Existence and multiplicity of positive solutions to Schrödinger–Poisson type systems with critical nonlocal term

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Received: 15 July 2015 / Accepted: 30 July 2017
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Abstract The existence, nonexistence and multiplicity of positive radially symmetric solutions to a class of Schrödinger–Poisson type systems with critical nonlocal term are studied with variational methods. The existence of both the ground state solution and mountain pass type solutions are proved. It is shown that the parameter ranges of existence and nonexistence of positive solutions for the critical nonlocal case are completely different from the ones for the subcritical nonlocal system.

Mathematics Subject Classification 35J50 · 35J47 · 35J61 · 47J30

1 Introduction

In this paper, we consider positive solutions to the nonlinear Schrödinger–Poisson type system

\[
\begin{align*}
-\Delta u + u + \lambda \phi |u|^3 u &= \mu |u|^{p-1} u, \quad \text{in } \mathbb{R}^3, \\
-\Delta \phi &= |u|^5, \quad \text{in } \mathbb{R}^3,
\end{align*}
\]

where \( \lambda \in \mathbb{R} \) and \( \mu \geq 0 \) are physical parameters. The investigation of (1.1) is motivated by recent studies of Schrödinger–Poisson system
\[
\begin{cases}
-\Delta u + bu + \lambda \phi g(u) = f(u), & \text{in } \mathbb{R}^3, \\
-\Delta \phi = 2G(u), & \text{in } \mathbb{R}^3,
\end{cases}
\] (1.2)

where the functions \( g(u) \) and \( G(u) \) satisfy \(|g(u)| \leq C(|u| + |u|^q) \) for some \( q \in [1, 4) \), \( G(u) = \int_0^u g(t) dt \), and \( f(u) \) satisfies \(|f(u)| \leq C(|u| + |u|^p) \) for some \( p \in (1, 5) \). For subcritical nonlinearity \( f \) with \( p \in (1, 5) \) and subcritical nonlocal term \( g \) with \( q \in [1, 4) \), the problem (1.2) was studied in, for example, [4,16]. In [4], system (1.2) on a bounded domain \( \Omega \subset \mathbb{R}^3 \) was studied for positive and negative values of \( \lambda \). In [16], system (1.2) was considered and it was shown that there exists a positive solution for small \( \lambda \geq 0 \). For the special case \( g(u) = u \), there have been many studies of system (1.2), see for example, [2,10,13,18,21–23,28,29,31,33]. In [2,10,22], the existence and multiplicity of positive radial solutions was considered for various \( \lambda \) and \( p \); when \( b \) depends on \( x \) and is not radial, and \( f \) is asymptotically linear, the existence of positive solution for small \( \lambda > 0 \) and the nonexistence of nontrivial solution for large \( \lambda > 0 \) were obtained in [28]; when \( b \) depends on \( x \) and is sign-changing, the existence and multiplicity was shown in [31]; when \( b \) depends on \( x \), the existence of a sign-changing solution was proved in [29]; existence of a nontrivial solution and concentration results for \( p \in (3, 5) \) were shown in [27,33]. Moreover the ground state solution for system (1.2) with \( g(u) = u \) have been considered in [5,6,26,27], and the ground and bound states for system (1.2) with \( g(u) = u \) and \( b = 0 \) were studied in [13,23].

However, all the papers mentioned above studied the subcritical nonlocal term. The critical case for the nonlocal term corresponds to \( q = 4 \) like in (1.1). In this paper, the nonlocal term is of critical growth, and it is much more difficult to obtain the existence of positive solutions. Firstly the critical growth causes a lack of compactness; secondly the critical nonlocal term case competes with the critical nonlinearity case which has been studied previously in [32,34].

To the best of our knowledge, the system (1.1) with critical nonlocal term has only been studied in [3,15]. In [3], the authors studied the system (1.1) in a ball domain \( B_R(0) \subset \mathbb{R}^3 \). In [15], we studied the system (1.1) for general nonlinearity in \( \mathbb{R}^3 \). By using monotonicity technique, we obtained the existence of positive solutions for positive and negative \( \lambda \) separately [15]. These results show the subtle differences between the critical nonlocal case and the subcritical nonlocal case. In this paper we continue our studies in [15] to show some further properties of the set of positive solutions to (1.1). In particular, we prove the existence of ground state solutions and mountain-pass type solutions of (1.1) for certain parameter ranges, and we also show the nonexistence of positive solutions for other parameter ranges. These results provide a much better understanding of the solution set of (1.1).

To describe our results, we first introduce some notations. In this paper, \( H^1(\mathbb{R}^3) \) and \( \mathcal{D}^{1,2}(\mathbb{R}^3) \) are the usual Sobolev spaces defined by \( H^1(\mathbb{R}^3) = \{u \in L^2(\mathbb{R}^3) : |\nabla u| \in L^2(\mathbb{R}^3)\} \) and \( \mathcal{D}^{1,2}(\mathbb{R}^3) = \{u \in L^6(\mathbb{R}^3) : |\nabla u| \in L^2(\mathbb{R}^3)\} \). The corresponding norms and inner products are defined by

\[
\|u\|_{H^1(\mathbb{R}^3)} = \left( \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) \right)^{1/2}, \quad (u, v) = \int_{\mathbb{R}^3} \nabla u \cdot \nabla v + uv, \quad u, v \in H^1(\mathbb{R}^3),
\]

\[
\|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^3)} = \left( \int_{\mathbb{R}^3} |\nabla u|^2 \right)^{1/2}, \quad (u, v) = \int_{\mathbb{R}^3} \nabla u \cdot \nabla v, \quad u, v \in \mathcal{D}^{1,2}(\mathbb{R}^3).
\]

Here we use the well-known Sobolev embedding in the definition of \( \mathcal{D}^{1,2}(\mathbb{R}^3) \). Indeed we consider the system (1.1) in the subspace \( H^1_+(\mathbb{R}^3) \) of \( H^1(\mathbb{R}^3) \) which consists of all radial functions of \( H^1(\mathbb{R}^3) \). In the following we use \( \| \cdot \| \) as the norm of \( H^1_+(\mathbb{R}^3) \), \( (\cdot, \cdot) \) as the inner product.
product in $H^1_r(\mathbb{R}^3)$, $\langle \cdot, \cdot \rangle$ as the pairing of $[H^1_r(\mathbb{R}^3)]^*$ (the dual space) and $H^1_r(\mathbb{R}^3)$, and $| \cdot |_p$ is the norm of $L^p(\mathbb{R}^3)$ for $p \in [1, \infty)$.

First we have the following nonexistence results of nontrivial solutions of (1.1).

**Theorem 1.1** Suppose that $\lambda \in \mathbb{R}$ and $\mu \in [0, \infty)$.

1. For $p = 1$, system (1.1) has no nontrivial solution for $\lambda \geq 0$ and $\mu \geq 0$, or $\lambda < 0$ and $\mu \neq 1$.
2. For $p = 5$, system (1.1) has no nontrivial solution for $\lambda \in \mathbb{R}$ and $\mu \geq 0$.
3. For $p \in (1, 5)$, system (1.1) has no nontrivial solution for $\lambda \geq \mu^8/(p-1)/4$ and $\mu > 0$, or $\lambda \in \mathbb{R}$ and $\mu = 0$.

In particular, this implies the following result for the case of $\mu = 0$.

**Corollary 1.2** For $p \in [1, 5]$, system (1.1) has no nontrivial solution for $\lambda \in \mathbb{R}$ and $\mu = 0$.

For the nonlocal system (1.2) with subcritical growth function $g(u)$, it is well-known that there exists a positive solution when $\lambda < 0$ and $\mu = 0$ [17]. So the nonexistence result for the critical nonlocal system in Corollary 1.2 shows that the critical case is quite different from the subcritical nonlocal system. On the other hand, when $\lambda = 0$ and $\mu > 0$, the nonexistence result for $p = 5$ is well known [19].

Next we define the energy functional corresponding to (1.1), the notion of ground state solution and mountain-pass solution of system (1.1). Define a functional $J_\lambda : H^1(\mathbb{R}^3) \rightarrow \mathbb{R}$ by

$$J_\lambda(u) = \frac{1}{2} ||u||^2 + \frac{\lambda}{10} \int_{\mathbb{R}^3} \phi_u |u|^5 - \frac{\mu}{p + 1} \int_{\mathbb{R}^3} |u|^{p+1}, \quad u \in H^1(\mathbb{R}^3),$$

where $\phi_u$ is the unique solution of $-\Delta \phi = |u|^5$ in $\mathbb{R}^3$. Then $J_\lambda$ is well-defined on $H^1(\mathbb{R}^3)$ and is of $C^1$ class for any $\lambda \in \mathbb{R}$ and $\mu \geq 0$, and

$$\langle J'_\lambda(u), v \rangle = (u, v) + \lambda \int_{\mathbb{R}^3} \phi_u |u|^3 uv - \mu \int_{\mathbb{R}^3} |u|^{p-1} uv, \quad u, v \in H^1(\mathbb{R}^3).$$

It is standard to verify that a critical point $u$ of the functional $J_\lambda$ corresponds to a weak solution $(u, \phi_u)$ of system (1.1). Thus, in what follows, we consider critical points of $J_\lambda$ using variational methods.

We recall the following definitions of types of critical points of functionals.

**Definition 1.3** Let $X$ be a Banach space and let $J \in C^1(X)$ be a functional.

1. If $u_0 \in X$ is a critical point of $J$ and $u_0 \neq 0$ such that $J(u_0) = \inf_{u \in X} J(u)$, then we say that $u_0$ is a ground state solution of $J' = 0$. So the ground state realizes the minimum of the energy among critical points.
2. Let $\Omega$ be a bounded open subset of $X$, and let $e_1, e_2 \in X$ with $e_1 \in \Omega$ and $e_2 \notin \overline{\Omega}$. If $\inf_{u \in \partial \Omega} J(u) > \max\{J(e_1), J(e_2)\}$, we say that $J$ satisfies the mountain-pass geometric structure. Let

$$\Gamma = \{ \gamma \in C[0, 1] : \gamma(0) = e_1, \gamma(1) = e_2 \},$$

and

$$c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} J(\gamma(t)).$$

If $u_0$ is a critical point of $J$ with $J(u) = c$, we say that $u_0$ is a mountain-pass type solution of $J' = 0$. 

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Theorem 1.4 For any $\mu > 0$ and $p \in (1,5)$, there exists $\lambda_0 \in (0, \mu^{8/(p-1)}/4)$ such that system (1.1) has a positive radially symmetric ground state solution $u_{1,\lambda}$ for $\lambda \in (0, \lambda_0]$. Furthermore, the solutions $(\lambda, u_{1,\lambda})$ satisfy that $\lim_{\lambda \to 0^+} ||u_{1,\lambda}|| = \infty$, that is, $(\lambda, u) = (0, \infty)$ is a bifurcation point for system (1.1) (see Fig. 1).

Note that $u_{1,\lambda}$ found here has the smallest energy among all non-trivial radially symmetric solutions, but it is not known whether its energy is the smallest among all non-trivial solutions. On the other hand we show that for a wider range of parameter values, a mountain-pass type solution exists.

Theorem 1.5 For any $\mu > 0$ and $p \in (1,5)$, there exists a $\lambda_1 \in (\lambda_0, \mu^{8/(p-1)}/4)$ such that system (1.1) has a positive radially symmetric mountain-pass type solution $u_{2,\lambda}$ for $\lambda \in (0, \lambda_1)$. Moreover,

1. When $\lambda \to \lambda_1^-$, the limit $\lim_{\lambda \to \lambda_1^-} u_{2,\lambda} = u_{2,\lambda_1}$ (strongly in $H^1(\mathbb{R}^3)$) exists, and $u_{2,\lambda_1}$ is a nontrivial positive solution of (1.1) for $\lambda = \lambda_1$.
2. When $\lambda \to 0^+$, the following alternatives hold:

   (2a) either $u_{2,\lambda}$ is bounded in $H^1(\mathbb{R}^3)$ as $\lambda \to 0^+$, and $\lim_{\lambda \to 0^+} u_{2,\lambda} = u_0$ is the unique positive solution of the equation

   $$-\Delta u + u = \mu |u|^{p-1}u, \quad u \in H^1(\mathbb{R}^3);$$

   (1.3)

   (2b) or $\lim_{\lambda \to 0^+} ||u_{2,\lambda}|| = \infty$, and there exists a $\lambda_2 \in (0, \lambda_0)$ such that system (1.1) has at least three positive solutions for $\lambda \in (0, \lambda_2)$.

The two diagrams in Fig. 1 show the two alternatives in Theorem 1.5 (2a) and (2b). It would be interesting to know which diagram cannot occur, and it is also interesting to see whether the branch of ground state solutions is connected to the branch of mountain pass type solutions. For the special case $\lambda = 0$, the system (1.1) is reduced to (1.3). The existence of a positive solution $u_0$ to (1.3) is well-known [19,24], and it is known that any positive solution $u_0$ of (1.3) must be radially symmetric [12]. The uniqueness of $u_0$ was proved in [14], and it is also known that $u_0$ is the ground state solution of (1.3) [8]. This positive solution $u_0$ can be obtained by using the mountain pass theorem in the space of radially symmetric functions, which is the strategy used in the proof of Theorem 1.5. So in some sense, the results in Theorem 1.5 is an analog of the classical results for the case $\lambda = 0$. But here for (1.1), the radial symmetry of all positive solutions for $\lambda > 0$ is still not known, and our result implies that the uniqueness of positive solution does not hold for (1.1) and $\lambda > 0$.

The results in Theorem 1.5 imply that system (1.1) possesses multiple positive solutions for small $\lambda > 0$ and fixed $\mu > 0$. In the case (2a), system (1.1) has at least two positive solutions for $\lambda \in (0, \lambda_0]$ and has at least one positive solution for $\lambda \in (\lambda_0, \lambda_1]$. See Fig. 1, left panel. On the other hand, from the pictures: an analogue in the case (2b), system (1.1) has at least three positive solutions for $\lambda \in (0, \lambda_2]$, has at least two positive solutions for $\lambda \in (\lambda_2, \lambda_0]$, and has at least one positive solution for $\lambda \in (\lambda_0, \lambda_1]$, see Fig. 1 right panel.

We emphasize some distinctive characters of our results in the following remarks.
Remark 1.6 The existence/non-existence results for positive solutions for (1.1) are different from the ones for the system with subcritical nonlocal term

\[
\begin{aligned}
-\Delta u + u + \lambda \phi u &= |u|^{p-1} u, &\text{in } \mathbb{R}^3, \\
-\Delta \phi &= |u|^2, &\text{in } \mathbb{R}^3,
\end{aligned}
\]  

(1.4)

which was studied in [22]. In [22], the author obtained the existences of two positive solutions of (1.4) for \( p \in (1, 2) \) and small \( \lambda > 0 \), and one positive solution for \( p \in [2, 5) \) and all \( \lambda > 0 \). For (1.1), Theorem 1.1 shows that no positive solution of (1.1) exists for large \( \lambda > 0 \) and \( p \in (1, 5) \). On the other hand, Theorems 1.4 and 1.5 show that (1.1) always have two positive solutions for small \( \lambda > 0 \). Thus the parameter ranges for existence and nonexistence for (1.1) and (1.4) are very different.

Remark 1.7 From the proof of Lemma 3.9, for the case of \( \lambda < 0 \), we can obtain the existence of one positive solution for (1.1) for small \(|\lambda|\) and \( \lambda < 0 \) or \( \lambda = -1 \) (see [15]). However, we do not know whether a positive solution exists for all \( \lambda \in (-1, 0) \) or for \( \lambda \in (-\infty, -1) \). In [15], we have shown that for any \( \lambda < 0 \), there exists a positive solution to the system with subcritical nonlocal term

\[
\begin{aligned}
-\Delta u + u + \lambda \phi |u|^{q-1} u &= |u|^{p-1} u, &\text{in } \mathbb{R}^3, \\
-\Delta \phi &= |u|^{q+1}, &\text{in } \mathbb{R}^3,
\end{aligned}
\]  

(1.5)

where \( q \in [1, 4) \). This again shows the difference between the system with critical nonlocal term and the system with subcritical nonlocal term.

The remaining part of the paper is structured as follows: In Sect. 2, we give some preliminaries including (i) provide some basic setup of the problem studied, (ii) recall some convergence lemmas, (iii) introduce the Pohozaev identity and give the proof of nonexistence results, (iv) prove the boundedness of Palais–Smale sequences. In Sect. 3, we prove all other main results.

In what follows we frequently use the following inequality

\[
S|u|^2_{L^6(\mathbb{R}^3)} \leq \|u\|^2_{D^{1,2}(\mathbb{R}^3)}, \quad u \in D^{1,2}(\mathbb{R}^3),
\]  

(1.6)

and

\[
S|u|^2_{L^6(\mathbb{R}^3)} \leq \|u\|^2_{D^{1,2}(\mathbb{R}^3)} \leq \|u\|^2_{H^1(\mathbb{R}^3)}, \quad u \in H^1(\mathbb{R}^3).
\]  

(1.7)

Note that the best constant \( S \) can be achieved (see [25,30]).
2 Preliminaries

2.1 Variational setup

For a given $u \in H^1(\mathbb{R}^3)$, the second equation of (1.1) is a Poisson equation for $\phi$ which is uniquely solvable. Then (1.1) can be reduced to the first equation with $\phi$ represented by the solution of the Poisson equation. This is a basic strategy of solving (1.1). To be more precise about the solution $\phi$ of the Poisson equation, we recall the following lemma from [15, Lemma 2.1].

**Lemma 2.1** For every $u \in L^6(\mathbb{R}^3)$, there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ which is the solution of

$$-\Delta \phi = |u|^5, \quad \text{in } \mathbb{R}^3,$$

(2.1)

here $\phi_u$ can be expressed by the form

$$\phi_u(x) = \int_{\mathbb{R}^3} \frac{|u|^5(y)}{|x-y|} \, dy.$$

Moreover,

(i) $\|\phi_u\|_{D^{1,2}(\mathbb{R}^3)}^2 = \int_{\mathbb{R}^3} \phi_u |u|^5$;

(ii) $\phi_u(x) > 0$ for $x \in \mathbb{R}^3$;

(iii) for any $\theta > 0$, $\phi_u = \theta^2 (\phi_u)_\theta$, where $u_\theta(\cdot) = u(\cdot/\theta)$;

(iv) for any $t > 0$, $\phi_{tu} = t^5 \phi_u$;

(v) for any $u \in L^6(\mathbb{R}^3)$,

$$\|\phi_u\|_{D^{1,2}(\mathbb{R}^3)} \leq S^{-1/2} |u|_6^5, \quad \int_{\mathbb{R}^3} \phi_u |u|^5 \leq S^{-1} |u|_6^{10},$$

where $S$ is defined in (1.6);

(vi) if $u$ is a radial function, then $\phi_u$ is also radial;

(vii) if $u_n \rightharpoonup u$ in $L^6(\mathbb{R}^3)$ and $u_n \to u$ a.e. in $\mathbb{R}^3$ as $n \to \infty$, then $\phi_{u_n} \rightharpoonup \phi_u$ in $D^{1,2}(\mathbb{R}^3)$.

The result in Lemma 2.1 implies that the system (1.1) can be reduced to a semilinear nonlocal elliptic equation

$$-\Delta u + u + \lambda \phi_u |u|^3 u = \mu |u|^{p-1} u, \quad \text{in } \mathbb{R}^3,$$

(2.2)

where $\phi_u$ is defined in Lemma 2.1.

We say a few words about the regularity of solutions to (1.1). Let $u$ be a critical point of $J_\lambda$. Then $u \in H^1(\mathbb{R}^3)$, and from the Sobolev embedding theorem, $u \in L^6(\mathbb{R}^3)$. We then deduce that $\phi \in L^6(\mathbb{R}^3)$ from Lemma 2.1. It follows that $-\Delta u = a(x)u$, where $a(x) = \mu |u|^{p-1} - 1 - \lambda \phi |u|^3$. Since $p \in [1, 5]$, then $\mu |u|^{p-1} - 1 \in L^{3/2}_{loc}(\mathbb{R}^3)$ and $\lambda \phi |u|^3 \in L^{3/2}_{loc}(\mathbb{R}^3)$. The Brézis-Kato Theorem [9] implies that $u \in L^{r}_{loc}(\mathbb{R}^3)$ for all $r \in [1, \infty)$. Therefore, $u \in W^{2,r}_{loc}(\mathbb{R}^3)$ for all $r \in [1, \infty)$. From elliptic regularity theory, it follows that $u, \phi \in C^2(\mathbb{R}^3)$. This shows that a weak solution $u \in H^1(\mathbb{R}^3)$ is indeed a classical solution of (2.2), and in the following we look for positive solutions of the Eq. (2.2) in the space $H^1(\mathbb{R}^3)$. 

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2.2 Brézis–Lieb type convergence lemma

To prove the convergence of the nonlocal term in (1.1), we recall the following Brézis–Lieb type lemmas which are proved in [15, Lemmas 2.2, 2.3].

**Lemma 2.2** Let \( r \geq 1 \) and let \( \Omega \) be an open subset of \( \mathbb{R}^N \). Suppose that \( u_n \rightharpoonup u \) in \( L^r(\Omega) \), and \( u_n \rightarrow u \) a.e. in \( \Omega \) as \( n \rightarrow \infty \), then for \( p \in [1, r] \), as \( n \rightarrow \infty \),

\[
|u_n|^p - |u_n - u|^p - |u|^p \rightarrow 0, \quad \text{in } L^{r/p}(\Omega), \quad (2.3)
\]

\[
|u_n|^{p-1}u_n - |u_n - u|^{p-1}(u_n - u) - |u|^{p-1}u \rightarrow 0, \quad \text{in } L^{r/p}(\Omega). \quad (2.4)
\]

**Lemma 2.3** If \( u_n \rightharpoonup u \) in \( L^6(\mathbb{R}^3) \) and \( u_n \rightarrow u \) a.e. in \( \mathbb{R}^3 \), then as \( n \rightarrow \infty \),

\[
|u_n|^3 - |u_n - u|^3 - |u|^3 \rightarrow 0, \quad \text{in } L^{6/5}(\mathbb{R}^3), \quad (2.5)
\]

\[
|u_n|^3u_n - |u_n - u|^3(u_n - u) - |u|^3u \rightarrow 0, \quad \text{in } L^{3/2}(\mathbb{R}^3), \quad (2.6)
\]

\[
\phi_{u_n} - \phi_{u_n-u} - \phi_u \rightarrow 0, \quad \text{in } D^{1,2}(\mathbb{R}^3), \quad (2.7)
\]

\[
\int_{\mathbb{R}^3} \phi_{u_n} |u_n|^5 - \int_{\mathbb{R}^3} \phi_{u_n-u} |u_n - u|^5 - \int_{\mathbb{R}^3} \phi_u |u|^5 \rightarrow 0. \quad (2.8)
\]

2.3 Pohozaev identity and nonexistence of solutions

In [15], a general Pohozaev identity is established, and the following one is a special case.

**Lemma 2.4** If \( u \in H^1(\mathbb{R}^3) \) is a weak solution of (1.1) with \( p \in [1, 5] \), then the following Pohozaev type identity holds

\[
\frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{3}{2} \int_{\mathbb{R}^3} |u|^2 + \frac{\lambda}{2} \int_{\mathbb{R}^3} \phi |u|^5 = \frac{3}{p+1} \mu \int_{\mathbb{R}^3} |u|^{p+1}. \quad (2.9)
\]

**Proof** Applying Lemma 2.8 of [15] with \( f(u) = \mu |u|^{p-1}u \), we obtain (2.9). \( \square \)

By using the Pohozaev identity (2.9), we prove Theorem 1.1 in the following four theorems.

**Theorem 2.5** Suppose that \( \lambda \in \mathbb{R}, \mu \geq 0 \) and \( p = 5 \), then there is no nontrivial solution for (1.1).

**Proof** Let \( u \) be a weak solution of (1.1). Then by the Pohozaev identity (2.9), we have that

\[
\frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{3}{2} \int_{\mathbb{R}^3} |u|^2 + \frac{\lambda}{2} \int_{\mathbb{R}^3} \phi |u|^5 = \frac{\mu}{2} \int_{\mathbb{R}^3} |u|^6. \quad (2.10)
\]

On the other hand, it follows from \( J'_u(u) = 0 \) that

\[
\int_{\mathbb{R}^3} |\nabla u|^2 + \int_{\mathbb{R}^3} |u|^2 + \lambda \int_{\mathbb{R}^3} \phi |u|^5 = \mu \int_{\mathbb{R}^3} |u|^6. \quad (2.11)
\]

Subtracting (2.10) and one half of (2.11), we obtain that \( \int_{\mathbb{R}^3} |u|^2 = 0 \), which implies that \( u = 0 \). \( \square \)

**Theorem 2.6** For \( \lambda \geq 0 \) and \( \mu \geq 0 \) or \( \lambda < 0 \) and \( \mu \neq 1 \), there is no nontrivial solution for

\[
\begin{cases}
-\Delta u + u + \lambda \phi |u|^3 u = \mu u, & \text{in } \mathbb{R}^3, \\
-\Delta \phi = |u|^5, & \text{in } \mathbb{R}^3.
\end{cases}
\quad (2.12)
\]
Proof Let \( u \) be a weak solution of (2.12). By the Pohozaev identity (2.9), we have that
\[
\frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{3}{2} \int_{\mathbb{R}^3} u^2 + \frac{\lambda}{2} \int_{\mathbb{R}^3} \phi |u|^5 = \frac{3}{2} \mu \int_{\mathbb{R}^3} u^2.
\] (2.13)
On the other hand, it follows from \( J(u) \) bounded in \( H \) that
\[
\int_{\mathbb{R}^3} |\nabla u|^2 + \int_{\mathbb{R}^3} u^2 + \lambda \int_{\mathbb{R}^3} \phi |u|^5 = \mu \int_{\mathbb{R}^3} u^2.
\] (2.14)
Subtracting (2.13) and (3/2) times of (2.14), we have
\[
\int_{\mathbb{R}^3} |\nabla u|^2 + \int_{\mathbb{R}^3} u^2 + \lambda \int_{\mathbb{R}^3} \phi |u|^5 = 0,
\]
which implies that \( u = 0 \) when \( \lambda \geq 0 \). On the other hand, subtracting (2.13) and one half of (2.14), we have
\[
(1 - \mu) \int_{\mathbb{R}^3} u^2 = 0,
\]
which implies that \( u = 0 \) for \( \lambda < 0 \) and \( \mu \neq 1 \). \( \square \)

**Theorem 2.7** If \( \mu = 0 \), then for any \( p \in [1, 5] \), the system (1.1) has no nontrivial solution.

**Proof** We can prove it with the same proof as the one for Theorem 2.5 with \( \mu = 0 \). \( \square \)

**Theorem 2.8** Let \( \lambda \geq \mu^{8/(p-1)}/4 \), \( \mu \geq 0 \) and \( p \in (1, 5) \). Then the system (1.1) has no nontrivial solution.

**Proof** Let \( \alpha = 4/(p - 1) \). If \( (u, \phi_u) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3) \) is a solution of (1.1), then it follows from (i) of Lemma 2.1 that for all \( u \in H^1(\mathbb{R}^3) \),
\[
\mu^\alpha \int_{\mathbb{R}^3} |u|^6 = -\mu^\alpha \int_{\mathbb{R}^3} |u| \Delta \phi_u = \mu^\alpha \int_{\mathbb{R}^3} \nabla \phi_u \cdot \nabla |u| \leq \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{1}{4} \mu^{2\alpha} \int_{\mathbb{R}^3} \phi_u |u|^5.
\] (2.15)
On the other hand, multiplying the first equation of (1.1) by \( u \) and integrating, we obtain that
\[
\int_{\mathbb{R}^3} \left( |\nabla u|^2 + u^2 + \lambda \phi_u |u|^5 - \mu |u|^{p+1} \right) = 0.
\] (2.16)
Thus, it follows from (2.15) and (2.16) that
\[
0 = \int_{\mathbb{R}^3} \left( |\nabla u|^2 + u^2 + \lambda \phi_u |u|^5 - \mu |u|^{p+1} \right) \geq \int_{\mathbb{R}^3} \left( u^2 + \mu^\alpha |u|^6 - \mu |u|^{p+1} \right).
\]
Since the function \( f(x) = x^2 + \mu^\alpha x^6 - \mu |x|^{p+1} \geq 0 \) for all \( x \in \mathbb{R} \), then we must have \( u = 0 \). \( \square \)

From the nonexistence results proved in this subsection, system (1.1) can only have nontrivial solutions when \( \lambda \in \mathbb{R}, \mu > 0 \) and \( p \in (1, 5) \).

### 2.4 The boundedness of Palais–Smale sequence

The following lemma is useful in the variational setting involving the Palais–Smale condition.

**Lemma 2.9** Fix \( \lambda > 0, \mu > 0 \) and \( p \in (1, 5) \). Let \( \{u_n\} \subset H^1_r(\mathbb{R}^3) \) be a sequence satisfying \( J_n(u_n) \leq C \) or \( \langle J'_n(u_n), u_n \rangle \leq C \) for all \( n \in \mathbb{N} \), where \( C \) is a positive constant. Then \( \{u_n\} \) is uniformly bounded in \( H^1_r(\mathbb{R}^3) \).
Proof First we assume that $J_{\lambda}(u_n) \leq C$. Since
\[
\alpha_\lambda \int_{\mathbb{R}^3} |u_n|^6 = \alpha_\lambda \int_{\mathbb{R}^3} \nabla \phi_{u_n} \cdot \nabla |u_n| \leq \int_{\mathbb{R}^3} \left( \frac{1}{4} |\nabla u_n|^2 + \frac{\lambda}{20} |\nabla \phi_{u_n}|^2 \right),
\]
where $\alpha_\lambda = \sqrt{\lambda/20} > 0$, then it follows that
\[
C \geq J_{\lambda}(u_n) \geq \int_{\mathbb{R}^3} \left( \frac{1}{4} |\nabla u_n|^2 + \frac{1}{2} u_n^2 + \frac{\lambda}{20} |\phi_{u_n}|^2 + \alpha_\lambda |u_n|^6 - \frac{\mu}{p+1} |u_n|^{p+1} \right).
\]
Let the function $h : \mathbb{R}_+ \to \mathbb{R}$ be defined by
\[
h(s) := \frac{1}{4} s^2 + \alpha_\lambda s^6 - \frac{\mu}{p+1} s^{p+1}, \quad s \in [0, \infty).
\]
It is clear that for $\lambda > 0$ sufficiently large, $h(s) \geq 0$ for all $s \geq 0$. Since $p \in (1, 5)$, $h(0) = 0$ and $h(s) \to \infty$ as $s \to \infty$, and then the infimum $h_0 := \inf_{s \in [0, \infty)} h(s) = h(s_0)$ is attained at some $s_0 \in [0, \infty)$. If $h_0 = 0$, then it follows from (2.17) that $\|u_n\| = \left( \int_{\mathbb{R}^3} (|\nabla u_n|^2 + u_n^2) \right)^{1/2} \leq 4C$. If $h_0 = -m < 0$, then we can obtain that $\{s > 0 : h(s) < 0\} = (a, b)$ for some $b > a > 0$. Thus, by (2.17), we have that
\[
C \geq J_{\lambda}(u_n) \geq \frac{1}{4} \int_{\mathbb{R}^3} |\nabla u_n|^2 + \frac{1}{4} \int_{\mathbb{R}^3} u_n^2 + \frac{\lambda}{20} \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^5 + \int_{\mathbb{R}^3} h(|u_n|)
\]
\[
\geq \frac{1}{4} \|u_n\|^2 + \frac{\lambda}{20} \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^5 - \frac{\mu}{p+1} \|u_n\| - m |A_n|,
\]
where $A_n = \{x \in \mathbb{R}^3 : |u_n(x)| \in (a, b)\}$ and $|A_n|$ is its Lebesgue measure. Note that $A_n$ is spherically symmetric since $u_n$ is radial. Suppose that $\{u_n\}$ is unbounded in $H^1_\gamma(\mathbb{R}^3)$. Without loss of generality, we may assume that $\|u_n\| \to \infty$ as $n \to \infty$. Since $J_{\lambda}(u_n) \leq C$, then we have from (2.18) that
\[
m |A_n| \geq \frac{1}{4} \|u_n\|^2 - C,
\]
which implies that $|A_n| \to \infty$. From the Strauss inequality [30, Lemma 4.5], for any $u \in H^1_\gamma(\mathbb{R}^3)$, we have that
\[
|u(x)| \leq c_0 |x|^{-1} \|u\|, \quad \text{a.e. } x \in \mathbb{R}^3,
\]
for some $c_0 > 0$. Therefore, for every $n$, there exists $E_n$ with $|E_n| = 0$ such that
\[
|u_n(x)| \leq c_0 |x|^{-1} \|u_n\|, \quad x \in \mathbb{R}^3 \setminus E_n.
\]
Define $B_n = A_n \setminus E_n$. Then $|A_n| = |B_n|$. Note that $B_n$ is also spherically symmetric. Let $r_n = \sup\{|x| : x \in B_n\}$. It follows from (2.19) that $r_n < \infty$ for fixed $n$. By the definition of $r_n$, there exists $x_n \in B_n$ such that $r_n/2 \leq |x_n|$ and $|u_n(x_n)| \leq c_0 |x_n|^{-1} \|u_n\|$. Thus we obtain from (2.19) that
\[
0 < a \leq |u_n(x_n)| \leq 2c_0 r_n^{-1} \|u_n\| \leq 4c_0 r_n^{-1} (m |A_n| + C)^{1/2},
\]
and consequently
\[
c_1 r_n \leq |A_n|^{1/2}
\]
for some $c_1 > 0$ and sufficiently large $n$. 

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On the other hand, since \( J_\lambda(u_n) \leq C \), then we have from (2.18) that
\[
\frac{\lambda}{20} \int_{\mathbb{R}^3} |\phi_{u_n}|^5 u_n |u_n|^5 \leq m|A_n| + C.
\]

However, by using Lemma 2.1 part (v), we have
\[
\text{Proof} \quad \frac{20}{\lambda} (m|A_n| + C) \geq \int_{\mathbb{R}^3} |\phi_{u_n}|^5 u_n |u_n|^5 = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |u_n|^5(x)|u_n|^5(y) \frac{dxdy}{|x-y|} \geq \int_{B_n} \int_{B_n} |u_n|^5(x)|u_n|^5(y) \frac{dxdy}{|x-y|} \geq a^{10} |A_n|^2 2r_n^{-1},
\]
which means that \( c_2 r_n \geq |A_n| \) for some \( c_2 > 0 \) and sufficiently large \( n \). This is a contradiction to (2.21). This proves the lemma when \( J_\lambda(u_n) \leq C \), and the proof is similar for the case \( \langle J'_\lambda(u_n), u_n \rangle \leq C \). \( \square \)

3 Proofs of main results

In this section, we prove Theorems 1.4 and 1.5. We define the ground state energy to be
\[
m_\lambda = \inf_{u \in H^1_\lambda(\mathbb{R}^3)} J_\lambda(u).
\] (3.1)

First we prove that \( m_\lambda \) is well-defined and that \( J_\lambda \) satisfies Palais–Smale condition.

**Lemma 3.1** Suppose that \( p \in (1, 5) \) and \( \mu > 0 \). Then for any \( \lambda > 0 \), the ground state energy \( m_\lambda \) satisfies \(-\infty < m_\lambda \leq 0\), and \( J_\lambda \) satisfies the Palais–Smale condition in \( H^1_p(\mathbb{R}^3) \).

**Proof** Since \( J_\lambda(0) = 0 \) then clearly \( m_\lambda \leq 0 \). Let \( \{u_n\} \subset H^1_p(\mathbb{R}^3) \) be a sequence satisfying \( J_\lambda(u_n) \rightarrow m_\lambda \leq 0 \) as \( n \rightarrow \infty \). Then \( J_\lambda(u_n) \leq C \) for some positive \( C \) and all \( n \). By Lemma 2.9, we have that \( \{u_n\} \) is bounded, and consequently \( \{J_\lambda(u_n)\} \) is also bounded which implies that \( m_\lambda > -\infty \).

Let \( \{u_n\} \subset H^1_p(\mathbb{R}^3) \) be a Palais–Smale sequence. It follows from Lemma 2.9 that \( \{u_n\} \) is bounded. Since \( p \in (1, 5) \), by using the compactness of the embedding \( H^1_p(\mathbb{R}^3) \hookrightarrow L^{p+1}(\mathbb{R}^3) \) [30, Corollary 1.26] or [24], we can assume that \( u_n \rightharpoonup u \) in \( H^1_p(\mathbb{R}^3) \), \( u_n \rightarrow u \) in \( L^{p+1}(\mathbb{R}^3) \) and \( u_n \rightarrow u \) a.e. on \( \mathbb{R}^3 \) as \( n \rightarrow \infty \) for some \( u \in H^1_p(\mathbb{R}^3) \). This implies that
\[
\int_{\mathbb{R}^3} \left( |u_n|^p - |u|^{p-1}u_n - |u|^{p-1}u \right) (u_n - u) \rightarrow 0, \quad \text{as} \quad n \rightarrow \infty.
\] (3.2)

Since \( u_n \rightarrow u \) in \( H^1_p(\mathbb{R}^3) \) as \( n \rightarrow \infty \), then in view of the Sobolev’s embedding theorem, we have that \( u_n \rightarrow u \) in \( L^6(\mathbb{R}^3) \) as \( n \rightarrow \infty \). It follows from Lemma 2.1 part (vi) that
\[
\phi_{u_n} \rightarrow \phi_u \quad \text{in} \quad D^{1,2}(\mathbb{R}^3), \quad \text{as} \quad n \rightarrow \infty.
\]

Let \( v_n = u_n - u \). Since \( \{\phi_{u_n}|u_n|^3u_n\} \) is bounded in \( L^{6/5}(\mathbb{R}^3) \) and \( \phi_{u_n}|u_n|^3u_n \rightarrow \phi_u|u|^3u \) a.e. on \( \mathbb{R}^3 \), then
\[
\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^3u_n \rightarrow \int_{\mathbb{R}^3} \phi_u|u|^3, \quad \text{as} \quad n \rightarrow \infty.
\]
By using Lemma 2.3, we obtain that, as $n \to \infty$,
\[
\int_{\mathbb{R}^3} \phi_{u_n} |u_n|^3 u_n (u_n - u) = \int_{\mathbb{R}^3} \phi_{v_n} |v_n|^5 + \int_{\mathbb{R}^3} \phi_u |u|^5 - \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^3 u_n u + o(1)
\]
\[
= \int_{\mathbb{R}^3} \phi_{v_n} |v_n|^5 + o(1).
\]

Thus, as $n \to \infty$,
\[
\langle J'_\lambda(u_n) - J'_\lambda(u), u_n - u \rangle
\]
\[
= \|u_n - u\|^2 + \lambda \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^3 u_n (u_n - u) - \lambda \int_{\mathbb{R}^3} \phi_u |u|^3 u (u_n - u)
\]
\[
- \mu \int_{\mathbb{R}^3} (|u_n|^{p-1} u_n - |u|^{p-1} u) (u_n - u)
\]
\[
= \|v_n\|^2 + \lambda \int_{\mathbb{R}^3} \phi_{v_n} |v_n|^5 + o(1).
\]

We then conclude that $u_n \to u$ strongly in $H^1_r(\mathbb{R}^3)$ as $n \to \infty$. The proof is completed. ☐

We also prove that $J_0$ satisfies the Palais–Smale condition in $H^1_r(\mathbb{R}^3)$, which is useful in our proof.

**Lemma 3.2** Suppose that $p \in (1, 5)$ and $\mu > 0$. Then $J_0$ satisfies the Palais–Smale condition in $H^1_r(\mathbb{R}^3)$.

**Proof** The proof is well known. But for the convenience of readers, we recall it. Suppose that $\{u_n\} \subset H^1_r(\mathbb{R}^3)$, $\{J_0(u_n)\}$ is bounded and $J_0'(u_n) \to 0$ as $n \to \infty$. Then
\[
J_0(u_n) - \frac{1}{p+1} \langle J_0'(u_n), u_n \rangle = \left( \frac{1}{2} - \frac{1}{p+1} \right) \|u_n\|^2,
\]
which implies that $\{u_n\}$ is bounded in $H^1_r(\mathbb{R}^3)$. Thus, we may assume that $u_n \to u$ for some $u \in H^1_r(\mathbb{R}^3)$. It follows from $J_0'(u_n) \to 0$ and $u_n \to u$ that $\langle J_0'(u_n), u_n - u \rangle \to 0$ and $\langle J_0'(u), u_n - u \rangle \to 0$ as $n \to \infty$. Then we have that
\[
\|u_n - u\| = \langle J_0'(u_n) - J_0'(u), u_n - u \rangle + \int_{\mathbb{R}^3} |u_n|^{p-1} u_n (u_n - u) - \int_{\mathbb{R}^3} |u|^{p-1} u (u_n - u).
\]

By using the compactness of the embedding $H^1_r(\mathbb{R}^3) \hookrightarrow L^{p+1}(\mathbb{R}^3)$ [24], we conclude that $u_n \to u$ strongly in $H^1_r(\mathbb{R}^3)$. ☐

Now we show the existence of the ground state positive solution of (1.1).

**Theorem 3.3** Suppose that $p \in (1, 5)$ and $\mu > 0$. Then there exists $\lambda_0 \in (0, \mu^{8/(p-1)}/4)$ such that the system (1.1) has at least one positive radially symmetric solution $u_{1,\lambda} \in H^1_r(\mathbb{R}^3)$ satisfying $J_\lambda(u_{1,\lambda}) = m_\lambda$ for $\lambda \in (0, \lambda_0)$.

**Proof** For $\lambda > 0$, note that $u = 0$ is an isolated local minimum of the functional $J_\lambda$ and $J_\lambda(0) = 0$. Let $u_0$ be the unique positive solution of (1.3). Since
\[
J_0(tu_0) = \frac{1}{2} t^2 \|u_0\|^2 - \frac{1}{p+1} t^{p+1} \int_{\mathbb{R}^3} |u_0|^{p+1}, \quad t > 0,
\]

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then $J_0(tu_0) → −∞$ as $t → ∞$. From Lemma 3.1 $m_λ$ exists for $λ > 0$. Therefore, $m_λ ≤ J_λ(tu_0) < 0$ for $λ > 0$ small. Since $m_λ$ is nondecreasing in $λ$, then we have that

$$λ_0 = \text{sup}\{λ > 0 : m_λ < 0\}$$  \hspace{1cm} (3.3)

exists. For $λ ∈ (0, λ_0)$, we have $m_λ < 0$, and from the Ekeland variational principle [11], a Palais–Smale sequence exists. From Lemma 3.1, $J_λ$ satisfies the Palais–Smale condition, and consequently the infimum $m_λ$ is attained by some $u_{1,λ} ∈ H^1_r(\mathbb{R}^3)$. This yields a nonzero solution $u_{1,λ}$ with $J_λ(u_{1,λ}) = m_λ < 0$. It follows from Theorem 2.8 that $λ_0 ∈ (0, μ^{8/(p−1)/4})$. By considering

$$J_λ^+(u) = \frac{1}{2}∥u∥^2 + \frac{λ}{10} \int_{\mathbb{R}^3} φ_0 |u|^5 - \frac{1}{p + 1}μ \int_{\mathbb{R}^3} (u^+)^{p+1},$$  \hspace{1cm} (3.4)

we can obtain the desired positive solution $u_{1,λ} ∈ H^1_r(\mathbb{R}^3)$. \hspace{1cm} □

Next we show that $m_λ$ is achieved by a nonzero solution not only for $λ ∈ (0, λ_0)$, but also at $λ = λ_0$, but for $λ ∈ (λ_0, ∞)$, $m_λ = 0$ and it is achieved by $u = 0$.

**Lemma 3.4** Suppose that $p ∈ (1, 5)$ and $μ > 0$. Let $m_λ$ be defined as in (3.1) and let $λ_0$ be defined as in (3.3).

1. For $0 < λ < λ_0$, $m_λ < 0$ and there exists $r_0 > 0$ such that the solution $u_{1,λ} ∈ H^1_r(\mathbb{R}^3)$ of (1.1) obtained in Theorem 3.3 satisfies $∥u_{1,λ}∥ ≥ r_0$, $J_λ(u_{1,λ}) = m_λ$ and $J'_\lambda(u_{1,λ}) = 0$.
2. For $λ = λ_0$, there exists a positive radially symmetric solution $u_{1,λ_0} ∈ H^1_r(\mathbb{R}^3)$ with $∥u_{1,λ_0}∥ ≥ r_0$ such that $J_λ(u_{1,λ_0}) = m_λ = 0$ and $J'_\lambda(u_{1,λ_0}) = 0$.
3. For $λ > λ_0$, $m_λ = 0$ and there is no nonzero critical point $u ∈ H^1_r(\mathbb{R}^3)$ such that $J_λ(u) = m_λ$.

**Proof** If $u ≠ 0$ is a solution of (1.1), then $⟨J'_\lambda(u), u⟩ = 0$ and consequently

$$∥u∥^2 ≤ ∥u∥^2 + λ \int_{\mathbb{R}^3} φ_0 |u|^5 = μ|u|^{p+1}_p ≤ C∥u∥^{p+1}.$$  \hspace{1cm} (3.5)

Thus, $J_λ$ has no nontrivial critical points with $∥u∥ ≤ C$ for some constant $C > 0$. On the other hand, for all $λ > 0$,

$$J_λ(u) = \frac{1}{2}∥u∥^2 + \frac{λ}{10} \int_{\mathbb{R}^3} φ_0 |u|^5 - \frac{μ}{p + 1} |u|^{p+1}_{p+1} \geq \frac{1}{2}∥u∥^2 - \frac{μ}{p + 1} |u|^{p+1}_{p+1} \geq \frac{1}{2}∥u∥^2 - C_μ∥u∥^{p+1}.$$  \hspace{1cm} (3.6)

Therefore, there exists an $r_0 ∈ (0, C)$ such that $J_λ(u) > 0$ for any $u$ satisfying $0 < ∥u∥ ≤ r_0$.

From the definition of $m_λ$, we know that $−∞ < m_λ < 0$ for $λ ∈ (0, λ_0)$. We claim that $m_{λ_0} = 0$. If $m_{λ_0} < 0$, then there exists $u$ with $∥u∥ ≥ r_0$ such that $J_{λ_0}(u) < 0$, and then there exists $ε > 0$ such that $J_{λ_0+ε}(u) < 0$, which contradicts with the definition of $λ_0$. From the definition of $m_λ$, $m_λ$ is nondecreasing for $λ ∈ (0, ∞)$. Then we have that $m_λ = 0$ for $λ ≥ λ_0$ since $J_λ(0) = 0$. By using Theorem 3.3, we have that $J_λ(u_{1,λ}) = m_λ < 0$ and $∥u_{1,λ}∥ ≥ r_0$. Since $⟨J'_\lambda(u_{1,λ}), u_{1,λ}⟩ = 0$, then $⟨J'_{λ_0/2}(u_{1,λ}), u_{1,λ}⟩ ≤ 0$ for $λ > λ_0/2$. Thus it follows from Lemma 2.9 that $u_{1,λ}$ is bounded as $λ → λ_0^−$, and for $φ ∈ H^1_r(\mathbb{R}^3)$,

$$⟨J'_{λ_0}(u_{1,λ}), φ⟩ = ⟨J'_\lambda(u_{1,λ}), φ⟩ + \frac{1}{10}(λ_0 - λ) \int_{\mathbb{R}^3} φ_0 |u_{1,λ}|^3 u_{1,λ} φ → 0, \text{ as } λ → λ_0^−.$$
Therefore, $J'_{\lambda_0}(u_{1,\lambda}) \to 0$ as $\lambda \to \lambda_0^-$. By Lemma 3.1, $J_{\lambda_0}$ satisfies the Palais–Smale condition, hence there is a subsequence $\{u_{1,\lambda_n}\}$ of $\{u_{1,\lambda}\}$ such that $u_{1,\lambda_n} \to u_{1,\lambda_0}$ for some $u_{1,\lambda_0} \in H^1_0(\mathbb{R}^3)$. It is obvious that $J'_{\lambda_0}(u_{1,\lambda_0}) = 0$, $J_{\lambda_0}(u_{1,\lambda_0}) = 0$ and $\|u_{1,\lambda_0}\| \geq r_0$. This proves the existence of a nonzero solution $u_{1,\lambda_0} \in H^1_0(\mathbb{R}^3)$ for $\lambda = \lambda_0$. Moreover $u_{1,\lambda_0}$ is also nonnegative and radially symmetric since each $u_{1,\lambda}$ is positive and radially symmetric, and indeed $u_{1,\lambda_0}$ is positive since $u_{1,\lambda_0} \neq 0$ and by the strong maximum principle. If $\lambda > \lambda_0$, then $m_\lambda \geq m_{\lambda_0} = 0$ and hence $m_\lambda = 0$. We show that there is no nonzero critical point $u$ such that $J_\lambda(u) = 0$ for $\lambda > \lambda_0$. Suppose that there is a nonzero critical point $u_\lambda$ of $J_\lambda$ such that $J_\lambda(u_\lambda) = 0$, then $J_\lambda_0(u_\lambda) < 0$ which contradicts that $m_{\lambda_0} = 0$. Hence there is no $u \neq 0$ such that $J_\lambda(u) = 0$ for $\lambda > \lambda_0$. \hfill $\Box$

Next we show some additional properties of ground state energy level $m_\lambda$ and the asymptotic behavior when $\lambda \to 0^+$.

**Proposition 3.5** Let $m_\lambda$ be defined as in (3.1) and let $u_{1,\lambda}$ be a positive ground state solution of (1.1) for $\lambda \in (0, \lambda_0)$, i.e., $J_\lambda(u_{1,\lambda}) = m_\lambda$. Then

1. The mapping $\lambda \mapsto m_\lambda$ is continuous for $\lambda \in (0, \infty)$.
2. \( \lim_{\lambda \to 0^+} \|u_{1,\lambda}\| = \infty \) and \( \lim_{\lambda \to 0^+} m_\lambda = -\infty \).

**Proof** 1. Suppose that $\lambda > 0$. Let $\{\lambda_n\}$ be a sequence satisfying $\lambda_n > \lambda$ and $\lambda_n \to \lambda$ as $n \to \infty$. Then we get from $J_{\lambda_n}(u) \leq J_\lambda(u)$ for any $u \in H^1_0(\mathbb{R}^3)$ that $m_{\lambda_n} \geq m_\lambda$. On the other hand, for any given $\varepsilon > 0$, there exists a $u \in H^1_0(\mathbb{R}^3)$ such that $J_\lambda(u) < m_\lambda + \varepsilon/2$. Thus, for $n$ large,

$$J_{\lambda_n}(u) = J_\lambda(u) + \frac{1}{10}(\lambda_n - \lambda) \int_{\mathbb{R}^3} \phi_u |u|^5 < m_\lambda + \varepsilon,$$

hence $m_{\lambda_n} < m_\lambda + \varepsilon$. Therefore, $m_\lambda$ is continuous from the right. Next let $\{\lambda_n\}$ be a sequence satisfying $\lambda_n < \lambda$ and $\lambda_n \to \lambda$ as $n \to \infty$. Then $m_{\lambda_n} \leq m_\lambda$. Since $m_\lambda$ is achieved by some $u_\lambda \in H^1_0(\mathbb{R}^3)$ for all $\lambda \in (0, \infty)$, then we may assume that $J_{\lambda_n}(u_{1,\lambda_n}) = m_{\lambda_n}$ for some $u_{1,\lambda_n} \in H^1_0(\mathbb{R}^3)$. Since for $n$ large,

$$\langle J_{\lambda_n/2}'(u_{1,\lambda_n}), u_{1,\lambda_n} \rangle \leq \langle J_{\lambda_n}'(u_{1,\lambda_n}), u_{1,\lambda_n} \rangle = 0,$$

then it follows from Lemma 2.9 that $\{u_{1,\lambda_n}\}$ is bounded. Thus,

$$m_{\lambda_n} = J_{\lambda_n}(u_{1,\lambda_n}) = J_\lambda(u_{1,\lambda_n}) - \frac{1}{10}(\lambda_n - \lambda) \int_{\mathbb{R}^3} \phi_{u_{1,\lambda_n}} |u_{1,\lambda_n}|^5 \geq m_\lambda - \varepsilon,$$

for $n$ large. Therefore $m_\lambda$ is also continuous from left. This proves the continuity of $m_\lambda$ in $\lambda$.

2. We claim that $\|u_{1,\lambda_n}\| \to \infty$ as $\lambda \to 0^+$. If it is not true, then there is a sequence $\{\lambda_n\}$ such that $\lambda_n \to 0^+$ and $\|u_{1,\lambda_n}\| \leq C$ for some constant $C$, then $\{J_0(u_{1,\lambda_n})\}$ is bounded. Since for any $v \in H^1_0(\mathbb{R}^3)$,

$$\langle J_0'(u_{1,\lambda_n}), v \rangle = \langle J_{\lambda_n}'(u_{1,\lambda_n}), v \rangle - \lambda_n \int_{\mathbb{R}^3} \phi_{u_{1,\lambda_n}} |u_{1,\lambda_n}|^3 u_{1,\lambda_n} v,$$

it follows that $J_0'(u_{1,\lambda_n}) \to 0$. According to Lemma 3.2, we may assume that $u_{1,\lambda_n} \to u$ and then $u$ is a nonnegative solution of (1.3). Since (3.5) and (3.6) still hold when $\lambda = 0$, then $\|u\| \geq r_0 > 0$ so $u \neq 0$ and $u$ is positive and radially symmetric. Therefore, $u = u_0$ from the uniqueness of the positive solution to (1.3). Since $u_{1,\lambda_n} \to u$ as $n \to \infty$ and $m_{\lambda_n} = J_{\lambda_n}(u_{1,\lambda_n}) < 0$, then

$$J_0(u_0) = \lim_{n \to \infty} J_{\lambda_n}(u_{1,\lambda_n}) \leq 0.$$
But on the other hand,
$$J_0(u_0) = \left(\frac{1}{2} - \frac{1}{p+1}\right)\|u_0\|^2 > 0.$$ That is a contradiction. Hence we must have \(\|u_{1,\lambda}\| \to \infty\) as \(\lambda \to 0^+\). Also from the proof of Theorem 3.3, \(J_0(tu_0) \to -\infty\) as \(t \to \infty\), then we have that \(m_{\lambda} \to -\infty\) as \(\lambda \to 0^+\). □

The conclusions of Theorem 1.4 are obtained from Lemma 3.4 and Proposition 3.5. Now we turn to the proof of a second positive solution of (1.1) by using the the mountain pass theorem of Ambrosetti and Rabinowitz [1]. To show that (1.1) has a mountain pass structure, first we note the following fact.

**Lemma 3.6** Suppose that \(p \in (1, 5)\) and \(\mu > 0\). Let \(r_0 > 0\) be defined as in Lemma 3.4.

1. For all \(\lambda > 0\), \(u = 0\) is an isolated local minimizer of the energy functional \(J_\lambda\).
2. For all \(\lambda > 0\), there exist \(r_1 \in (0, r_0)\) and \(\alpha > 0\) (both independent of \(\lambda\)) such that
   $$\inf_{\|u\|=r_1} J_\lambda(u) \geq \alpha.$$

**Proof** From (3.6), for \(\|u\| \neq 0\) small, \(J_\lambda(u) > 0\). Hence \(u = 0\) is an isolated local minimizer. And (3.6) also implies that there exist \(r_1 > 0\) and \(\alpha > 0\) such that \(\inf_{\|u\|=r_1} J_\lambda(u) \geq \alpha > 0\).

For \(\lambda \in (0, \infty)\), define
$$\Gamma_\lambda = \{\gamma \in C([0, 1], H^1_p(\mathbb{R}^3)) : \gamma(0) = 0, J_\lambda(\gamma(1)) < \alpha\}. \quad (3.7)$$
If \(\Gamma_\lambda \neq \emptyset\), we set
$$c_\lambda = \min_{\gamma \in \Gamma_\lambda} \max_{t \in [0, 1]} J_\lambda(\gamma(t)). \quad (3.8)$$
We show that \(c_\lambda\) satisfies the following properties.

**Lemma 3.7** Suppose that \(p \in (1, 5)\) and \(\mu > 0\). Then there exists \(\lambda_1 > \lambda_0\) such that for \(\lambda \in (0, \lambda_1), \Gamma_\lambda \neq \emptyset\), the mapping \(\lambda \mapsto c_\lambda\) is continuous from right and nondecreasing, that is, if \(\lambda_n \to \lambda^+\) as \(n \to \infty\), then \(c_{\lambda_n} \to c_\lambda\), and if \(0 < \lambda' < \lambda'' < \lambda_1\), then \(c_{\lambda'} \leq c_{\lambda''}\).

**Proof** Let \(u_{1,\lambda}\) be a positive ground state solution of (1.1) for \(\lambda \in (0, \lambda_0)\). Then from Lemma 3.4, \(m_\lambda = J_\lambda(u_{1,\lambda}) < 0 < \alpha\) and \(\|u_{1,\lambda}\| \geq r_0 > r_1\) for \(\lambda \in (0, \lambda_0)\). So \(\Gamma_\lambda \neq \emptyset\) for \(\lambda \in (0, \lambda_0)\). Moreover \(J_{\lambda_0}(u_{1,\lambda_0}) < 0 < \alpha\) and that implies \(J_{\lambda_0}(u_{1,\lambda_0}) < \alpha\) for \(\lambda \in (\lambda_0, \lambda_0 + \varepsilon)\) for some \(\varepsilon > 0\).

Let \(\lambda_1 = \sup\{\lambda \geq 0\ : \text{there is a } u \text{ with } \|u\| \geq r_0 \text{ such that } J_\lambda(u) < \alpha\}\). Then from the above arguments, \(\lambda_1 > \lambda_0\) and \(\inf_{\|u\|=r_0} J_{\lambda_1}(u) = \alpha\). We also have \(\Gamma_{\lambda_1} \neq \emptyset\) and \(c_{\lambda_1}\) is well-defined for \(\lambda \in (0, \lambda_1)\).

We prove that \(c_{\lambda_1}\) is continuous from the right and nondecreasing for \(\lambda \in (0, \lambda_1)\). Let \(\{\lambda_n\}\) be a sequence and \(\lambda_n \to \lambda^+\). Fix \(n\), for each \(\gamma \in \Gamma_{\lambda_n}\), we have that \(J_{\lambda_n}(\gamma(1)) < \alpha\). Since \(J_{\lambda} \leq J_{\lambda_n}\), then it follows that \(J_{\lambda}(\gamma(1)) < \alpha\). Therefore, \(\gamma \in \Gamma_{\lambda}\) and thus \(\Gamma_{\lambda_n} \subset \Gamma_{\lambda}\) for any \(n \in \mathbb{N}\). It also follows from \(J_{\lambda} \leq J_{\lambda_n}\) that for \(\gamma \in \Gamma_{\lambda_n}\),
$$\max_{t \in [0, 1]} J_{\lambda}(\gamma(t)) \leq \max_{t \in [0, 1]} J_{\lambda_n}(\gamma(t)).$$
and then
\[
\min_{y \in \Gamma_n} \max_{t \in [0,1]} J_\lambda(y(t)) \leq \min_{y \in \Gamma_n} \max_{t \in [0,1]} J_\lambda(y(t)) \leq \min_{y \in \Gamma_n} \max_{t \in [0,1]} J_{\lambda_n}(y(t)).
\]
Thus \(c_\lambda \leq c_{\lambda_n}\) for any \(n \in \mathbb{N}\).

On the other hand, let \(\varepsilon > 0\) be an arbitrary constant, and take \(\gamma \in \Gamma_\lambda\) such that \(J_\lambda'(y(t)) < c_\lambda + \varepsilon/2\) for all \(t \in [0,1]\). Since for \(t \in [0,1]\),
\[
|J_{\lambda_n}(\gamma(t)) - J_\lambda(\gamma(t))| = (\lambda_n - \lambda) \int_{\mathbb{R}^3} \phi''(\gamma(t)) |\gamma(t)|^5 \leq C S^{-1}(\lambda_n - \lambda) \|\gamma(t)\|^{10},
\]
then for \(n\) large, \(J_{\lambda_n}(\gamma(1)) < \alpha\) and \(J_{\lambda_n}(\gamma(t)) < c_\lambda + \varepsilon\). Thus, \(c_{\lambda_n} < c_\lambda + \varepsilon\) for large \(n\). The nondecreasing property of \(c_\lambda\) can be proved similarly as the proof of \(c_\lambda \leq c_{\lambda_n}\) above. \(\square\)

Now we are ready to prove the existence of a second positive solution of (1.1) for \(\lambda \in (0, \lambda_1)\).

**Proposition 3.8** Suppose that \(p \in (1, 5)\) and \(\mu > 0\). Let \(\lambda_1\) be defined as in Lemma 3.7. Then

1. For \(\lambda \in (0, \lambda_1)\), \(J_\lambda\) has a mountain pass type critical point \(u_{2,\lambda} \in H^1_r(\mathbb{R}^3)\) such that \(u_{2,\lambda} > 0\) and \(u_{2,\lambda} \neq u_{1,\lambda}\). Moreover \(0 < \lambda_1 < \mu^{8/(p-1)}/4\).
2. For \(\lambda = \lambda_1\), there exists \(u_{2,\lambda_1} \in H^1_r(\mathbb{R}^3)\) which is a critical point of \(J_\lambda\) and \(u_{2,\lambda_1} > 0\).

**Proof** Suppose that \(\lambda \in (0, \lambda_1)\). From Lemma 3.1, \(J_\lambda\) satisfies the Palais–Smale condition. From Lemma 3.7, the set \(\Gamma_\lambda \neq \emptyset\) and \(c_\lambda\) is well-defined. Now we can apply the mountain pass theorem of Ambrosetti and Rabinowitz [1] to conclude the existence of a nonzero mountain pass type critical point \(u_{2,\lambda}\) such that \(J_\lambda(u_{2,\lambda}) = c_\lambda > 0\) for \(\lambda \in (0, \lambda_1)\). Apparently \(u_{2,\lambda} \neq u_{1,\lambda}\) and again we may assume that \(u_{2,\lambda}\) is nonnegative by considering the functional in (3.4). Since \(u_{2,\lambda} \neq 0\) then \(u_{2,\lambda} > 0\) from the maximum principle. For \(\lambda = 0\), the proof is similar. We have \(\lambda_1 < \mu^{8/(p-1)}/4\) from Theorem 2.8.

To prove the existence of a positive solution for \(\lambda = \lambda_1\), let \(\{\lambda_n\}\) be a sequence such that \(\lambda_n \to \lambda_1^-\) as \(n \to \infty\). Then there exists an \(\varepsilon_0 > 0\) such that \(\lambda_n > \lambda_1 - \varepsilon_0 =: \theta > 0\) for larger \(n\). Since \(\langle J_{\lambda_n}'(u_{2,\lambda_n}), u_{2,\lambda_n} \rangle = 0\) and \(\langle J_{\lambda_n}'(u_{2,\lambda_n}), u_{2,\lambda_n} \rangle \leq 0\), then by Lemma 2.9, \(\{u_{2,\lambda_n}\}\) is bounded. Moreover
\[
J_{\lambda_1}(u_{2,\lambda_n}) = J_{\lambda_n}(u_{2,\lambda_n}) + \frac{1}{10} (\lambda_1 - \lambda_n) \int_{\mathbb{R}^3} \phi_{u_{2,\lambda_n}} |u_{2,\lambda_n}|^5 = c_{\lambda_n} + o(1),
\]
\[
\langle J_{\lambda_1}'(u_{2,\lambda_n}), v \rangle = \langle J_{\lambda_n}'(u_{2,\lambda_n}), v \rangle + (\lambda_1 - \lambda_n) \int_{\mathbb{R}^3} \phi_{u_{2,\lambda_n}} |u_{2,\lambda_n}|^3 u_{2,\lambda_n} v = o(\|v\|), \quad \forall v \in H^1_r(\mathbb{R}^3).
\]
It follows that \(\{J_{\lambda_1}(u_{2,\lambda_n})\}\) is bounded and \(J_{\lambda_1}'(u_{2,\lambda_n}) \to 0\). By Lemma 3.1, \(J_{\lambda_1}\) satisfies the Palais–Smale condition. Therefore there exists a subsequence \(\{v_n\}\) of \(\{u_{2,\lambda_n}\}\) such that \(v_n \to u_{2,\lambda_1}\) for some \(u_{2,\lambda_1} \in H^1_r(\mathbb{R}^3)\). It is clear that \(J_{\lambda_1}'(u_{2,\lambda_1}) = 0\) and \(u_{2,\lambda_1} \neq 0\) since \(\|u_{2,\lambda_n}\| \geq r_0\). Again \(u_{2,\lambda_1} > 0\) from the maximum principle and \(u_{2,\lambda_1} > 0\). \(\square\)

Finally we consider the behavior of the mountain pass type solution \(u_{2,\lambda}\) as \(\lambda \to 0^+\). We first show the following perturbation type result from the implicit function theorem.

**Lemma 3.9** Suppose that \(p \in (1, 5)\) and \(\mu > 0\). Then there exists a \(\lambda_2 > 0\) such that for \(\lambda \in (-\lambda_2, \lambda_2)\), \(J_\lambda\) has a unique critical point \(u_\lambda\) such that \(u_\lambda \to u_0\) as \(\lambda \to 0\).
Proof The proof is similar to that of Proposition 2.1 in [21]. Let \( F : \mathbb{R}^1 \times H^1_\text{r}(\mathbb{R}^3) \rightarrow [H^1_\text{r}(\mathbb{R}^3)]^* \) be defined by \( F(\lambda, u) = J''_\text{r}(u) \). We prove the existence of unique solution to the equation \( F(\lambda, u) = 0 \) near \((\lambda, u) = (0, u_0)\) by the implicit function theorem. In fact, \( F(0, u) = 0 \) has a unique positive solution \( u_0 \in H^1_\text{r}(\mathbb{R}^3) \), and for \( v \in H^1_\text{r}(\mathbb{R}^3) \),

\[
F_u(0, u_0)[v] = J''_\text{r}(0, u_0)[v] = \Delta v - v + \mu p|u_0|^{p-1} v,
\]
in the weak sense. Then \( F_u(0, u_0) : H^1_\text{r}(\mathbb{R}^3) \rightarrow [H^1_\text{r}(\mathbb{R}^3)]^* \) is a Fredholm operator, and \( F_u(0, u_0) = J''_\text{r}(0, u_0) \) is nondegenerate in \( H^1_\text{r}(\mathbb{R}^3) \) since the kernel of \( J''_\text{r}(0, u_0) \) in the space \( H^1(\mathbb{R}^3) \) is spanned by the functions \( \frac{\partial}{\partial r} u_0 \), which are orthogonal to the space \( H^1_\text{r}(\mathbb{R}^3) \) (see [7, 20]). Now from the implicit function theorem, for \(|\lambda|\) small, there is a continuous map \( \lambda \mapsto u_\lambda \) such that \( F(\lambda, u_\lambda) = 0 \) for \( u_\lambda \) near \( u_0 \).

Now we prove the part 2 of Theorem 1.5.

Theorem 3.10 Suppose that \( p \in (1, 5) \) and \( \mu > 0 \). Let \( u_{2, \lambda} \) be defined as in Proposition 3.8.

1. If \( u_{2, \lambda} \) is bounded as \( \lambda \to 0^+ \), then \( u_{2, \lambda} = u_\lambda \) for \( \lambda \in (0, \lambda_2) \) where \( u_\lambda \) is the unique positive solution approaching \( u_0 \) as defined in Lemma 3.9.
2. If \( u_{2, \lambda} \) is unbounded as \( \lambda \to 0^+ \), then the system (1.1) has at least three positive solutions for \( \lambda \in (0, \lambda_2) \).

Proof If \( u_{2, \lambda} \) is bounded as \( \lambda \to 0^+ \), then \( J_0(u_{2, \lambda}) \) is bounded and \( J_0'(u_{2, \lambda}) \to 0 \). Since \( J_0 \) satisfies the Palais–Smale condition, we may assume that \( u_{2, \lambda} \to u \) for some \( u \neq 0 \). Thus \( u \) is a positive solution of (1.3). By the uniqueness of the solution to (1.3), we have that \( u = u_0 \).

It follows from Lemma 3.9 that we must have \( u_{2, \lambda} = u_\lambda \). If \( u_{2, \lambda} \) is unbounded as \( \lambda \to 0^+ \), then \( u_{2, \lambda} \neq u_1 \) (the unique positive solution approaching \( u_0 \) as defined in Lemma 3.9).

Therefore we have three positive solutions: \( u_{1, \lambda}, u_{2, \lambda}, \) and \( u_\lambda \).

The proof of Theorem 1.5 is obtained from Proposition 3.8 and Theorem 3.10.

Acknowledgements We would like to thank the anonymous reviewer’s helpful comments, which greatly improve the presentation of our manuscript. This work was done when the first author visited Department of Mathematics, College of William and Mary during the academic year 2015–2016, and she would like to thank Department of Mathematics, College of William and Mary for their support and kind hospitality.

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