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Calculus of Variations

Existence and multiplicity of positive solutions to Schrödinger–Poisson type systems with critical nonlocal term

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Abstract The existence, nonexistence and multiplicity of positive radially symmetric solutions to a class of Schrödinger–Poisson type systems with critical nonlocal term are studied with variational methods. The existence of both the ground state solution and mountain pass type solutions are proved. It is shown that the parameter ranges of existence and nonexistence of positive solutions for the critical nonlocal case are completely different from the ones for the subcritical nonlocal system.

Mathematics Subject Classification 35J50 · 35J47 · 35J61 · 47J30

1 Introduction

In this paper, we consider positive solutions to the nonlinear Schrödinger–Poisson type system

$$\begin{cases} -\Delta u + u + \lambda\phi|u|^3 u = \mu|u|^{p-1}u, & \text{in } \mathbb{R}^3, \\ -\Delta\phi = |u|^5, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\lambda \in \mathbb{R}$ and $\mu \geqslant 0$ are physical parameters. The investigation of (1.1) is motivated by recent studies of Schrödinger–Poisson system

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$$\begin{cases} -\Delta u + bu + \lambda\phi g(u) = f(u), & \text{in } \mathbb{R}^3, \\ -\Delta\phi = 2G(u), & \text{in } \mathbb{R}^3, \end{cases} \tag{1.2}$$

where the functions $g(u)$ and $G(u)$ satisfy $|g(u)| \leqslant C(|u| + |u|^q)$ for some $q \in [1, 4)$, $G(u) = \int_0^u g(t)dt$, and $f(u)$ satisfies $|f(u)| \leqslant C(|u| + |u|^p)$ for some $p \in (1, 5]$. For subcritical nonlinearity f with $p \in (1, 5)$ and subcritical nonlocal term g with $q \in [1, 4)$, the problem (1.2) was studied in, for example, [4,16]. In [4], system (1.2) on a bounded domain $\Omega \subset \mathbb{R}^3$ was studied for positive and negative values of λ. In [16], system (1.2) was considered and it was shown that there exists a positive solution for small $\lambda \geqslant 0$. For the special case $g(u) = u$, there have been many studies of system (1.2), see for example, [2,10,13,18,21–23,28,29,31,33]. In [2,10,22], the existence and multiplicity of positive radial solutions was considered for various λ and p; when b depends on x and is not radial, and f is asymptotically linear, the existence of positive solution for small $\lambda > 0$ and the nonexistence of nontrivial solution for large $\lambda > 0$ were obtained in [28]; when b depends on x and is sign-changing, the existence and multiplicity was shown in [31]; when b depends on x, the existence of a sign-changing solution was proved in [29]; existence of a nontrivial solution and concentration results for $p \in (3, 5)$ were shown in [27,33]. Moreover the ground state solution for system (1.2) with $g(u) = u$ have been considered in [5,6,26,27], and the ground and bound states for system (1.2) with $g(u) = u$ and $b = 0$ were studied in [13,23].

However, all the papers mentioned above studied the subcritical nonlocal term. The critical case for the nonlocal term corresponds to $q = 4$ like in (1.1). In this paper, the nonlocal term is of critical growth, and it is much more difficult to obtain the existence of positive solutions. Firstly the critical growth causes a lack of compactness; secondly the critical nonlocal term case competes with the critical nonlinearity case which has been studied previously in [32,34].

To the best of our knowledge, the system (1.1) with critical nonlocal term has only been studied in [3,15]. In [3], the authors studied the system (1.1) in a ball domain $B_R(0) \subset \mathbb{R}^3$. In [15], we studied the system (1.1) for general nonlinearity in $\mathbb{R}^3$. By using monotonicity technique, we obtained the existence of positive solutions for positive and negative λ separately [15]. These results show the subtle differences between the critical nonlocal case and the subcritical nonlocal case. In this paper we continue our studies in [15] to show some further properties of the set of positive solutions to (1.1). In particular, we prove the existence of ground state solutions and mountain-pass type solutions of (1.1) for certain parameter ranges, and we also show the nonexistence of positive solutions for other parameter ranges. These results provide a much better understanding of the solution set of (1.1).

To describe our results, we first introduce some notations. In this paper, $H^1(\mathbb{R}^3)$ and $\mathcal{D}^{1,2}(\mathbb{R}^3)$ are the usual Sobolev spaces defined by $H^1(\mathbb{R}^3) = \{u \in L^2(\mathbb{R}^3) : |\nabla u| \in L^2(\mathbb{R}^3)\}$ and $\mathcal{D}^{1,2}(\mathbb{R}^3) = \{u \in L^6(\mathbb{R}^3) : |\nabla u| \in L^2(\mathbb{R}^3)\}$. The corresponding norms and inner products are defined by

$$\|u\|_{H^1(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} \left[|\nabla u|^2 + u^2\right]\right)^{1/2}, \quad (u, v) = \int_{\mathbb{R}^3} [\nabla u \cdot \nabla v + uv], \quad u, v \in H^1(\mathbb{R}^3),$$

$$\|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^{1/2}, \quad (u, v) = \int_{\mathbb{R}^3} \nabla u \cdot \nabla v, \quad u, v \in \mathcal{D}^{1,2}(\mathbb{R}^3).$$

Here we use the well-known Sobolev embedding in the definition of $\mathcal{D}^{1,2}(\mathbb{R}^3)$. Indeed we consider the system (1.1) in the subspace $H_r^1(\mathbb{R}^3)$ of $H^1(\mathbb{R}^3)$ which consists of all radial functions of $H^1(\mathbb{R}^3)$. In the following we use $\|\cdot\|$ as the norm of $H_r^1(\mathbb{R}^3)$, $(\cdot, \cdot)$ as the inner product in $H_r^1(\mathbb{R}^3)$, $\langle\cdot,\cdot\rangle$ as the pairing of $[H_r^1(\mathbb{R}^3)]^*$ (the dual space) and $H_r^1(\mathbb{R}^3)$, and $|\cdot|_p$ is the norm of $L^p(\mathbb{R}^3)$ for $p \in [1,\infty)$.

First we have the following nonexistence results of nontrivial solutions of (1.1).

Theorem 1.1 *Suppose that $\lambda \in \mathbb{R}$ and $\mu \in [0,\infty)$.*

1. *For $p = 1$, system (1.1) has no nontrivial solution for $\lambda \geqslant 0$ and $\mu \geqslant 0$, or $\lambda < 0$ and $\mu \neq 1$.*
2. *For $p = 5$, system (1.1) has no nontrivial solution for $\lambda \in \mathbb{R}$ and $\mu \geqslant 0$.*
3. *For $p \in (1,5)$, system (1.1) has no nontrivial solution for $\lambda \geqslant \mu^{8/(p-1)}/4$ and $\mu > 0$, or $\lambda \in \mathbb{R}$ and $\mu = 0$.*

In particular, this implies the following result for the case of $\mu = 0$.

Corollary 1.2 *For $p \in [1,5]$, system (1.1) has no nontrivial solution for $\lambda \in \mathbb{R}$ and $\mu = 0$.*

For the nonlocal system (1.2) with subcritical growth function $g(u)$, it is well-known that there exists a positive solution when $\lambda < 0$ and $\mu = 0$ [17]. So the nonexistence result for the critical nonlocal system in Corollary 1.2 shows that the critical case is quite different from the subcritical nonlocal system. On the other hand, when $\lambda = 0$ and $\mu > 0$, the nonexistence result for $p = 5$ is well known [19].

Next we define the energy functional corresponding to (1.1), the notion of ground state solution and mountain-pass solution of system (1.1). Define a functional $J_\lambda : H^1(\mathbb{R}^3) \to \mathbb{R}$ by

$$J_\lambda(u) = \frac{1}{2}\|u\|^2 + \frac{\lambda}{10}\int_{\mathbb{R}^3}\phi_u|u|^5 - \frac{\mu}{p+1}\int_{\mathbb{R}^3}|u|^{p+1},\quad u \in H^1(\mathbb{R}^3),$$

where ϕ_u is the unique solution of $-\Delta\phi = |u|^5$ in $\mathbb{R}^3$. Then J_λ is well-defined on $H^1(\mathbb{R}^3)$ and is of C^1 class for any $\lambda \in \mathbb{R}$ and $\mu \geqslant 0$, and

$$\langle J_\lambda'(u), v\rangle = (u,v) + \lambda\int_{\mathbb{R}^3}\phi_u|u|^3uv - \mu\int_{\mathbb{R}^3}|u|^{p-1}uv,\quad u,v \in H^1(\mathbb{R}^3).$$

It is standard to verify that a critical point u of the functional J_λ corresponds to a weak solution (u,ϕ_u) of system (1.1). Thus, in what follows, we consider critical points of J_λ using variational methods.

We recall the following definitions of types of critical points of functionals.

Definition 1.3 Let X be a Banach space and let $J \in C^1(X)$ be a functional.

1. If $u_0 \in X$ is a critical point of J and $u_0 \neq 0$ such that $J(u_0) = \inf_{u\in X} J(u)$, then we say that u_0 is a ground state solution of $J' = 0$. So the ground state realizes the minimum of the energy among critical points.
2. Let Ω be a bounded open subset of X, and let $e_1, e_2 \in X$ with $e_1 \in \Omega$ and $e_2 \notin \overline{\Omega}$. If $\inf_{u\in\partial\Omega} J(u) > \max\{J(e_1), J(e_2)\}$, we say that J satisfies the mountain-pass geometric structure. Let

$$\Gamma = \{\gamma \in C[0,1] : \gamma(0) = e_1, \gamma(1) = e_2\},$$

and

$$c = \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} J(\gamma(t)).$$

If u_0 is a critical point of J with $J(u) = c$, we say that u_0 is a mountain-pass type solution of $J' = 0$.

In the following existence results, we work in the space $X = H_r^1(\mathbb{R}^3)$. Next we have the following existence result of positive radially symmetric ground state solutions which are the global minimizers of the energy functional in the space X of radial functions.

Theorem 1.4 *For any $\mu > 0$ and $p \in (1, 5)$, there exists $\lambda_0 \in (0, \mu^{8/(p-1)}/4)$ such that system (1.1) has a positive radially symmetric ground state solution $u_{1,\lambda}$ for $\lambda \in (0, \lambda_0]$. Furthermore, the solutions $(\lambda, u_{1,\lambda})$ satisfy that $\lim_{\lambda\to 0^+} ||u_{1,\lambda}|| = \infty$, that is, $(\lambda, u) = (0, \infty)$ is a bifurcation point for system (1.1) (see Fig. 1).*

Note that $u_{1,\lambda}$ found here has the smallest energy among all non-trivial radially symmetric solutions, but it is not known whether its energy is the smallest among all non-trivial solutions. On the other hand we show that for a wider range of parameter values, a mountain-pass type solution exists.

Theorem 1.5 *For any $\mu > 0$ and $p \in (1, 5)$, there exists a $\lambda_1 \in (\lambda_0, \mu^{8/(p-1)}/4)$ such that system (1.1) has a positive radially symmetric mountain-pass type solution $u_{2,\lambda}$ for $\lambda \in (0, \lambda_1)$. Moreover,*

1. *When $\lambda \to \lambda_1^-$, the limit $\lim_{\lambda\to\lambda_1^-} u_{2,\lambda} = u_{2,\lambda_1}$ (strongly in $H^1(\mathbb{R}^3)$) exists, and u_{2,λ_1} is a nontrivial positive solution of (1.1) for $\lambda = \lambda_1$.*
2. *When $\lambda \to 0^+$, the following alternatives hold:*

(2a) *either $u_{2,\lambda}$ is bounded in $H^1(\mathbb{R}^3)$ as $\lambda \to 0^+$, and $\lim\limits_{\lambda\to 0^+} u_{2,\lambda} = u_0$ is the unique positive solution of the equation*

$$-\Delta u + u = \mu|u|^{p-1}u, \quad u \in H^1(\mathbb{R}^3); \tag{1.3}$$

(2b) *or $\lim\limits_{\lambda\to 0^+} ||u_{2,\lambda}|| = \infty$, and there exists a $\lambda_2 \in (0, \lambda_0)$ such that system (1.1) has at least three positive solutions for $\lambda \in (0, \lambda_2)$.*

The two diagrams in Fig. 1 show the two alternatives in Theorem 1.5 (2a) and (2b). It would be interesting to know which diagram cannot occur, and it is also interesting to see whether the branch of ground state solutions is connected to the branch of mountain pass type solutions. For the special case $\lambda = 0$, the system (1.1) is reduced to (1.3). The existence of a positive solution u_0 to (1.3) is well-known [19,24], and it is known that any positive solution u_0 of (1.3) must be radially symmetric [12]. The uniqueness of u_0 was proved in [14], and it is also known that u_0 is the ground state solution of (1.3) [8]. This positive solution u_0 can be obtained by using the mountain pass theorem in the space of radially symmetric functions, which is the strategy used in the proof of Theorem 1.5. So in some sense, the results in Theorem 1.5 is an analog of the classical results for the case $\lambda = 0$. But here for (1.1), the radial symmetry of all positive solutions for $\lambda > 0$ is still not known, and our result implies that the uniqueness of positive solution does not hold for (1.1) and $\lambda > 0$.

The results in Theorem 1.5 imply that system (1.1) possesses multiple positive solutions for small $\lambda > 0$ and fixed $\mu > 0$. In the case (2a), system (1.1) has at least two positive solutions for $\lambda \in (0, \lambda_0]$ and has at least one positive solution for $\lambda \in (\lambda_0, \lambda_1]$. See Fig. 1, left panel. On the other hand, from the pictures: an analogue in the case (2b), system (1.1) has at least three positive solutions for $\lambda \in (0, \lambda_2)$, has at least two positive solutions for $\lambda \in (\lambda_2, \lambda_0]$, and has at least one positive solution for $\lambda \in (\lambda_0, \lambda_1]$, see Fig. 1 right panel.

We emphasize some distinctive characters of our results in the following remarks.

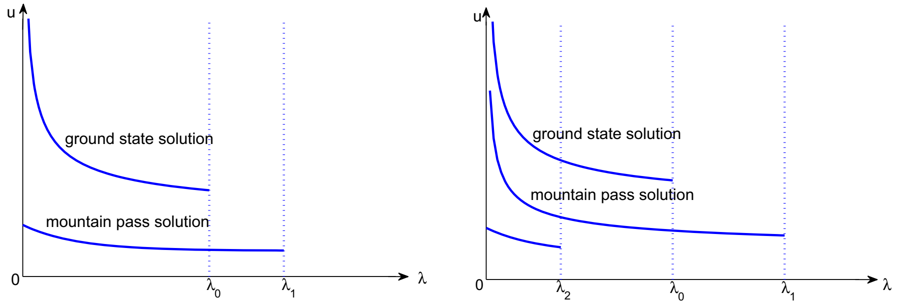

Fig. 1 Illustration of the set of positive solutions of (1.1)

Remark 1.6 The existence/non-existence results for positive solutions for (1.1) are different from the ones for the system with subcritical nonlocal term

$$\begin{cases} -\Delta u + u + \lambda\phi u = |u|^{p-1}u, & \text{in } \mathbb{R}^3, \\ -\Delta\phi = |u|^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.4}$$

which was studied in [22]. In [22], the author obtained the existences of two positive solutions of (1.4) for $p \in (1, 2)$ and small $\lambda > 0$, and one positive solution for $p \in [2, 5)$ and all $\lambda > 0$. For (1.1), Theorem 1.1 shows that no positive solution of (1.1) exists for large $\lambda > 0$ and $p \in (1, 5)$. On the other hand, Theorems 1.4 and 1.5 show that (1.1) always have two positive solutions for small $\lambda > 0$. Thus the parameter ranges for existence and nonexistence for (1.1) and (1.4) are very different.

Remark 1.7 From the proof of Lemma 3.9, for the case of $\lambda < 0$, we can obtain the existence of one positive solution for (1.1) for small $|\lambda|$ and $\lambda < 0$ or $\lambda = -1$ (see [15]). However, we do not know whether a positive solution exists for all $\lambda \in (-1, 0)$ or for $\lambda \in (-\infty, -1)$. In [15], we have shown that for any $\lambda < 0$, there exists a positive solution to the system with subcritical nonlocal term

$$\begin{cases} -\Delta u + u + \lambda\phi|u|^{q-1}u = |u|^{p-1}u, & \text{in } \mathbb{R}^3, \\ -\Delta\phi = |u|^{q+1}, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.5}$$

where $q \in [1, 4)$. This again shows the difference between the system with critical nonlocal term and the system with subcritical nonlocal term.

The remaining part of the paper is structured as follows: In Sect. 2, we give some preliminaries including (i) provide some basic setup of the problem studied, (ii) recall some convergence lemmas, (iii) introduce the Pohozaev identity and give the proof of nonexistence results, (iv) prove the boundedness of Palais–Smale sequences. In Sect. 3, we prove all other main results.

In what follows we frequently use the following inequality

$$S|u|^2_{L^6(\mathbb{R}^3)} \leqslant \|u\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^3)}, \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^3), \tag{1.6}$$

and

$$S|u|^2_{L^6(\mathbb{R}^3)} \leqslant \|u\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^3)} \leqslant \|u\|^2_{H^1(\mathbb{R}^3)}, u \in H^1(\mathbb{R}^3). \tag{1.7}$$

Note that the best constant S can be achieved (see [25,30]).

2 Preliminaries

2.1 Variational setup

For a given $u \in H^1(\mathbb{R}^3)$, the second equation of (1.1) is a Poisson equation for ϕ which is uniquely solvable. Then (1.1) can be reduced to the first equation with ϕ represented by the solution of the Poisson equation. This is a basic strategy of solving (1.1). To be more precise about the solution ϕ of the Poisson equation, we recall the following lemma from [15, Lemma 2.1].

Lemma 2.1 *For every $u \in L^6(\mathbb{R}^3)$, there exists a unique $\phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ which is the solution of*

$$-\Delta\phi = |u|^5, \quad \text{in } \mathbb{R}^3, \tag{2.1}$$

here ϕ_u can be expressed by the form

$$\phi_u(x) = \int_{\mathbb{R}^3} \frac{|u|^5(y)}{|x-y|} \mathrm{d}y.$$

Moreover,

(i) $\|\phi_u\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^3)} = \int_{\mathbb{R}^3} \phi_u |u|^5$;
(ii) $\phi_u(x) > 0$ *for* $x \in \mathbb{R}^3$;
(iii) *for any* $\theta > 0$, $\phi_{u_\theta} = \theta^2 (\phi_u)_\theta$, *where* $u_\theta(\cdot) = u(\cdot/\theta)$;
(iv) *for any* $t > 0$, $\phi_{tu} = t^5 \phi_u$;
(v) *for any* $u \in L^6(\mathbb{R}^3)$,

$$\|\phi_u\|_{\mathcal{D}^{1,2}(\mathbb{R}^3)} \leqslant S^{-1/2}|u|_6^5, \quad \int_{\mathbb{R}^3} \phi_u|u|^5 \leqslant S^{-1}|u|_6^{10},$$

where S is defined in (1.6);
(vi) *if u is a radial function, then ϕ_u is also radial;*
(vii) *if $u_n \rightharpoonup u$ in $L^6(\mathbb{R}^3)$ and $u_n \to u$ a.e. in $\mathbb{R}^3$ as $n \to \infty$, then $\phi_{u_n} \rightharpoonup \phi_u$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$.*

The result in Lemma 2.1 implies that the system (1.1) can be reduced to a semilinear nonlocal elliptic equation

$$-\Delta u + u + \lambda\phi_u|u|^3 u = \mu|u|^{p-1}u, \quad \text{in } \mathbb{R}^3, \tag{2.2}$$

where ϕ_u is defined in Lemma 2.1.

We say a few words about the regularity of solutions to (1.1). Let u be a critical point of J_λ. Then $u \in H^1(\mathbb{R}^3)$, and from the Sobolev embedding theorem, $u \in L^6(\mathbb{R}^3)$. We then deduce that $\phi \in L^6(\mathbb{R}^3)$ from Lemma 2.1. It follows that $-\Delta u = a(x)u$, where $a(x) = \mu|u|^{p-1} - 1 - \lambda\phi|u|^3$. Since $p \in [1, 5]$, then $\mu|u|^{p-1} - 1 \in L^{3/2}_{\text{loc}}(\mathbb{R}^3)$ and $\lambda\phi|u|^3 \in L^{3/2}(\mathbb{R}^3)$. The Brézis-Kato Theorem [9] implies that $u \in L^r_{\text{loc}}(\mathbb{R}^3)$ for all $r \in [1, \infty)$. Therefore, $u \in W^{2,r}_{\text{loc}}(\mathbb{R}^3)$ for all $r \in [1, \infty)$. From elliptic regularity theory, it follows that $u, \phi \in C^2(\mathbb{R}^3)$. This shows that a weak solution $u \in H^1(\mathbb{R}^3)$ is indeed a classical solution of (2.2), and in the following we look for positive solutions of the Eq. (2.2) in the space $H^1(\mathbb{R}^3)$.

2.2 Brézis–Lieb type convergence lemma

To prove the convergence of the nonlocal term in (1.1), we recall the following Brézis–Lieb type lemmas which are proved in [15, Lemmas 2.2, 2.3].

Lemma 2.2 *Let $r \geqslant 1$ and let Ω be an open subset of $\mathbb{R}^N$. Suppose that $u_n \rightharpoonup u$ in $L^r(\Omega)$, and $u_n \to u$ a.e. in Ω as $n \to \infty$, then for $p \in [1, r]$, as $n \to \infty$,*

$$|u_n|^p - |u_n - u|^p - |u|^p \to 0, \quad \text{in } L^{r/p}(\Omega), \tag{2.3}$$

$$|u_n|^{p-1}u_n - |u_n - u|^{p-1}(u_n - u) - |u|^{p-1}u \to 0, \quad \text{in } L^{r/p}(\Omega). \tag{2.4}$$

Lemma 2.3 *If $u_n \rightharpoonup u$ in $L^6(\mathbb{R}^3)$ and $u_n \to u$ a.e. in $\mathbb{R}^3$, then as $n \to \infty$,*

$$|u_n|^5 - |u_n - u|^5 - |u|^5 \to 0, \quad \text{in } L^{6/5}(\mathbb{R}^3), \tag{2.5}$$

$$|u_n|^3 u_n - |u_n - u|^3(u_n - u) - |u|^3 u \to 0, \quad \text{in } L^{3/2}(\mathbb{R}^3), \tag{2.6}$$

$$\phi_{u_n} - \phi_{u_n - u} - \phi_u \to 0, \quad \text{in } \mathcal{D}^{1,2}(\mathbb{R}^3), \tag{2.7}$$

$$\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^5 - \int_{\mathbb{R}^3} \phi_{u_n - u}|u_n - u|^5 - \int_{\mathbb{R}^3} \phi_u |u|^5 \to 0. \tag{2.8}$$

2.3 Pohozaev identity and nonexistence of solutions

In [15], a general Pohozaev identity is established, and the following one is a special case.

Lemma 2.4 *If $u \in H^1(\mathbb{R}^3)$ is a weak solution of (1.1) with $p \in [1, 5]$, then the following Pohozaev type identity holds*

$$\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{3}{2}\int_{\mathbb{R}^3} u^2 + \frac{\lambda}{2}\int_{\mathbb{R}^3} \phi_u |u|^5 = \frac{3}{p+1}\mu \int_{\mathbb{R}^3} |u|^{p+1}. \tag{2.9}$$

Proof Applying Lemma 2.8 of [15] with $f(u) = \mu|u|^{p-1}u$, we obtain (2.9). □

By using the Pohozaev identity (2.9), we prove Theorem 1.1 in the following four theorems.

Theorem 2.5 *Suppose that $\lambda \in \mathbb{R}$, $\mu \geqslant 0$ and $p = 5$, then there is no nontrivial solution for (1.1).*

Proof Let u be a weak solution of (1.1). Then by the Pohozaev identity (2.9), we have that

$$\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{3}{2}\int_{\mathbb{R}^3} u^2 + \frac{\lambda}{2}\int_{\mathbb{R}^3} \phi |u|^5 = \frac{\mu}{2}\int_{\mathbb{R}^3} |u|^6. \tag{2.10}$$

On the other hand, it follows from $J_\lambda'(u) = 0$ that

$$\int_{\mathbb{R}^3} |\nabla u|^2 + \int_{\mathbb{R}^3} u^2 + \lambda \int_{\mathbb{R}^3} \phi |u|^5 = \mu \int_{\mathbb{R}^3} |u|^6. \tag{2.11}$$

Subtracting (2.10) and one half of (2.11), we obtain that $\int_{\mathbb{R}^3} u^2 = 0$, which implies that $u = 0$. □

Theorem 2.6 *For $\lambda \geqslant 0$ and $\mu \geqslant 0$ or $\lambda < 0$ and $\mu \neq 1$, there is no nontrivial solution for*

$$\begin{cases} -\Delta u + u + \lambda \phi |u|^3 u = \mu u, & \text{in } \mathbb{R}^3, \\ -\Delta \phi = |u|^5, & \text{in } \mathbb{R}^3. \end{cases} \tag{2.12}$$

Proof Let u be a weak solution of (2.12). By the Pohozaev identity (2.9), we have that

$$\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2+\frac{3}{2}\int_{\mathbb{R}^3}u^2+\frac{\lambda}{2}\int_{\mathbb{R}^3}\phi|u|^5=\frac{3}{2}\mu\int_{\mathbb{R}^3}u^2. \tag{2.13}$$

On the other hand, it follows from $J'_\lambda(u)=0$ that

$$\int_{\mathbb{R}^3}|\nabla u|^2+\int_{\mathbb{R}^3}u^2+\lambda\int_{\mathbb{R}^3}\phi|u|^5=\mu\int_{\mathbb{R}^3}u^2. \tag{2.14}$$

Subtracting (2.13) and (3/2) times of (2.14), we have $\int_{\mathbb{R}^3}\left(|\nabla u|^2+\lambda\phi_u|u|^5\right)=0$, which implies that $u=0$ when $\lambda\geqslant 0$. On the other hand, subtracting (2.13) and one half of (2.14), we have $(1-\mu)\int_{\mathbb{R}^3}u^2=0$, which implies that $u=0$ for $\lambda<0$ and $\mu\neq 1$. □

Theorem 2.7 *If $\mu=0$, then for any $p\in[1,5]$, the system (1.1) has no nontrivial solution.*

Proof We can prove it with the same proof as the one for Theorem 2.5 with $\mu=0$. □

Theorem 2.8 *Let $\lambda\geqslant\mu^{8/(p-1)}/4$, $\mu\geqslant 0$ and $p\in(1,5)$. Then the system (1.1) has no nontrivial solution.*

Proof Let $\alpha=4/(p-1)$. If $(u,\phi_u)\in H^1(\mathbb{R}^3)\times\mathcal{D}^{1,2}(\mathbb{R}^3)$ is a solution of (1.1), then it follows from (i) of Lemma 2.1 that for all $u\in H^1(\mathbb{R}^3)$,

$$\mu^\alpha\int_{\mathbb{R}^3}|u|^6=-\mu^\alpha\int_{\mathbb{R}^3}|u|\Delta\phi_u=\mu^\alpha\int_{\mathbb{R}^3}\nabla\phi_u\cdot\nabla|u|\leqslant\int_{\mathbb{R}^3}|\nabla u|^2+\frac{1}{4}\mu^{2\alpha}\int_{\mathbb{R}^3}\phi_u|u|^5. \tag{2.15}$$

On the other hand, multiplying the first equation of (1.1) by u and integrating, we obtain that

$$\int_{\mathbb{R}^3}\left(|\nabla u|^2+u^2+\lambda\phi_u|u|^5-\mu|u|^{p+1}\right)=0. \tag{2.16}$$

Thus, it follows from (2.15) and (2.16) that

$$0=\int_{\mathbb{R}^3}\left(|\nabla u|^2+u^2+\lambda\phi_u|u|^5-\mu|u|^{p+1}\right)\geqslant\int_{\mathbb{R}^3}\left(u^2+\mu^\alpha|u|^6-\mu|u|^{p+1}\right).$$

Since the function $f(x)=x^2+\mu^\alpha x^6-\mu|x|^{p+1}\geqslant 0$ for all $x\in\mathbb{R}$, then we must have $u=0$. □

From the nonexistence results proved in this subsection, system (1.1) can only have nontrivial solutions when $\lambda\in\mathbb{R}$, $\mu>0$ and $p\in(1,5)$.

2.4 The boundedness of Palais–Smale sequence

The following lemma is useful in the variational setting involving the Palais–Smale condition.

Lemma 2.9 *Fix $\lambda>0$, $\mu>0$ and $p\in(1,5)$. Let $\{u_n\}\subset H^1_r(\mathbb{R}^3)$ be a sequence satisfying $J_\lambda(u_n)\leqslant C$ or $\langle J'_\lambda(u_n),u_n\rangle\leqslant C$ for all $n\in\mathbb{N}$, where C is a positive constant. Then $\{u_n\}$ is uniformly bounded in $H^1_r(\mathbb{R}^3)$.*

Proof First we assume that $J_\lambda(u_n) \leqslant C$. Since

$$\alpha_\lambda \int_{\mathbb{R}^3} |u_n|^6 = \alpha_\lambda \int_{\mathbb{R}^3} \nabla\phi_{u_n} \cdot \nabla|u_n| \leqslant \int_{\mathbb{R}^3} \left(\frac{1}{4}|\nabla u_n|^2 + \frac{\lambda}{20}|\nabla\phi_{u_n}|^2\right),$$

where $\alpha_\lambda = \sqrt{\lambda/20} > 0$, then it follows that

$$C \geqslant J_\lambda(u_n) \geqslant \int_{\mathbb{R}^3} \left(\frac{1}{4}|\nabla u_n|^2 + \frac{1}{2}u_n^2 + \frac{\lambda}{20}\phi_{u_n}|u_n|^5 + \alpha_\lambda|u_n|^6 - \frac{\mu}{p+1}|u_n|^{p+1}\right). \tag{2.17}$$

Let the function $h : \mathbb{R}_+ \to \mathbb{R}$ be defined by

$$h(s) := \frac{1}{4}s^2 + \alpha_\lambda s^6 - \frac{\mu}{p+1}s^{p+1}, \quad s \in [0,\infty).$$

It is clear that for $\lambda > 0$ sufficiently large, $h(s) \geqslant 0$ for all $s \geqslant 0$. Since $p \in (1,5)$, $h(0) = 0$ and $h(s) \to \infty$ as $s \to \infty$, and then the infimum $h_0 := \inf_{s\in[0,\infty)} h(s) = h(s_0)$ is attained at some $s_0 \in [0,\infty)$. If $h_0 = 0$, then it follows from (2.17) that $\|u_n\| = [\int_{\mathbb{R}^3}(|\nabla u_n|^2 + u_n^2)]^{1/2} \leqslant 4C$. If $h_0 = -m < 0$, then we can obtain that $\{s > 0 : h(s) < 0\} = (a,b)$ for some $b > a > 0$. Thus, by (2.17), we have that

$$\begin{aligned} C \geqslant J_\lambda(u_n) &\geqslant \frac{1}{4}\int_{\mathbb{R}^3} |\nabla u_n|^2 + \frac{1}{4}\int_{\mathbb{R}^3} u_n^2 + \frac{\lambda}{20}\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^5 + \int_{\mathbb{R}^3} h(|u_n|) \\ &\geqslant \frac{1}{4}\|u_n\|^2 + \frac{\lambda}{20}\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^5 + \int_{A_n} h(|u_n|) \\ &\geqslant \frac{1}{4}\|u_n\|^2 + \frac{\lambda}{20}\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^5 - m|A_n|, \end{aligned} \tag{2.18}$$

where $A_n = \{x \in \mathbb{R}^3 : |u_n(x)| \in (a,b)\}$ and $|A_n|$ is its Lebesgue measure. Note that A_n is spherically symmetric since u_n is radial. Suppose that $\{u_n\}$ is unbounded in $H_r^1(\mathbb{R}^3)$. Without loss of generality, we may assume that $\|u_n\| \to \infty$ as $n \to \infty$. Since $J_\lambda(u_n) \leqslant C$, then we have from (2.18) that

$$m|A_n| \geqslant \frac{1}{4}\|u_n\|^2 - C, \tag{2.19}$$

which implies that $|A_n| \to \infty$. From the Strauss inequality [30, Lemma 4.5], for any $u \in H_r^1(\mathbb{R}^3)$, we have that

$$|u(x)| \leqslant c_0|x|^{-1}\|u\|, \quad \text{a.e. } x \in \mathbb{R}^3,$$

for some $c_0 > 0$. Therefore, for every n, there exists E_n with $|E_n| = 0$ such that

$$|u_n(x)| \leqslant c_0|x|^{-1}\|u_n\|, \quad x \in \mathbb{R}^3\backslash E_n. \tag{2.20}$$

Define $B_n = A_n\backslash E_n$. Then $|A_n| = |B_n|$. Note that B_n is also spherically symmetric. Let $r_n = \sup\{|x| : x \in B_n\}$. It follows from (2.20) that $r_n < \infty$ for fixed n. By the definition of r_n, there exists $x_n \in B_n$ such that $r_n/2 \leqslant |x_n|$ and $|u_n(x_n)| \leqslant c_0|x_n|^{-1}\|u_n\|$. Thus we obtain from (2.19) that

$$0 < a \leqslant |u_n(x_n)| \leqslant 2c_0 r_n^{-1}\|u_n\| \leqslant 4c_0 r_n^{-1}(m|A_n| + C)^{1/2},$$

and consequently

$$c_1 r_n \leqslant |A_n|^{1/2} \tag{2.21}$$

for some $c_1 > 0$ and sufficiently large n.

On the other hand, since $J_\lambda(u_n) \leqslant C$, then we have from (2.18) that

$$\frac{\lambda}{20}\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^5 \leqslant m|A_n| + C.$$

However, by using Lemma 2.1 part (v), we have

$$\begin{aligned}\frac{20}{\lambda}(m|A_n| + C) &\geqslant \int_{\mathbb{R}^3}\phi_{u_n}|u_n|^5 = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u_n|^5(x)|u_n|^5(y)}{|x-y|}\mathrm{d}x\mathrm{d}y\\ &\geqslant \int_{B_n}\int_{B_n}\frac{|u_n|^5(x)|u_n|^5(y)}{|x-y|}\mathrm{d}x\mathrm{d}y \geqslant a^{10}\frac{|A_n|^2}{2r_n},\end{aligned}$$

which means that $c_2 r_n \geqslant |A_n|$ for some $c_2 > 0$ and sufficiently large n. This is a contradiction to (2.21). This proves the lemma when $J_\lambda(u_n) \leqslant C$, and the proof is similar for the case $\langle J'_\lambda(u_n), u_n\rangle \leqslant C$. □

3 Proofs of main results

In this section, we prove Theorems 1.4 and 1.5. We define the ground state energy to be

$$m_\lambda = \inf_{u\in H_r^1(\mathbb{R}^3)} J_\lambda(u). \tag{3.1}$$

First we prove that m_λ is well-defined and that J_λ satisfies Palais–Smale condition.

Lemma 3.1 *Suppose that $p \in (1,5)$ and $\mu > 0$. Then for any $\lambda > 0$, the ground state energy m_λ satisfies $-\infty < m_\lambda \leqslant 0$, and J_λ satisfies the Palais–Smale condition in $H_r^1(\mathbb{R}^3)$.*

Proof Since $J_\lambda(0) = 0$ then clearly $m_\lambda \leqslant 0$. Let $\{u_n\} \subset H_r^1(\mathbb{R}^3)$ be a sequence satisfying $J_\lambda(u_n) \to m_\lambda \leqslant 0$ as $n \to \infty$. Then $J_\lambda(u_n) \leqslant C$ for some positive C and all n. By Lemma 2.9, we have that $\{u_n\}$ is bounded, and consequently $\{J_\lambda(u_n)\}$ is also bounded which implies that $m_\lambda > -\infty$.

Let $\{u_n\} \subset H_r^1(\mathbb{R}^3)$ be a Palais–Smale sequence. It follows from Lemma 2.9 that $\{u_n\}$ is bounded. Since $p \in (1,5)$, by using the compactness of the embedding $H_r^1(\mathbb{R}^3) \hookrightarrow L^{p+1}(\mathbb{R}^3)$ [30, Corollary 1.26] or [24], we can assume that $u_n \rightharpoonup u$ in $H_r^1(\mathbb{R}^3)$, $u_n \to u$ in $L^{p+1}(\mathbb{R}^3)$ and $u_n \to u$ a.e. on $\mathbb{R}^3$ as $n \to \infty$ for some $u \in H_r^1(\mathbb{R}^3)$. This implies that

$$\int_{\mathbb{R}^3}\left(|u_n|^{p-1}u_n - |u|^{p-1}u\right)(u_n - u) \to 0, \quad \text{as} \quad n \to \infty. \tag{3.2}$$

Since $u_n \rightharpoonup u$ in $H_r^1(\mathbb{R}^3)$ as $n \to \infty$, then in view of the Sobolev's embedding theorem, we have that $u_n \rightharpoonup u$ in $L^6(\mathbb{R}^3)$ as $n \to \infty$. It follows from Lemma 2.1 part (vii) that

$$\phi_{u_n} \rightharpoonup \phi_u, \quad \text{in } \mathcal{D}^{1,2}(\mathbb{R}^3), \quad \text{as} \quad n \to \infty.$$

Let $v_n = u_n - u$. Since $\{\phi_{u_n}|u_n|^3u_n\}$ is bounded in $L^{6/5}(\mathbb{R}^3)$ and $\phi_{u_n}|u_n|^3u_n \to \phi_u|u|^3u$ a.e. on $\mathbb{R}^3$, then

$$\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^3u_n u \to \int_{\mathbb{R}^3}\phi_u|u|^5, \quad \text{as} \quad n \to \infty.$$

By using Lemma 2.3, we obtain that, as $n \to \infty$,

$$\begin{aligned}\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^3 u_n(u_n-u) &= \int_{\mathbb{R}^3} \phi_{v_n}|v_n|^5 + \int_{\mathbb{R}^3} \phi_u|u|^5 - \int_{\mathbb{R}^3} \phi_{u_n}|u_n|^3 u_n u + o(1) \\ &= \int_{\mathbb{R}^3} \phi_{v_n}|v_n|^5 + o(1).\end{aligned}$$

Thus, as $n \to \infty$,

$$\begin{aligned}&\langle J'_\lambda(u_n) - J'_\lambda(u), u_n - u\rangle \\ &= \|u_n-u\|^2 + \lambda \int_{\mathbb{R}^3} \phi_{u_n}|u_n|^3 u_n(u_n-u) - \lambda \int_{\mathbb{R}^3} \phi_u|u|^3 u(u_n-u) \\ &\quad - \mu \int_{\mathbb{R}^3} \left(|u_n|^{p-1}u_n - |u|^{p-1}u\right)(u_n-u) \\ &= \|v_n\|^2 + \lambda \int_{\mathbb{R}^3} \phi_{v_n}|v_n|^5 + o(1).\end{aligned}$$

We then conclude that $u_n \to u$ strongly in $H_r^1(\mathbb{R}^3)$ as $n \to \infty$. The proof is completed. □

We also prove that J_0 satisfies the Palais–Smale condition in $H_r^1(\mathbb{R}^3)$, which is useful in our proof.

Lemma 3.2 *Suppose that $p \in (1,5)$ and $\mu > 0$. Then J_0 satisfies the Palais–Smale condition in $H_r^1(\mathbb{R}^3)$.*

Proof The proof is well known. But for the convenience of readers, we recall it. Suppose that $\{u_n\} \subset H_r^1(\mathbb{R}^3)$, $\{J_0(u_n)\}$ is bounded and $J'_0(u_n) \to 0$ as $n \to \infty$. Then

$$J_0(u_n) - \frac{1}{p+1}\langle J'_0(u_n), u_n\rangle = \left(\frac{1}{2} - \frac{1}{p+1}\right)\|u_n\|^2,$$

which implies that $\{u_n\}$ is bounded in $H_r^1(\mathbb{R}^3)$. Thus, we may assume that $u_n \rightharpoonup u$ for some $u \in H_r^1(\mathbb{R}^3)$. It follows from $J'_0(u_n) \to 0$ and $u_n \rightharpoonup u$ that $\langle J'_0(u_n), u_n - u\rangle \to 0$ and $\langle J'_0(u), u_n - u\rangle \to 0$ as $n \to \infty$. Then we have that

$$\|u_n - u\| = \langle J'_0(u_n) - J'_0(u), u_n - u\rangle + \int_{\mathbb{R}^3} |u_n|^{p-1}u_n(u_n-u) - \int_{\mathbb{R}^3} |u|^{p-1}u(u_n-u).$$

By using the compactness of the embedding $H_r^1(\mathbb{R}^3) \hookrightarrow L^{p+1}(\mathbb{R}^3)$ [24], we conclude that $u_n \to u$ strongly in $H_r^1(\mathbb{R}^3)$. □

Now we show the existence of the ground state positive solution of (1.1).

Theorem 3.3 *Suppose that $p \in (1,5)$ and $\mu > 0$. Then there exists $\lambda_0 \in (0, \mu^{8/(p-1)}/4)$ such that the system (1.1) has at least one positive radially symmetric solution $u_{1,\lambda} \in H_r^1(\mathbb{R}^3)$ satisfying $J_\lambda(u_{1,\lambda}) = m_\lambda$ for $\lambda \in (0, \lambda_0)$.*

Proof For $\lambda > 0$, note that $u = 0$ is an isolated local minimum of the functional J_λ and $J_\lambda(0) = 0$. Let u_0 be the unique positive solution of (1.3). Since

$$J_0(tu_0) = \frac{1}{2}t^2\|u_0\|^2 - \frac{1}{p+1}t^{p+1}\int_{\mathbb{R}^3} |u_0|^{p+1}, \quad t > 0,$$

then $J_0(tu_0) \to -\infty$ as $t \to \infty$. From Lemma 3.1 m_λ exists for $\lambda > 0$. Therefore, $m_\lambda \leqslant J_\lambda(tu_0) < 0$ for $\lambda > 0$ small. Since m_λ is nondecreasing in λ, then we have that

$$\lambda_0 = \sup\{\lambda > 0 : m_\lambda < 0\} \tag{3.3}$$

exists. For $\lambda \in (0, \lambda_0)$, we have $m_\lambda < 0$, and from the Ekeland variational principle [11], a Palais–Smale sequence exists. From Lemma 3.1, J_λ satisfies the Palais–Smale condition, and consequently the infimum m_λ is attained by some $u_{1,\lambda} \in H_r^1(\mathbb{R}^3)$. This yields a nonzero solution $u_{1,\lambda}$ with $J_\lambda(u_{1,\lambda}) = m_\lambda < 0$. It follows from Theorem 2.8 that $\lambda_0 \in (0, \mu^{8/(p-1)}/4)$. By considering

$$J_\lambda^+(u) = \frac{1}{2}\|u\|^2 + \frac{\lambda}{10}\int_{\mathbb{R}^3} \phi_u |u|^5 - \frac{1}{p+1}\mu \int_{\mathbb{R}^3} (u^+)^{p+1}, \tag{3.4}$$

we can obtain the desired positive solution $u_{1,\lambda} \in H_r^1(\mathbb{R}^3)$. □

Next we show that m_λ is achieved by a nonzero solution not only for $\lambda \in (0, \lambda_0)$, but also at $\lambda = \lambda_0$, but for $\lambda \in (\lambda_0, \infty)$, $m_\lambda = 0$ and it is achieved by $u = 0$.

Lemma 3.4 *Suppose that $p \in (1, 5)$ and $\mu > 0$. Let m_λ be defined as in (3.1) and let λ_0 be defined as in (3.3).*

1. *For $0 < \lambda < \lambda_0$, $m_\lambda < 0$ and there exists $r_0 > 0$ such that the solution $u_{1,\lambda} \in H_r^1(\mathbb{R}^3)$ of (1.1) obtained in Theorem 3.3 satisfies $\|u_{1,\lambda}\| \geqslant r_0$, $J_\lambda(u_{1,\lambda}) = m_\lambda$ and $J'_\lambda(u_{1,\lambda}) = 0$.*
2. *For $\lambda = \lambda_0$, there exists a positive radially symmetric solution $u_{1,\lambda_0} \in H_r^1(\mathbb{R}^3)$ with $\|u_{1,\lambda_0}\| \geqslant r_0$ such that $J_{\lambda_0}(u_{1,\lambda_0}) = m_{\lambda_0} = 0$ and $J'_\lambda(u_{1,\lambda_0}) = 0$.*
3. *For $\lambda > \lambda_0$, $m_\lambda = 0$ and there is no nonzero critical point $u \in H_r^1(\mathbb{R}^3)$ such that $J_\lambda(u) = m_\lambda$.*

Proof If $u \neq 0$ is a solution of (1.1), then $\langle J'_\lambda(u), u\rangle = 0$ and consequently

$$\|u\|^2 \leqslant \|u\|^2 + \lambda \int_{\mathbb{R}^3} \phi_u |u|^5 = \mu |u|_{p+1}^{p+1} \leqslant C\|u\|^{p+1}. \tag{3.5}$$

Thus, J_λ has no nontrivial critical points with $\|u\| \leqslant C$ for some constant $C > 0$. On the other hand, for all $\lambda > 0$,

$$\begin{aligned} J_\lambda(u) &= \frac{1}{2}\|u\|^2 + \frac{\lambda}{10}\int_{\mathbb{R}^3} \phi_u |u|^5 - \frac{\mu}{p+1}|u|_{p+1}^{p+1} \\ &\geqslant \frac{1}{2}\|u\|^2 - \frac{\mu}{p+1}|u|_{p+1}^{p+1} \geqslant \frac{1}{2}\|u\|^2 - C_p\|u\|^{p+1}. \end{aligned} \tag{3.6}$$

Therefore, there exists an $r_0 \in (0, C)$ such that $J_\lambda(u) > 0$ for any u satisfying $0 < \|u\| \leqslant r_0$.

From the definition of m_λ, we know that $-\infty < m_\lambda < 0$ for $\lambda \in (0, \lambda_0)$. We claim that $m_{\lambda_0} = 0$. If $m_{\lambda_0} < 0$, then there exists u with $\|u\| \geqslant r_0$ such that $J_{\lambda_0}(u) < 0$, and then there exists $\varepsilon > 0$ such that $J_{\lambda_0+\varepsilon}(u) < 0$, which contradicts with the definition of λ_0. From the definition of m_λ, m_λ is nondecreasing for $\lambda \in (0, \infty)$. Then we have that $m_\lambda = 0$ for $\lambda \geqslant \lambda_0$ since $J_\lambda(0) = 0$. By using Theorem 3.3, we have that $J_\lambda(u_{1,\lambda}) = m_\lambda < 0$ and $\|u_{1,\lambda}\| \geqslant r_0$. Since $\langle J'_\lambda(u_{1,\lambda}), u_{1,\lambda}\rangle = 0$, then $\langle J'_{\lambda_0/2}(u_{1,\lambda}), u_{1,\lambda}\rangle \leqslant 0$ for $\lambda > \lambda_0/2$. Thus it follows from Lemma 2.9 that $\{u_{1,\lambda}\}$ is bounded as $\lambda \to \lambda_0^-$, and for $\varphi \in H_r^1(\mathbb{R}^3)$,

$$\langle J'_{\lambda_0}(u_{1,\lambda}), \varphi\rangle = \langle J'_\lambda(u_{1,\lambda}), \varphi\rangle + \frac{1}{10}(\lambda_0 - \lambda)\int_{\mathbb{R}^3} \phi_{u_{1,\lambda}} |u_{1,\lambda}|^3 u_{1,\lambda}\varphi \to 0, \quad \text{as } \lambda \to \lambda_0^-.$$

Therefore, $J'_{\lambda_0}(u_{1,\lambda}) \to 0$ as $\lambda \to \lambda_0^-$. By Lemma 3.1, J_{λ_0} satisfies the Palais–Smale condition, hence there is a subsequence $\{u_{1,\lambda_n}\}$ of $\{u_{1,\lambda}\}$ such that $u_{1,\lambda_n} \to u_{1,\lambda_0}$ for some $u_{1,\lambda_0} \in H_r^1(\mathbb{R}^3)$. It is obvious that $J'_{\lambda_0}(u_{1,\lambda_0}) = 0$, $J_{\lambda_0}(u_{1,\lambda_0}) = 0$ and $\|u_{1,\lambda_0}\| \geqslant r_0$. This proves the existence of a nonzero solution $u_{1,\lambda_0} \in H_r^1(\mathbb{R}^3)$ for $\lambda = \lambda_0$. Moreover u_{1,λ_0} is also nonnegative and radially symmetric since each $u_{1,\lambda}$ is positive and radially symmetric, and indeed u_{1,λ_0} is positive since $u_{1,\lambda_0} \neq 0$ and by the strong maximum principle. If $\lambda > \lambda_0$, then $m_\lambda \geqslant m_{\lambda_0} = 0$ and hence $m_\lambda = 0$. We show that there is no nonzero critical point u such that $J_\lambda(u) = 0$ for $\lambda > \lambda_0$. Suppose that there is a nonzero critical point u_λ of J_λ such that $J_\lambda(u_\lambda) = 0$, then $J_{\lambda_0}(u_\lambda) < 0$ which contradicts that $m_{\lambda_0} = 0$. Hence there is no $u \neq 0$ such that $J_\lambda(u) = 0$ for $\lambda > \lambda_0$. □

Next we show some additional properties of ground state energy level m_λ and the asymptotic behavior when $\lambda \to 0^+$.

Proposition 3.5 *Let m_λ be defined as in (3.1) and let $u_{1,\lambda}$ be a positive ground state solution of (1.1) for $\lambda \in (0, \lambda_0)$, i.e., $J_\lambda(u_{1,\lambda}) = m_\lambda$. Then*

1. *The mapping $\lambda \mapsto m_\lambda$ is continuous for $\lambda \in (0, \infty)$.*
2. $\lim_{\lambda\to 0^+} \|u_{1,\lambda}\| = \infty$, *and* $\lim_{\lambda\to 0^+} m_\lambda = -\infty$.

Proof 1. Suppose that $\lambda > 0$. Let $\{\lambda_n\}$ be a sequence satisfying $\lambda_n > \lambda$ and $\lambda_n \to \lambda$ as $n \to \infty$. Then we get from $J_{\lambda_n}(u) \leqslant J_\lambda(u)$ for any $u \in H_r^1(\mathbb{R}^3)$ that $m_{\lambda_n} \geqslant m_\lambda$. On the other hand, for any given $\varepsilon > 0$, there exists a $u \in H_r^1(\mathbb{R}^3)$ such that $J_\lambda(u) < m_\lambda + \varepsilon/2$. Thus, for n large,

$$J_{\lambda_n}(u) = J_\lambda(u) + \frac{1}{10}(\lambda_n - \lambda)\int_{\mathbb{R}^3} \phi_u |u|^5 < m_\lambda + \varepsilon,$$

hence $m_{\lambda_n} < m_\lambda + \varepsilon$. Therefore, m_λ is continuous from the right. Next let $\{\lambda_n\}$ be a sequence satisfying $\lambda_n < \lambda$ and $\lambda_n \to \lambda$ as $n \to \infty$. Then $m_{\lambda_n} \leqslant m_\lambda$. Since m_λ is achieved by some $u_\lambda \in H_r^1(\mathbb{R}^3)$ for all $\lambda \in (0, \infty)$, then we may assume that $J_{\lambda_n}(u_n) = m_{\lambda_n}$ for some $u_n \in H_r^1(\mathbb{R}^3)$. Since for n large,

$$\langle J'_{\lambda/2}(u_n), u_n\rangle \leqslant \langle J'_{\lambda_n}(u_n), u_n\rangle = 0,$$

then it follows from Lemma 2.9 that $\{u_n\}$ is bounded. Thus,

$$m_{\lambda_n} = J_{\lambda_n}(u_n) = J_\lambda(u_n) - \frac{1}{10}(\lambda - \lambda_n)\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^5 \geqslant m_\lambda - \varepsilon,$$

for n large. Therefore m_λ is also continuous from left. This proves the continuity of m_λ in λ.

2. We claim that $\|u_{1,\lambda}\| \to \infty$ as $\lambda \to 0^+$. If it is not true, then there is a sequence $\{\lambda_n\}$ such that $\lambda_n \to 0^+$ and $\|u_{1,\lambda_n}\| \leqslant C$ for some constant C, then $\{J_0(u_{1,\lambda_n})\}$ is bounded. Since for any $v \in H_r^1(\mathbb{R}^3)$,

$$\langle J'_0(u_{1,\lambda_n}), v\rangle = \langle J'_{\lambda_n}(u_{1,\lambda_n}), v\rangle - \lambda_n \int_{\mathbb{R}^3} \phi_{u_{1,\lambda_n}} |u_{1,\lambda_n}|^3 u_{1,\lambda_n} v,$$

it follows that $J'_0(u_{1,\lambda_n}) \to 0$. According to Lemma 3.2, we may assume that $u_{1,\lambda_n} \to u$ and then u is a nonnegative solution of (1.3). Since (3.5) and (3.6) still hold when $\lambda = 0$, then $\|u\| \geqslant r_0 > 0$ so $u \neq 0$ and u is positive and radially symmetric. Therefore, $u = u_0$ from the uniqueness of the positive solution to (1.3). Since $u_{1,\lambda_n} \to u$ as $n \to \infty$ and $m_{\lambda_n} = J_{\lambda_n}(u_{1,\lambda_n}) < 0$, then

$$J_0(u_0) = \lim_{n\to\infty} J_{\lambda_n}(u_{1,\lambda_n}) \leqslant 0.$$

But on the other hand,

$$J_0(u_0) = \left(\frac{1}{2} - \frac{1}{p+1}\right) \|u_0\|^2 > 0.$$

That is a contradiction. Hence we must have $\|u_{1,\lambda}\| \to \infty$ as $\lambda \to 0^+$. Also from the proof of Theorem 3.3, $J_0(tu_0) \to -\infty$ as $t \to \infty$, then we have that $m_\lambda \to -\infty$ as $\lambda \to 0^+$. □

The conclusions of Theorem 1.4 are obtained from Lemma 3.4 and Proposition 3.5. Now we turn to the proof of a second positive solution of (1.1) by using the the mountain pass theorem of Ambrosetti and Rabinowitz [1]. To show that (1.1) has a mountain pass structure, first we note the following fact.

Lemma 3.6 *Suppose that $p \in (1, 5)$ and $\mu > 0$. Let $r_0 > 0$ be defined as in Lemma 3.4.*

1. *For all $\lambda > 0$, $u = 0$ is an isolated local minimizer of the energy functional J_λ.*
2. *For all $\lambda > 0$, there exist $r_1 \in (0, r_0)$ and $\alpha > 0$ (both independent of λ) such that $\inf_{\|u\|=r_1} J_\lambda(u) \geqslant \alpha$.*

Proof From (3.6), for $\|u\| \neq 0$ small, $J_\lambda(u) > 0$. Hence $u = 0$ is an isolated local minimizer. And (3.6) also implies that there exist $r_1 > 0$ and $\alpha > 0$ such that $\inf_{\|u\|=r_1} J_\lambda(u) \geqslant \alpha > 0$.

For $\lambda \in [0, \infty)$, define □

$$\Gamma_\lambda = \{\gamma \in C([0,1], H_r^1(\mathbb{R}^3)) : \gamma(0) = 0, J_\lambda(\gamma(1)) < \alpha\}. \tag{3.7}$$

If $\Gamma_\lambda \neq \emptyset$, we set

$$c_\lambda = \min_{\gamma \in \Gamma_\lambda} \max_{t \in [0,1]} J_\lambda(\gamma(t)). \tag{3.8}$$

We show that c_λ satisfies the following properties.

Lemma 3.7 *Suppose that $p \in (1, 5)$ and $\mu > 0$. Then there exists $\lambda_1 > \lambda_0$ such that for $\lambda \in (0, \lambda_1)$, $\Gamma_\lambda \neq \emptyset$, the mapping $\lambda \mapsto c_\lambda$ is continuous from right and nondecreasing, that is, if $\lambda_n \to \lambda^+$ as $n \to \infty$, then $c_{\lambda_n} \to c_\lambda$, and if $0 < \lambda' < \lambda'' < \lambda_1$, then $c_{\lambda'} \leqslant c_{\lambda''}$.*

Proof Let $u_{1,\lambda}$ be a positive ground state solution of (1.1) for $\lambda \in (0, \lambda_0]$. Then from Lemma 3.4, $m_\lambda = J_\lambda(u_{1,\lambda}) < 0 < \alpha$ and $\|u_{1,\lambda}\| \geqslant r_0 > r_1$ for $\lambda \in (0, \lambda_0)$. So $\Gamma_\lambda \neq \emptyset$ for $\lambda \in (0, \lambda_0)$. Moreover $J_{\lambda_0}(u_{1,\lambda_0}) = 0 < \alpha$ and that implies $J_\lambda(u_{1,\lambda_0}) < \alpha$ for $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$ for some $\varepsilon > 0$.

Let $\lambda_1 = \sup\{\lambda \geqslant 0 :$ there is a u with $\|u\| \geqslant r_0$ such that $J_\lambda(u) < \alpha\}$. Then from the above arguments, $\lambda_1 > \lambda_0$ and $\inf_{\|u\| \geqslant r_0} J_{\lambda_1}(u) = \alpha$. We also have $\Gamma_\lambda \neq \emptyset$ and c_λ is well-defined for $\lambda \in (0, \lambda_1)$.

We prove that c_λ is continuous from the right and nondecreasing for $\lambda \in (0, \lambda_1)$. Let $\{\lambda_n\}$ be a sequence and $\lambda_n \to \lambda^+$. Fix n, for each $\gamma \in \Gamma_{\lambda_n}$, we have that $J_{\lambda_n}(\gamma(1)) < \alpha$. Since $J_\lambda \leqslant J_{\lambda_n}$, then it follows that $J_\lambda(\gamma(1)) < \alpha$. Therefore, $\gamma \in \Gamma_\lambda$ and thus $\Gamma_{\lambda_n} \subset \Gamma_\lambda$ for any $n \in \mathbb{N}$. It also follows from $J_\lambda \leqslant J_{\lambda_n}$ that for $\gamma \in \Gamma_{\lambda_n}$,

$$\max_{t \in [0,1]} J_\lambda(\gamma(t)) \leqslant \max_{t \in [0,1]} J_{\lambda_n}(\gamma(t)),$$

and then

$$\min_{\gamma\in\Gamma_\lambda}\max_{t\in[0,1]} J_\lambda(\gamma(t)) \leqslant \min_{\gamma\in\Gamma_{\lambda_n}}\max_{t\in[0,1]} J_\lambda(\gamma(t)) \leqslant \min_{\gamma\in\Gamma_{\lambda_n}}\max_{t\in[0,1]} J_{\lambda_n}(\gamma(t)).$$

Thus $c_\lambda \leqslant c_{\lambda_n}$ for any $n \in \mathbb{N}$.

On the other hand, let $\varepsilon > 0$ be an arbitrary constant, and take $\gamma \in \Gamma_\lambda$ such that $J_\lambda(\gamma(t)) < c_\lambda + \varepsilon/2$ for all $t \in [0, 1]$. Since for $t \in [0, 1]$,

$$|J_{\lambda_n}(\gamma(t)) - J_\lambda(\gamma(t))| = (\lambda_n - \lambda)\int_{\mathbb{R}^3} \phi_{\gamma(t)}|\gamma(t)|^5 \leqslant CS^{-1}(\lambda_n - \lambda)\|\gamma(t)\|^{10},$$

then for n large, $J_{\lambda_n}(\gamma(1)) < \alpha$ and $J_{\lambda_n}(\gamma(t)) < c_\lambda + \varepsilon$. Thus, $c_{\lambda_n} < c_\lambda + \varepsilon$ for large n. The nondecreasing property of c_λ can be proved similarly as the proof of $c_\lambda \leqslant c_{\lambda_n}$ above. □

Now we are ready to prove the existence of a second positive solution of (1.1) for $\lambda \in (0, \lambda_1)$.

Proposition 3.8 *Suppose that $p \in (1, 5)$ and $\mu > 0$. Let λ_1 be defined as in Lemma 3.7. Then*

1. *For $\lambda \in [0, \lambda_1)$, J_λ has a mountain pass type critical point $u_{2,\lambda} \in H^1_r(\mathbb{R}^3)$ such that $u_{2,\lambda} > 0$ and $u_{2,\lambda} \neq u_{1,\lambda}$. Moreover $0 < \lambda_1 < \mu^{8/(p-1)}/4$.*
2. *For $\lambda = \lambda_1$, there exists $u_{2,\lambda_1} \in H^1_r(\mathbb{R}^3)$ which is a critical point of J_{λ_1} and $u_{2,\lambda_1} > 0$.*

Proof Suppose that $\lambda \in (0, \lambda_1)$. From Lemma 3.1, J_λ satisfies the Palais–Smale condition. From Lemma 3.7, the set $\Gamma_\lambda \neq \emptyset$ and c_λ is well-defined. Now we can apply the mountain pass theorem of Ambrosetti and Rabinowitz [1] to conclude the existence of a nonzero mountain pass type critical point $u_{2,\lambda}$ such that $J_\lambda(u_{2,\lambda}) = c_\lambda > 0$ for $\lambda \in (0, \lambda_1)$. Apparently $u_{2,\lambda} \neq u_{1,\lambda}$ and again we may assume that $u_{2,\lambda}$ is nonnegative by considering the functional in (3.4). Since $u_{2,\lambda} \neq 0$ then $u_{2,\lambda} > 0$ from the maximum principle. For $\lambda = 0$, the proof is similar. We have $\lambda_1 < \mu^{8/(p-1)}/4$ from Theorem 2.8.

To prove the existence of a positive solution for $\lambda = \lambda_1$, let $\{\lambda_n\}$ be a sequence such that $\lambda_n \to \lambda_1^-$ as $n \to \infty$. Then there exists a $\varepsilon_0 > 0$ such that $\lambda_n > \lambda_1 - \varepsilon_0 := \theta > 0$ for lager n. Since $\langle J'_{\lambda_n}(u_{2,\lambda_n}), u_{2,\lambda_n}\rangle = 0$ and $\langle J'_\theta(u_{2,\lambda_n}), u_{2,\lambda_n}\rangle \leqslant 0$, then by Lemma 2.9, $\{u_{2,\lambda_n}\}$ is bounded. Moreover

$$J_{\lambda_1}(u_{2,\lambda_n}) = J_{\lambda_n}(u_{2,\lambda_n}) + \frac{1}{10}(\lambda_1 - \lambda_n)\int_{\mathbb{R}^3} \phi_{u_{2,\lambda_n}}|u_{2,\lambda_n}|^5 = c_{\lambda_n} + o(1),$$

$$\langle J'_{\lambda_1}(u_{2,\lambda_n}), v\rangle = \langle J'_{\lambda_n}(u_{2,\lambda_n}), v\rangle + (\lambda_1 - \lambda_n)\int_{\mathbb{R}^3} \phi_{u_{2,\lambda_n}}|u_{2,\lambda_n}|^3 u_{2,\lambda_n} v = o(\|v\|), \quad \forall v \in H^1_r(\mathbb{R}^3).$$

It follows that $\{J_{\lambda_1}(u_{2,\lambda_n})\}$ is bounded and $J'_{\lambda_1}(u_{2,\lambda_n}) \to 0$. By Lemma 3.1, J_{λ_1} satisfies the Palais–Smale condition. Therefore there exists a subsequence $\{v_n\}$ of $\{u_{2,\lambda_n}\}$ such that $v_n \to u_{2,\lambda_1}$ for some $u_{2,\lambda_1} \in H^1_r(\mathbb{R}^3)$. It is clear that $J'_{\lambda_1}(u_{2,\lambda_1}) = 0$ and $u_{2,\lambda_1} \neq 0$ since $\|u_{2,\lambda_n}\| \geqslant r_0$. Again $u_{2,\lambda_1} > 0$ from the maximum principle and $u_{2,\lambda_n} > 0$. □

Finally we consider the behavior of the mountain pass type solution $u_{2,\lambda}$ as $\lambda \to 0^+$. We first show the following perturbation type result from the implicit function theorem.

Lemma 3.9 *Suppose that $p \in (1, 5)$ and $\mu > 0$. Then there exists a $\lambda_2 > 0$ such that for $\lambda \in (-\lambda_2, \lambda_2)$, J_λ has a unique critical point u_λ such that $u_\lambda \to u_0$ as $\lambda \to 0$.*

Proof The proof is similar to that of Proposition 2.1 in [21]. Let $F : \mathbb{R}^1 \times H_r^1(\mathbb{R}^3) \to [H_r^1(\mathbb{R}^3)]^*$ be defined by $F(\lambda, u) = J_\lambda'(u)$. We prove the existence of unique solution to the equation $F(\lambda, u) = 0$ near $(\lambda, u) = (0, u_0)$ by the implicit function theorem. In fact, $F(0, u) = 0$ has a unique positive solution $u_0 \in H_r^1(\mathbb{R}^3)$, and for $v \in H_r^1(\mathbb{R}^3)$,

$$F_u(0, u_0)[v] = J_0''(u_0)[v] = \Delta v - v + \mu p|u_0|^{p-1}v,$$

in the weak sense. Then $F_u(0, u_0) : H_r^1(\mathbb{R}^3) \to [H_r^1(\mathbb{R}^3)]^*$ is a Fredholm operator, and $F_u(0, u_0) = J_0''(u_0)$ is nondegenerate in $H_r^1(\mathbb{R}^3)$ since the kernel of $J_0''(u_0)$ in the space $H^1(\mathbb{R}^3)$ is spanned by the functions $\frac{\partial}{\partial x_i}u_0$, which are orthogonal to the space $H_r^1(\mathbb{R}^3)$ (see [7,20]). Now from the implicit function theorem, for $|\lambda|$ small, there is a continuous map $\lambda \mapsto u_\lambda$ such that $F(\lambda, u_\lambda) = 0$ for u_λ near u_0. □

Now we prove the part 2 of Theorem 1.5.

Theorem 3.10 *Suppose that $p \in (1, 5)$ and $\mu > 0$. Let $u_{2,\lambda}$ be defined as in Proposition 3.8.*

1. *If $u_{2,\lambda}$ is bounded as $\lambda \to 0^+$, then $u_{2,\lambda} = u_\lambda$ for $\lambda \in (0, \lambda_2)$ where u_λ is the unique positive solution approaching u_0 as defined in Lemma 3.9.*
2. *If $u_{2,\lambda}$ is unbounded as $\lambda \to 0^+$, then the system (1.1) has at least three positive solutions for $\lambda \in (0, \lambda_2)$.*

Proof If $u_{2,\lambda}$ is bounded as $\lambda \to 0^+$, then $J_0(u_{2,\lambda})$ is bounded and $J_0'(u_{2,\lambda}) \to 0$. Since J_0 satisfies the Palais–Smale condition, we may assume that $u_{2,\lambda} \to u$ for some $u \neq 0$. Thus u is a positive solution of (1.3). By the uniqueness of the solution to (1.3), we have that $u = u_0$. It follows from Lemma 3.9 that we must have $u_{2,\lambda} = u_\lambda$. If $u_{2,\lambda}$ is unbounded as $\lambda \to 0^+$, then $u_{2,\lambda} \neq u_\lambda$ (the unique positive solution approaching u_0 as defined in Lemma 3.9). Therefore we have three positive solutions: $u_{1,\lambda}$, $u_{2,\lambda}$ and u_λ. □

The proof of Theorem 1.5 is obtained from Proposition 3.8 and Theorem 3.10.

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