lambda))$ contains only a finite number of points of $P_\sigma(A_{n\tau}(\lambda))$ which is the point spectrum of $A_{n\tau}(\lambda)$, and all of these points are of finite multiplicity. Then, when $\tau \to 0$, the conclusion in (ii) holds.

Then, we show that $\sigma_b(A_{n0}(\lambda)) \ne \emptyset$ by proving $\mu_1 \in \sigma_b(A_{n0}(\lambda))$. By the variational method, we can also write the λ_* which is the principal eigenvalue of $-d\Delta$ into the following form:

$$\lambda_* = \inf_{0 \ne y \in H_0^1(\Omega)} \frac{d\int_\Omega |\nabla y|^2}{\int_\Omega y^2}. \tag{2.14}$$

So, we can calculate that

$$\mu_1 - (-\lambda_*) = \inf_{0\neq y\in H_0^1(\Omega)} \frac{\lambda\int_\Omega f(u_\lambda)y^2 + \lambda\int_\Omega u_\lambda f'(u_\lambda)y^2}{\int_\Omega y^2}$$
$$= \inf_{0\neq y\in H_0^1(\Omega)} \frac{\lambda\int_\Omega [f(u_\lambda) + u_\lambda f'(u_\lambda)]y^2}{\int_\Omega y^2}.$$

When $\lambda \to \lambda_*$, that is $u_\lambda \to 0$, by the assumptions (H1) and (H2), we know that $f(u_\lambda) \to 1$ and $u_\lambda f'(u_\lambda) \to 0$. Therefore, we have $\mu_1 > -\lambda_*$ which means that $\mu_1 > b$ and thus $\mu_1 \in \sigma_b(A_{n0}(\lambda))$.

Part (iii) is a direct consequence of (i) and (ii). From part (iii), we know that all the eigenvalues of $A_{n\tau}(\lambda)$ have negative real part for sufficient small $\tau > 0$, which implies that the steady state u_λ of Eq. (1.1) is locally asymptotically stable. □

In order to investigate Hopf bifurcations in system (1.1), we consider the case that $A_{n\tau}(\lambda)$ has a pair of purely imaginary eigenvalues $\mu = \pm i\omega_n$ $(\omega_n > 0)$ for some $\tau > 0$. From (2.9), we know that the operator $A_{n\tau}(\lambda)$ has an eigenvalue $i\omega_n$ is equivalent to

$$\left[A(\lambda) + \lambda u_\lambda f'(u_\lambda)\frac{1}{(1+i\theta_n)^{n+1}} - i\omega_n\right]\psi_n = 0, \quad \psi_n \in X_{\mathbb{C}} \setminus \{0\}, \tag{2.15}$$

where $\theta_n := \omega_n\tau$.

Next we will show that there exist some triples $(\omega_n, \theta_n, \psi_n)$ which solve Eq. (2.15) for $n > 0$. For further discussion, we need the following lemma.

Lemma 2.3. *Recall that λ_* is the principal eigenvalue of $-d\Delta$, we have*

(i) if $z \in X_{\mathbb{C}}$ and $\langle\phi, z\rangle = 0$, then $|\langle(d\Delta + \lambda_*)z, z\rangle| \geq (\lambda_2 - \lambda_*)\|z\|^2_{Y_{\mathbb{C}}}$, where λ_2 is the second eigenvalue of $-d\Delta$;
(ii) for each $n \geq 0$, if there exist some $(\omega_n, \theta_n, \psi_n)$ solve Eq. (2.15) with $\psi_n \in X_{\mathbb{C}}$, then $\omega_n/(\lambda - \lambda_*)$ is uniformly bounded for $\lambda \in (\lambda_*, \lambda^*]$.

Proof. Part (i) is the same as [5, Lemma 2.3]. We prove part (ii). By Eq. (2.15), we have

$$\left\langle\left[A(\lambda) + \lambda u_\lambda f'(u_\lambda)\frac{1}{(1+i\theta_n)^{n+1}} - i\omega_n\right]\psi_n, \psi_n\right\rangle = 0. \tag{2.16}$$

Since $A(\lambda)$ is self-adjoint, then $\langle A(\lambda)\psi_n, \psi_n\rangle$ is real. And by using $1 + i\theta_n = \sqrt{1+\theta_n^2}\, e^{i\eta_n}$ with $\tan\eta_n = \theta_n$, Eq. (2.16) can be rewritten as

$$\left\langle\left[A(\lambda) + \lambda u_\lambda f'(u_\lambda)(1+\theta_n^2)^{-(n+1)/2}e^{-i(n+1)\eta_n} - i\omega_n\right]\psi_n, \psi_n\right\rangle = 0. \tag{2.17}$$

Separating the real and imaginary parts of Eq. (2.17), we have

$$\omega_n\langle\psi_n, \psi_n\rangle = -\left\langle(1+\theta_n^2)^{-(n+1)/2}\sin((n+1)\eta_n)\lambda u_\lambda f'(u_\lambda)\psi_n, \psi_n\right\rangle.$$

Therefore, we obtain

$$\frac{|\omega_n|}{\lambda-\lambda_*}=\frac{\lambda\alpha_\lambda\left|(1+\theta_n^2)^{-(n+1)/2}\sin((n+1)\eta_n)\langle f'(u_\lambda)[\phi+(\lambda-\lambda_*)\xi_\lambda]\psi_n,\psi_n\rangle\right|}{\|\psi_n\|_{Y_{\mathbb{C}}}^2}.$$

According to the boundedness of f' and θ_n, we know that there is a constant $M_1>0$ such that $\left|\left((1+\theta_n^2)^{-(n+1)/2}\sin((n+1)\eta_n)\right)f'(u_\lambda)\right|\le M_1$, which implies that

$$\frac{|\omega_n|}{\lambda-\lambda_*}\le\lambda\alpha_\lambda M_1\left(1+(\lambda-\lambda_*)\|\xi_\lambda\|_\infty\right),\quad \lambda\in(\lambda_*,\lambda^*].$$

The boundedness of $\omega_n/(\lambda-\lambda_*)$ follows from the continuity of $\lambda\mapsto(\alpha_\lambda,\|\xi_\lambda\|_\infty)$. □

Suppose that $(\omega_n,\theta_n,\psi_n)$ is a solution of Eq. (2.15) with $\psi_n\in X_{\mathbb{C}}$, then ψ_n can be decomposed and normalized as

$$\begin{aligned}&\psi_n=\beta_n\phi+(\lambda-\lambda_*)z_n,\quad \langle\phi,z_n\rangle=0,\\&\|\psi_n\|_{Y_{\mathbb{C}}}^2=\beta_n^2\|\phi\|_{Y_{\mathbb{C}}}^2+(\lambda-\lambda_*)^2\|z_n\|_{Y_{\mathbb{C}}}^2=\|\phi\|_{Y_{\mathbb{C}}}^2.\end{aligned}\tag{2.18}$$

Substituting Eqs. (2.2), (2.18) and $\omega_n=(\lambda-\lambda_*)h_n$ into Eq. (2.15), we get the following system which is equivalent to Eq. (2.15):

$$\begin{aligned}g_1(z_n,\beta_n,h_n,\theta_n,\lambda):=&(d\Delta+\lambda_*)z_n+(\beta_n\phi+(\lambda-\lambda_*)z_n)\Big\{1+\lambda m_1(\xi_\lambda,\alpha_\lambda,\lambda)\\&+\lambda\alpha_\lambda[\phi+(\lambda-\lambda_*)\xi_\lambda]f'(u_\lambda)\frac{1}{(1+i\theta_n)^{n+1}}-ih_n\Big\}=0,\\g_2(\beta_n,z_n,\lambda):=&\left(\beta_n^2-1\right)\|\phi\|_{Y_{\mathbb{C}}}^2+(\lambda-\lambda_*)^2\|z_n\|_{Y_{\mathbb{C}}}^2=0,\end{aligned}\tag{2.19}$$

where

$$m_1(\xi_\lambda,\alpha_\lambda,\lambda)=\begin{cases}\dfrac{f(u_\lambda)-1}{\lambda-\lambda_*}, & \lambda\ne\lambda_*,\\ f'(0)\alpha_{\lambda_*}\phi, & \lambda=\lambda_*.\end{cases}\tag{2.20}$$

We define $G:X_{\mathbb{C}}\times\mathbb{R}^3\times\mathbb{R}\to Y_{\mathbb{C}}\times\mathbb{R}$ as

$$G(z_n,\beta_n,h_n,\theta_n,\lambda):=(g_1,g_2).$$

We will show that the equation $G=0$ can be solved for λ near λ_*, and we first solve the limiting equation when $\lambda=\lambda_*$ in the following lemma.

Lemma 2.4. *When $\lambda=\lambda_*$, for $n\ge0$ and $m\in\mathbb{N}\cup\{0\}$, we define*

$$\eta_{n\lambda_*}^m:=\frac{(4m+1)\pi}{2(n+1)},\quad \Sigma_n:=\left\{\theta_{n\lambda_*}^m=\tan\left(\eta_{n\lambda_*}^m\right):\sin\left(\eta_{n\lambda_*}^m\right)>0,\cos\left(\eta_{n\lambda_*}^m\right)>0\right\}.\tag{2.21}$$

Then $G(z_n, \beta_n, h_n, \theta_n, \lambda_*) = 0$ *has exactly* $|\Sigma_n|$ *solutions given by*

$$\begin{aligned} W^m_{n\lambda_*} &:= \left(z^m_{n\lambda_*}, \beta^m_{n\lambda_*}, h^m_{n\lambda_*}, \theta^m_{n\lambda_*}\right) \\ &= \left(\left(1 - i\cos^{n+1}\left(\eta^m_{n\lambda_*}\right)\right)\xi_{\lambda_*}, 1, \cos^{n+1}\left(\eta^m_{n\lambda_*}\right), \tan\left(\eta^m_{n\lambda_*}\right)\right), \end{aligned} \tag{2.22}$$

where $\theta^m_{n\lambda_*} \in \Sigma_n$ *(here* $|\Sigma_n|$ *is the number of elements in the set* Σ_n*). Moreover,*

(i) if $n = 0$, $|\Sigma_n| = 0$;
(ii) if $n \in \mathbb{N}$, $|\Sigma_n| = m_n + 1$ *where* $m_n := [(n-1)/4]$ *(here* $[k]$ *is the integer part of* $k \in \mathbb{R}$*);*
(iii) if $n \in \mathbb{Q}^+ - \mathbb{N}$, $|\Sigma_n| < \infty$;
(iv) if $n \in \mathbb{R}^+ - \mathbb{Q}^+$, $|\Sigma_n| = \infty$.

Proof. Our purpose is to solve $G(z_n, \beta_n, h_n, \theta_n, \lambda) = (g_1, g_2) = 0$ when $\lambda = \lambda_*$. Firstly, we have $\beta^m_{n\lambda_*} = 1$ through solving $g_2|_{\lambda=\lambda_*} = 0$. When $\lambda = \lambda_*$, $g_1 = 0$ is equivalent to

$$(d\Delta + \lambda_*)z_n + (1 - ih_n)\phi + \lambda_*\alpha_{\lambda_*} f'(0)\left(1 + \frac{1}{(1+i\theta_n)^{n+1}}\right)\phi^2 = 0. \tag{2.23}$$

Multiplying Eq. (2.23) with ϕ and integrating over Ω, we obtain that

$$\begin{aligned} &(1 - ih_n)\int_\Omega \phi^2 dx + \lambda_*\alpha_{\lambda_*} f'(0)\left(1 + \frac{1}{(1+i\theta_n)^{n+1}}\right)\int_\Omega \phi^3 dx \\ &= \left(1 - ih_n - \left(1 + \frac{1}{(1+i\theta_n)^{n+1}}\right)\right)\int_\Omega \phi^2 dx = 0, \end{aligned}$$

which implies that

$$\frac{1}{(1+i\theta_n)^{n+1}} + ih_n = 0. \tag{2.24}$$

Separating the real and imaginary parts of Eq. (2.24), we have

$$\begin{cases} (1+\theta_n^2)^{-(n+1)/2}\cos((n+1)\eta_n) = 0, \\ -(1+\theta_n^2)^{-(n+1)/2}\sin((n+1)\eta_n) + h_n = 0, \end{cases} \tag{2.25}$$

where $\tan\eta_n = \theta_n$. Since $\theta_n = \omega_n\tau > 0$ and $h_n > 0$ which is from the second equation of (2.25), then we have

$$\eta_n = \frac{(4m+1)\pi}{2(n+1)}, \quad m = \mathbb{N}\cup\{0\}, \ \theta_n = \tan(\eta_n), \ \text{with } \sin(\eta_n) > 0, \cos(\eta_n) > 0. \tag{2.26}$$

Hence we have $\theta_n = \theta^m_{n\lambda_*} \in \Sigma_n$ defined in (2.21) when $\lambda = \lambda_*$. When $n = 0$, from (2.25), there is no solution satisfying $G = 0$. When $n \in \mathbb{N}$, since $k(m) = \tan\left(\frac{(4m+1)\pi}{2(n+1)}\right)$ is periodic in m, and from (2.26), the number of $\theta^m_{n\lambda_*}$ is determined by $0 < \frac{(4m+1)\pi}{2(n+1)} < \frac{\pi}{2}$ or equivalently $-1 <$

$4m < n$. It is easy to see that the number of integers m satisfying $-1 < 4m < n$ is $m_n + 1$ for $n \in \mathbb{N}$. When n is rational but not an integer, that is $n \in \mathbb{Q}^+ - \mathbb{N}$, then we can write n as a fraction. By an analogous argument of the $n \in \mathbb{N}$ case, it can be shown that the number of $\theta^m_{n\lambda_*}$ is still finite. When n is irrational, the function $k(m)$ defined above is not periodic, so the number of solutions of $G = 0$ is infinite.

By the second equation of Eq. (2.25), h_n is obtained when $\lambda = \lambda_*$:

$$h^m_{n\lambda_*} = \cos^{n+1}\left(\eta^m_{n\lambda_*}\right).$$

Substituting Eq. (2.24) into Eq. (2.23), we get

$$(d\Delta + \lambda_*)z^m_{n\lambda_*} + (1 - ih^m_{n\lambda_*})\left(\phi + \lambda_*\alpha_{\lambda_*} f'(0)\phi^2\right) = 0. \tag{2.27}$$

Because $(d\Delta + \lambda_*)^{-1}$ is bijective in $(X_1)_{\mathbb{C}}$, so the solution of (2.27) is given by

$$z^m_{n\lambda_*} = -\left(1 - i\cos^{n+1}\left(\eta^m_{n\lambda_*}\right)\right)(d\Delta + \lambda_*)^{-1}\left(\phi + \lambda_*\alpha_{\lambda_*} f'(0)\phi^2\right) = \left(1 - i\cos^{n+1}\left(\eta^m_{n\lambda_*}\right)\right)\xi_{\lambda_*},$$

where ξ_{λ_*} is the unique solution of Eq. (2.3). This completes the proof. □

Lemma 2.4 shows that θ_n (frequency parameter) can be explicitly solved as in (2.21) for any real-valued shape parameter $n > 0$. For simplicity, in the following we will only consider the case of n being a positive integer, although the rational n case can also be established with similar approach. Note that in the integer or rational case, the number of potential critical frequencies θ^m_n is finite, which is different from previous case of discrete delay case in [36] (which corresponds to the distribution function is a Dirac delta function). For that case, it was shown that $\theta^m = \dfrac{(4m+1)\pi}{2}$ is a critical frequency for any $m \in \mathbb{N} \cup \{0\}$.

Now by applying the implicit function theorem, we obtain the following result regarding the eigenvalue problem for λ near λ_*.

Theorem 2.5. *For $n \in \mathbb{N}$ and $0 \le m \le m_n$, and with $W^m_{n\lambda_*}$ defined in Eq. (2.22), we have*

(i) for each m, there is a unique continuously differentiable map $W^m : [\lambda_*, \lambda^*] \to (X_1)_{\mathbb{C}} \times \mathbb{R}^3$ defined by $W^m(\lambda) := \left(z^m_{n\lambda}, \beta^m_{n\lambda}, h^m_{n\lambda}, \theta^m_{n\lambda}\right)$ such that $G(W^m(\lambda), \lambda) = 0$ and $W^m(\lambda_*) = W^m_{n\lambda_*}$. Moreover, if there exists $W^m_1(\lambda) := \left(\tilde{z}^m_{n\lambda}, \tilde{\beta}^m_{n\lambda}, \tilde{h}^m_{n\lambda}, \tilde{\theta}^m_{n\lambda}\right)$ such that $G\left(W^m_1(\lambda), \lambda\right) = 0$ with $\tilde{h}^m_{n\lambda} > 0$, $\tilde{\theta}^m_{n\lambda} > 0$, then $W^m_1(\lambda) = W^m(\lambda)$;

(ii) for $\lambda \in (\lambda_*, \lambda^*]$, the eigenvalue problem

$$\Lambda(\lambda, i\omega_n, \tau_n)\psi_n = 0,\ \tau_n > 0,\ \psi_n \in X_{\mathbb{C}} \setminus \{0\}$$

with Λ defined in (2.9) has solutions, that is, $i\omega_n \in \sigma(A_{n\tau}(\lambda))$ if and only if

$$\begin{aligned} &\omega_n = \omega^m_{n\lambda} := (\lambda - \lambda_*)h^m_{n\lambda},\ \tau_n = \tau^m_{n\lambda} := \theta^m_{n\lambda}/\omega^m_{n\lambda}, \\ &\psi_n = r_n\psi^m_{n\lambda} \text{ with } \psi^m_{n\lambda} := \beta^m_{n\lambda}\phi + (\lambda - \lambda_*)z^m_{n\lambda}, \end{aligned} \tag{2.28}$$

where r_n is a nonzero constant and $\left(z^m_{n\lambda}, \beta^m_{n\lambda}, h^m_{n\lambda}, \theta^m_{n\lambda}\right) = W^m(\lambda)$ is defined in part (i).

Proof. We define $T = (T_1, T_2) : (X_1)_{\mathbb{C}} \times \mathbb{R}^3 \mapsto Y_{\mathbb{C}} \times \mathbb{R}$ by $T := D_{(z_n,\beta_n,h_n,\theta_n)} G\left(W^m_{n\lambda_*}, \lambda_*\right)$, that is the Fréchet derivative of G with respect to $(z_n, \beta_n, h_n, \theta_n)$ at $(z^m_{n\lambda_*}, \beta^m_{n\lambda_*}, h^m_{n\lambda_*}, \theta^m_{n\lambda_*})$. Then we have

$$\begin{aligned} T_1(\chi, \kappa, \epsilon, \vartheta) &= (d\Delta + \lambda_*)\chi + \left(1 - ih^m_{n\lambda_*}\right)\kappa\phi\left[1 + \lambda_*\alpha_{\lambda_*} f'(0)\phi\right] - i\epsilon\phi \\ &\quad - \frac{i(n+1)\vartheta\lambda_*\alpha_{\lambda_*} f'(0)\phi^2}{\left(1 + i\theta^m_{n\lambda_*}\right)^{n+2}}, \\ T_2(\kappa) &= 2\kappa\|\phi\|^2_{Y_{\mathbb{C}}}, \end{aligned}$$

where $\alpha_{\lambda*}$ is defined in Lemma 2.1. It can be verified that T is bijective from $(X_1)_{\mathbb{C}} \times \mathbb{R}^3$ to $Y_{\mathbb{C}} \times \mathbb{R}$. By the implicit function theorem, for each m, there exists a unique continuously differentiable mapping $W^m(\lambda) : [\lambda_*, \lambda^*] \to (X_1)_{\mathbb{C}} \times \mathbb{R}^3$ such that $G(W^m(\lambda), \lambda) = 0$ with $W^m(\lambda_*) = W^m_{n\lambda_*}$. This completes the proof of existence. And we need also to prove the uniqueness of the solution.

From the implicit function theorem, we need to verify that if $G\left(W^m_1(\lambda), \lambda\right) = 0$, then $W^m_1(\lambda) \to W^m_{n\lambda_*}$ as $\lambda \to \lambda_*$ in the norm of $X_{\mathbb{C}} \times \mathbb{R}^3$. From Lemma 2.3 and Eq. (2.23), we see that $\{\tilde{h}^m_{n\lambda}\}$, $\{\tilde{\beta}^m_{n\lambda}\}$ and $\{\tilde{\theta}^m_{n\lambda}\}$ are bounded for each n. From Lemma 2.3 and the first equation of Eq. (2.23), we have

$$\begin{aligned} \|\tilde{z}^m_{n\lambda}\|^2_{Y_{\mathbb{C}}} &\le \frac{1}{\lambda_2 - \lambda_*}\left[\left|\left\langle\left(1 + \lambda m_1(\alpha_\lambda, \xi_\lambda, \lambda) - i\tilde{h}^m_{n\lambda}\right)\left(\tilde{\beta}^m_{n\lambda}\phi + (\lambda - \lambda_*)\tilde{z}^m_{n\lambda}\right), \tilde{z}^m_{n\lambda}\right\rangle\right|\right. \\ &\quad \left. + \left|\left\langle\frac{\lambda\alpha_\lambda(\phi + (\lambda - \lambda_*)\xi_\lambda) f'(u_\lambda)}{\left(1 + i\tilde{\theta}^m_{n\lambda}\right)^{n+1}}\left(\tilde{\beta}^m_{n\lambda}\phi + (\lambda - \lambda_*)\tilde{z}^m_{n\lambda}\right), \tilde{z}^m_{n\lambda}\right\rangle\right|\right] \\ &= \frac{1}{\lambda_2 - \lambda_*}\left|\left\langle\left(1 + \lambda m_1(\alpha_\lambda, \xi_\lambda, \lambda) - i\tilde{h}^m_{n\lambda}\right.\right.\right. \\ &\quad \left.\left.\left. - i\lambda\alpha_\lambda(\phi + (\lambda - \lambda_*)\xi_\lambda) f'(u_\lambda)\tilde{h}^m_{n\lambda}\right)\left(\tilde{\beta}^m_{n\lambda}\phi + (\lambda - \lambda_*)\tilde{z}^m_{n\lambda}\right), \tilde{z}^m_{n\lambda}\right\rangle\right|. \end{aligned}$$

The boundedness of $\{\tilde{h}^m_{n\lambda}\}$, $\left\{\tilde{\alpha}^m_{n\lambda}\right\}$, $\{f'(u_\lambda)\}$ and $\{\xi_\lambda\}$ implies that there exists $M_2 > 0$ such that

$$\frac{1}{\lambda_2 - \lambda_*}\left\|1 + \lambda m_1(\alpha_\lambda, \xi_\lambda, \lambda) - i\tilde{h}^m_{n\lambda} - i\lambda\alpha_\lambda(\phi + (\lambda - \lambda_*)\xi_\lambda) f'(u_\lambda)\tilde{h}^m_{n\lambda}\right\|_\infty \le M_2,$$

then we have

$$\left\|\tilde{z}^m_{n\lambda}\right\|^2_{Y_{\mathbb{C}}} \le M_2\|\phi\|_{Y_{\mathbb{C}}}\left|\tilde{\beta}^m_{n\lambda}\right|\left\|\tilde{z}^m_{n\lambda}\right\|_{Y_{\mathbb{C}}} + M_2(\lambda - \lambda_*)\left\|\tilde{z}^m_{n\lambda}\right\|^2_{Y_{\mathbb{C}}}. \tag{2.29}$$

We can choose proper M_2 such that $M_2(\lambda - \lambda_*) < 1/2$, then Eq. (2.29) implies that

$$\left\|\tilde{z}^m_{n\lambda}\right\|^2_{Y_{\mathbb{C}}} \le 2M_2\left|\tilde{\beta}^m_{n\lambda}\right|\|\phi\|^2_{Y_{\mathbb{C}}}. \tag{2.30}$$

Hence, $\left\{\tilde{z}_{n\lambda}^m\right\}$ is bounded in $Y_{\mathbb{C}}$ when $\lambda \in [\lambda_*, \lambda^*]$. Since the operator $d\Delta + \lambda_* : (X_1)_{\mathbb{C}} \mapsto (Y_1)_{\mathbb{C}}$ has a bounded inverse, by applying $(d\Delta + \lambda_*)^{-1}$ on $g_1\left(\tilde{z}_{n\lambda}^m, \tilde{\beta}_{n\lambda}^m, \tilde{h}_{n\lambda}^m, \tilde{\theta}_{n\lambda}^m, \lambda\right) = 0$, we find that $\left\{\tilde{z}_{n\lambda}^m\right\}$ is also bounded in $X_{\mathbb{C}}$, and hence $\left\{W_1^m(\lambda) : \lambda \in (\lambda_*, \lambda^*]\right\}$ is precompact in $Y_{\mathbb{C}} \times \mathbb{R}^3$. Therefore, there is a subsequence $\left\{W_1^m(\lambda^j) := \left(\tilde{z}_{n\lambda^j}^m, \tilde{\beta}_{n\lambda^j}^m, \tilde{h}_{n\lambda^j}^m, \tilde{\theta}_{n\lambda^j}^m\right)\right\}$ such that

$$W_1^m(\lambda^j) \to W_1^m(\lambda_*), \quad \lambda^j \to \lambda_* \text{ as } j \to \infty.$$

By taking the limit of the equation $(d\Delta + \lambda_*)^{-1} G\left(W_1^m(\lambda^j), \lambda^j\right) = 0$ as $j \to \infty$, we have that $G\left(W_1^m(\lambda_*), \lambda_*\right) = 0$. Also, by Lemma 2.4, we know that $G(z_n, \beta_n, h_n, \theta_n, \lambda_*) = 0$ has a unique solution given by $(z_n, \beta_n, h_n, \theta_n) = W_{n\lambda_*}^m$, thus $W_1^m(\lambda_*) = W_{n\lambda_*}^m$. Hence, $W_1(\lambda) \to W_{n\lambda_*}^m$ as $\lambda \to \lambda_*$ in the norm of $X_{\mathbb{C}} \times \mathbb{R}^3$. This proves part (i), and part (ii) is immediately observed from part (i). □

When using the delay τ as bifurcation parameter, Theorem 2.5 identifies $m_n + 1$ possible Hopf bifurcation values $\tau_{n\lambda}^m$ for $0 \le m \le m_n$. We show that the bifurcation values $\tau_{n\lambda}^m$ are monotonic with respect to n and m.

Proposition 2.6. *For $\lambda \in (\lambda_*, \lambda^*]$, $n \in \mathbb{N}$ and $0 \le m \le m_n$, we have the following results:*

(i) for a fixed n, $\tau_{n\lambda}^m$ is strictly increasing with respect to m;
(ii) for a fixed m, $\tau_{n\lambda}^m$ is strictly decreasing with respect to n.

Proof. It is sufficient to show that the monotonicity holds for

$$q(m,n) := \lim_{\lambda \to \lambda_*} (\lambda - \lambda_*)\tau_{n\lambda}^m = \frac{\tan\left(\eta_{n\lambda_*}^m\right)}{\cos^{n+1}\left(\eta_{n\lambda_*}^m\right)}. \tag{2.31}$$

From $\eta_{n\lambda_*}^m = \dfrac{(4m+1)\pi}{2(n+1)}$, we see that $\eta_{n\lambda_*}^m$ is strictly increasing in m and strictly decreasing in n. Moreover, by definition $0 < \eta_{n\lambda_*}^m < \pi/2$, and the function $G_1(x) = \dfrac{\tan(x)}{\cos^{n+1}(x)}$ is strictly increasing in $(0, \pi/2)$. This shows that $q(m,n)$ is strictly increasing with respect to m and strictly decreasing with respect to n. □

Proposition 2.6 shows that the possible Hopf bifurcation values satisfy

$$0 < \tau_{n\lambda}^0 < \tau_{n\lambda}^1 < \cdots < \tau_{n\lambda}^{m_n},$$

and the minimum value $\tau_{n\lambda}^0$ is where the steady state u_λ of Eq. (1.1) loses stability.

Next we verify that the simplicity and transversality conditions for Hopf bifurcation are satisfied.

Lemma 2.7. *For each $\lambda \in (\lambda_*, \lambda^*]$, $n \in \mathbb{N}$ and $0 \le m \le m_n$, we have*

(i) $S^m_{n\lambda} := \int\limits_\Omega \left(1 + \frac{(n+1)\lambda\tau^m_{n\lambda} u_\lambda f'(u_\lambda)}{\left(1+i\theta^m_{n\lambda}\right)^{n+2}}\right)\left(\psi^m_{n\lambda}(x)\right)^2 dx \neq 0;$

(ii) $\mu = \mu\left(\tau^m_{n\lambda}\right) := i\omega^m_{n\lambda}$ *is a simple eigenvalue of* $A_{n\tau}(\lambda)$ *when* $\tau = \tau^m_{n\lambda}$*;*

(iii) $\mathcal{R}e\left(\frac{d\mu}{d\tau}\left(\tau^m_{n\lambda}\right)\right) > 0.$

Proof. For part (i), substituting Eq. (2.24) into the definition of $S^m_{n\lambda}$, we have

$$S^m_{n\lambda} = \int\limits_\Omega \left(\psi^m_{n\lambda}(x)\right)^2 dx - \int\limits_\Omega \frac{i(n+1)\lambda f'(u_\lambda)u_\lambda\theta^m_{n\lambda}}{(\lambda-\lambda_*)\left(1+i\theta^m_{n\lambda}\right)}\left(\psi^m_{n\lambda}(x)\right)^2 dx.$$

Let $\lambda \to \lambda_*$, we obtain that

$$\begin{aligned}\lim_{\lambda\to\lambda_*} S^m_{n\lambda} &= \int\limits_\Omega \phi^2(x)dx - \int\limits_\Omega \frac{i(n+1)\lambda_* f'(0)\alpha_{\lambda_*}\theta^m_{n\lambda_*}}{1+i\theta^m_{n\lambda_*}}\phi^3(x)dx \\ &= \left(\frac{1+(n+2)\left(\theta^m_{n\lambda_*}\right)^2}{1+\left(\theta^m_{n\lambda_*}\right)^2} + i\frac{(n+1)\theta^m_{n\lambda_*}}{1+\left(\theta^m_{n\lambda_*}\right)^2}\right)\int\limits_\Omega \phi^2(x)dx \\ &= 1+(n+1)\sin^2\left(\eta^m_{n\lambda_*}\right) + i\frac{n+1}{2}\sin\left(2\eta^m_{n\lambda_*}\right).\end{aligned} \tag{2.32}$$

This shows that $S^m_{n\lambda} \neq 0$ and it proves part (i). The proof of part (ii) is similar to that of Theorem 3.5 in [36], so we omit it here.

Now we come to the proof of part (iii), by applying the implicit function theorem, we obtain that there exists a neighborhood $O \times D \times H \subset \mathbb{R}\times\mathbb{C}\times X_{\mathbb{C}}$ of $\left(\tau^m_{n\lambda},\ i\omega^m_{n\lambda},\ \psi^m_{n\lambda}\right)$ and a continuous differential function $(\mu, \psi) : O \to D \times H$ such that, for each $\tau \in O$, $\mu(\tau)$ is the only eigenvalue of $A_{n\tau}(\lambda)$ with its associated eigenfunction $\psi(\tau)$ and the following equalities hold:

$$\begin{aligned}&\mu\left(\tau^m_{n\lambda}\right) = i\omega^m_{n\lambda},\quad \psi\left(\tau^m_{n\lambda}\right) = \psi^m_{n\lambda}, \\ &\Lambda(\lambda,\mu(\tau),\tau) = \left[A(\lambda) + \lambda u_\lambda f'(u_\lambda)\textstyle\int_{-\infty}^0 g_n(\tau,-s)e^{\mu(\tau)s}ds - \mu(\tau)\right]\psi(\tau) = 0,\quad \tau \in O.\end{aligned} \tag{2.33}$$

Differentiating Eq. (2.33) with respect to τ at $\tau = \tau^m_{n\lambda}$, we get

$$\begin{aligned}&\frac{d\mu}{d\tau}\left(\tau^m_{n\lambda}\right)\left[1 - \lambda u_\lambda f'(u_\lambda)\int\limits_{-\infty}^0 s g_n(\tau,-s)e^{i\omega^m_{n\lambda}s}ds\right]\psi^m_{n\lambda} \\ &+ \Lambda\left(\lambda, i\omega^m_{n\lambda}, \tau^m_{n\lambda}\right)\frac{d\psi}{d\tau}\left(\tau^m_{n\lambda}\right) + \lambda u_\lambda f'(u_\lambda)\int\limits_{-\infty}^0 \frac{\partial g_n(\tau,-s)}{\partial\tau}e^{i\omega^m_{n\lambda}s}ds\,\psi^m_{n\lambda} = 0.\end{aligned}$$

Multiplying the equation by $\psi_{n\lambda}^m$ and integrating over Ω, we obtain

$$\frac{d\mu}{d\tau}\left(\tau_{n\lambda}^m\right)=\frac{\int_{-\infty}^{0}\frac{\partial g_n(\tau,-s)}{\partial\tau}e^{i\omega_{n\lambda}^m s}ds\int_\Omega \lambda u_\lambda f'(u_\lambda)\left(\psi_{n\lambda}^m\right)^2dx}{\int_\Omega\left(1+\frac{(n+1)\lambda\tau_{n\lambda}^m u_\lambda f'(u_\lambda)}{\left(1+i\theta_{n\lambda}^m\right)^{n+2}}\right)\left(\psi_{n\lambda}^m\right)^2dx}$$

$$=\frac{\frac{-i(n+1)\theta_{n\lambda}^m}{\tau_{n\lambda}^m\left(1+i\theta_{n\lambda}^m\right)^{n+2}}\int_\Omega \lambda u_\lambda f'(u_\lambda)\left(\psi_{n\lambda}^m\right)^2dx}{\int_\Omega\left(1-\frac{(n+1)\lambda\tau_{n\lambda}^m u_\lambda f'(u_\lambda)ih_{n\lambda}^m}{1+i\theta_{n\lambda}^m}\right)\left(\psi_{n\lambda}^m\right)^2dx}$$

$$=\frac{1}{\left|S_{n\lambda}^m\right|^2}\left(\frac{-(n+1)\theta_{n\lambda}^m h_{n\lambda}^m}{\tau_{n\lambda}^m\left(1+i\theta_{n\lambda}^m\right)}\int_\Omega\left(\psi_{n\lambda}^m\right)^2dx\int_\Omega \lambda u_\lambda f'(u_\lambda)\left(\psi_{n\lambda}^m\right)^2dx\right.$$
$$\left.-\,i\theta_{n\lambda}^m\left|\frac{(n+1)h_{n\lambda}^m}{1+i\theta_{n\lambda}^m}\int_\Omega \lambda u_\lambda f'(u_\lambda)\left(\psi_{n\lambda}^m\right)^2dx\right|^2\right).$$

Then, we have

$$\mathcal{R}e\left(\frac{d\mu}{d\tau}\left(\tau_{n\lambda}^m\right)\right)=\frac{-(\lambda-\lambda_*)^2(n+1)\left(h_{n\lambda}^m\right)^2}{\left(1+\left(\theta_{n\lambda}^m\right)^2\right)\left|S_{n\lambda}^m\right|^2}\int_\Omega\left(\psi_{n\lambda}^m\right)^2dx\int_\Omega\frac{\lambda u_\lambda f'(u_\lambda)\left(\psi_{n\lambda}^m\right)^2}{\lambda-\lambda_*}dx.$$

When $\lambda\to\lambda_*$,

$$\lim_{\lambda\to\lambda_*}\int_\Omega\frac{\lambda u_\lambda f'(u_\lambda)\left(\psi_{n\lambda}^m\right)^2}{\lambda-\lambda_*}dx=-\int_\Omega\phi^2(x)dx=-1,$$

so we have

$$\lim_{\lambda\to\lambda_*}\frac{1}{(\lambda-\lambda_*)^2}\mathcal{R}e\left(\frac{d\mu}{d\tau}\left(\tau_{n\lambda}^m\right)\right)=\frac{(n+1)\cos^{2(n+1)}\left(\eta_{n\lambda_*}^m\right)}{\sin^2\left(\eta_{n\lambda_*}^m\right)\left|S_{n\lambda_*}^m\right|^2}>0,$$

where

$$\left|S_{n\lambda_*}^m\right|=\sqrt{\left(1+(n+1)\sin^2\left(\eta_{n\lambda_*}^m\right)\right)^2+\left((n+1)\sin\left(2\eta_{n\lambda_*}^m\right)\right)^2\Big/4}.$$

This implies that $\mathcal{R}e\left(\frac{d\mu}{d\tau}\left(\tau_{n\lambda}^m\right)\right)>0$ for $\lambda\in(\lambda_*,\lambda^*]$. □

Now from Theorem 2.5, Proposition 2.6 and Lemma 2.7, we have the following results for the Hopf bifurcations near the positive steady state u_λ of (1.1) when the delay is near the critical value $\tau = \tau_{n\lambda}^m$.

Theorem 2.8. *Suppose that f satisfies $(H1)$ and $(H2)$, $d > 0$ and g_n is Gamma distribution function with shape parameter $n \in \mathbb{N} \cup \{0\}$. For each $\lambda \in (\lambda_*, \lambda^*]$,*

(i) when $n = 0$ (weak kernel), all the eigenvalues of $A_{n\tau}(\lambda)$ have negative real parts for all $\tau > 0$, and the positive steady state u_λ of (1.1) is locally asymptotically stable for all $\tau > 0$;
(ii) when $n \in \mathbb{N}$, there exists an increasing finite sequence $\tau_{n\lambda}^m > 0$ for $0 \le m \le m_n$ such that all the eigenvalues of $A_{n\tau}(\lambda)$ have negative real parts when $\tau \in \left(0, \tau_{n\lambda}^0\right)$, $A_{n\tau}(\lambda)$ has a pair of purely imaginary eigenvalues $\pm i\omega_{n\lambda}^m$ $\left(\omega_{n\lambda}^m > 0\right)$ when $\tau = \tau_{n\lambda}^m$, $A_{n\tau}(\lambda)$ has $2(m+1)$ eigenvalues with positive real parts when $\tau \in \left(\tau_{n\lambda}^m, \tau_{n\lambda}^{m+1}\right)$, and $A_{n\tau}(\lambda)$ has $2(m_n+1)$ eigenvalues with positive real parts when $\tau \in \left(\tau_{n\lambda}^{m_n}, \infty\right)$;
(iii) for $n \in \mathbb{N}$ and $0 \le m \le m_n$, a Hopf bifurcation occurs at $\tau = \tau_{n\lambda}^m$ for (1.1) so that there is a continuous family of periodic orbits of (1.1) in form of

$$\left\{\left(\tau_n^m(s), u_n^m(x,t,s), T_n^m(s)\right) : s \in (0, \delta_1)\right\},$$

where $u_n^m(x,t,s)$ is a $T_n^m(s)$-periodic solution of (1.1) with $\tau = \tau_n^m(s)$, and $\tau_n^m(0) = \tau_{n\lambda}^m$, $\lim_{s\to 0^+} u^m(x,t,s) = u_\lambda(x)$ and $\lim_{s\to 0^+} T_n^m(s) = 2\pi/\omega_{n\lambda}^m$;
(iv) for $n \in \mathbb{N}$, the positive steady state u_λ of Eq. (1.1) is locally asymptotically stable when $\tau \in \left(0, \tau_{n\lambda}^0\right)$, and it is unstable when $\tau \in \left(\tau_{n\lambda}^0, \infty\right)$.

Proof. Part (ii) and (iii) follow from Theorem 2.5, Proposition 2.6, Lemma 2.7 and the Hopf Bifurcation Theorem [22]. Part (iv) is a straightforward corollary of part (ii). We prove part (i) by modifying an approach in [5,7].

Assume that the conclusion of (i) is not true, then there exist two sequences $\{\lambda^j\}_{j=1}^\infty$ and $\{\tau^j\}_{j=1}^\infty$, satisfying $\lambda^j > \lambda_*$ for $j \ge 1$, $\lim_{j\to\infty} \lambda^j = \lambda_*$ and $\tau^j > 0$, and for each j, the eigenvalue problem

$$\begin{cases} A(\lambda^j)\psi + \lambda^j u_{\lambda^j} f'(u_{\lambda^j}) \displaystyle\int_{-\infty}^0 \frac{1}{\tau^j} e^{s/\tau^j} e^{\mu s} ds \psi = \mu\psi, & x \in \Omega, \\ \psi(x) = 0, & x \in \partial\Omega, \end{cases} \tag{2.34}$$

has an eigenvalue μ_{λ^j} with nonnegative real part and corresponding eigenfunction ψ_{λ^j} satisfying $\|\psi_{\lambda^j}\|_{Y_\mathbb{C}} = 1$. Then, we write ψ_{λ^j} as $\psi_{\lambda^j} = c_{\lambda^j} u_{\lambda^j} + \phi_{\lambda^j}$, where $c_{\lambda^j} \in \mathbb{C}$ and $c_{\lambda^j} = \langle u_{\lambda^j}, \psi_{\lambda^j}\rangle / \langle u_{\lambda^j}, u_{\lambda^j}\rangle$. Here u_{λ^j} is the positive steady state of Eq. (1.1) for $\lambda = \lambda^j$, and $\phi_{\lambda^j} \in X_\mathbb{C}$ satisfies $\langle \phi_{\lambda^j}, u_{\lambda^j}\rangle = 0$.

Substituting $\psi = \psi_{\lambda^j} = c_{\lambda^j} u_{\lambda^j} + \phi_{\lambda^j}$ and $\mu = \mu_{\lambda^j}$ into Eq. (2.34), multiplying by ψ_{λ^j} and integrating, we have

$$\langle A(\lambda^j)\phi_{\lambda^j}, \phi_{\lambda^j}\rangle = \mu_{\lambda^j} - \lambda^j \left\langle u_{\lambda^j} f'(u_{\lambda^j}) \int_{-\infty}^{0} \frac{1}{\tau^j} e^{s/\tau^j} e^{\mu_{\lambda^j} s} ds \psi_{\lambda^j}, \psi_{\lambda^j} \right\rangle, \tag{2.35}$$

from that $\langle A(\lambda^j)\phi_{\lambda^j}, u_{\lambda^j}\rangle = \langle \phi_{\lambda^j}, A(\lambda^j)u_{\lambda^j}\rangle$ and $A(\lambda^j)u_{\lambda^j} = 0$. Define

$$D_j = \lambda^j \left\langle u_{\lambda^j} f'(u_{\lambda^j}) \int_{-\infty}^{0} \frac{1}{\tau^j} e^{s/\tau^j} e^{\mu_{\lambda^j} s} ds \psi_{\lambda^j}, \psi_{\lambda^j} \right\rangle,$$

then we can obtain that

$$|D_j| \le \lambda^j \|u_{\lambda^j}\|_\infty \frac{|f'(u_{\lambda^j})|}{1+\mu_{\lambda^j}\tau^j} \to 0, \text{ as } j \to \infty. \tag{2.36}$$

From Eq. (2.35) and the fact that $\langle A(\lambda^j)\phi_{\lambda^j}, \phi_{\lambda^j}\rangle < 0$, it can be inferred that

$$0 \le \mathcal{R}e(\mu_{\lambda^j}) \le |D_j|, \quad 0 \le \left|\mathcal{I}m(\mu_{\lambda^j})\right| \le |D_j|,$$

hence by (2.36), we have

$$\lim_{j\to\infty} \mathcal{R}e(\mu_{\lambda^j}) = \lim_{j\to\infty} \left|\mathcal{I}m(\mu_{\lambda^j})\right| = 0.$$

From (2.35) and using similar argument as in the proof of Lemma 2.3 part (i), we have

$$\left|D_j\right| + |\mu_{\lambda^j}| \ge |\langle A(\lambda^j)\phi_{\lambda^j}, \phi_{\lambda^j}\rangle| \ge |\lambda_2(\lambda^j)| \cdot \|\phi_{\lambda^j}\|^2_{Y_{\mathbb{C}}}, \tag{2.37}$$

where $\lambda_2(\lambda^j)$ is the second eigenvalue of $A(\lambda^j)$. When $j \to \infty$, both $|D_j|$ and $|\mu_{\lambda^j}|$ go to zero because of $\lim_{j\to\infty} \|u_{\lambda^j}\|_\infty = 0$, so the inequality (2.37) implies that $\lim\limits_{j\to\infty} \|\phi_{\lambda^j}\|_{Y_{\mathbb{C}}} = 0$.

Since $\psi_{\lambda^j} = c_{\lambda^j} u_{\lambda^j} + \phi_{\lambda^j}$ and $\|\psi_{\lambda^j}\|_{Y_{\mathbb{C}}} = 1$, then we obtain

$$\lim_{n\to\infty} |c_{\lambda^j}|(\lambda^j - \lambda_*) \lim_{j\to\infty} \left\| \frac{u_{\lambda^j}}{\lambda^j - \lambda_*} \right\|_{Y_{\mathbb{C}}} = 1,$$

and hence $\lim\limits_{j\to\infty} |c_{\lambda^j}|(\lambda^j - \lambda_*) = \dfrac{1}{\alpha_{\lambda_*}} > 0$. Now we calculate that

$$\begin{aligned} \frac{D_j}{\lambda^j - \lambda_*} &= \frac{1}{\lambda^j - \lambda_*} \lambda^j \left\langle u_{\lambda^j} f'(u_{\lambda^j}) \int_{-\infty}^{0} \frac{1}{\tau^j} e^{s/\tau^j} e^{\mu_{\lambda^j} s} ds (c_{\lambda^j} u_{\lambda^j} + \phi_{\lambda^j}), (c_{\lambda^j} u_{\lambda^j} + \phi_{\lambda^j}) \right\rangle \\ &= \frac{\lambda^j}{1+\mu_{\lambda^j}\tau^j} (J_1 + J_2 + J_3 + J_4), \end{aligned} \tag{2.38}$$

where

$$J_1 = |c_{\lambda^j}|^2(\lambda^j - \lambda_*)^2 \int\limits_\Omega \frac{u_{\lambda^j}^3(x) f'(u_{\lambda^j}(x))}{(\lambda^j - \lambda_*)^3} dx,$$

$$J_2 = c_{\lambda^j}(\lambda^j - \lambda_*) \int\limits_\Omega \frac{u_{\lambda^j}^2(x) f'(u_{\lambda^j}(x)) \overline{\phi_{\lambda^j}}(x)}{(\lambda^j - \lambda_*)^2} dx,$$

$$J_3 = \overline{c_{\lambda^j}}(\lambda^j - \lambda_*) \int\limits_\Omega \frac{u_{\lambda^j}^2(x) f'(u_{\lambda^j}(x)) \phi_{\lambda^j}(x)}{(\lambda^j - \lambda_*)^2} dx,$$

$$J_4 = \int\limits_\Omega \frac{\phi_{\lambda^j}(x) \overline{\phi_{\lambda^j}}(x) u_{\lambda^j}(x)}{\lambda^j - \lambda_*} dx.$$

Since $\lim\limits_{j\to\infty} \|\phi_{\lambda^j}\|_{Y_{\mathbb{C}}} = 0$, then $\lim\limits_{j\to\infty} \|\phi_{\lambda^j}\|_{L^1} = 0$, so

$$\lim_{j\to\infty} J_1 = \alpha_{\lambda_*} f'(0) \int\limits_\Omega \phi^3 dx = -\frac{1}{\lambda_*}, \quad \lim_{j\to\infty} J_i = 0, i = 2, 3, 4.$$

Therefore, by letting $\mu_{\lambda^j} = \mu_{\lambda^j}^R + i\mu_{\lambda^j}^I$, when $j \to \infty$,

$$\frac{D_j}{\lambda^j - \lambda_*} = -\frac{1 + o(1)}{1 + \mu_{\lambda^j}\tau^j} = -\frac{1 + o(1)}{1 + \mu_{\lambda^j}^R \tau^j + i\mu_{\lambda^j}^I \tau^j} = \frac{-(1 + \mu_{\lambda^j}^R \tau^j) + i\mu_{\lambda^j}^I \tau^j + o(1)}{(1 + \mu_{\lambda^j}^R \tau^j)^2 + (\mu_{\lambda^j}^I \tau^j)^2}.$$

So, $\mathcal{R}e(D_j) < 0$ which implies that

$$\mathcal{R}e(\mu_{\lambda^j}) = \langle A(\lambda^j)\phi_{\lambda^j}, \phi_{\lambda^j} \rangle + \mathcal{R}e(D_j) < 0. \tag{2.39}$$

That is a contradiction with $\mathcal{R}e(\mu_{\lambda^j}) \geq 0$ for $j \geq 1$. Therefore, all the eigenvalues of $A_{0\tau}(\lambda)$ have negative real parts for all $\tau > 0$, which implies that the steady state of Eq. (1.1) is locally asymptotically stable when $n = 0$ for any $\tau > 0$. □

3. Stability and direction of Hopf bifurcation

In the previous section, when $n \in \mathbb{N}$, we obtain conditions under which system (1.1) undergoes a Hopf bifurcation near the positive steady state u_λ at $\tau = \tau_{n\lambda}^m$. In this section, we apply the theory in [10,11,38] to compute the normal form of the Hopf bifurcation to determine the bifurcation direction and the stability of bifurcating periodic orbits. Firstly, by letting $U(t) = u(\cdot, t) - u_\lambda$ and $U_t = U(t + a) \in \mathcal{C} = C((-\infty, 0], Y_{\mathbb{C}})$, $\alpha = \tau - \tau_{n\lambda}^m$ and $t \to t/\tau$, for each $\lambda \in (\lambda_*, \lambda^*]$, we translate the steady state and parameter τ to the origin, then $\alpha = 0$ is the Hopf bifurcation value. For the simplicity of writing, we give the following new notations:

$$\tau_{n\lambda} := \tau_{n\lambda}^m,\ \psi_{n\lambda} := \psi_{n\lambda}^m,\ \theta_{n\lambda} := \theta_{n\lambda}^m,\ \eta_{n\lambda} := \eta_{n\lambda}^m,\ \omega_{n\lambda} := \omega_{n\lambda}^m,\ S_{n\lambda} := S_{n\lambda}^m, \tag{3.1}$$

for $\lambda \in [\lambda_*, \lambda^*]$, where $\theta_{n\lambda}^m$, $\tau_{n\lambda}^m$, $\psi_{n\lambda}^m$, $\omega_{n\lambda}^m$ are defined in Theorem 2.5, $S_{n\lambda}^m$ is defined in Lemma 2.7. Also we recall the following limits for the subsequent computation:

$$\lim_{\lambda\to\lambda_*}\frac{u_\lambda}{\lambda-\lambda_*}=\alpha_{\lambda_*}\phi,\quad \lim_{\lambda\to\lambda_*}\psi_{n\lambda}=\phi,$$
$$\lim_{\lambda\to\lambda_*}S_{n\lambda}=S_{n\lambda_*}:=1+(n+1)\sin^2\left(\eta_{n\lambda_*}\right)+i\frac{n+1}{2}\sin\left(2\eta_{n\lambda_*}\right) \tag{3.2}$$

with $u_\lambda, \alpha_{\lambda_*}$ defined in Lemma 2.1 and ϕ being the eigenfunction of $-d\Delta$ with eigenvalue λ_* and norm $\int_\Omega \phi^2 dx = 1$. For the simplicity of writing, we define

$$\mathcal{T}(\mathcal{F})=\int_{-\infty}^{0}\frac{(-s)^n e^s\mathcal{F}(s)}{n!}ds,$$

where $\mathcal{F}(s)$ can be a function or a vector-valued function defined on $(-\infty, 0]$. Then, we rewrite Eq. (1.1) as follows:

$$\frac{dU(t)}{dt}=\tau_{n\lambda}d\Delta U_t(0)+L_0(U_t)+F(U_t,\alpha), \tag{3.3}$$

where

$$\begin{aligned}
L_0(U_t)=&\lambda\tau_{n\lambda}\left(f(u_\lambda)U(t)+u_\lambda f'(u_\lambda)\mathcal{T}(U_t)\right),\\
F(U_t,\alpha)=&\alpha\left((d\Delta+\lambda f(u_\lambda))U(t)+\lambda u_\lambda f'(u_\lambda)\mathcal{T}(U_t)\right)+\lambda(\alpha+\tau_{n\lambda})f'(u_\lambda)U(t)\mathcal{T}(U_t)\\
&+\lambda(\alpha+\tau_{n\lambda})(U(t)+u_\lambda)\left[f\left(\mathcal{T}(U_t)+u_\lambda\right)-f(u_\lambda)-f'(u_\lambda)\mathcal{T}(U_t)\right].
\end{aligned}$$

For the convenience of computation, we rewrite $F(U_t,\alpha)$ as

$$F(U_t,\alpha)=\frac{1}{2!}F_2(U_t),\alpha)+\frac{1}{3!}F_3(U_t,\alpha)+h.o.t., \tag{3.4}$$

where $h.o.t.$ stands for "high order terms", and

$$\begin{aligned}
F_2(U_t,\alpha)=&2!\alpha\left((d\Delta+\lambda f(u_\lambda))U(t)+\lambda u_\lambda f'(u_\lambda)\mathcal{T}(U_t)\right)\\
&+2!\lambda\tau_{n\lambda}\left(f'(u_\lambda)U(t)\mathcal{T}(U_t)+\frac{u_\lambda f''(u_\lambda)}{2!}\left(\mathcal{T}(U_t)\right)^2\right),\\
F_3(U_t,\alpha)=&3!\left[\alpha\lambda f'(u_\lambda)U(t)\mathcal{T}(U_t)+\frac{1}{2!}\lambda(\tau_{n\lambda}U(t)+\alpha u_\lambda)f''(u_\lambda)\left(\mathcal{T}(U_t)\right)^2\right.\\
&\left.+\frac{1}{3!}\lambda\tau_{n\lambda}u_\lambda f'''(u_\lambda)\left(\mathcal{T}(U_t)\right)^3\right].
\end{aligned}$$

The linearized equation of (3.3) is

$$\frac{dU(t)}{dt}=\tau_{n\lambda}d\Delta U(t)+L_0(U_t). \tag{3.5}$$

Denote the infinitesimal generator of Eq. (3.5) by $\mathcal{A}_\tau$, we have $\mathcal{A}_\tau \psi = \dot{\psi}$, with its domain

$$\mathfrak{D}(\mathcal{A}_\tau) = \left\{ \psi \in \mathcal{C}_{\mathbb{C}} \bigcap \mathcal{C}^1_{\mathbb{C}} : \dot{\psi}(0) = \tau_{n\lambda} \left(A(\lambda)\psi(0) + \lambda u_\lambda f'(u_\lambda)\mathcal{T}(\psi) \right) \right\}.$$

Then Eq. (3.3) can be rewritten in the abstract form:

$$\frac{dU(t)}{dt} = \mathcal{A}_\tau U_t + \chi_0 F(U_t, \alpha) \tag{3.6}$$

with

$$\chi_0(a) = \begin{cases} 0, & a \in (-\infty, 0), \\ I, & a = 0. \end{cases}$$

For $\psi \in \mathcal{C}, \varphi \in \mathcal{C}^* = C((-\infty, 0], X^*)$, we introduce a bilinear functional $\langle\langle \cdot, \cdot \rangle\rangle$ defined by

$$\langle\langle \varphi, \psi \rangle\rangle = \langle \varphi(0), \psi(0) \rangle - \int_{-\infty}^{0} \int_{0}^{a} \langle \varphi(\xi - a), d\delta(a)\psi(\xi) \rangle d\xi, \tag{3.7}$$

where $\langle \cdot, \cdot \rangle$ is the formal duality between X and X^* and $\delta(a)$ is a bounded variation function such that

$$L_0\psi = \int_{-\infty}^{0} d\delta(a)\psi(a), \ \psi \in \mathcal{C}.$$

By Theorem 2.8, $\pm i\omega_{n\lambda}\tau_{n\lambda}$ are a pair of simple purely imaginary eigenvalues of $\mathcal{A}_\tau$ and we denote $\omega_{n\lambda}\tau_{n\lambda}$ as $\theta_{n\lambda}$. Let $\Lambda = \{i\theta_{n\lambda}, -i\theta_{n\lambda}\}$, then the eigenspace associated with Λ is

$$P = \text{Span}\{\Phi\}, \quad \Phi(a) = \left(\psi_{n\lambda} e^{i\theta_{n\lambda} a}, \bar{\psi}_{n\lambda} e^{-i\theta_{n\lambda} a} \right), \ a \in (-\infty, 0].$$

Then $\mathcal{C}$ can be decomposed as $\mathcal{C} = P \bigoplus Q$ with

$$Q = \{\varphi \in \mathcal{C} : \langle\langle \psi, \varphi \rangle\rangle = 0, \ \text{for all } \psi \in P^*\},$$

where P^* is the generalized eigenspace of the adjoint equation associated with Λ and

$$P^* = \text{Span}\{\Psi\}, \quad \Psi(s) = \left(\frac{1}{S_{n\lambda}} \psi_{n\lambda} e^{-i\theta_{n\lambda} s}, \frac{1}{\bar{S}_{n\lambda}} \bar{\psi}_{n\lambda} e^{i\theta_{n\lambda} s} \right)^T, \ s \in [0, +\infty).$$

In order to apply the method in [10] to compute the normal form, we consider the enlarged phase space

$$\mathcal{BC} = \{\psi : (-\infty, 0] \to X : \psi \text{ is continuous on } (-\infty, 0) \text{ with a positive jump discontinuity at } 0\},$$

and $\mathcal{BC}$ can be decomposed as $\mathcal{BC} = P \bigoplus \mathrm{Ker}(\pi)$ with π is the projection from $\mathcal{BC}$ to P and defined by

$$\pi(\varphi + \chi_0 y) = \Phi(\langle\langle \Psi, \varphi\rangle\rangle + \langle \Psi(0), y\rangle),\ \varphi \in \mathcal{C},\ \ y \in X.$$

The extension of $\mathcal{A}_\tau : \mathcal{C}_0^1 \to \mathcal{BC}$ is

$$\mathcal{A}_\tau v = \dot{v}(a) + \chi_0[\tau_{n\lambda} d\Delta v(0) + L_0 v - \dot{v}(0)]$$

with $v(a) = U_t(a)$ and $\mathcal{C}_0^1 = \{\varphi \in \mathcal{C} : \dot{\varphi} \in \mathcal{C},\ \ \varphi(0) \in X\} \subset \mathcal{BC}$. So Eq. (3.6) can be rewritten as

$$\frac{dv(0)}{dt} = \mathcal{A}_\tau v + \chi_0 F(v, \alpha). \tag{3.8}$$

By letting $v = \Phi z(t) + y(t)$ and following the process of [10], Eq. (3.8) can be decomposed as

$$\begin{cases} \dot{z}(t) = Bz(t) + \langle \Psi(0), F(\Phi z(t) + y(t), \alpha)\rangle, \\ \dot{y}(t) = \mathcal{A}_{\tau 1} y(t) + (I - \pi)\chi_0 F(\Phi z(t) + y(t), \alpha), \end{cases} \tag{3.9}$$

where $z(t) \in \mathcal{C}^2,\ \ y(t) \in Q_0^1 = Q \cap \mathcal{C}_0^1$,

$$B = \begin{pmatrix} i\theta_{n\lambda} & 0 \\ 0 & -i\theta_{n\lambda} \end{pmatrix},$$

and $\mathcal{A}_{\tau 1} : Q_0^1 \to \mathrm{Ker}(\pi)$ with $\mathcal{A}_{\tau 1} v = \mathcal{A}_\tau v,\ \ v \in Q_0^1$.

We write the Taylor expansion of the high order terms of Eq. (3.9) as follows:

$$\begin{cases} \langle \Psi(0), F(\Phi z(t) + y(t), \alpha)\rangle = \dfrac{1}{2!} f_2^1(z, y, \alpha) + \dfrac{1}{3!} f_3^1(z, y, \alpha) + h.o.t., \\ (I - \pi)\chi_0 F(\Phi z(t) + y(t), \alpha) = \dfrac{1}{2!} f_2^2(z, y, \alpha) + \dfrac{1}{3!} f_3^2(z, y, \alpha) + h.o.t., \end{cases}$$

where

$$f_j^1(z, y, \alpha) = \langle \Psi(0), F_j(\Phi z + y, \alpha)\rangle,\ \ f_j^2(z, y, \alpha) = (I - \pi)\chi_0 F_j(\Phi z + y, \alpha) \tag{3.10}$$

are homogeneous polynomials in (z, y, α) of degree j with $j = 2, 3$.

Then, by using the transformation $(z, y) = (\bar{z}, \bar{y}) + \frac{1}{j!}(U_j^1(\bar{z}), U_j^2(\bar{z}))$, we obtain the normal form of Eq. (3.9)

$$\begin{cases} \dot{z}(t) = Bz(t) + \dfrac{1}{2!} g_2^1(z, y, \alpha) + \dfrac{1}{3!} g_3^1(z, y, \alpha) + h.o.t., \\ \dot{y}(t) = \mathcal{A}_{\tau 1} y(t) + \dfrac{1}{2!} g_2^2(z, y, \alpha) + \dfrac{1}{3!} g_3^2(z, y, \alpha) + h.o.t., \end{cases} \tag{3.11}$$

where

$$\begin{cases} g_j^1(z,y,\alpha) = f_j^1(z,y,\alpha) - [D_z U_j^1 Bz - BU_j^1(z)], \\ g_j^2(z,y,\alpha) = f_j^2(z,y,\alpha) - [D_z U_j^2 Bz - \mathcal{A}_{\tau 1}(U_j^2(z))]. \end{cases}$$

Define

$$\begin{aligned} M_j^1(p)(z,\alpha) &= D_z p(z,\alpha)Bz - Bp(z,\alpha), \ \ j \geq 2, \\ M_j^2(p)(z,\alpha) &= D_z p(z,\alpha)Bz - \mathcal{A}_{\tau 1}(p(z,\alpha)), \ \ j \geq 2, \end{aligned} \tag{3.12}$$

then we have

$$\begin{aligned} g_j^1(z,y,\alpha) &= Proj_{\mathrm{Ker}(M_2^1)} f_j^1(z,y,\alpha), \\ g_j^2(z,y,\alpha) &= Proj_{\mathrm{Ker}(M_2^2)} f_j^2(z,y,\alpha). \end{aligned} \tag{3.13}$$

By the results of Faria [10], $g_j^2 = 0$ for any $j \geq 2$. So on the center manifold, we have $y = 0$ and Eq. (3.11) has the following form:

$$\dot{z}(t) = Bz(t) + \frac{1}{2!} g_2^1(z,0,\alpha) + \frac{1}{3!} g_3^1(z,0,\alpha) + h.o.t.. \tag{3.14}$$

And we can calculate that

$$\mathrm{Ker}(M_2^1) = \mathrm{Span}\left\{ \begin{pmatrix} z_1\alpha \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ z_2\alpha \end{pmatrix} \right\},$$

$$\mathrm{Ker}(M_3^1) = \mathrm{Span}\left\{ \begin{pmatrix} z_1^2 z_2 \\ 0 \end{pmatrix}, \begin{pmatrix} z_1\alpha^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ z_1 z_2^2 \end{pmatrix}, \begin{pmatrix} 0 \\ z_2\alpha^2 \end{pmatrix} \right\}.$$

First we compute $g_2^1(z,0,\alpha)$. From (3.13), we need to compute $f_2^1(z,0,\alpha)$ which is defined in Eq. (3.10). According to its definition, we have

$$\begin{aligned} &\frac{1}{2!} f_2^1(z,0,\alpha) = \frac{1}{2!} \langle \Psi(0), F_2(\Phi z + y, \alpha) \rangle \big|_{y=0} \\ &= \Big\langle \Psi(0), \alpha \left[d\Delta\Phi(0)z + \lambda f(u_\lambda)\Phi(0)z + \lambda u_\lambda f'(u_\lambda)\mathcal{T}(\Phi)z \right] \\ &\quad = \lambda\tau_{n\lambda} f''(u_\lambda)(\Phi(0)z)(\mathcal{T}(\Phi)z) + \frac{1}{2}\lambda\tau_{n\lambda} u_\lambda f''(u_\lambda)\left(\mathcal{T}(\Phi)z\right)^2 \Big\rangle \\ &= \left\langle \Psi(0), \frac{\alpha\dot{\Phi}(0)z}{\tau_{n\lambda}} \right\rangle + \lambda\tau_{n\lambda} \left\langle \Psi(0), f'(u_\lambda)(\Phi(0)z)(\mathcal{T}(\Phi)z) \right\rangle \\ &\quad + \frac{1}{2}\lambda\tau_{n\lambda} \left\langle \Psi(0), u_\lambda f''(u_\lambda)\left(\mathcal{T}(\Phi)z\right)^2 \right\rangle \end{aligned}$$

$$= \begin{pmatrix} \frac{i\omega_{n\lambda}}{S_{n\lambda}} \langle \psi_{n\lambda}, \psi_{n\lambda} \rangle z_1 \alpha \\ \frac{-i\omega_{n\lambda}}{\bar{S}_{n\lambda}} \langle \bar{\psi}_{n\lambda}, \bar{\psi}_{n\lambda} \rangle z_2 \alpha \end{pmatrix} + \lambda \tau_{n\lambda} \left\langle \Psi(0), f'(u_\lambda)(\Phi(0)z)(\mathcal{T}(\Phi)z) \right\rangle$$
$$+ \frac{1}{2} \lambda \tau_{n\lambda} \left\langle \Psi(0), u_\lambda f''(u_\lambda)\, (\mathcal{T}(\Phi)z)^2 \right\rangle,$$

so by the definition of $\operatorname{Ker}(M_2^1)$ we can get

$$\frac{1}{2!} g_2^1(z, 0, \alpha) = Proj_{\operatorname{Ker}(M_2^1)} \frac{1}{2!} f_2^1(z, 0, \alpha) = \begin{pmatrix} \frac{i\omega_{n\lambda}}{S_{n\lambda}} \langle \psi_{n\lambda}, \psi_{n\lambda} \rangle z_1 \alpha \\ \frac{-i\omega_{n\lambda}}{\bar{S}_{n\lambda}} \langle \bar{\psi}_{n\lambda}, \bar{\psi}_{n\lambda} \rangle z_2 \alpha \end{pmatrix} \triangleq \begin{pmatrix} A_1 z_1 \alpha \\ \bar{A}_1 z_2 \alpha \end{pmatrix} \tag{3.15}$$

with

$$A_1 = \frac{i\omega_{n\lambda}}{S_{n\lambda}} \langle \psi_{n\lambda}, \psi_{n\lambda} \rangle. \tag{3.16}$$

Let $K_1 := \mathcal{R}e(A_1)$, then from (3.2), we have

$$\lim_{\lambda \to \lambda_*} \tau_{n\lambda} K_1 = \theta_{n\lambda_*} \mathcal{R}e \left(\frac{i}{S_{n\lambda_*}} \right) = \frac{n+1}{2} \sin\left(2\eta_{n\lambda_*}\right) \theta_{n\lambda_*} = (n+1) \sin^2\left(\eta_{n\lambda_*}\right) > 0. \tag{3.17}$$

Note that K_1 is an important number in determining the direction of Hopf bifurcation and $K_1 = O(|\lambda - \lambda_*|)$.

Next, we calculate $g_3^1(z, 0, \alpha)$. By Eq. (3.13) and the definition of $\operatorname{Ker}(M_3^1)$, we know that

$$\frac{1}{3!} g_3^1(z, 0, \alpha) = Proj_{\operatorname{Ker}(M_3^1)} \frac{1}{3!} \tilde{f}_3^1(z, 0, \alpha) = Proj_S \frac{1}{3!} \tilde{f}_3^1(z, 0, 0) + O(|z|\alpha^2) \tag{3.18}$$

with

$$S = \operatorname{Span} \left\{ \begin{pmatrix} z_1^2 z_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ z_1 z_2^2 \end{pmatrix} \right\},$$

and $\tilde{f}_3^1(z, 0, 0)$ is given in [10] as follows,

$$\tilde{f}_3^1(z, 0, 0) = f_3^1(z, 0, 0) + \frac{3}{2} [D_z f_2^1(z, 0, 0) U_2^1(z, 0) - D_z U_2^1(z, 0) g_2^1(z, 0, 0)$$
$$+ D_y f_2^1(z, 0, 0) U_2^2(z, 0)],$$

where

$$U_2^1(z, 0) = U_2^1(z, \alpha)\big|_{\alpha=0} = (M_2^1)^{-1} Proj_{\operatorname{Ker}(M_2^1)} f_2^1(z, 0, 0), \tag{3.19}$$

and $U_2^2(z, 0)$ is determined by

$$(M_2^2 U_2^2)(z, 0) = f_2^2(z, 0, 0). \tag{3.20}$$

By Eq. (3.15), we know that $g_2^1(z,0,0)=0$. Therefore, $\tilde{f}_3^1(z,0,0)$ has there terms left: $f_3^1(z,0,0)$, $D_z f_2^1(z,0,0)U_2^1(z,0)$ and $D_y f_2^1(z,0,0)U_2^2(z,0)$ which will be computed one by one in the following.

By the definition of $f_3^1(z,0,0)$ which is in Eq. (3.10), we can get

$$\begin{aligned}&\frac{1}{3!}f_3^1(z,0,0)=\langle\Psi(0),\frac{1}{3!}F_3(\Phi z,0)\rangle\\ &=\left\langle\Psi(0),\frac{1}{2}\lambda\tau_{n\lambda}f''(u_\lambda)\Phi(0)z\,(\mathcal{T}(\Phi)z)^2+\frac{1}{3!}\lambda\tau_{n\lambda}u_\lambda f'''(u_\lambda)\,(\mathcal{T}(\Phi)z)^3\right\rangle=(E,\bar{E})^T,\end{aligned}$$

where

$$\begin{aligned}E=\Big\langle&\frac{1}{S_{n\lambda}}\psi_{n\lambda},\frac{1}{2}\lambda\tau_{n\lambda}f''(u_\lambda)(\psi_{n\lambda}z_1+\bar{\psi}_{n\lambda}z_2)\left(\rho_{n\lambda}\psi_{n\lambda}z_1+\bar{\rho}_{n\lambda}\bar{\psi}_{n\lambda}z_2\right)^2\\ &+\frac{1}{3!}\lambda\tau_{n\lambda}u_\lambda f'''(u_\lambda)\left(\rho_{n\lambda}\psi_{n\lambda}z_1+\bar{\rho}_{n\lambda}\bar{\psi}_{n\lambda}z_2\right)^3\Big\rangle\end{aligned}$$

with $\rho_{n\lambda}:=\dfrac{1}{(1+i\theta_{n\lambda})^{n+1}}$. Therefore, we can get

$$Projs\left(\frac{1}{3!}f_3^1(z,0,0)\right)=\begin{pmatrix}C_0z_1^2z_2\\ \bar{C}_0z_1z_2^2\end{pmatrix}$$

with

$$\begin{aligned}C_0=\frac{\lambda\tau_{n\lambda}}{S_{n\lambda}}\Big(&\langle\psi_{n\lambda},f''(u_\lambda)\psi_{n\lambda}|\psi_{n\lambda}|^2\rangle|\rho_{n\lambda}|^2+\frac{1}{2}\langle\psi_{n\lambda},f''(u_\lambda)\bar{\psi}_{n\lambda}\psi_{n\lambda}^2\rangle\rho_{n\lambda}^2\\ &+\frac{1}{2}\langle\psi_{n\lambda},u_\lambda f'''(u_\lambda)\psi_{n\lambda}^2\bar{\psi}_{n\lambda}\rangle\rho_{n\lambda}^2\bar{\rho}_{n\lambda}\Big).\end{aligned}\tag{3.21}$$

When $\lambda\to\lambda_*$, according to Eq. (3.2), we have

$$\lim_{\lambda\to\lambda_*}\frac{C_0}{\tau_{n\lambda}}=\frac{\lambda_*f''(0)h_{n\lambda_*}^2\int_\Omega\phi^4dx}{2S_{n\lambda_*}},$$

which means that $C_0=O(|\lambda-\lambda_*|^{-1})$, thus we have

$$\lim_{\lambda\to\lambda_*}\frac{C_0}{\tau_{n\lambda}^2}=0.\tag{3.22}$$

Next we calculate $D_z f_2^1(z,0,0)U_2^1(z,0)$. First of all, we have

$$f_2^1(z,0,0)=\langle\Psi(0),F_2^1(\Phi z,0)\rangle$$

$$=\lambda\tau_{n\lambda}\left\langle\Psi(0),2f'(u_\lambda)(\Phi(0)z)(\mathcal{T}(\Phi)z)+u_\lambda f'(u_\lambda)\,(\mathcal{T}(\Phi)z)^2\right\rangle$$
$$=\lambda\tau_{n\lambda}(H,\bar{H})^T$$

with

$$\begin{aligned}H=\frac{2}{S_{n\lambda}}\Bigg[&\left(\rho_{n\lambda}\langle\psi_{n\lambda},f'(u_\lambda)\psi_{n\lambda}^2\rangle+\frac{1}{2}\rho_{n\lambda}^2\langle\psi_{n\lambda},u_\lambda f''(u_\lambda)\psi_{n\lambda}^2\rangle\right)z_1^2\\&+\left(2\mathcal{R}e\,(\rho_{n\lambda})\,\langle\psi_{n\lambda},f'(u_\lambda)|\psi_{n\lambda}|^2\rangle+|\rho_{n\lambda}|^2\langle\psi_{n\lambda},u_\lambda f''(u_\lambda)|\psi_{n\lambda}|^2\rangle\right)z_1z_2\\&+\left(\bar{\rho}_{n\lambda}\langle\psi_{n\lambda},f'(u_\lambda)\bar{\psi}_{n\lambda}^2\rangle+\frac{1}{2}\bar{\rho}_{n\lambda}^2\langle\psi_{n\lambda},u_\lambda f'(u_\lambda)\bar{\psi}_{n\lambda}^2\rangle\right)z_2^2\Bigg].\end{aligned}$$

Hence, by the definition of $U_2^1(z,0)$ and M_2^1 which are in Eqs. (3.19) and (3.12) respectively, we obtain

$$U_2^1(z,0)=(M_2^1)^{-1}f_2^1(z,0,0)=\frac{2\lambda\tau_{n\lambda}}{i\theta_{n\lambda}}(U_1,U_2)^T$$

with

$$\begin{aligned}U_1=\frac{2}{S_{n\lambda}}\Bigg[&\left(\rho_{n\lambda}\langle\psi_{n\lambda},f'(u_\lambda)\psi_{n\lambda}^2\rangle+\frac{1}{2}\rho_{n\lambda}^2\langle\psi_{n\lambda},u_\lambda f''(u_\lambda)\psi_{n\lambda}^2\rangle\right)z_1^2\\&-\left(2\mathcal{R}e\,(\rho_{n\lambda})\,\langle\psi_{n\lambda},f'(u_\lambda)|\psi_{n\lambda}|^2\rangle+|\rho_{n\lambda}|^2\langle\psi_{n\lambda},u_\lambda f''(u_\lambda)|\psi_{n\lambda}|^2\rangle\right)z_1z_2\\&-\frac{1}{3}\left(\bar{\rho}_{n\lambda}\langle\psi_{n\lambda},f'(u_\lambda)\bar{\psi}_{n\lambda}^2\rangle+\frac{1}{2}\bar{\rho}_{n\lambda}^2\langle\psi_{n\lambda},u_\lambda f'(u_\lambda)\bar{\psi}_{n\lambda}^2\rangle\right)z_2^2\Bigg],\\U_2=\frac{2}{\bar{S}_{n\lambda}}\Bigg[&\frac{1}{3}\left(\rho_{n\lambda}\langle\bar{\psi}_{n\lambda},f'(u_\lambda)\psi_{n\lambda}^2\rangle+\frac{1}{2}\rho_{n\lambda}^2\langle\bar{\psi}_{n\lambda},u_\lambda f''(u_\lambda)\psi_{n\lambda}^2\rangle\right)z_1^2\\&+\left(2\mathcal{R}e\,(\rho_{n\lambda})\,\langle\bar{\psi}_{n\lambda},f'(u_\lambda)|\psi_{n\lambda}|^2\rangle+|\rho_{n\lambda}|^2\langle\bar{\psi}_{n\lambda},u_\lambda f''(u_\lambda)|\psi_{n\lambda}|^2\rangle\right)z_1z_2\\&-\left(\bar{\rho}_{n\lambda}\langle\bar{\psi}_{n\lambda},f'(u_\lambda)\bar{\psi}_{n\lambda}^2\rangle+\frac{1}{2}\bar{\rho}_{n\lambda}^2\langle\bar{\psi}_{n\lambda},u_\lambda f'(u_\lambda)\bar{\psi}_{n\lambda}^2\rangle\right)z_2^2\Bigg].\end{aligned}$$

Therefore,

$$Proj_S(D_zf_2^1(z,0,0)U_2^1(z,0))=\begin{pmatrix}C_1z_1^2z_2\\\bar{C}_1z_1z_2^2\end{pmatrix},$$

where

$$\begin{aligned}C_1=\frac{8\lambda^2\tau_{n\lambda}^2}{i\theta_{n\lambda}}\Bigg[&\frac{\mathcal{R}e\,(\rho_{n\lambda})}{S_{n\lambda}^2}\langle\psi_{n\lambda},f'(u_\lambda)|\psi_{n\lambda}|^2\rangle\\&\times\left(\rho_{n\lambda}\langle\psi_{n\lambda},f'(u_\lambda)\psi_{n\lambda}^2\rangle+\frac{\rho_{n\lambda}^2}{2}\langle\psi_{n\lambda},u_\lambda f''(u_\lambda)\psi_{n\lambda}^2\rangle\right)\end{aligned}$$

$$- \frac{\rho_{n\lambda}}{S_{n\lambda}^2}\langle \psi_{n\lambda}, f'(u_\lambda)\psi_{n\lambda}^2\rangle \Big(2\mathcal{R}e\,(\rho_{n\lambda})\,\langle \psi_{n\lambda}, f'(u_\lambda)|\psi_{n\lambda}|^2\rangle + |\rho_{n\lambda}|^2\langle \psi_{n\lambda}, u_\lambda f''(u_\lambda)|\psi_{n\lambda}|^2\rangle\Big)$$
$$+ \frac{\bar{\rho}_{n\lambda}}{3|S_{n\lambda}|^2}\langle \psi_{n\lambda}, f'(u_\lambda)\bar{\psi}_{n\lambda}^2\rangle \left(\rho_{n\lambda}\langle \bar{\psi}_{n\lambda}, f'(u_\lambda)\psi_{n\lambda}^2\rangle + \frac{1}{2}\rho_{n\lambda}^2\langle \bar{\psi}_{n\lambda}, u_\lambda f''(u_\lambda)\psi_{n\lambda}^2\rangle\right)$$
$$+ \frac{\mathcal{R}e\,(\rho_{n\lambda})}{|S_{n\lambda}|^2}\langle \psi_{n\lambda}, f'(u_\lambda)|\psi_{n\lambda}|^2\rangle \Big(2\mathcal{R}e\,(\rho_{n\lambda})\,\langle \bar{\psi}_{n\lambda}, f'(u_\lambda)|\psi_{n\lambda}|^2\rangle$$
$$+ |\rho_{n\lambda}|^2\langle \bar{\psi}_{n\lambda}, u_\lambda f''(u_\lambda)|\psi_{n\lambda}|^2\rangle\Big)\Big].$$

With the fact that $\mathcal{R}e\,(\rho_{n\lambda}) = \mathcal{R}e(-ih_{n\lambda}) = 0$, C_1 can be reduced as:

$$\begin{aligned} C_1 = \frac{8i\lambda^2\tau_{n\lambda}^2}{\theta_{n\lambda}} \Bigg[& \frac{1}{S_{n\lambda}^2}|\rho_{n\lambda}|^2\rho_{n\lambda}\langle \psi_{n\lambda}, f'(u_\lambda)\psi_{n\lambda}^2\rangle\langle \psi_{n\lambda}, u_\lambda f''(u_\lambda)|\psi_{n\lambda}|^2\rangle \\ & - \frac{1}{3|S_{n\lambda}|^2}\Big(|\rho_{n\lambda}|^2\langle \bar{\psi}_{n\lambda}, f'(u_\lambda)\psi_{n\lambda}^2\rangle\langle \psi_{n\lambda}, f'(u_\lambda)\bar{\psi}_{n\lambda}^2\rangle \\ & + \frac{1}{2}|\rho_{n\lambda}|^2\rho_{n\lambda}\langle \psi_{n\lambda}, f'(u_\lambda)\bar{\psi}_{n\lambda}\rangle\langle \bar{\psi}_{n\lambda}, u_\lambda f''(u_\lambda)\psi_{n\lambda}^2\rangle\Big)\Bigg]. \end{aligned} \tag{3.23}$$

When $\lambda \to \lambda_*$, we have

$$\lim_{\lambda\to\lambda_*} \frac{C_1}{\tau_{n\lambda}^2} = \frac{-8i\lambda_* h_{n\lambda_*}^2 \int_\Omega \phi^3 dx}{3\alpha_{\lambda_*}\theta_{n\lambda_*}|S_{n\lambda_*}|^2}. \tag{3.24}$$

Therefore,

$$\lim_{\lambda\to\lambda_*} \mathcal{R}e\left(\frac{C_1}{\tau_{n\lambda_*}^2}\right) = 0. \tag{3.25}$$

Finally, we determine $Proj_S\big(D_y f_2^1(z,0,0)U_2^2(z,0)\big)$, and define $h(z,a) = U_2^2(z,0)$ with $h(z,a) = h_{20}(a)z_1^2 + h_{11}(a)z_1z_2 + h_{02}(a)z_2^2$. Then, $h_{20}(a)$, $h_{11}(a)$, $h_{02}(a)$ can be uniquely determined by

$$(M_2^2 h)(z) = f_2^2(z,0,0), \tag{3.26}$$

which is equivalent to

$$\begin{aligned} & D_z h(z,a)Bz - \mathcal{A}_{\tau 1}(h(z,a)) = (I-\pi)\chi_0 F_2(\Phi z, 0) \\ = & 2\lambda\tau_{n\lambda}\left[\chi_0 f'(u_\lambda)(\Phi(0)z)(\mathcal{T}(\Phi)z) - \Phi\big\langle \Psi(0), f'(u_\lambda)(\Phi(0)z)(\mathcal{T}(\Phi)z)\big\rangle\right] \\ & + \lambda\tau_{n\lambda}\left[\chi_0 u_\lambda f''(u_\lambda)\,(\mathcal{T}(\Phi)z)^2 - \Phi\Big\langle \Psi(0), u_\lambda f''(u_\lambda)\,(\mathcal{T}(\Phi)z)^2\Big\rangle\right]. \end{aligned} \tag{3.27}$$

Applying the definition of $\mathcal{A}_{\tau 1}$ and χ_0, we have

$$\begin{cases} \dot{h}(z,a) - D_z h(z,a)Bz = 2\lambda\tau_{n\lambda}\Phi\left\langle \Psi(0), f'(u_\lambda)(\psi_{n\lambda}z_1 + \bar{\phi}_{n\lambda}z_2)(\psi_{n\lambda}\rho_{n\lambda}z_1 + \bar{\phi}_{n\lambda}\bar{\rho}_{n\lambda}z_2)\right\rangle \\ \qquad + \lambda\tau_{n\lambda}\Phi\left\langle \Psi(0), u_\lambda f''(u_\lambda)(\psi_{n\lambda}\rho_{n\lambda}z_1 + \bar{\phi}_{n\lambda}\bar{\rho}_{n\lambda}z_2)^2\right\rangle, \\ \dot{h}(z,0) - \tau_{n\lambda}d\Delta h(z,0) - L_0 h(z,a) = 2\lambda\tau_{n\lambda}f'(u_\lambda)(\psi_{n\lambda}z_1 + \bar{\phi}_{n\lambda}z_2)(\psi_{n\lambda}\rho_{n\lambda}z_1 + \bar{\phi}_{n\lambda}\bar{\rho}_{n\lambda}z_2) \\ \qquad + \lambda\tau_{n\lambda}(\psi_{n\lambda}\rho_{n\lambda}z_1 + \bar{\phi}_{n\lambda}\bar{\rho}_{n\lambda}z_2)^2. \end{cases} \tag{3.28}$$

Matching the coefficients of z_1^2 and z_1z_2 of Eq. (3.28), we can get the following equations about h_{20} and h_{11},

$$\begin{cases} \dot{h}_{20}(a) - 2i\theta_{n\lambda}h_{20}(a) \\ \quad = 2\lambda\tau_{n\lambda}\rho_{n\lambda}\left(\dfrac{\langle \psi_{n\lambda}, f'(u_\lambda)\psi_{n\lambda}^2\rangle\psi_{n\lambda}e^{i\theta_{n\lambda}a}}{S_{n\lambda}} + \dfrac{\langle \bar{\psi}_{n\lambda}, f'(u_\lambda)\psi_{n\lambda}^2\rangle\bar{\psi}_{n\lambda}e^{-i\theta_{n\lambda}a}}{\bar{S}_{n\lambda}}\right) \\ \qquad + \lambda\tau_{n\lambda}\rho_{n\lambda}^2\left(\dfrac{\langle \psi_{n\lambda}, u_\lambda f''(u_\lambda)\psi_{n\lambda}^2\rangle\psi_{n\lambda}e^{i\theta_{n\lambda}a}}{S_{n\lambda}} + \dfrac{\langle \bar{\psi}_{n\lambda}, u_\lambda f''(u_\lambda)\psi_{n\lambda}^2\rangle\bar{\psi}_{n\lambda}e^{-i\theta_{n\lambda}a}}{\bar{S}_{n\lambda}}\right), \\ \dot{h}_{20}(0) - \tau_{n\lambda}\left[A(\lambda)h_{20}(0) + \lambda u_\lambda f'(u_\lambda)\mathcal{T}(h_{20})\right] = \lambda\tau_{n\lambda}(2f'(u_\lambda)\rho_{n\lambda} + u_\lambda f''(u_\lambda)\rho_{n\lambda}^2)\psi_{n\lambda}^2, \end{cases} \tag{3.29}$$

$$\begin{cases} \dot{h}_{11}(a) = 4\lambda\tau_{n\lambda}\mathcal{R}e\,(\bar{\rho}_{n\lambda})\left(\dfrac{\langle \psi_{n\lambda}, f'(u_\lambda)|\psi_{n\lambda}|^2\rangle\psi_{n\lambda}e^{i\theta_{n\lambda}a}}{S_{n\lambda}} + \dfrac{\langle \bar{\psi}_{n\lambda}, f'(u_\lambda)|\psi_{n\lambda}|^2\rangle\bar{\psi}_{n\lambda}e^{-i\theta_{n\lambda}a}}{\bar{S}_{n\lambda}}\right) \\ \quad + 2\lambda\tau_{n\lambda}|\rho_{n\lambda}|^2\left(\dfrac{\langle \psi_{n\lambda}, u_\lambda f''(u_\lambda)|\psi_{n\lambda}|^2\rangle\psi_{n\lambda}e^{i\theta_{n\lambda}a}}{S_{n\lambda}} + \dfrac{\langle \bar{\psi}_{n\lambda}, u_\lambda f''(u_\lambda)|\psi_{n\lambda}|^2\rangle\bar{\psi}_{n\lambda}e^{-i\theta_{n\lambda}a}}{\bar{S}_{n\lambda}}\right), \\ \dot{h}_{11}(0) - \tau_{n\lambda}\left[A(\lambda)h_{11}(0) + \lambda u_\lambda f'(u_\lambda)\mathcal{T}(h_{11})\right] = 2\lambda\tau_{n\lambda}(2f'(u_\lambda)\mathcal{R}e\,(\rho_{n\lambda}) \\ \qquad + u_\lambda f''(u_\lambda)|\rho_{n\lambda}|^2)|\psi_{n\lambda}|^2, \end{cases} \tag{3.30}$$

where $A(\lambda)$ is defined by (2.5). Moreover,

$$\begin{aligned} f_2^1(z,y,0) = {} & 2\lambda\tau_{n\lambda}\left\langle \Psi(0), f'(u_\lambda)(\Phi(0)z + y(0))(\mathcal{T}(\Phi)z + \mathcal{T}(y))\right\rangle \\ & + \lambda\tau_{n\lambda}\left\langle \Psi(0), u_\lambda f''(u_\lambda)\,(\mathcal{T}(\Phi)z + \mathcal{T}(y))^2\right\rangle, \end{aligned}$$

so we have

$$\begin{aligned} D_y f_2^1(z,0,0)y = {} & 2\lambda\tau_{n\lambda}\left[\left\langle \Psi(0), f'(u_\lambda)\mathcal{T}(\Phi)zy(0) + f'(u_\lambda)\Phi(0)z\mathcal{T}(y)\right\rangle\right. \\ & \left. + \left\langle \Psi(0), u_\lambda f''(u_\lambda)\mathcal{T}(\Phi)z\mathcal{T}(y)\right\rangle\right] = 2\lambda\tau_{n\lambda}(J, \bar{J})^T, \end{aligned}$$

where

$$\begin{aligned} J = {} & \frac{1}{S_{n\lambda}}\left\langle \psi_{n\lambda}, f'(u_\lambda)(\psi_{n\lambda}z_1 + \bar{\psi}_{n\lambda}z_2)\mathcal{T}(y) + f'(u_\lambda)\left(\rho_{n\lambda}\psi_{n\lambda}z_1 + \bar{\rho}_{n\lambda}\bar{\psi}_{n\lambda}z_2\right)y(0)\right. \\ & \left. + u_\lambda f''(u_\lambda)\left(\rho_{n\lambda}\psi_{n\lambda}z_1 + \bar{\rho}_{n\lambda}\bar{\psi}_{n\lambda}z_2\right)\mathcal{T}(y)\right\rangle. \end{aligned}$$

Therefore,

$$Proj_S(D_y f_2^1(z,0,0)h) = \begin{pmatrix} C_2 z_1^2 z_2 \\ \bar{C}_2 z_1 z_2^2 \end{pmatrix}$$

with

$$\begin{aligned} C_2 = \frac{2\lambda\tau_{n\lambda}}{S_{n\lambda}} & \left[\left\langle \psi_{n\lambda}, f'(u_\lambda)\psi_{n\lambda} \left(\rho_{n\lambda} h_{11}(0) + \mathcal{T}(h_{11})\right) \right\rangle \right. \\ & + \left\langle \psi_{n\lambda}, f'(u_\lambda)\bar{\psi}_{n\lambda} \left(\bar{\rho}_{n\lambda} h_{20}(0) + \mathcal{T}(h_{20})\right) \right\rangle \\ & \left. + \left\langle \psi_{n\lambda}, u_\lambda f''(u_\lambda) \left(\bar{\psi}_{n\lambda}\bar{\rho}_{n\lambda}\mathcal{T}(h_{20}) + \psi_{n\lambda}\rho_{n\lambda}\mathcal{T}(h_{11})\right) \right\rangle \right]. \end{aligned} \tag{3.31}$$

We define

$$w_{20}(a) := \frac{h_{20}(a)}{\tau_{n\lambda}}, \quad w_{11} := \frac{h_{11}(a)}{\tau_{n\lambda}}, \quad a \in (-\infty, 0], \tag{3.32}$$

then when $\lambda \to \lambda_*$, we have

$$\lim_{\lambda\to\lambda_*} \frac{C_2}{\tau_{n\lambda}^2} = \frac{2\lambda_* f'(0)}{S_{n\lambda_*}} \int_\Omega \phi^2 \left[\left(\rho_{n\lambda_*} w_{11}^*(0) + \mathcal{T}(w_{11}^*)\right) + \left(\bar{\rho}_{n\lambda_*} w_{20}^*(0) + \mathcal{T}(w_{20}^*)\right) \right] dx, \tag{3.33}$$

where $w_{20}^* = \lim\limits_{\lambda\to\lambda_*} w_{20}$, $w_{11}^* = \lim\limits_{\lambda\to\lambda_*} w_{11}$. For Eqs. (3.29) and (3.30), dividing them by $\tau_{n\lambda}$ and letting $\lambda \to \lambda_*$, we can get the following limit equations:

$$\begin{cases} \dot{w}_{20}^*(a) - 2i\theta_{n\lambda_*} w_{20}^*(a) = \dfrac{2ih_{n\lambda_*}\phi}{\alpha_{\lambda_*}} \left(\dfrac{e^{i\theta_{n\lambda_*}a}}{S_{n\lambda_*}} + \dfrac{e^{-i\theta_{n\lambda_*}a}}{\bar{S}_{n\lambda_*}} \right), \\ \dot{w}_{20}^*(0) - \tau_{n\lambda_*}[A(\lambda_*)w_{20}^*(0) + \lambda_* u_{\lambda_*} f'(0)\mathcal{T}(w_{20}^*)] = -2ih_{n\lambda_*}\lambda_* f'(0)\phi^2, \end{cases} \tag{3.34}$$

and

$$\begin{cases} \dot{w}_{11}^*(a) = 0, \\ \dot{w}_{11}^*(0) - \tau_{n\lambda_*}[A(\lambda_*)w_{11}^*(0) + \lambda_* u_{\lambda_*} f'(0)\mathcal{T}(w_{11}^*)] = 0. \end{cases} \tag{3.35}$$

In Eq. (3.35), we know that $\dot{w}_{11}^*(0) = 0$ from the first equation, and we substitute it into the second equation and obtain

$$A(\lambda_*)w_{11}(0) = (d\Delta + \lambda_*)w_{11}^*(0) = 0.$$

Hence we have $w_{11}^*(0) = c_n\phi$ for some constant c_n and ϕ is the eigenfunction of $-d\Delta$ corresponding to the eigenvalue λ_*. Again, we use the first equation of Eq. (3.35) and obtain that $w_{11}^*(a) = w_{11}^*(0)$ which implies that $w_{11}^*(a)$ is constant for $a \in (-\infty, 0]$. Therefore, $w_{11}^*(0) = 0$ and $w_{11}^*(a) = 0$. Then, the following conclusion can be reached:

$$\rho_{n\lambda_*} w_{11}^*(0) + \mathcal{T}(w_{11}^*) = 0. \tag{3.36}$$

With the same method, we solve Eq. (3.34). By the first equation of Eq. (3.34), we get the expression of $\dot{w}^*_{20}(0)$, submit it into the second equation and have

$$A(\lambda_*)w^*_{20}(0) = L_{\lambda_*} := -\lambda_* u_{\lambda_*} f'(0)\mathcal{T}(w^*_{20}) + \frac{2i\theta_{n\lambda_*}}{\tau_{n\lambda_*}} w^*_{20}(0) + \frac{2ih_{n\lambda_*}\phi}{\alpha_{\lambda_*}\tau_{n\lambda_*}}\left(\frac{1}{S_{n\lambda_*}} + \frac{1}{\bar{S}_{n\lambda_*}}\right) + \frac{2i\lambda_* f'(0)h_{n\lambda_*}\phi^2}{\tau_{n\lambda_*}}. \tag{3.37}$$

It can be inferred that $L_{\lambda_*} = 0$ from the definition of u_λ and $\tau_{n\lambda}$, so Eq. (3.37) becomes $(d\Delta + \lambda_*)w^*_{20}(0) = 0$ which implies that $w^*_{20}(0) = d_n\phi$. Here d_n is a constant depending on $n \in \mathbb{N}$. On the other hand, by solving the first equation of Eq. (3.34), we can obtain

$$w^*_{20}(a) = e^{2i\theta_{n\lambda_*}a} w^*_{20}(0) - \frac{2h_{n\lambda_*}\phi}{\alpha_{\lambda_*}\theta_{n\lambda_*}}\left(\frac{e^{i\theta_{n\lambda_*}a} - e^{2i\theta_{n\lambda_*}a}}{S_{n\lambda_*}} + \frac{e^{-i\theta_{n\lambda_*}a} - e^{2i\theta_{n\lambda_*}a}}{3\bar{S}_{n\lambda_*}}\right). \tag{3.38}$$

By using the integral (2.10), we can compute $\mathcal{T}(w^*_{20})$ as follows,

$$\mathcal{T}(w^*_{20}) = \int_{-\infty}^{0} \frac{(-s)^n e^s w^*_{20}(s)}{n!} ds = \gamma_{n\lambda_*} w^*_{20}(0) - \frac{2h_{n\lambda_*}\phi}{\alpha_{\lambda_*}\theta_{n\lambda_*}}\left(\frac{\rho_{n\lambda_*} - \gamma_{n\lambda_*}}{S_{n\lambda_*}} + \frac{\bar{\rho}_{n\lambda_*} - \gamma_{n\lambda_*}}{3\bar{S}_{n\lambda_*}}\right),$$

then we have

$$\bar{\rho}_{n\lambda_*} w^*_{20}(0) + \mathcal{T}(w^*_{20}) = (\bar{\rho}_{n\lambda_*} + \gamma_{n\lambda_*})w^*_{20}(0) - \frac{2h_{n\lambda_*}\phi}{\alpha_{\lambda_*}\theta_{n\lambda_*}}\left(\frac{\rho_{n\lambda_*} - \gamma_{n\lambda_*}}{S_{n\lambda_*}} + \frac{\bar{\rho}_{n\lambda_*} - \gamma_{n\lambda_*}}{3\bar{S}_{n\lambda_*}}\right), \tag{3.39}$$

where $\gamma_{n\lambda_*} := (1 + 2i\theta_{n\lambda_*})^{-(n+1)}$. Substituting Eqs. (3.36) and (3.39) into Eq. (3.33), we obtain

$$\lim_{\lambda\to\lambda_*} \frac{C_2}{\tau^2_{n\lambda}} = -\frac{2d_n(\bar{\rho}_{n\lambda_*} + \gamma_{n\lambda_*})}{\alpha_{\lambda_*} S_{n\lambda_*}} + \frac{4h_{n\lambda_*}}{\alpha^2_{\lambda_*} S_{n\lambda_*}\theta_{n\lambda_*}}\left(\frac{\rho_{n\lambda_*} - \gamma_{n\lambda_*}}{S_{n\lambda_*}} + \frac{\bar{\rho}_{n\lambda_*} - \gamma_{n\lambda_*}}{3\bar{S}_{n\lambda_*}}\right). \tag{3.40}$$

Now we determine the value of d_n. Applying the duality (3.7) on w_{20}, we have

$$0 = \langle\langle \Psi, w_{20}\rangle\rangle = \langle \Psi(0), w_{20}(0)\rangle + \lambda\tau_{n\lambda} \int_{-\infty}^{0} \frac{(-s)^n e^s}{n!} \int_0^s \langle \Psi(\xi - s), u_\lambda w_{20}(\xi)\rangle d\xi ds.$$

When $\lambda \to \lambda_*$, by using

$$\lim_{\lambda\to\lambda_*} \frac{u_\lambda}{\lambda - \lambda_*} = \alpha_{\lambda_*}\phi,$$

we have

$$\langle \phi, w_{20}^*(0)\rangle = -\frac{\lambda_* \theta_{n\lambda_*}\alpha_{\lambda_*}}{h_{n\lambda_*}} \int_{-\infty}^{0} \frac{(-s)^n e^s e^{i\theta_{n\lambda} s}}{n!} \int_0^s \int_\Omega \phi^2 w_{20}^*(\xi) e^{-i\theta_{n\lambda}\xi} d\xi dx ds. \tag{3.41}$$

Substituting $w_{20}^*(0) = d_n\phi$ into Eq. (3.41), we have

$$\begin{aligned} d_n \int_\Omega \phi^2 dx = \langle \phi, d_n\phi\rangle &= \frac{\lambda_* \theta_{n\lambda_*}\alpha_{\lambda_*} f'(0)}{h_{n\lambda_*}} \int_{-\infty}^{0} \frac{(-s)^n e^s e^{i\theta_{n\lambda_*} s}}{n!} \int_0^s \int_\Omega \phi^2 e^{-i\theta_{n\lambda_*}\xi} \Big[d_n \phi e^{2i\theta_{n\lambda_*}\xi} \\ &\quad - \frac{2h_{n\lambda_*}\phi}{\alpha_{\lambda_*}\theta_{n\lambda_*}} \left(\frac{1}{S_{n\lambda_*}} (e^{i\theta_{n\lambda_*}\xi} - e^{2i\theta_{n\lambda_*}\xi}) + \frac{1}{3\bar{S}_{n\lambda_*}} (e^{-i\theta_{n\lambda_*}\xi} - e^{2i\theta_{n\lambda_*}\xi}) \right) \Big] dx d\xi ds \\ &= \frac{i d_n}{h_{n\lambda_*}} (\gamma_{n\lambda_*} - \rho_{n\lambda_*}) + \frac{2}{\alpha_{\lambda_*}} \Big[\frac{1}{S_{n\lambda_*}} \left(\frac{i(n+1)h_{n\lambda_*}}{1 + i\theta_{n\lambda_*}} - \frac{1}{i\theta_{n\lambda_*}} (\gamma_{n\lambda_*} - \rho_{n\lambda_*}) \right) \\ &\quad - \frac{1}{3\bar{S}_{n\lambda_*}} \left(\frac{1}{2i\theta_{n\lambda_*}} (\bar{\rho}_{n\lambda_*} - \rho_{n\lambda_*}) + \frac{1}{i\theta_{n\lambda_*}} (\gamma_{n\lambda_*} - \rho_{n\lambda_*}) \right) \Big]. \end{aligned} \tag{3.42}$$

Note that d_n here should be d_n^m, but because we use the abbreviated notations which are stated in (3.1) in this chapter, so here we also simplify d_n^m as d_n. Then d_n^m is uniquely determined by Eq. (3.42) for each $n \in \mathbb{N}$ and $m \in [0, m_n]$. Later we will show that (3.42) can be simplified for more specific n and m. Finally we obtain

$$\frac{1}{3!} g_3^1(z, 0, 0) = \begin{pmatrix} A_2 z_1^2 z_2 \\ \bar{A}_2 z_1 z_2^2 \end{pmatrix},$$

where

$$A_2 = C_0 + \frac{1}{4}(C_1 + C_2) \tag{3.43}$$

with $C_0,\ C_1,\ C_2$ defined in Eqs. (3.21), (3.23) and (3.31), respectively. On the center manifold, the normal form of Eq. (3.3) is given by

$$\dot{z} = Bz + \begin{pmatrix} A_1 z_1 \alpha \\ \bar{A}_1 z_2 \alpha \end{pmatrix} + \begin{pmatrix} A_2 z_1^2 z_2 \\ \bar{A}_2 z_1 z_2^2 \end{pmatrix} + h.o.t. \tag{3.44}$$

By letting $z_1 = \omega_1 - i\omega_2,\ z_2 = \omega_1 + i\omega_2$ and $\omega_1 = \rho\cos\xi,\ \omega_2 = \rho\sin\xi$, we transform Eq. (3.44) into Eq. (3.45).

In the beginning of this section, we change some notations into an easier form for the simplicity of the writing. But now, in order to avoid confusion, we use the notations in Section 2 to elaborate our results.

Theorem 3.1. *For each $\lambda \in (\lambda_*, \lambda^*)$ and $n \in \mathbb{N}$, with f satisfying $(H1)$, $(H2)$, Eq. (3.3) has a 2-dimensional local center manifold near the positive steady state u_λ at $\tau = \tau_{n\lambda}^m$ for*

$0 \le m \le m_n := [\frac{n-1}{4}]$. *On the center manifold, the reduced flow is given by a norm form ordinary differential equation in polar coordinates* (ρ, ξ) *as*

$$\begin{cases} \dot{\rho} = K_1 \left(\tau - \tau_{n\lambda}^m\right) \rho + K_2 \rho^3 + O\left(\left(\tau - \tau_{n\lambda}^m\right)^2 \rho + \left|\left(\tau - \tau_{n\lambda}^m, \rho\right)\right|^4\right), \\ \dot{\xi} = -i\theta_{n\lambda}^m + O\left(\left|\left(\tau - \tau_{n\lambda}^m, \rho\right)\right|\right), \end{cases} \tag{3.45}$$

where $K_1 = \mathcal{R}e(A_1)$, $K_2 = \mathcal{R}e(A_2)$ *and*

$$A_1 = \frac{i\omega_{n\lambda}^m}{S_{n\lambda}^m} \langle \psi_{n\lambda}^m, \psi_{n\lambda}^m \rangle, \quad A_2 = C_0 + \frac{1}{4}(C_1 + C_2) \tag{3.46}$$

with C_0, C_1, C_2 *given by Eqs.* (3.21), (3.23) *and* (3.31), *respectively. Especially, when* $\lambda \to \lambda_*$, *we have the following results for the limits of* K_1 *and* K_2:

$$\begin{aligned} &\lim_{\lambda \to \lambda_*} \tau_{n\lambda}^m K_1 = (n+1) \sin^2\left(\eta_{n\lambda_*}^m\right) > 0, \\ &\lim_{\lambda \to \lambda_*} \frac{K_2}{\left(\tau_{n\lambda}^m\right)^2} = \frac{1}{4} \lim_{\lambda \to \lambda_*} \mathcal{R}e\left(\frac{C_2}{\left(\tau_{n\lambda}^m\right)^2}\right), \end{aligned} \tag{3.47}$$

and

$$\begin{aligned} \lim_{\lambda \to \lambda_*} \frac{C_2}{\left(\tau_{n\lambda}^m\right)^2} = &-\frac{2d_n^m}{\alpha_{\lambda_*} S_{n\lambda_*}^m} \left[\left(1 - i\theta_{n\lambda_*}^m\right)^{-(n+1)} + \left(1 + 2i\theta_{n\lambda_*}^m\right)^{-(n+1)}\right] \\ &+ \frac{4h_{n\lambda_*}^m}{\alpha_{\lambda_*}^2 S_{n\lambda_*}^m \theta_{n\lambda_*}^m} \left[\frac{1}{S_{n\lambda_*}^m} \left(\left(1 + i\theta_{n\lambda_*}^m\right)^{-(n+1)} - \left(1 + 2i\theta_{n\lambda_*}^m\right)^{-(n+1)}\right)\right. \\ &+ \left.\frac{1}{3\bar{S}_{n\lambda_*}^m} \left(\left(1 - i\theta_{n\lambda_*}^m\right)^{-(n+1)} - \left(1 + 2i\theta_{n\lambda_*}^m\right)^{-(n+1)}\right)\right] \end{aligned} \tag{3.48}$$

with d_n^m *being a complex number and uniquely determined by Eq.* (3.42) *and* α_{λ_*}, $\theta_{n\lambda_*}$, $S_{n\lambda_*}$ *defined in Lemmas 2.1, 2.4 and 2.7, respectively. Moreover, the direction of Hopf bifurcation near* u_λ *at* $\tau = \tau_{n\lambda}^m$ *and the bifurcating periodic orbit can be determined according to the following rules:*

(i) when $K_2 < 0$, *the periodic orbit is locally asymptotically stable. Then, if* $K_1 > 0$, *the direction of Hopf bifurcation is forward; if* $K_1 < 0$, *the bifurcation direction is backward;*
(ii) when $K_2 > 0$, *the periodic orbit is always unstable. If* $K_1 > 0$, *the direction of Hopf bifurcation is backward; if* $K_1 < 0$, *the bifurcation direction is forward.*

Theorem 3.1 holds for any $n \in \mathbb{N}$. To conclude this section, we show that when $n = 1$ (the strong kernel case), we can more concretely determine the direction of Hopf bifurcation and the stability of bifurcating periodic orbits of Eq. (1.1), as given in the following corollary.

Corollary 3.2. *For each* $\lambda \in (\lambda_*, \lambda^*]$ *and* f *satisfying* $(H1)$, $(H2)$, *when* $n = 1$, *there is a unique bifurcation value* $\tau = \tau_{1\lambda}^0 \approx \frac{2}{\lambda - \lambda_*}$ *where a Hopf bifurcation from the positive steady state* u_λ

occurs. Moreover the direction of the Hopf bifurcation at $\tau = \tau_{1\lambda}^0$ *is forward, and the bifurcating periodic orbit is locally asymptotically stable.*

Proof. The occurrence of Hopf bifurcation at the bifurcating critical value $\tau_{1\lambda}^0$ is proved in Theorem 2.8. Moreover $\tau_{1\lambda}^0$ is the unique bifurcating value for the case $n = 1$ by Lemma 2.4. From Theorem 3.1, we determine the constants d_1^0 and K_1 to completely determine the bifurcation direction and stability of periodic orbits. Substituting

$$\theta_{1\lambda_*}^0 = 1,\ h_{1\lambda_*}^0 = \frac{1}{2},\ S_{1\lambda_*}^0 = 2 + i,$$

into Eq. (3.42), we obtain that

$$d_1^0 = \frac{1+i}{20\alpha_{\lambda_*}}. \tag{3.49}$$

Then by Eqs. (3.40) and (3.49), we know that

$$\lim_{\lambda\to\lambda_*} \mathcal{R}e\left(\frac{C_2}{\left(\tau_{1\lambda}^0\right)^2}\right) = -\frac{1}{20\alpha_{\lambda_*}} < 0. \tag{3.50}$$

Thus, by Eq. (3.47), we have

$$\lim_{\lambda\to\lambda_*} \frac{K_2}{\left(\tau_{1\lambda}^0\right)^2} = -\frac{1}{80\alpha_{\lambda_*}} < 0$$

Then, by the continuity of K_1, K_2 in λ, we have $K_1 > 0,\ K_2 < 0$ in a small neighbor of λ_*. By applying the results in Theorem 3.1, we know that the Hopf bifurcation is forward, and the bifurcating periodic orbit is locally asymptotically stable. □

From our results in this section, when λ is close enough to λ_*, the direction of the Hopf bifurcations for (1.1) is determined by only $f'(0)$ and does not depend on the higher order derivative of f. Because we always have $f'(0) < 0$ which is the condition for the existence of locally stable steady state without delay, so the Hopf bifurcation in Eq. (1.1) is always forward and bifurcating periodic orbits are stable. This is in consistence with the local delay case considered in [3,36]. This is precisely verified for the most typical case of strong kernel case ($n = 1$), and the higher but specific n case can also be calculated from the formulas given in (3.46), (3.47) and (3.48).

4. Examples and simulations

In this section, we apply our general results to two population models and perform some numerical simulations. From Theorem 2.8, the critical delay value of stability switching for Eq. (1.1) is $\tau_{n\lambda}^0$ which is independent of geometry of the function $f(u)$, and the direction of Hopf bifurcation depends only on $f'(u)$ which is negative in our model. Therefore, the Hopf bifurcations in the two following examples are both forward and stable periodic orbits arise. However the parameters in $f(u)$ will affect the amplitude of the steady state and bifurcating periodic orbits.

4.1. A diffusive logistic model

When $f(u) = 1 - u/K$, we have the diffusive logistic model with a distributed delayed growth rate per capita:

$$\begin{cases} u_t(x,t) = d\Delta u(x,t) + \lambda u(x,t)\left(1 - \dfrac{\int_{-\infty}^t g_n(\tau, t-s)u(x,s)ds}{K}\right) = 0, & x \in (0,\pi),\ t > 0, \\ u(x,t) = 0, & x = 0, \pi,\ t > 0, \end{cases} \tag{4.1}$$

where $u(x,t)$ is the population density of a biological species, $\lambda > 0$ is the maximum intrinsic growth rate and $K > 0$ denotes the carrying capacity. And for the purpose of numerical simulation, we use the one-dimensional spatial domain $\Omega = (0, \pi)$, so that we can compute the exact values of λ_* and $\tau^0_{n\lambda_*}$. The kernel function g_n is the Gamma distribution function defined in (1.2). This is exactly the example considered in Busenberg and Huang [3] but the delay is a local one there. Then all results proved in Theorems 2.8, 3.1 and Corollary 3.2 hold for (4.1).

Numerical simulations of (1.1) or (4.1) is challenging as the delay is an integral over an infinite interval. Here we use a method motivated by Gourley and So [16] by defining

$$v(x,t) = \int_{-\infty}^{t} g_0(\tau, t-s)u(x,s)ds = \int_{-\infty}^{0} g_0(\tau, -s)u(x,s)ds,$$

then the equation (4.1) when $n = 0$ becomes an equivalent new system:

$$\begin{cases} u_t(x,t) = d\Delta u(x,t) + \lambda u(x,t)\left(1 - \dfrac{v(x,t)}{K}\right), & x \in (0,\pi),\ t > 0, \\ v_t(x,t) = \dfrac{1}{\tau}(u(x,t) - v(x,t)), & x \in [0,\pi],\ t > 0, \\ u(0,t) = u(\pi,t) = 0, & t > 0, \\ u(x,0) = u_0(x,0), & x \in (0,\pi), \\ v(x,0) = \displaystyle\int_{-\infty}^{0} g_0(\tau,-s)u_0(x,s)ds, & x \in (0,\pi). \end{cases} \tag{4.2}$$

The simulation of (4.2) can be treated as a regular reaction–diffusion system with a single evaluation of an integral in the initial condition of $v(x, 0)$. Especially converging to a steady state or periodic orbit for (4.2) is equivalent to the same convergence for the original system (4.1). For each $n \in \mathbb{N}$, a similar change of variables can generate a new system with $n + 1$ variables which does not have an explicit delay in the system, and the system consists of one diffusive equation and n linear equations without diffusion. So numerical simulations of (1.1) or (4.1) can be achieved through integrating the new $(n + 1)$-variable system.

In Figs. 1 and 2, the parameter values are $\lambda = 1.1,\ d = 1,\ K = 1$, with $\lambda_* = 1$ here. For $\tau = 22$, when $n = 0$, Fig. 1 shows the convergence to the positive steady state u_λ, but for $n = 1$, the solution with same initial value converges to a periodic orbit (see Fig. 2). In this case when $n = 1$,

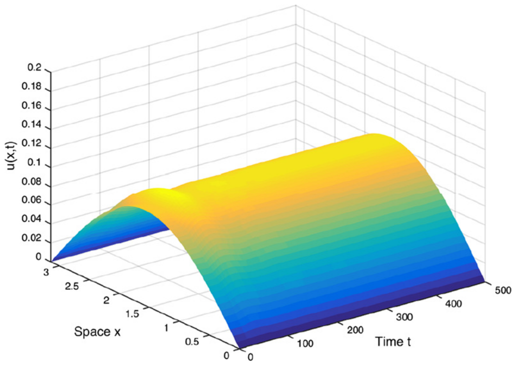

Fig. 1. Dynamic behavior of Eq. (4.1) with weak kernel ($n = 0$). Here $\lambda = 1.1,\ d = 1,\ K = 1$ and $\tau = 22$.

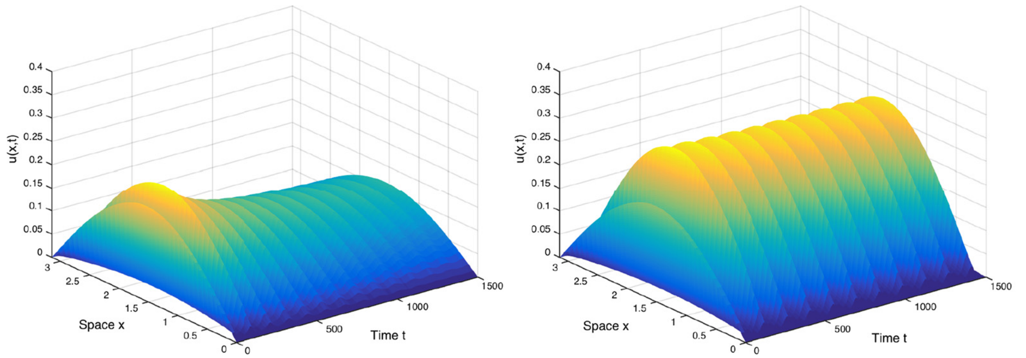

Fig. 2. Dynamic behavior of Eq. (4.1) with strong kernel ($n = 1$). Here $\lambda = 1.1,\ d = 1,\ K = 1$, and the critical value of Hopf bifurcation $\tau^0_{1\lambda} \approx 20$. (Left): $\tau = 17 < \tau^0_{1\lambda}$, convergence of a positive steady state; (Right): $\tau = 22 > \tau^0_{1\lambda}$, convergence to a stable periodic orbit.

we can compute the critical delay value to be $\tau^0_{1\lambda} \approx 2/(\lambda - \lambda_*) \approx 20$. Then, the positive steady state is stable for $\tau \in [0, 20)$ and unstable for $\tau > 20$. For $\tau = 17$, from the left panel of Fig. 2, we can see that the solution of Eq. (4.1) with strong kernel converges to the positive steady state, while when $\tau = 22$, as depicted in the right panel of Fig. 2, a stable spatially nonhomogeneous periodic orbit arises.

4.2. *A reaction–diffusion food-limited population model*

Here we consider the food-limited model which is the case $f(u) = \dfrac{1-u}{1+cu}$. The model is as follows:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = d\Delta u(x,t) + \lambda u(x,t)\dfrac{1 - g_n * u(x,s)}{1 + cg_n * u(x,s)} = 0, & x \in (0,\pi),\ t > 0, \\ u(x,t) = 0, & x = 0, \pi,\ t > 0, \end{cases} \tag{4.3}$$

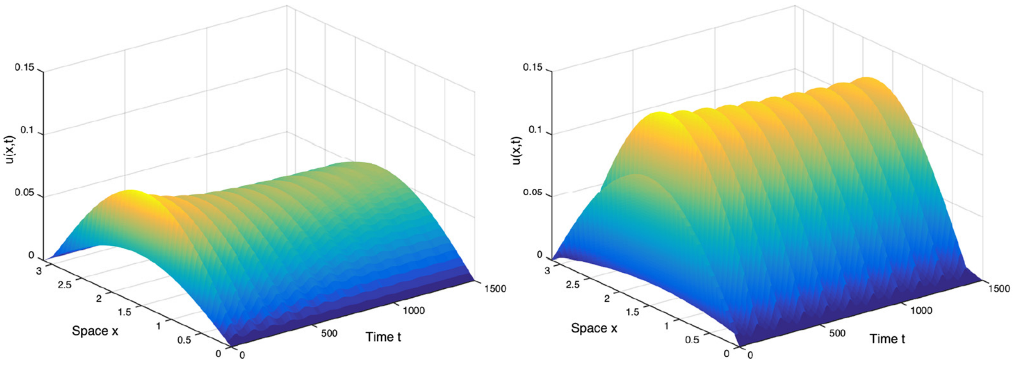

Fig. 3. Dynamic behavior of Eq. (4.3) when $n = 1$. With $\lambda = 1.1,\ d = 1,\ c = 1$, the critical value of Hopf bifurcation $\tau_{1\lambda}^0 \approx 20$. (Left): $\tau = 17 < \tau_{1\lambda}^0$, convergence of a positive steady state; (Right): $\tau = 22 > \tau_{1\lambda}^0$, convergence to a stable periodic orbit.

where $g_n * u(x,s) = \int_{-\infty}^{t} g_n(\tau, t-s)u(x,s)ds$. Here, $u(x,t),\ d,\ \lambda,\ g_n$ have the same meaning as in logistic case (4.1), and $c > 0$ is the replacement of mass in the population at saturation. The model without delay was originally proposed by Smith [33] who argued that a food-limited species demands food for both maintenance and growth in its growing stage, while food is needed for maintenance only when the population has reached saturation level. In [9], the dynamics of (4.3) with a single discrete delay is considered. They proved the existence of the spatially nonhomogeneous steady state and derived the condition under which the steady state loses its stability. Su et al. [35] revisited this model and gave another method to prove the existence of steady state. And in [36], Su et al. rigorously proved the occurrence of Hopf bifurcation for the discrete delay case.

Again when the distributed delay is incorporated in this model, Theorems 2.8, 3.1 and Corollary 3.2 can be applied to obtain the occurrence and stability switch of Hopf bifurcation. The stability switching point is still $\tau_{1\lambda}^0 \approx 2/(\lambda - \lambda_*)$. Then, we choose the parameters as $\lambda = 1.1,\ d = 1,\ c = 1$, and we have $\lambda_* = 1$. The dynamics of this model is demonstrated in Fig. 3 which is similar to the logistic case: when $\tau < \tau_{1\lambda}^0$, the solution of Eq. (4.3) converges to the stable steady state (see the left panel); when $\tau > \tau_{1\lambda}^0$, the steady state u_λ loses its stability and the solution will eventually converge to a periodic orbit (see the right panel).

5. Conclusion

In this paper we consider a general reaction–diffusion equation with distributed delay under Dirichlet boundary condition. The delay feedback effect which reflects the dependence of growth rate on the past time states are of significance in biological and physical systems. And it is reasonable to consider dynamical system which is influenced not only by the information of a particular past temporal point, but also the whole historical information of the system. Hence a distributed delay is a more general setting for considering the delay effect and it can be used to incorporate different biological situations. In the present paper, we consider a distributed delay tuned by a Gamma distribution function and we use the average delay as a parameter to investigate the stability and bifurcation of spatially nonhomogeneous steady state of this system.

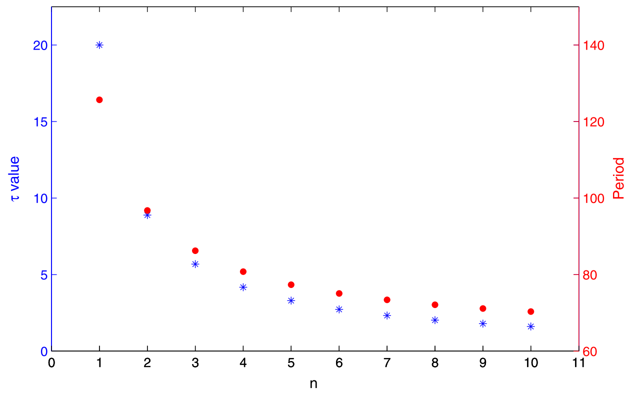

Fig. 4. The effect of the shape parameter n on the critical delay values and critical periods of (4.1). Here $1 \le n \le 10$ and $\lambda = 1.1$, $d = 1$, $K = 1$. The star: the smallest Hopf bifurcation value $\tau^0_{n\lambda}$; the dot: the period of bifurcating periodic orbits.

Our analytic results show that the shape parameter n of Gamma kernel function affects the dynamics of system (1.1) significantly. Firstly, the parameter n determines whether a Hopf bifurcation occurs or not in system (1.1): when $n = 0$, that is the weak kernel case, there is no Hopf bifurcation and the steady state u_λ is always locally asymptotically stable for any $\tau > 0$; when $n \ge 1$, Hopf bifurcations can occur and the critical bifurcating values can be obtained for each n. Secondly, the number of Hopf bifurcation values is $m_n = [(n-1)/4]$ which depends on n: the bigger n, the more bifurcation points. Moreover, n also effects the smallest critical values of Hopf bifurcation and the values of periods of the bifurcating periodic orbits. We have obtained the smallest critical values for Hopf bifurcation is $\tau^0_{n\lambda} \approx \tan\left(\frac{\pi}{2(n+1)}\right) \Big/ \left((\lambda - \lambda_*)\cos^{n+1}\left(\frac{\pi}{2(n+1)}\right)\right)$ which is strictly decreasing with respect to n, which means that it become easier for the occurrence of Hopf bifurcation in system (1.1) with a larger shape parameter n. Also the period of the bifurcating periodic orbits can be calculated as $T = 2\pi \Big/ \left((\lambda - \lambda_*)h^0_{n\lambda_*}\right)$ is also declining (see Fig. 4).

Acknowledgments

We would like to thank the anonymous reviewer for helpful comments which improve the manuscript. This work was done when the first author visited Department of Mathematics, College of William and Mary during the academic year 2016–2017, and she would like to thank Department of Mathematics, College of William and Mary for their support and kind hospitality.

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