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Positive solutions of Kirchhoff-type non-local elliptic equation: a bifurcation approach

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Positive solutions of a Kirchhoff-type nonlinear elliptic equation with a non-local integral term on a bounded domain in $\mathbb{R}^N$, $N \geqslant 1$, are studied by using bifurcation theory. The parameter regions of existence, non-existence and uniqueness of positive solutions are characterized by the eigenvalues of a linear eigenvalue problem and a nonlinear eigenvalue problem. Local and global bifurcation diagrams of positive solutions for various parameter regions are obtained.

Keywords: Kirchhoff-type equation; positive solution; bifurcation theory

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1. Introduction

In 1876, as a generalization of the classical wave equations describing a vibrating string, Kirchhoff [23] introduced a nonlinear wave equation of the following form:

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 \mathrm{d}x\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.1}$$

to describe the transversal oscillations of a stretched string and consider the effect of the change in the length of the string during vibration. Here $u(x,t)$ is the displacement of the string at location $x \in [0,L]$ and time t, L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density and P_0 is the initial tension. Since (1.1) contains an integral over $[0,L]$, it is no longer a pointwise identity, and therefore is often called a non-local Kirchhoff wave equation. Corresponding higher-dimensional and non-homogeneous models were developed later in, for example, [3, 30, 38], and the well-posedness, global existence of dynamical solutions of the non-local Kirchhoff wave equations have been well studied in [2, 11–13, 18] and the references cited therein.

In the last decade or so, there has also been an extensive effort to study the steady-state solutions of the non-local Kirchhoff wave equation, which satisfy a nonlinear elliptic equation with a non-local integral term:

$$\left.\begin{aligned} -\left(a+b\int_\Omega |\nabla u|^2\,\mathrm{d}x\right)\Delta u &= f(x,u) \quad \text{in } \Omega, \\ u &= 0 \qquad\quad\ \ \text{on } \partial\Omega. \end{aligned}\right\} \tag{1.2}$$

Here $a, b > 0$ are positive constants, Ω is a domain in $\mathbb{R}^N$ and $f(x,u)$ represents an external force that may depend on the vibration itself. The existence and multiplicity of solutions of (1.2) have been obtained in, for example, [1, 6, 7, 16, 19, 20, 28, 32, 37, 44, 46, 47, 50] for Ω a bounded smooth domain in $\mathbb{R}^N$ with $N = 1, 2, 3$ and f having subcritical growth. On the other hand, the case when f is subcritical and $\Omega = \mathbb{R}^N$ with $N = 3$ has been studied in [5, 15, 21, 25–27, 45, 48]. Furthermore, the critical growth case of f has been considered in [14, 26, 45], and recently the critical case for $N = 4$ was considered in [33]. In most early works, it is assumed that $N = 1, 2, 3$, and most of these works use variational methods and topological degree arguments.

Liang *et al*. [28] and Perera and Zhang [37] considered (1.2) with nonlinearity f having a prescribed asymptotic behaviour near $u = 0$ and $u = \infty$. More precisely, they assumed that, for $f_0, f_\infty < \infty$, the limits

$$\lim_{u\to 0^+}\frac{f(x,u)}{u} = f_0, \qquad \lim_{|u|\to\infty}\frac{f(x,u)}{u^3} = f_\infty \tag{1.3}$$

exist uniformly for $x \in \bar{\Omega}$. Similar problems were also considered in [6, 46, 47]. Motivated by the results in [28, 37], in this paper we study a canonical version of (1.2) with nonlinearity satisfying (1.3):

$$\left.\begin{aligned} -\left(1+\int_\Omega |\nabla u|^2\,\mathrm{d}x\right)\Delta u &= \lambda u + \mu u^3 \quad \text{in } \Omega, \\ u &= 0 \qquad\qquad\ \ \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.4}$$

where $(\lambda, \mu) \in \mathbb{R}^2$ is a parameter pair, and Ω is a smooth bounded domain in $\mathbb{R}^N$ for $N \geqslant 1$. Such a specific form of $f(x,u)$ was also used in [33] for the $N = 4$ case. We emphasize that the only restriction on the spatial dimension is $N \geqslant 1$, unless otherwise specified.

In order to state the main results of this paper, we introduce some notation and basic facts. We define

$$\lambda_1 = \inf\left\{\int_\Omega |\nabla u|^2\,\mathrm{d}x\colon u \in H_0^1(\Omega),\ \int_\Omega |u|^2\,\mathrm{d}x = 1\right\}, \tag{1.5}$$

where $H_0^1(\Omega)$ is the usual Sobolev space defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\| = \left(\int_\Omega |\nabla u|^2\,\mathrm{d}x\right)^{1/2}.$$

Then it is well known that λ_1 is the principal eigenvalue of the problem

$$\left.\begin{aligned} -\Delta\varphi &= \lambda\varphi \quad \text{in } \Omega, \\ \varphi &= 0 \qquad \text{on } \partial\Omega. \end{aligned}\right\} \tag{1.6}$$

Moreover, λ_1 is a simple eigenvalue of (1.6); the associated eigenfunction φ_1 can be chosen as positive in Ω and any eigenfunction corresponding to an eigenvalue larger than λ_1 must change sign. In the following we also assume that φ_1 is scaled so that $\int_\Omega \varphi_1^2 \, \mathrm{d}x = 1$. On the other hand, we define

$$\mu_1 = \inf \left\{ \left(\int_\Omega |\nabla u|^2 \, \mathrm{d}x \right)^2 : u \in H_0^1(\Omega), \int_\Omega |u|^4 \, \mathrm{d}x = 1 \right\} \geqslant 0. \tag{1.7}$$

If $N = 1, 2, 3$, as shown in theorem 2.4, $\mu_1 > 0$ is the principal eigenvalue of the problem

$$\left.\begin{aligned} -\left(\int_\Omega |\nabla \phi|^2 \right) \Delta\phi &= \mu\phi^3 \quad \text{in } \Omega, \\ \phi &= 0 \qquad \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.8}$$

and there exists a corresponding eigenfunction $\phi_1 > 0$ in Ω. We assume that ϕ_1 is scaled so that $\int_\Omega \phi_1^4 \, \mathrm{d}x = 1$. When $N = 4$ we still have $\mu_1 > 0$, but it cannot be achieved by any $\phi \in H_0^1(\Omega)$, and when $N \geqslant 5$ we have $\mu_1 = 0$ (see theorem 2.4 for more details). Another critical threshold value for the existence of positive solutions of (1.4) is

$$\bar{\mu} = \frac{\lambda_1^2}{\int_\Omega \varphi_1^4 \, \mathrm{d}x} > 0, \tag{1.9}$$

where (λ_1, φ_1) is the principal eigenpair of (1.6) defined above. Indeed, we can show that $\bar{\mu} > \mu_1$ (see lemma 3.3).

Equation (1.4) can be viewed as a linear combination of the linear eigenvalue equation (1.6) and the nonlinear eigenvalue equation (1.8). Hence, the values $\lambda = \lambda_1$, $\mu = \mu_1$ and $\mu = \bar{\mu}$ are important for the existence of positive solutions of (1.4). Our existence, non-existence and uniqueness results for the positive solutions of (1.4) are summarized as follows.

(1) Assume that $N \geqslant 1$. Then (1.4) has only the trivial solution $u = 0$ when $(\lambda, \mu) \in \mathrm{I} = \{(\lambda, \mu) \in \mathbb{R}^2 : \lambda \leqslant \lambda_1, \ \mu \leqslant \mu_1\}$ (proposition 2.2).

(2) Assume that $N \geqslant 1$. Then (1.4) has a unique positive solution when $(\lambda, \mu) \in \mathrm{IV} = \{(\lambda, \mu) \in \mathbb{R}^2 : \lambda > \lambda_1, \ \mu \leqslant 0\}$ (theorem 3.6).

(3) Assume that $N \geqslant 1$. Then (1.4) has at least one positive solution when $(\lambda, \mu) \in A = \{(\lambda, \mu) \in \mathbb{R}^2 : \lambda_1 < \lambda < \lambda_1 + \varepsilon^*(\mu), \ \mu_1 < \mu < \bar{\mu}\}$ or when $(\lambda, \mu) \in B = \{(\lambda, \mu) \in \mathbb{R}^2 : \lambda_1 - \varepsilon^*(\mu) < \lambda < \lambda_1, \ \mu > \bar{\mu}\}$ (theorem 3.4).

(4) Assume that $N = 1, 2, 3$. Then (1.4) has at least one positive solution when $(\lambda, \mu) \in \mathrm{V} = \{(\lambda, \mu) \in \mathbb{R}^2 : \lambda > \lambda_1, \ 0 < \mu < \mu_1\}$ (theorem 3.5 or [28, theorem 1.1]).

(5) Assume that $N \geqslant 4$ and Ω is star-shaped. Then (1.4) has only the trivial solution $u = 0$ when $(\lambda, \mu) \in \mathrm{II} = \{(\lambda, \mu) \in \mathbb{R}^2 : \lambda \leqslant 0, \ \mu \geqslant 0\}$ (proposition 2.2).

The parameter regions I–VI defined above are illustrated in figure 1. Note that the parameter region V does not exist for $N \geqslant 5$ as $\mu_1 = 0$, and the non-existence of

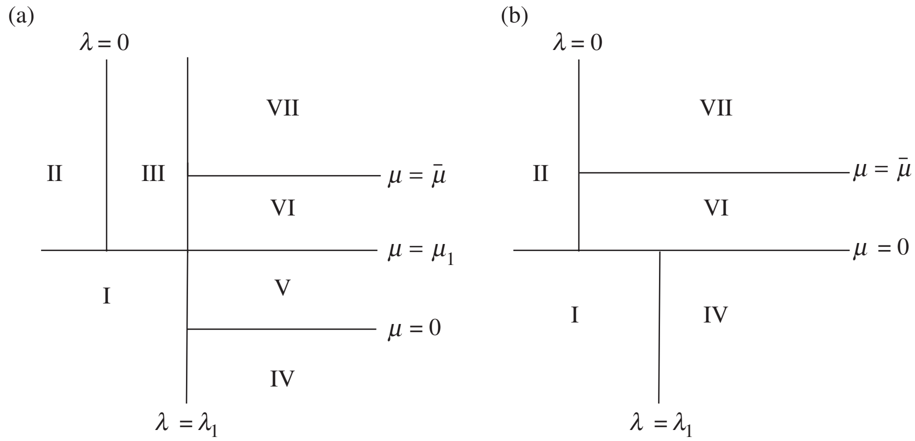

Figure 1. Parameter regions (a) $N = 1, 2, 3, 4$ and (b) $N \geqslant 5$ for existence, non-existence and uniqueness of positive solutions of (1.4). Here the horizontal coordinate is λ and the vertical coordinate is μ.

positive solutions for region II only holds when $N \geqslant 4$. For $(\lambda, \mu) \in$ III and $N = 1, 2, 3$ some partial existence results were proved in [6, 28], while for $(\lambda, \mu) \in$ III and $N = 4$ the existence of a positive solution was shown in [33]. In theorem 3.5, some additional results for global bifurcation of positive solutions when $(\lambda, \mu) \in \mathrm{VI} \cup \mathrm{VII}$ are also obtained. From these results, we have a clear but still incomplete picture of the existence, non-existence and uniqueness of the positive solutions of (1.4).

The existence of a positive solution of (1.4) for $N = 1, 2, 3$ and $(\lambda, \mu) \in \mathrm{V} = \{\lambda > \lambda_1,\ 0 < \mu < \mu_1\}$ has been proved in [28, theorem 1.1]. Here we give a different proof with the view of bifurcation theory given in theorem 3.5. The case when $N = 1, 2, 3$ and $(\lambda, \mu) \in \mathrm{III} = \{0 < \lambda < \lambda_1,\ \mu > \mu_1\}$ was also considered in [28, theorem 1.2], and it was shown that either (1.4) has a positive solution or a bifurcation from infinity occurs at this (λ, μ). The local bifurcation result in theorem 3.4 confirms the existence of a positive solution when $(\lambda, \mu) \in A$, which is a subset of III. Together with the global bifurcation results given in theorem 3.5, a new perspective on the positive solutions of (1.4) is gained here by using a bifurcation approach, while most previous works use variational methods.

The parameter region diagrams in figure 1 also clearly show the parameter regions where the existence/non-existence of positive solutions of (1.4) is still unknown. For region II and $N = 1, 2, 3$, the existence of positive solutions is not known; and in regions VI and VII for all $N \geqslant 1$, the existence/non-existence of positive solutions is still unclear despite some global bifurcation results in theorem 3.5 for the case $N = 1, 2, 3$. When $N = 4$ and in region V, the existence/non-existence of positive solutions is not known. All these require further investigations.

Note that the solutions of (1.4) satisfy

$$\left.\begin{aligned} -\Delta u &= \lambda' u + \mu' u^3 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.10}$$

where (λ', μ') are rescaled parameter pair. The semilinear elliptic equation (1.10) has been studied extensively since the seminal work [4]. We use the uniqueness of positive solution when $\mu' < 0$ in proving the uniqueness of positive solution of (1.4) when $\mu < 0$. The uniqueness of positive solution of (1.10) is also known when Ω is a ball and $1 \leqslant N \leqslant 4$ [34–36, 49]. Thus, another interesting open question is the uniqueness of positive solution of (1.4) when $\mu > 0$ and Ω is a ball.

In § 2 we prove some preliminary results regarding (1.4): we use Pohožaev's identity in § 2.1 to prove the non-existence of positive solutions in certain cases; some previous results on the nonlinear eigenvalue problem (1.8) are reviewed in § 2.2; explicit solutions when $\lambda = 0$ or $\mu = 0$ is discussed in § 2.3; and some abstract local and global bifurcation theorems are reviewed in § 2.4. In § 3, we prove our main results on the existence and bifurcation of positive solutions of (1.4).

2. Preliminaries

In this section we provide some preliminary results regarding (1.4).

2.1. Pohožaev's identity

First, we state a Pohožaev-type identity for solutions of (1.4). The proof is standard and it is omitted here.

Lemma 2.1. *Assume that u is a classical solution of (1.4). Then*

$$\lambda \int_\Omega u^2 \,\mathrm{d}x + \frac{\mu}{4}(4-N)\int_\Omega u^4 \,\mathrm{d}x = \frac{1}{2}\left(1 + \int_\Omega |\nabla u|^2 \,\mathrm{d}x\right)\int_{\partial\Omega} |\nabla u|^2 (\nu \cdot x)\,\mathrm{d}\sigma, \quad (2.1)$$

where ν is the unit outer normal to $\partial\Omega$.

By using the definitions of λ_1 and μ_1 as well as Pohožaev's identity, we prove the following non-existence of positive solutions for (1.4) in some parameter regions of (λ, μ).

Proposition 2.2. *Equation (1.4) has only the trivial solution if one of the following holds:*

(1) *$N \geqslant 1$, $\lambda \leqslant \lambda_1$ and $\mu \leqslant \mu_1$; or*

(2) *$N \geqslant 4$, Ω is star-shaped, $\lambda \leqslant 0$ and $\mu \geqslant 0$.*

Proof.
(1) Suppose that u is a non-trivial solution to (1.4) with $\lambda \leqslant \lambda_1$ and $\mu \leqslant \mu_1$. Then

$$\begin{aligned}\int_\Omega |\nabla u|^2 \,\mathrm{d}x + \left(\int_\Omega |\nabla u|^2 \,\mathrm{d}x\right)^2 &= \lambda \int_\Omega u^2 \,\mathrm{d}x + \mu \int_\Omega u^4 \,\mathrm{d}x \\ &\leqslant \lambda_1 \int_\Omega u^2 \,\mathrm{d}x + \mu_1 \int_\Omega u^4 \,\mathrm{d}x. \quad (2.2)\end{aligned}$$

On the other hand, it follows from (1.5) and (1.7) that

$$\int_\Omega |\nabla u|^2 \,\mathrm{d}x + \left(\int_\Omega |\nabla u|^2 \,\mathrm{d}x\right)^2 \geqslant \lambda_1 \int_\Omega u^2 \,\mathrm{d}x + \mu_1 \int_\Omega u^4 \,\mathrm{d}x. \quad (2.3)$$

Assume that u is not identically zero. If $\lambda < \lambda_1$ or $\mu < \mu_1$, then (2.2) is a strict inequality. Hence, a contradiction is reached from (2.3) and (2.2). If $\lambda = \lambda_1$ and $\mu = \mu_1$, then from (2.2) and (2.3) we must have $u = k\varphi_1$. From (1.4), we must also have that $\mu_1\varphi_1^2 \equiv \lambda_1^2$, which is impossible. Hence, $u \equiv 0$ if $\lambda \leqslant \lambda_1$ and $\mu \leqslant \mu_1$.

(2) Suppose that u is a solution to (1.4). Then from Pohožaev's identity in lemma 2.1 and the assumption that Ω is star-shaped, we know that

$$\lambda \int_\Omega u^2 \,\mathrm{d}x + \frac{\mu}{4}(4-N)\int_\Omega u^4 \,\mathrm{d}x = \frac{1}{2}\left(1 + \int_\Omega |\nabla u|^2 \,\mathrm{d}x\right)\int_{\partial\Omega} |\nabla u|^2(\nu\cdot x)\,\mathrm{d}\sigma \geqslant 0. \tag{2.4}$$

On the other hand, from $N \geqslant 4$, $\lambda \leqslant 0$ and $\mu \geqslant 0$, we get

$$\lambda \int_\Omega u^2 \,\mathrm{d}x + \frac{\mu}{4}(4-N)\int_\Omega u^4 \,\mathrm{d}x \leqslant 0. \tag{2.5}$$

Combining (2.4) and (2.5) we get $u \equiv 0$. □

2.2. A nonlinear eigenvalue problem

The existence and properties of positive solutions of (1.4) are closely related to the parameter values of λ and μ and also the nonlinear eigenvalue problem (1.8). Here we recall some well-known and some lesser known results about the solutions of (1.8). The positive solutions of (1.8) are rescaled solutions of the following equation:

$$\left.\begin{aligned} -\Delta u &= u^3 \quad \text{in } \Omega,\\ u &= 0 \quad\ \ \text{on } \partial\Omega.\end{aligned}\right\} \tag{2.6}$$

We have the following results for the positive solutions of (2.6).

Proposition 2.3. *Assume that Ω is a bounded domain in $\mathbb{R}^N$ for $N \geqslant 1$ with a smooth boundary $\partial\Omega$.*

(1) *If $N = 1, 2, 3$, then (2.6) possesses at least one positive solution u_1; if $N \geqslant 4$ and Ω is star-shaped, then the only non-negative solution of (2.6) is $u = 0$.*

(2) *The positive solution of (2.6) is unique if one of the following holds:*

  (i) *Ω is an open ball in $\mathbb{R}^N$ for $N = 1, 2, 3$;*

  (ii) *$\Omega \subset \mathbb{R}^2$ is symmetric in x and y, and is convex in the x- and y-directions.*

(3) *The positive least energy solution of (2.6) is unique if $\Omega \subset \mathbb{R}^2$ is convex.*

(4) *Suppose that $N = 1, 2, 3$. Then for any $k \in \mathbb{N}$, there exists a bounded smooth domain Ω_k such that (2.6) has at least $2^k - 1$ positive solutions.*

(5) *Suppose that $\alpha > 0$. A function u_1 is a positive solution of (2.6) if and only if $u_\alpha = \alpha^{-1/2}u_1$ is a positive solution of*

$$\left.\begin{aligned} -\Delta u &= \alpha u^3 \quad \text{in } \Omega,\\ u &= 0 \qquad \text{on } \partial\Omega.\end{aligned}\right\} \tag{2.7}$$

Proof. The existence result in (1) is a standard result in variational methods (see, for example, [43, theorem I.2.1]), and the non-existence one follows from the standard Pohožaev's identity [43, lemma III.1.4].

Part (2)(i) is a classical result in [17], and part (2)(ii) was proved in [9, 10]. The result in (3) was proved in [29]. Here, the least energy solution is the one which achieves

$$\inf_{u\in H_0^1(\Omega),\ u\not\equiv 0}\bigg(\int_\Omega |\nabla u|^2\,\mathrm{d}x\bigg)\bigg(\int_\Omega |u|^4\,\mathrm{d}x\bigg)^{-1/2}. \tag{2.8}$$

Finally, (4) is a result in [10], and (5) can be obtained with a simple calculation. □

We now turn to the nonlinear eigenvalue problem (1.8). A real number $\mu_* \in \mathbb{R}$ is an eigenvalue of (1.8) if there exists a $u(\not\equiv 0) \in H_0^1(\Omega)$ satisfying (1.8) for $\mu = \mu_*$, and $\mu = u_*$ is called a principal eigenvalue of (1.8) if μ_* is the least element of the set of eigenvalues of (1.8). An eigenvalue μ_* of (1.8) is simple if any two solutions u_1 and u_2 of (1.8) with $\mu = \mu_*$ satisfies $u_1 = ku_2$ for $k \in \mathbb{R}$. By using the results in proposition 2.3, we prove the following results on the nonlinear eigenvalue problem (1.8).

THEOREM 2.4. *Assume that Ω is a bounded domain in $\mathbb{R}^N$ for $N \geqslant 1$ with a smooth boundary $\partial\Omega$, and let μ_1 be defined as in (1.7).*

(1) *If $1 \leqslant N \leqslant 4$, then $\mu_1 > 0$, and when $N = 1, 2, 3$ μ_1 is the principal eigenvalue of (1.8) and there exists a corresponding eigenfunction $\phi_1 > 0$ with $\int_\Omega \phi_1^4\,\mathrm{d}x = 1$.*

(2) *If $N \geqslant 5$, then $\mu_1 = 0$.*

(3) *$\mu = \mu_1$ is a simple eigenvalue of (1.8) if one of conditions (2) or (3) in proposition 2.3 is satisfied. Moreover, if one of the conditions in proposition 2.3(2) is satisfied, then any eigenfunction of (1.8) corresponding to an eigenvalue larger than μ_1 must change sign.*

(4) *Suppose that $N = 1, 2, 3$. Then, for any $k \in \mathbb{N}$, there exists a bounded smooth domain Ω_k such that (1.8) possesses at least $2^k - 1$ eigenvalues μ_i satisfying $\mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_{2^k-1}$ and the corresponding eigenfunction ϕ_i is positive, $i = 1, 2, 3, \dots, 2^k - 1$.*

Proof.
(1) If $1 \leqslant N \leqslant 4$, then the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$ implies that $\mu_1 > 0$. Moreover, when $N = 1, 2, 3$, the embedding $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$ is also compact. Hence, μ_1 defined in (1.7) can be achieved by some $u \in H_0^1(\Omega)$. Then it is standard to prove that μ_1 is the principal eigenvalue of (1.8) and there exists a corresponding eigenfunction $\phi_1 > 0$ with $\int_\Omega \phi_1^4\,\mathrm{d}x = 1$. It is also known that, when $N = 4$, $\mu_1 > 0$ but this cannot be achieved in $H_0^1(\Omega)$ (see, for example, [43, remark I.4.7, theorem III.1.2]).

(2) To prove that $\mu_1 = 0$ when $N \geqslant 5$, without loss of generality we may assume that $0 \in \Omega$. Define

$$U(x) = \frac{(N(N-2))^{(N-2)/4}}{(1+|x|^2)^{(N-2)/2}}$$

and let $\xi \in C_0^\infty(\Omega)$ be a non-negative function satisfying $\xi \equiv 1$ in $B_\rho(0) \subset \Omega$ for some $\rho > 0$. We define, for $0 < \varepsilon < \rho$,

$$U_\varepsilon(x) = \varepsilon^{(2-N)/2} U(x/\varepsilon), \quad u_\varepsilon(x) = \xi(x) U_\varepsilon(x).$$

By computation, we have

$$\int_\Omega |\nabla u_\varepsilon(x)|^2 \,\mathrm{d}x = S^{N/2} + O(\varepsilon^{N-2}),$$

where

$$S = \inf_{u \in D^{1,2}(\mathbb{R}^N), u \not\equiv 0} \left(\int_{\mathbb{R}^N} |\nabla u|^2 \,\mathrm{d}x \right) \left(\int_{\mathbb{R}^N} |u|^{2N/(N-2)} \,\mathrm{d}x \right)^{-(N-2)/2}.$$

On the other hand, we have

$$\int_\Omega |u_\varepsilon(x)|^4 \,\mathrm{d}x \geqslant \int_{B(0,\varepsilon)} U_\varepsilon^4(x) \,\mathrm{d}x = c\varepsilon^{4-N},$$

for some $c > 0$. Hence, for $N \geqslant 5$, we know that

$$0 \leqslant \mu_1 \leqslant \left(\int_\Omega |\nabla u_\varepsilon(x)|^2 \,\mathrm{d}x \right)^2 \left(\int_\Omega |u_\varepsilon(x)|^4 \,\mathrm{d}x \right)^{-1} \to 0 \quad \text{as } \varepsilon \to 0.$$

This proves that $\mu_1 = 0$ when $N \geqslant 5$.

(3) We prove only the claim that $\mu = \mu_1$ is a simple eigenvalue of (1.8) if the conditions in proposition 2.3(3) are satisfied. The proof of other claims here is similar to that in [28, lemma 5.3]. Suppose that u_1 and u_2 are positive solutions of (1.8) with $\mu = \mu_1$. Then both u_1 and u_2 are minimizers of the minimization problem (2.8). Then, from proposition 2.3(3) or [29, theorem 1], $u_1 \equiv k u_2$ for some $k > 0$, as the minimizer of (2.8) is unique in this case. Hence, μ_1 is a simple eigenvalue of (1.8) if the conditions in proposition 2.3(3) are satisfied.

(4) Assume that $N = 1, 2, 3$. For given $k \in \mathbb{N}$, it follows from proposition 2.3(4) that there exists a bounded smooth domain Ω_k such that (2.6) has at least $2^k - 1$ positive solutions v_i, $i = 1, 2, \dots, 2^k - 1$. Without loss of generality, we may assume that

$$\mu_i = \left(\int_\Omega |\nabla v_i|^2 \,\mathrm{d}x \right)^2 \left(\int_\Omega |v_i|^4 \,\mathrm{d}x \right)^{-1}, \quad i = 2, \dots, 2^k - 1,$$

satisfying $\mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_{2^k-1}$. Then μ_i is an eigenvalue of (1.8) and the corresponding eigenfunction $\phi_i = v_i$ is positive, $i = 1, 2, \dots, 2^k - 1$. □

Remark 2.5. The results stated in theorem 2.4(1) when $N = 1, 2, 3$ are shown in [37]. Other results collected here are also more or less known, and we recall them here for applications. The results in theorem 2.4(4) suggest that in general μ_1 may not be simple, and there may also exist eigenvalues $\mu_i > \mu_1$ that still have positive eigenfunctions.

2.3. Degenerate cases

In this subsection we consider the cases when either $\lambda = 0$ or $\mu = 0$ in (1.4). In these cases, positive solutions of (1.4) are constant multiples of positive eigenfunctions φ_1 of (1.6) or ϕ_1 of (1.8).

THEOREM 2.6. *Assume that Ω is a bounded domain in $\mathbb{R}^N$ for $N \geqslant 1$ with a smooth boundary $\partial\Omega$.*

(1) *If $\lambda = 0$ and $N = 1, 2, 3$, then (1.4) has a positive solution u_μ when*

$$\mu > \mu_1 \quad \text{and} \quad u_\mu = \frac{\mu_1^{1/4}}{\sqrt{\mu - \mu_1}}\phi_1. \tag{2.9}$$

(2) *If $\mu = 0$ and $N \geqslant 1$, then (1.4) has a unique positive solution u_λ when*

$$\lambda > \lambda_1 \quad \text{and} \quad u_\lambda = \frac{\sqrt{\lambda - \lambda_1}}{\lambda_1}\varphi_1. \tag{2.10}$$

Proof.
(1) Assume that $\lambda = 0$ and $N = 1, 2, 3$. Then (1.4) possesses a positive solution u_μ if and only if u_μ is a positive solution of

$$\left.\begin{aligned} -\Delta u = \mu\left(1 + \int_\Omega |\nabla u|^2 \,\mathrm{d}x\right)^{-1} u^3 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{2.11}$$

Hence,

$$\mu\left(1 + \int_\Omega |\nabla u_\mu|^2 \,\mathrm{d}x\right)^{-1} = \mu_1\left(\int_\Omega |\nabla u_\mu|^2 \,\mathrm{d}x\right)^{-1}$$

and $u_\mu = k\phi_1$. From theorem 2.4, (1.8) has a positive solution ϕ_1 with $\mu = \mu_1$ and $\int_\Omega \phi_1^4 \,\mathrm{d}x = 1$. Then, by a direct calculation, we know that (μ, u_μ) given as in (2.9) is a positive solution of (1.4) with $\lambda = 0$.

(2) Assume that $\mu = 0$ and $\lambda > \lambda_1$. Then (1.4) possesses a positive solution u_λ if and only if u_λ is a positive solution of

$$\left.\begin{aligned} -\Delta u = \lambda\left(1 + \int_\Omega |\nabla u|^2 \,\mathrm{d}x\right)^{-1} u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{2.12}$$

Hence,

$$\lambda\left(1 + \int_\Omega |\nabla u_\lambda|^2 \,\mathrm{d}x\right)^{-1} = \lambda_1$$

and $u_\lambda = k\varphi_1$ with $k > 0$, which imply that (1.4) possesses exactly one positive solution u_λ when (2.10) is satisfied. $\square$

We note that the solutions given in (2.9) are on the boundary between regions II and III, while those given in (2.10) are on the boundary between regions V and IV when $N = 1, 2, 3, 4$, or between IV and VI when $N \geqslant 5$. The form in (2.9) suggests that $(\lambda, \mu) = (0, \mu_1)$ is a parameter point where a bifurcation from infinity occurs. In the case when $\lambda = 0$ and $\mu > \mu_1$, if (1.8) has another solution (μ_2, ϕ_2) with $\mu_2 \geqslant \mu_1$ and $\phi_2 > 0$, then another solution of (1.4) can be obtained, as in (2.9) with (μ_1, ϕ_1) replaced by (μ_2, ϕ_2).

2.4. Bifurcation theorems

Before concluding this section, we recall some global and local bifurcation theorems that will be used to prove some of the main theorems in our paper. The following are [42, theorems 4.3 and 4.4], respectively, and generalize earlier results in [8, 39].

Theorem 2.7. *Assume that X and Y are Banach spaces. Let U be an open connected subset of $\mathbb{R} \times X$ and $(\lambda_0, u_0) \in U$, and let F be a continuously differentiable mapping from U into Y. Suppose that*

(1) *$F(\lambda, u_0) = 0$ for $(\lambda, u_0) \in U$,*

(2) *the partial derivative $F_{\lambda u}(\lambda, u)$ exists and is continuous in (λ, u) near (λ_0, u_0),*

(3) *$F_u(\lambda_0, u_0)$ is a Fredholm operator with index 0, and $\dim \mathcal{N}(F_u(\lambda_0, u_0)) = 1$,*

(4) *$F_{\lambda u}(\lambda_0, u_0)[w_0] \notin \mathcal{R}(F_u(\lambda_0, u_0))$, where $w_0 \in X$ spans $\mathcal{N}(F_u(\lambda_0, u_0))$.*

Let Z be any complement of $\operatorname{span}\{w_0\}$ in X. Then there exist an open interval $I = (-\varepsilon, \varepsilon)$ and continuous functions $\lambda\colon I \to \mathbb{R}$, $\psi\colon I \to Z$, such that $\lambda(0) = \lambda_0$, $\psi(0) = 0$, and if $u(s) = u_0 + sw_0 + s\psi(s)$ for $s \in \mathrm{I}$, then $F(\lambda(s), u(s)) = 0$. Moreover, $F^{-1}(\{0\})$ near (λ_0, u_0) consists precisely of the curves $u = u_0$ and $\Gamma = \{(\lambda(s), u(s))\colon s \in \mathrm{I}\}$. If, in addition,

(5) *$F_u(\lambda, u)$ is a Fredholm operator for all $(\lambda, u) \in U$,*

then the curve Γ is contained in $\mathcal{C}$, which is a connected component of $\bar{S}$ where $S = \{(\lambda, u) \in U\colon F(\lambda, u) = 0,\ u \neq u_0\}$; and either $\mathcal{C}$ is not compact in U or $\mathcal{C}$ contains a point (λ_*, u_0) with $\lambda_* \neq \lambda_0$.

Theorem 2.8. *Suppose that all the conditions in theorem 2.7 are satisfied. Let $\mathcal{C}$ be defined as in theorem 2.7. We define $\Gamma_+ = \{(\lambda(s), u(s))\colon s \in (0, \epsilon)\}$ and $\Gamma_- = \{(\lambda(s), u(s))\colon s \in (-\epsilon, 0)\}$. In addition, we assume that*

(1) *$F_u(\lambda, u_0)$ is continuously differentiable in λ for $(\lambda, u_0) \in U$,*

(2) *the norm function $u \mapsto \|u\|$ in X is continuously differentiable for any $u \neq 0$,*

(3) *for $k \in (0, 1)$, if (λ, u_0) and (λ, u) are both in U, then $(1 - k)F_u(\lambda, u_0) + kF_u(\lambda, u)$ is a Fredholm operator.*

Let $\mathcal{C}^+$ (respectively, $\mathcal{C}^-$) be the connected component of $\mathcal{C} \setminus \Gamma_-$ that contains Γ_+ (respectively, the connected component of $\mathcal{C} \setminus \Gamma_+$ that contains Γ_-). Then each of *the sets $\mathcal{C}^+$ and $\mathcal{C}^-$ satisfies one of the following:*

(i) *it is not compact;*

(ii) *it contains a point (λ_*, u_0) with $\lambda_* \neq \lambda_0$; or*

(iii) *it contains a point $(\lambda, u_0 + z)$, where $z \neq 0$ and $z \in Z$.*

3. Bifurcation

In this section, we use a bifurcation approach to study the positive solutions of (1.4). For that purpose we set $X = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and $Y = L^p(\Omega)$ for $p > \max\{N, 2\}$. Because of the smoothness of nonlinearities in (1.4), the weak solution $u \in X$ is indeed a classical solution. Assuming $\mu \in \mathbb{R}$ is fixed, we define a nonlinear operator $F\colon \mathbb{R} \times X \to Y$ by

$$F(\lambda, u) = \left(1 + \int_\Omega |\nabla u|^2 \, \mathrm{d}x\right) \Delta u + \lambda u + \mu u^3. \tag{3.1}$$

First, we show that the nonlinear mapping F is continuously differentiable and we calculate the Fréchet derivatives of F. The proof is standard and omitted here.

LEMMA 3.1. *$F \in C^3(\mathbb{R} \times X, Y)$ and for $(\lambda, \mu) \in \mathbb{R}^2$ and $u, v, w, z \in X$ we have the following:*

$$\begin{aligned}
F_u(\lambda, u)[w] &= \left(1 + \int_\Omega |\nabla u|^2 \, \mathrm{d}x\right) \Delta w \\
&\quad + 2\left(\int_\Omega \nabla u \nabla w \, \mathrm{d}x\right) \Delta u + \lambda w + 3\mu u^2 w; \\
F_\lambda(\lambda, u) &= u, \qquad F_{\lambda u}(\lambda, u)[w] = w; \\
F_{uu}(\lambda, u)[w, v] &= 2\left(\int_\Omega \nabla u \nabla v \, \mathrm{d}x\right) \Delta w + 2\left(\int_\Omega \nabla u \nabla w \, \mathrm{d}x\right) \Delta v \\
&\quad + 2\left(\int_\Omega \nabla w \nabla v \, \mathrm{d}x\right) \Delta u + 6\mu u w v; \\
F_{uuu}(\lambda, u)[w, v, z] &= 2\left(\int_\Omega \nabla z \nabla v \, \mathrm{d}x\right) \Delta w + 2\left(\int_\Omega \nabla z \nabla w \, \mathrm{d}x\right) \Delta v \\
&\quad + 2\left(\int_\Omega \nabla w \nabla v \, \mathrm{d}x\right) \Delta z + 6\mu w v z.
\end{aligned}$$

Second, we prove that F is a Fredholm operator of index 0 for any $(\lambda, u) \in \mathbb{R} \times X$, and F_u can be extended to a self-adjoint operator.

LEMMA 3.2. *Let $\mu \in \mathbb{R}$ be fixed. Then for every $(\lambda, u) \in \mathbb{R} \times X$,*

(1) *$F_u(\lambda, u)\colon X \to Y$ is a Fredholm operator of index zero,*

(2) *$F_u(\lambda, u)\colon L^2(\Omega) \to L^2(\Omega)$ is a densely defined closed symmetric operator,*

(3) *$F_u(\lambda, u)$ can be extended to a self-adjoint operator from $H^2(\Omega) \cap H_0^1(\Omega)$ to $L^2(\Omega)$, and the spectrum of $F_u(\lambda, u)$ consists of real eigenvalues.*

Proof.
(1) For given $\lambda, \mu \in \mathbb{R}$ and $u \in X$, define $q(x) = \lambda + 3\mu u^2$. Then $q \in L^\infty(\Omega)$. Hence, $L_q \colon X \to Y$ defined by

$$L_q[w] = \left(1 + \int_\Omega |\nabla u|^2 \,\mathrm{d}x\right)\Delta w + q(x)w$$

is a Fredholm operator of index zero. Moreover, the map $L_1 \colon X \to Y$ defined by

$$L_1[w] = 2\left(\int_\Omega \nabla u \nabla w \,\mathrm{d}x\right)\Delta u$$

is a rank-1 operator whose range is one dimensional. Hence, $F_u(\lambda, u) = L_q + L_1$ is a Fredholm operator of index zero, as it is a compact perturbation of a Fredholm operator (see [22, theorem IV.5.26]).

(2) We denote by $\langle \cdot, \cdot \rangle$ the inner product of $L^2(\Omega)$. Then, for given $\lambda \in \mathbb{R}$, $u, w, v \in D \equiv H^2(\Omega) \cap H_0^1(\Omega)$, we have

$$\begin{aligned}\langle F_u(\lambda, u)[w], v\rangle &= \left(1 + \int_\Omega |\nabla u|^2 \,\mathrm{d}x\right)\langle \Delta w, v\rangle \\ &\quad + 2\left(\int_\Omega \nabla u \nabla w \,\mathrm{d}x\right)\langle \Delta u, v\rangle + \langle \lambda w + 3\mu u^2 w, v\rangle \\ &= \left(1 + \int_\Omega |\nabla u|^2 \,\mathrm{d}x\right)\langle \Delta v, w\rangle \\ &\quad + 2\left(\int_\Omega \nabla u \nabla v \,\mathrm{d}x\right)\langle \Delta u, w\rangle + \langle \lambda v + 3\mu u^2 v, w\rangle \\ &= \langle F_u(\lambda, u)[v], w\rangle.\end{aligned}$$

Hence, $F_u(\lambda, u)$ is a symmetric operator densely defined on $L^2(\Omega)$.

(3) We prove that $-F_u(\lambda, u)$ is bounded from below. Since $u \in X$, we assume that $\|u\|_\infty \leqslant M$ for some $M > 0$. Then, for $w \in D$,

$$\begin{aligned}\langle -F_u(\lambda, u)[w], w\rangle &= \left(1 + \int_\Omega |\nabla u|^2 \,\mathrm{d}x\right)\int_\Omega |\nabla w|^2 \,\mathrm{d}x \\ &\quad + 2\left(\int_\Omega \nabla u \nabla w \,\mathrm{d}x\right)^2 - \lambda \int_\Omega w^2 \,\mathrm{d}x - 3\mu \int_\Omega u^2 w^2 \\ &\geqslant -(\lambda + 3\mu M^2)\int_\Omega w^2 \,\mathrm{d}x = c\langle w, w\rangle,\end{aligned}$$

where $c = -(\lambda + 3\mu M^2)$. Now from the Friedrichs extension theorem [24, theorem 33.4], $F_u(\lambda, u)$ can be extended to a self-adjoint operator defined on $D = H^2(\Omega) \cap H_0^1(\Omega)$. In particular, the spectrum of $F_u(\lambda, u)$ is real valued. Also, since $L_c \equiv -F_u(\lambda, u) + (-c+1) \colon D \to L^2(\Omega)$ is positively definite, L_c is invertible by the Lax–Milgram theorem, and the inverse, $L_c^{-1} \colon L^2(\Omega) \to L^2(\Omega)$, is compact. Thus, the spectrum of L_c (and consequently the spectrum of $F_u(\lambda, u)$) consists only of eigenvalues. $\square$

Recalling $\bar{\mu}$ defined in (1.9), we have the following relation between $\bar{\mu}$ and μ_1 defined in (1.7).

LEMMA 3.3. *Let $\bar{\mu}$ and μ_1 be defined as in (1.9) and (1.7), respectively. Then $\bar{\mu} > \mu_1$.*

Proof. If $\mu_1 = 0$, then it is clear that $\bar{\mu} > \mu_1$. So we assume that $\mu_1 > 0$. Since $\bar{\mu} = \lambda_1^2 / \int_\Omega \varphi_1^4 \, \mathrm{d}x = \|\varphi_1\|^4 / \|\varphi_1\|_4^4$ and $\varphi_1 \in H_0^1(\Omega)$, we know that $\bar{\mu} \geqslant \mu_1$ by (1.7). Suppose that $\bar{\mu} = \mu_1$. Then φ_1 satisfies (1.8) with $\mu = \mu_1$, and a similar argument to that in the proof of proposition 2.2 leads to φ_1 equalling a constant, which is impossible. Hence, $\bar{\mu}$ cannot be achieved by φ_1, and consequently we have $\bar{\mu} > \mu_1$. □

Now we consider the bifurcation of positive solutions of (1.4) from the line of trivial solutions $\Gamma_0 = \{(\lambda, 0) \in \mathbb{R} \times X : \lambda \in \mathbb{R}\}$, where $\mu \in \mathbb{R}$ is fixed. First we have the following local bifurcation result for any bounded smooth domain $\Omega \subset \mathbb{R}^N$ for $N \geqslant 1$ and any fixed $\mu \in \mathbb{R}$.

THEOREM 3.4. *Assume that Ω is a bounded domain in $\mathbb{R}^N$ for $N \geqslant 1$ with a smooth boundary $\partial\Omega$. Let $\mu \in \mathbb{R}$ be a fixed constant. Then the principal eigenvalue $\lambda = \lambda_1$ of (1.6) is a bifurcation point of (1.4) where solutions of (1.4) bifurcate from the line of trivial solutions $\Gamma_0 = \{(\lambda, 0) : \lambda \in \mathbb{R}\}$; near $(\lambda_1, 0)$, all the solutions of (1.4) lie on a smooth curve $\Gamma_1 = \{(\lambda(s), u(s)) : s \in (-\delta, \delta)\}$ for some $\delta > 0$ such that $s \mapsto (\lambda(s), u(s))$ is a smooth function from $(-\delta, \delta)$ to $\mathbb{R} \times X$, $\lambda(0) = \lambda_1$, $\lambda'(0) = 0$, $u(s) = s\varphi_1 + o(s)$ and*

$$\lambda''(0) = 2\left(\lambda_1^2 - \mu \int_\Omega \varphi_1^4 \, \mathrm{d}x\right). \tag{3.2}$$

In particular, if $\mu > \bar{\mu}$, then there exists $\varepsilon_* = \varepsilon_*(\mu) > 0$ such that (1.4) has at least one positive solution when $\lambda \in (\lambda_1 - \varepsilon_*, \lambda_1)$, and if $\mu < \bar{\mu}$, then there exists $\varepsilon^* = \varepsilon_*(\mu) > 0$ such that (1.4) has at least one positive solution when $\lambda \in (\lambda_1, \lambda_1 + \varepsilon^*)$.

Proof. Fix $\mu \in \mathbb{R}$. Define a nonlinear operator $F : \mathbb{R} \times X \to Y$ as in (3.1). We consider the bifurcation of solutions to $F(\lambda, u) = 0$ at $(\lambda, u) = (\lambda_1, 0)$. By lemma 3.1, we know that

$$\left.\begin{aligned} F(\lambda, 0) &= 0, & F_u(\lambda_1, 0)[w] &= \Delta w + \lambda_1 w, \\ F_{\lambda u}(\lambda_1, 0)[w] &= w, & F_{uu}(\lambda_1, 0)[w, v] &= 0, \end{aligned}\right\} \tag{3.3}$$

and

$$\begin{aligned} F_{uuu}(\lambda_1, 0)[w, v, z] = 2\left(\int_\Omega \nabla z \nabla v \, \mathrm{d}x\right)\Delta w + 2\left(\int_\Omega \nabla z \nabla w \, \mathrm{d}x\right)\Delta v \\ + 2\left(\int_\Omega \nabla w \nabla v \, \mathrm{d}x\right)\Delta z + 6\mu z w v, \end{aligned} \tag{3.4}$$

where $\lambda \in \mathbb{R}, w, v, z \in X$. It is easy to verify that the kernel $\mathcal{N}(F_u(\lambda_1, 0)) = \operatorname{span}\{\varphi_1\}$, the range space

$$\mathcal{R}(F_u(\lambda_1, 0)) = \left\{ w \in Y : \int_\Omega \varphi_1(x) w(x) \, \mathrm{d}x = 0 \right\} \tag{3.5}$$

and $F_{\lambda u}(\lambda_1, 0)[\varphi_1] = \varphi_1$. It follows from $\int_\Omega \varphi_1^2(x)\,\mathrm{d}x = 1$ that $F_{\lambda u}(\lambda_1, 0)[\varphi_1] \notin \mathcal{R}(F_u(\lambda_1, 0))$. Thus, conditions (1)–(4) in theorem 2.7 are all satisfied, and we can apply the first part of theorem 2.7 to conclude that the set of solutions to (1.4) near $(\lambda_1, 0)$ is a smooth curve $\Gamma_1 = \{(\lambda(s), u(s)) \colon s \in (-\delta, \delta)\}$ such that $\lambda(0) = \lambda_1, u(s) = s\varphi_1 + o(s)$. Moreover, $\lambda'(0)$ and $\lambda''(0)$ can be calculated as in [31, 40]:

$$\lambda'(0) = -\frac{\langle l, F_{uu}(\lambda_1, 0)[\varphi_1, \varphi_1]\rangle}{2\langle l, F_{\lambda u}(\lambda_1, 0)[\varphi_1]\rangle} = 0, \tag{3.6}$$

and

$$\begin{aligned}\lambda''(0) &= -\frac{\langle l, F_{uuu}(\lambda_1, 0)[\varphi_1, \varphi_1, \varphi_1]\rangle + 3\langle l, F_{uu}(\lambda_1, 0)[\varphi_1, 0]\rangle}{3\langle l, F_{\lambda u}(\lambda_1, 0)[\varphi_1]\rangle} \\ &= -\frac{1}{\langle l, \varphi_1\rangle}\left(2\left\langle l, \int_\Omega |\nabla\varphi_1|^2\,\mathrm{d}x\,\Delta\varphi_1 + \mu\varphi_1^3\right\rangle\right) \\ &= 2\left[\left(\int_\Omega |\nabla\varphi_1|^2\,\mathrm{d}x\right)^2 - \mu\int_\Omega \varphi_1^4\,\mathrm{d}x\right] \\ &= 2\left(\lambda_1^2 - \mu\int_\Omega \varphi_1^4\,\mathrm{d}x\right), \end{aligned} \tag{3.7}$$

where l is a linear functional on Y defined as $\langle l, w\rangle = \int_\Omega \varphi_1(x)w(x)\,\mathrm{d}x$.

When $\mu > \bar{\mu}$, we have $\lambda''(0) < 0$ from (3.7). Then there exists some $\varepsilon_* > 0$ such that (1.4) has a pair of solutions when $\lambda \in (\lambda_1 - \varepsilon_*, \lambda_1)$. When $s \in (0, \delta)$, $u(s) = s\varphi_1 + o(s) > 0$ is a positive solution, and clearly $u(-s) = -u(s)$ from the oddness of (1.4). The case for $\mu < \bar{\mu}$ is similar. □

From the global bifurcation result in theorem 2.7, the set Γ_1 is indeed a part of global continuum. To further explore the global nature of the set of positive solutions of (1.4), we define

$$S_\mu = \{(\lambda, u) \in \mathbb{R} \times X \colon (\lambda, u) \text{ is a positive solution of (1.4)}\}. \tag{3.8}$$

We shall discuss the nature of the connected component Σ_1 of $\bar{S}_\mu$ containing $(\lambda_1, 0)$ for the following parameter ranges for μ:

- $0 < \mu < \mu_1$;
- $\mu > \mu_1$; and
- $\mu \leqslant 0$.

In the following result, we consider the $0 < \mu < \mu_1$ and $\mu > \mu_1$ cases, and in theorem 3.6 we consider the $\mu \leqslant 0$ case.

THEOREM 3.5. *Assume that Ω is a bounded domain in $\mathbb{R}^N$ for $N \geqslant 1$ with a smooth boundary $\partial\Omega$. Let $\Gamma_1 = \{(\lambda(s), u(s)) \colon s \in (-\delta, \delta)\}$ be the curve of non-trivial solutions of (1.4) in theorem 3.4, and let S_μ be defined as in (3.8). Then there exists a connected component Σ_1 of $\bar{S}_\mu$ containing $\{(\lambda(s), u(s)) \colon s \in (0, \delta)\}$, and Σ_1 is unbounded in $\mathbb{R} \times X$. Moreover, let $\mathrm{Proj}_\lambda\,\Sigma_1$ be the projection of Σ_1 into* *the λ-axis, and assume that $\mu_1 > 0$. Then*

(1) *for $0 < \mu < \mu_1$ and $N = 1, 2, 3$,* $\operatorname{Proj}_\lambda \Sigma_1 = (\lambda_1, \infty)$,

(2) *for $\mu > \mu_1$, if one of the following conditions is satisfied:*

(a) *Ω is an open ball in $\mathbb{R}^N$ for $N = 1, 2, 3$;*

(b) *$\Omega \subset \mathbb{R}^2$ is symmetric in x and y, and convex in x- and y-directions; or*

(c) *$\Omega \subset \mathbb{R}^2$ is convex,*

then $\operatorname{Proj}_\lambda \Sigma_1$ *is unbounded so that either*

$$(\lambda_1, \infty) \subset \operatorname{Proj}_\lambda \Sigma_1 \quad \textit{or} \quad (-\infty, \lambda_1) \subset \operatorname{Proj}_\lambda \Sigma_1. \tag{3.9}$$

Proof. We first we prove that Σ_1 is unbounded in $\mathbb{R} \times X$. We apply theorem 2.8 at $(\lambda_1, 0)$ with $U = \mathbb{R} \times X$. In the proof of theorem 3.4, we showed that conditions (1)–(4) of theorem 2.7 are satisfied. By lemma 3.2(1), theorem 2.7(5) is also met, and the proof of lemma 3.2 also implies that theorem 2.8(3) is satisfied. Theorem 2.8(1) is clearly satisfied by lemma 3.1, and finally theorem 2.8(2) is satisfied for X and Y chosen here from the remark after [42, theorem 4.4].

Now we can apply theorem 2.8 to obtain a connected component $\mathcal{C}^+$ of the set $\mathcal{C} \setminus \Gamma_-$ of solutions of (1.4) emanating from $(\lambda_1, 0)$, where $\Gamma_- = \{(\lambda(s), u(s))\colon s \in (-\delta, 0)\}$. Clearly, $\mathcal{C}^+$ contains $\Gamma_+ = \{(\lambda(s), u(s))\colon s \in (0, \delta)\}$. Indeed, from the maximum principle, any $(\lambda, u) \in \mathcal{C}^+$ satisfies $u > 0$ or $u \equiv 0$ for all $x \in \Omega$. Thus, $\mathcal{C}^+$ is a connected component of $\bar{S}_\mu$. In the following we call it Σ_1.

From theorem 2.8, we have the following possibilities:

(i) Σ_1 is not compact in U,

(ii) Σ_1 contains a point $(\lambda_*, 0)$ with $\lambda_* \neq \lambda_1$, or

(iii) Σ_1 contains (λ, z) for some $z(\neq 0) \in Z$, where Z is a complement of the kernel space $\operatorname{span}\{\varphi_1\}$.

The alternative (iii), is impossible as any $(\lambda, u) \in \Sigma_1$ satisfies $u > 0$ or $u \equiv 0$ for all $x \in \Omega$, while any element in Z is sign changing. Next we prove that alternative (ii) is also impossible. We claim that there is no $\lambda_* \in \mathbb{R}$ with $\lambda_* \neq \lambda_1$ such that Σ_1 contains $(\lambda_*, 0)$. Suppose on the contrary that there exists $\{(\lambda_n, u_n)\} \subset \Sigma_1$ such that $u_n \neq 0$ and

$$(\lambda_n, u_n) \to (\lambda_*, 0) \quad \text{in } \mathbb{R} \times X, \quad n \to \infty.$$

Hence, $u_n \to 0$ in $H_0^1(\Omega)$. Let $w_n = u_n/\|u_n\|$ for any n. Then we have, for any $v \in X$,

$$\int_\Omega \nabla w_n \cdot \nabla v \,\mathrm{d}x = \frac{\lambda_n}{1 + \|u_n\|^2} \int_\Omega w_n v \,\mathrm{d}x + \frac{\mu}{1 + \|u_n\|^2} \int_\Omega w_n u_n^2 v \,\mathrm{d}x. \tag{3.10}$$

Since $\{w_n\}$ is bounded in $H_0^1(\Omega)$, passing to a subsequence if necessary, we may assume that $w_n \rightharpoonup w_0 \in H_0^1(\Omega)$. Letting $n \to \infty$ in (3.10), we obtain

$$\int_\Omega \nabla w_0 \cdot \nabla v \,\mathrm{d}x = \lambda_* \int_\Omega w_0 v \,\mathrm{d}x. \tag{3.11}$$

Taking $v = w_n$ in (3.10) and again letting $n \to \infty$, we have $1 = \lambda_* \int_\Omega w_0^2 \,\mathrm{d}x$, which implies that $w_0 \not\equiv 0$ and w_0 is a non-trivial eigenfunction of (1.6) since $(0, 0)$ is not a bifurcation point. But the assumption $\lambda_* \neq \lambda_1$ implies that w_0 must be sign changing, which contradicts $w_0 \geqslant 0$. Therefore, the alternative (ii) is impossible, and we must have that Σ_1 is unbounded.

We now consider the projection of Σ_1 into the λ-axis in different cases. Here we assume that $\mu_1 > 0$. Since we have proved that Σ_1 is unbounded in $\mathbb{R} \times X$, then Σ_1 is unbounded in either $\mathbb{R}$ or X. We prove that, under some conditions, Σ_1 cannot be unbounded in X for a finite value λ_*. That is, we prove that bifurcation from infinity cannot occur at a finite $\lambda_* > 0$.

(1) Assuming that $0 < \mu < \mu_1$, we shall prove that $\mathrm{Proj}_\lambda \Sigma_1$ is unbounded. Suppose, to the contrary, $\mathrm{Proj}_\lambda \Sigma_1$ is bounded. Since Σ_1 is unbounded, there exist $(\lambda_n, u_n) \subset \Sigma_1$ such that $\lambda_n \to \lambda_0 \in \mathbb{R}$, $\|u_n\| \to \infty$. For any $v \in H_0^1(\Omega)$, we have

$$(1 + \|u_n\|^2) \int_\Omega \nabla u_n \nabla v \,\mathrm{d}x = \lambda_n \int_\Omega u_n v \,\mathrm{d}x + \mu \int_\Omega u_n^3 v \,\mathrm{d}x. \tag{3.12}$$

Let $v_n = u_n/\|u_n\|$. Then, for $v \in H_0^1(\Omega)$, we have

$$\frac{1 + \|u_n\|^2}{\|u_n\|^2} \int_\Omega \nabla v_n \nabla v \,\mathrm{d}x = \frac{\lambda_n}{\|u_n\|^2} \int_\Omega v_n v \,\mathrm{d}x + \mu \int_\Omega v_n^3 v \,\mathrm{d}x. \tag{3.13}$$

Since $\{v_n\}$ is bounded in $H_0^1(\Omega)$, we may assume, by passing to a subsequence if necessary, that $v_n \rightharpoonup v_0$ in $H_0^1(\Omega)$. Taking $v = v_n$ in (3.13) and letting $n \to \infty$, we have that $\mu \int_\Omega v_0^4 \,\mathrm{d}x = 1$. Hence, it follows from (1.7) that $\mu \geqslant \mu_1$, which contradicts $0 < \mu < \mu_1$. From elliptic regularity theory, $\mathrm{Proj}_\lambda \Sigma_1$ is unbounded in $\mathbb{R}$. By proposition 2.2, we know that $\mathrm{Proj}_\lambda \Sigma_1 = (\lambda_1, \infty)$.

(2) Similarly to (1), we have (3.13). Passing to the limit in (3.13) and passing to the limit with $v = v_n - v_0$ in (3.13), we know that

$$\int_\Omega \nabla v_0 \cdot \nabla v = \mu \int_\Omega v_0^3 v, \quad v \in H_0^1(\Omega), \tag{3.14}$$

and $\|v_0\| = 1$. Hence, v_0 is a positive solution to (1.8). Theorem 2.4 implies that $\mu = \mu_1$, which contradicts the condition $\mu > \mu_1$. Hence, $\mathrm{Proj}_\lambda \Sigma_1$ is unbounded in $\mathbb{R}$, i.e. either $(\lambda_1, \infty) \subset \mathrm{Proj}_\lambda \Sigma_1$ or $(-\infty, \lambda_1) \subset \mathrm{Proj}_\lambda \Sigma_1$. □

Finally, we consider the $\mu \leqslant 0$ case. Here we prove the existence and uniqueness of positive solution of (1.4).

THEOREM 3.6. *Assume that Ω is a bounded domain in $\mathbb{R}^N$ ($N \geqslant 1$) with a smooth boundary $\partial\Omega$. Suppose that $\mu \leqslant 0$. Then (1.4) has no positive solution if $\lambda \leqslant \lambda_1$, and it has exactly one positive solution, u_λ, if $\lambda > \lambda_1$.*

Proof. The $\mu = 0$ case has been considered in theorem 2.6. Hence, we only consider the $\mu < 0$ case. The non-existence of a positive solution of (1.4) when $\lambda \leqslant \lambda_1$ follows from proposition 2.2. So, in the following, we assume that $\mu < 0$ and $\lambda > \lambda_1$.

(1) We prove the existence of a positive solution to (1.4). Assume that $\lambda > \lambda_1$. For the existence of a positive solution of (1.4), we define $g\colon [0,\infty) \to \mathbb{R}$,

$$g(r) = \begin{cases} \dfrac{1}{1+r}\displaystyle\int_\Omega \lambda\left(1+\frac{\mu}{\lambda}u_r^2\right)u_r^2\,\mathrm{d}x, & 0 \leqslant r < \dfrac{\lambda-\lambda_1}{\lambda_1}, \\ 0, & r \geqslant \dfrac{\lambda-\lambda_1}{\lambda_1}, \end{cases} \tag{3.15}$$

where u_r is the unique positive solution of

$$\left.\begin{aligned} -(1+r)\Delta u_r = \lambda\left(1+\frac{\mu}{\lambda}u_r^2\right)u_r \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{3.16}$$

From [41, theroem 2.3], (3.16) has a unique positive solution u_r if and only if $\lambda/(1+r) > \lambda_1$, which is equivalent to $r < (\lambda-\lambda_1)/\lambda_1$. Moreover, the result in [41, theroem 2.3] implies that u_r is continuously differentiable and decreasing in r for $r \in [0,(\lambda-\lambda_1)/\lambda_1)$, and that for $r_0 = (\lambda-\lambda_1)/\lambda_1$, $\lim_{r\to r_0^-} u_r = 0$ uniformly for $x \in \bar{\Omega}$. Hence, $g(r)$ is well defined and continuous on $[0,\infty)$. Since $0 \leqslant u_r(x) \leqslant \sqrt{-\lambda/\mu}$ for any $r \in [0,r_0)$ and $x \in \bar{\Omega}$, we have

$$0 \leqslant g(r) = \frac{1}{1+r}\int_\Omega \lambda\left(1+\frac{\mu}{\lambda}u_r^2\right)u_r^2\,\mathrm{d}x \leqslant \frac{\lambda^2|\Omega|}{(1+r)|\mu|} \leqslant \frac{\lambda^2|\Omega|}{|\mu|}, \quad r \in [0,\infty). \tag{3.17}$$

Moreover, $g(0) > 0$, and hence there exists $R \in (0,r_0)$ such that $g(R) = R$ and $u_\lambda = u_R$ is a positive solution of (1.4) for the given λ since

$$R = g(R) = \frac{\lambda}{1+R}\int_\Omega \left(1+\frac{\mu}{\lambda}u_R^2\right)u_R^2\,\mathrm{d}x = \int_\Omega |\nabla u_R|^2\,\mathrm{d}x. \tag{3.18}$$

(2) We prove the uniqueness of the positive solution of (1.4) for given $\lambda > \lambda_1$. Assume that w and v are two positive solutions of (1.4). We claim that

$$\int_\Omega |\nabla w|^2\,\mathrm{d}x = \int_\Omega |\nabla v|^2\,\mathrm{d}x,$$

which guarantees that $w = v$ from the uniqueness of the positive solution of (3.16). Suppose, by contradiction, that

$$r_1 = \int_\Omega |\nabla w|^2\,\mathrm{d}x > \int_\Omega |\nabla v|^2\,\mathrm{d}x = r_2.$$

Then, from the decreasing property of positive solutions to (3.16), we have $u_{r_1} = w < v = u_{r_2}$. Hence,

$$\begin{aligned} 0 &< \int_\Omega (|\nabla w|^2 - |\nabla v|^2)\,\mathrm{d}x \\ &= \int_\Omega \nabla(w-v)\nabla(w+v)\,\mathrm{d}x \\ &= -\int_\Omega (w-v)\Delta(w+v)\,\mathrm{d}x \\ &\leqslant 0, \end{aligned} \tag{3.19}$$

which is a contradiction. Hence, we must have

$$\int_\Omega |\nabla w|^2 \, \mathrm{d}x = \int_\Omega |\nabla v|^2 \, \mathrm{d}x,$$

and consequently $w \equiv v$. □

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