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Standing waves of a weakly coupled Schrödinger system with distinct potential functions ☆

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Abstract

The standing wave solutions of a weakly coupled nonlinear Schrödinger system with distinct trapping potential functions in $\mathbb{R}^N$ $(1 \le N \le 3)$ are considered. This type of system arises from models in Bose–Einstein condensates theory and nonlinear optics. The existence of a positive ground state solution is shown when the coupling constant is larger than a sharp threshold value, which is explicitly defined in terms of potential functions and system parameters. It is also shown that such solutions concentrate near the minimum points of potential functions, and multiple positive concentration solutions exist when the topological structure of the set of minimum points satisfies certain condition. Variational approach is used for the existence and concentration of positive solutions.

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Keywords: Coupled nonlinear Schrödinger system; Positive ground solutions; Variational methods

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1. Introduction and main results

In this paper we consider the existence, multiplicity and nonexistence of standing wave solutions of the nonlinear Schrödinger system

$$(\mathscr{A}_\varepsilon) \qquad \begin{cases} -\varepsilon^2\Delta u + P(x)u = \mu u^3 + \beta v^2 u & \text{in } \mathbb{R}^N, \\ -\varepsilon^2\Delta v + Q(x)v = \nu v^3 + \beta u^2 v & \text{in } \mathbb{R}^N, \\ u, v > 0, \quad u, v \in H^1(\mathbb{R}^N), \end{cases}$$

where $1 \le N \le 3$, $\mu, \nu > 0$, ε is a small positive parameter, $P(x)$ and $Q(x)$ are positive potential functions, $\beta > 0$ is a coupling constant.

The nonlinear Schrödinger equation is a canonical and universal equation in physics which is of essential importance in condensed matter, nonlinear optics, continuum mechanics and plasma physics. The coupled nonlinear Schrödinger equations have been the focus of many recent theoretical studies because of recent experimental advances in multi-component Bose–Einstein condensates [1]. The two-component coupled nonlinear Schrödinger equations (also known as Gross–Pitaevskii equations) can be written in the following form:

$$\begin{cases} -i\hbar\dfrac{\partial}{\partial t}\Phi_1 + V_1(x)\Phi_1 = \dfrac{\hbar^2}{2m}\Delta\Phi_1 + \mu|\Phi_1|^2\Phi_1 + \beta|\Phi_2|^2\Phi_1, & \text{in } \mathbb{R}^N,\ t > 0, \\ -i\hbar\dfrac{\partial}{\partial t}\Phi_2 + V_2(x)\Phi_2 = \dfrac{\hbar^2}{2m}\Delta\Phi_2 + \nu|\Phi_2|^2\Phi_2 + \beta|\Phi_1|^2\Phi_2, & \text{in } \mathbb{R}^N,\ t > 0. \end{cases} \tag{1.1}$$

In the context of Bose–Einstein condensates, the complex-valued $\Phi_j(x,t)$ $(j = 1, 2)$ are the wave functions of two interacting condensates; $V_j(x)$ are the trapping potentials; the interaction strength parameters μ, ν and β are determined by the scattering lengths for binary collisions of like and unlike bosons. The physically realistic spatial dimensions are $1 \le N \le 3$. When $N = 2$, problem (1.1) arises in the Hartree–Fock theory for a double condensate, *i.e.*, a binary mixture of Bose–Einstein condensates in two different hyperfine states (see [2,3]). In the attractive case, the components of a vector solution tend to go along with each other, leading to synchronization. In the repulsive case, the components tend to segregate from each other, leading to phase separations. These phenomena have been documented in experiments as well as in numerical simulations (see [4,5] and references therein).

Another recent interest on coupled nonlinear Schrödinger equations is on the propagation of soliton-like pulses in birefringent nonlinear fibers. Experiments have proved the existence of self-trapping of incoherent beam in a nonlinear medium [6,5]. Such findings are significant since optical pulses propagating in a linear medium have a natural tendency to broaden in time (dispersion) and space (diffraction). In the context of optical propagation, Φ_j in (1.1) denotes the j-th component of the beam in Kerr-like photorefractive media; the positive constants μ, ν indicate the self-focusing strength in the component of the beam; and the coupling constant β measures the interaction between the two components of the beam. The sign of β determines whether the interactions of states are repulsive or attractive.

A standing wave solution of (1.1) is of the following form:

$$\Phi_1(x,t) = e^{-iEt/\hbar}u(x) \quad \text{and} \quad \Phi_2(x,t) = e^{-iEt/\hbar}v(x), \tag{1.2}$$

where $(u(x), v(x))$ describes the spatial profile of the wave functions. Typically the wave functions tend to zero when $|x| \to \infty$. Substituting (1.2) into (1.1), and renaming the parameters by

$$\varepsilon = \sqrt{\frac{\hbar^2}{2m}}, \quad P(x) = V_1(x) - E, \quad Q(x) = V_2(x) - E,$$

we obtain the elliptic system $(\mathscr{A}_\varepsilon)$.

Before going further, we point out that the system $(\mathscr{A}_\varepsilon)$ possesses a trivial solution $(0, 0)$ and semi-trivial solutions of type $(u, 0)$ or $(0, v)$. A solution (u, v) of $(\mathscr{A}_\varepsilon)$ is nontrivial if $u \neq 0$ and $v \neq 0$. A solution (u, v) with $u > 0$ and $v > 0$ is called a positive solution. A solution is called a ground state solution (or positive ground state solution) if its energy is minimal among all the nontrivial solutions (or all the nontrivial positive solutions) of $(\mathscr{A}_\varepsilon)$. Here the energy functional corresponding to $(\mathscr{A}_\varepsilon)$ is defined by

$$L(u, v) = \frac{1}{2}\int_{\mathbb{R}^N} (\varepsilon^2|\nabla u|^2 + P(x)u^2 + \varepsilon^2|\nabla v|^2 + Q(x)v^2) - \frac{1}{4}\int_{\mathbb{R}^N} (\mu u^4 + 2\beta u^2v^2 + \nu v^4),$$

for $(u, v) \in E \equiv H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$.

There have been extensive mathematical studies in recent years for the corresponding autonomous system with $\varepsilon = 1$, $P(x) \equiv \lambda_1 > 0$ and $Q(x) \equiv \lambda_2 > 0$:

$$\begin{cases} -\Delta u + \lambda_1 u = \mu u^3 + \beta v^2 u & \text{in } \mathbb{R}^N, \\ -\Delta v + \lambda_2 v = \nu v^3 + \beta u^2 v & \text{in } \mathbb{R}^N, \\ u, v > 0, \quad u, v \in H^1(\mathbb{R}^N). \end{cases} \tag{1.3}$$

The existence, multiplicity, bifurcation, concentration behavior of positive solutions of $(\mathscr{A}_\varepsilon)$ with $\varepsilon = 1$ and (1.3) (in $\mathbb{R}^N$ or a bounded domain of $\mathbb{R}^N$) have been considered in, for example, [7–25] and references therein. In particular for the autonomous case (1.3), the existence, uniqueness of positive solutions can be summarized as follows:

Theorem 1.1. *Suppose that* $\lambda_i, \mu, \nu, \beta > 0$, $\mu \neq \nu$ *and* $1 \leq N \leq 3$.

1. *If* $\lambda_1 = \lambda_2 = \lambda$, *then* (1.3) *has a positive ground state solution* (u, v) *when* $\beta \in (0, \min\{\mu, \nu\}) \bigcup (\max\{\mu, \nu\}, \infty)$, *which can be expressed by*

$$(u(x), v(x)) = \left(\sqrt{\frac{\lambda(\beta - \nu)}{\beta^2 - \mu\nu}} w_1(\sqrt{\lambda}x), \sqrt{\frac{\lambda(\beta - \mu)}{\beta^2 - \mu\nu}} w_1(\sqrt{\lambda}x)\right), \tag{1.4}$$

and w_1 *is the unique positive (radially symmetric) solution of the problem:*

$$\Delta w - w + w^3 = 0 \text{ in } \mathbb{R}^N, \quad w \in H^1(\mathbb{R}^N); \tag{1.5}$$

and (1.3) *has no positive solution when* $\beta \in [\min\{\mu, \nu\}, \max\{\mu, \nu\}]$. *Moreover when* $\beta \in (\max\{\mu, \nu\}, \infty)$, *the positive solution of* (1.3) *is unique up to a translation.*

2. *If* $\lambda_1 \neq \lambda_2$, *then there exist* $\beta_0 = \beta_0(\mu, \nu, \lambda_1, \lambda_2) \in (0, \min\{\mu, \nu\}]$ *and* $\beta_1 = \beta_1(\mu, \nu, \lambda_2/\lambda_1) \geq \max\{\mu, \nu\}$ *such that* (1.3) *has a positive ground state solution when* $\beta \in (0, \beta_0) \bigcup (\beta_1, \infty)$, *and* (1.3) *has no positive solution when* $\beta \in [\min\{\mu, \nu\}, \max\{\mu, \nu\}]$, *where*

$$\beta_1(\mu, \nu, z) = \max\left\{\mu z, \nu z^{\frac{N}{2}-1}, \nu z^{-1}, \mu z^{1-\frac{N}{2}}\right\}, \quad \text{for } z > 0,\ N = 1, 2, 3. \tag{1.6}$$

The existence of the positive ground state solution of (1.3) in Theorem 1.1 was proved in [23], and the uniqueness was proved in [26]. Partial uniqueness results for the case $\beta \in (0, \min\{\mu, \nu\})$ and $\lambda_1 = \lambda_2$ case were also proved in [10,11,26].

The semiclassical case $(\mathscr{A}_\varepsilon)$ with trapping potentials $P(x)$ and $Q(x)$ has been studied in [27, 13,16,20,21]. Lin and Wei [16] proved the existence and asymptotic concentration behavior of a ground state solution of $(\mathscr{A}_\varepsilon)$ with $-\infty < \beta < \beta_0$ for a small $\beta_0 > 0$. Pomponio [21] proved a similar result for $(\mathscr{A}_\varepsilon)$ with $\mu = \mu(x)$, $\nu = \nu(x)$ and $\beta < 0$. In [20], for small $\beta > 0$, Montefusco, Pellacci and Squassina showed the existence of nonnegative ground state solutions of $(\mathscr{A}_\varepsilon)$ concentrating around the local minimum (possibly degenerate) points of the potentials, which are in the same region. Moreover, if $\beta > 0$ is small, then one component of the ground state solution converges to zero. Ikoma and Tanaka [13] connected the solutions of $(\mathscr{A}_\varepsilon)$ with the limiting system (1.3) with $\lambda_1 = P(x_0)$ and $\lambda_2 = Q(x_0)$ for a fixed $x_0 \in \mathbb{R}^N$. Assume that there exists an open bounded set $\Lambda \subset \mathbb{R}^N$ such that

$$\inf_{x_0 \in \Lambda} m(x_0) < \inf_{x_0 \in \partial\Lambda} m(x_0), \tag{1.7}$$

where $m(x_0)$ is the ground state energy level of (1.3) with $(\lambda_1, \lambda_2) = (P(x_0), Q(x_0))$, they showed the existence of positive solution concentrating at an $x_0 \in \Lambda$ which achieves the minimal energy. Recently Long and Peng [28] proved that $(\mathscr{A}_\varepsilon)$ (with $n \geq 2$ variables) has a positive solution for $\beta > 0$ small, whose components may have spikes clustering at the same point as $\varepsilon \to 0$. Note that these results are all for β being negative or being positive but close to zero. We also mention that in the last 20 years, various existence and concentration results for the scalar nonlinear Schrödinger equations have also been obtained in, for example, [29–37].

For the large $\beta > 0$ case, Chen and Zou [27] proved the existence of positive solutions for $(\mathscr{A}_\varepsilon)$ which concentrate around local minima of the potentials for the case that P, Q may vanish in $\mathbb{R}^3$ and large β. More precisely, they assume that there exist a bounded open subset Λ of $\mathbb{R}^3$ which is similar to the one in [13] so that (1.7) holds, and

$$\beta > \tilde{\beta}_0 := \max\{\mu, \nu\} \cdot \max_{x \in \bar{\Lambda}} \left\{\frac{P(x)}{Q(x)}, \frac{Q(x)}{P(x)}\right\}, \tag{1.8}$$

then for small ε, a ground state solution $(\mathscr{A}_\varepsilon)$ exists, and it concentrates near the point x_0 where $m(x_0)$ achieves the minimal energy in (1.7).

In this paper we consider the semiclassical case of $(\mathscr{A}_\varepsilon)$ with positive trapping potential functions $P(x)$, $Q(x)$ and large coupling strength β. Throughout the paper we assume that

(PQ0) $P, Q \in C(\mathbb{R}^N, \mathbb{R}_+)$, and P, Q are bounded.

We also use the following notations.

$$P_0 = \inf_{x\in\mathbb{R}^N} P(x), \quad P_\infty = \liminf_{|x|\to\infty} P(x) \quad \text{and} \quad P^\infty = \limsup_{|x|\to\infty} P(x),$$

$$Q_0 = \inf_{x\in\mathbb{R}^N} Q(x), \quad Q_\infty = \liminf_{|x|\to\infty} Q(x) \quad \text{and} \quad Q^\infty = \limsup_{|x|\to\infty} Q(x),$$

$$\mathscr{P}_0 = \{x \in \mathbb{R}^N : P(x) = P_0\} \quad \text{and} \quad \mathscr{Q}_0 = \{x \in \mathbb{R}^N : Q(x) = Q_0\}.$$

Because of the assumption **(PQ0)**, P^∞, Q^∞ and P_∞, Q_∞ are all finite. By using the notation defined above, we define a critical value characterized by ν, μ, P_0, Q_0, P_∞ and Q_∞ to guarantee the existence of a positive ground state solution of $(\mathscr{A}_\varepsilon)$. Set

$$\hat{\beta}_1 = \max\left\{\beta_1\left(\frac{Q_0}{P_0}\right), \beta_1\left(\frac{Q_\infty}{P_\infty}\right)\right\}, \tag{1.9}$$

where the function $\beta_1(z) = \beta_1(\mu, \nu, z)$ is defined as in (1.6) with fixed μ and ν. Our main existence and concentration of a positive ground state solution of $(\mathscr{A}_\varepsilon)$ is as follows.

Theorem 1.2. *Suppose that* $1 \le N \le 3$, P, Q *satisfy* **(PQ0)** *and*

(PQ1) $0 < P_0 < P_\infty < \infty$, $0 < Q_0 < Q_\infty < \infty$ *and* $\mathscr{V} = \mathscr{P}_0 \cap \mathscr{Q}_0 \neq \emptyset$.

Let $\hat{\beta}_1$ *be defined as in* (1.9). *Then for each* $\beta > \hat{\beta}_1$ *and small* $\varepsilon > 0$,

(*i*) $(\mathscr{A}_\varepsilon)$ *possesses at least one positive ground state solution* $w_\varepsilon = (u_\varepsilon, v_\varepsilon)$ *in* E.
(*ii*) *Let* $\mathscr{B}_\varepsilon$ *be the set of all positive ground state solutions of* $(\mathscr{A}_\varepsilon)$. *Then* $\mathscr{B}_\varepsilon$ *is compact in* E.

If in addition, P, Q *are uniformly continuous in* $\mathbb{R}^N$, *then we have that*

(*iii*) *there exists a maximum point* x_ε *of* $u_\varepsilon + v_\varepsilon$ *such that* $\lim_{\varepsilon\to 0} dist(x_\varepsilon, \mathscr{V}) = 0$, *and subject to a subsequence,* $x_\varepsilon \to y_0 \in \mathscr{V}$ *and* $h_\varepsilon(x) = w_\varepsilon(\varepsilon x + x_\varepsilon) = (u_\varepsilon(\varepsilon x + x_\varepsilon), v_\varepsilon(\varepsilon x + x_\varepsilon))$ *converges in* E *to a positive ground state solution of* (1.3) *with* $\lambda_1 = P(y_0)$ *and* $\lambda_2 = Q(y_0)$ *as* $\varepsilon \to 0$. *Moreover* $w_\varepsilon \in [C^{1,\sigma}_{loc}(\mathbb{R}^N)]^2$ *with* $\sigma \in (0, 1)$, *and there exist* $C_1, C_2 > 0$ *such that*

$$\lim_{|x|\to\infty} |w_\varepsilon(x)| = \lim_{|x|\to\infty} |\nabla w_\varepsilon(x)| = 0, \;\; |w_\varepsilon(x)| \le C_1 e^{-\frac{C_2}{\varepsilon}|x-x_\varepsilon|}, \;\; x \in \mathbb{R}^N. \tag{1.10}$$

Our existence of positive ground state result in Theorem 1.2 is different from the one in [27] from several aspects. The result in [27] imposes weaker conditions on P, Q outside of open subset Λ and considers positive ground state solutions concentrating near $x_0 \in \Lambda$ where the ground state energy $m(x_0)$ for (1.3) with $(\lambda_1, \lambda_2) = (P(x_0), Q(x_0))$ achieves the minimum (see (1.7) and (1.8)). Our result considers positive ground state solutions concentrating near global minima points of P and Q (which necessarily achieves minimal $m(x_0)$) but with simplified conditions on P, Q as $m(x_0)$ in (1.7) cannot be explicitly expressed. Our result holds for $1 \le N \le 3$ and the one in [27] is only for $N = 3$ as they used Hardy inequality for the proof. Our condition (1.9) on β reveals the dependence on the spatial dimension N, and it is almost optimal compared with earlier result for autonomous system (see Theorem 1.1 and [23]).

Next we state the multiplicity and concentration of positive solutions of $(\mathscr{A}_\varepsilon)$. Here we first recall a definition from Ljusternik–Schnirelmann category theory. If Y is a closed subset of a topological space X, then the Ljusternik–Schnirelmann category $\mathrm{cat}_X(Y)$ is the least number of closed and contractible sets in X which cover Y. In view of the condition **(PQ1)**, the set $\mathscr{V}$ is compact. We also define $\mathscr{O}^\delta = \{x \in \mathbb{R}^N : dist(x, \mathscr{O}) \le \delta\}$ for any subset $\mathscr{O}$ of $\mathbb{R}^N$ and $\delta > 0$. Then we have the following result on the existence of multiple positive solutions.

Theorem 1.3. *Suppose that $1 \le N \le 3$, P, Q satisfy* **(PQ0)** *and* **(PQ1)**. *For each $\delta > 0$, there exist an $\varepsilon_\delta > 0$ such that for each $\beta > \hat{\beta}_1$ and $\varepsilon \in (0, \varepsilon_\delta)$, $(\mathscr{A}_\varepsilon)$ has at least $cat_{\mathscr{V}^\delta}(\mathscr{V})$ distinct nontrivial solutions. Additionally, if P, Q are uniformly continuous functions, and assume that $w_\varepsilon = (u_\varepsilon, v_\varepsilon)$ is any nontrivial solution of $(\mathscr{A}_\varepsilon)$, x_ε is a maximum point of $|u_\varepsilon| + |v_\varepsilon|$, then subject to a subsequence, $x_\varepsilon \to y_0 \in \mathscr{V}$ as $\varepsilon \to 0$, and the estimate (1.10) holds.*

The existence of multiple nontrivial solutions in Theorem 1.3 appears to be the first multiplicity result for solutions of $(\mathscr{A}_\varepsilon)$ with large β. Similar results for scalar nonlinear Schrödinger equations have been obtained in [30,38,39]. In the proof of Theorems 1.2 and 1.3, we overcome the difficulty of lack of compactness by combining the Nehari manifold methods and the condition **(PQ1)** to recover the local compactness. Secondly to exclude the semi-trivial solutions as the ground state of $(\mathscr{A}_\varepsilon)$, we carefully analyze the relationship between the ground state of scalar autonomous equation and the one for the nonautonomous equation, and prove that the ground state solutions in Theorems 1.2 and 1.3 are nontrivial when the parameter β satisfies the condition (1.9).

Finally we have the following nonexistence results.

Theorem 1.4. *Suppose that $1 \le N \le 3$, P, Q satisfy* **(PQ0)**.

(i) *If either (a) $P(x) \ge Q(x)$ for all $x \in \mathbb{R}^N$, $\mu < \nu$ and $\mu \le \beta \le \nu$, or (b) $P(x) \le Q(x)$ for all $x \in \mathbb{R}^N$, $\nu < \mu$ and $\nu \le \beta \le \mu$, then $(\mathscr{A}_\varepsilon)$ does not have any nontrivial positive solution for all $\varepsilon > 0$.*

(ii) *Suppose that P, Q satisfies*

(PQ2) *If $P(x) \ge P^\infty = P_\infty > 0$ and $Q(x) \ge Q^\infty = Q_\infty > 0$ for all $x \in \mathbb{R}^N$, $|\tilde{\mathscr{P}}| > 0$ or $|\tilde{\mathscr{Q}}| > 0$, where $\tilde{\mathscr{P}} = \{x \in \mathbb{R}^N : P(x) > P^\infty\}$ and $\tilde{\mathscr{Q}} = \{x \in \mathbb{R}^N : Q(x) > Q^\infty\}$.*

then there exist $0 < \hat{\beta}_2 \le \hat{\beta}_3 \le \hat{\beta}_1$ such that for $\beta \in (0, \hat{\beta}_2) \cup (\hat{\beta}_3, \infty)$, $(\mathscr{A}_\varepsilon)$ has no positive ground state solution for all $\varepsilon > 0$.

The nonexistence result in Theorem 1.4 completes the picture of existence/nonexistence of positive (ground state) solutions when β varies, similar to the one for autonomous system (1.3) given in Theorem 1.1. We also mention that when the two trapping potential functions $P(x)$ and $Q(x)$ are identical, then results in Theorems 1.2–1.4 can be simplified and strengthened with $\hat{\beta}_3 = \hat{\beta}_1 = \max\{\mu, \nu\}$ and $\hat{\beta}_2 = \min\{\mu, \nu\}$.

Corollary 1.5. *Suppose that $1 \le N \le 3$, $P(x) \equiv Q(x)$.*

(a) *If $P(x)$ satisfies*

(P1) $0 < P_0 < P_\infty < \infty$.

Then for $\beta > \max\{\mu, \nu\}$ and small $\varepsilon > 0$, the results in Theorems 1.2 and 1.3 hold. Moreover for $\beta > \max\{\mu, \nu\}$, each positive solution (u, v) of $(\mathscr{A}_\varepsilon)$ satisfies

$$v(x) = \sqrt{\frac{\beta - \mu}{\beta - \nu}} u(x), \ \ \varepsilon^2 \Delta u - P(x)u + \frac{\beta^2 - \mu\nu}{\beta - \nu} u^3 = 0, \ \ in\ \mathbb{R}^N, \ \ u \in H^1(\mathbb{R}^N).$$

(*b*) *$(\mathscr{A}_\varepsilon)$ has no positive ground state solution for all $\varepsilon > 0$ if either*
  (*b1*) *$\mu \neq \nu$ and $\beta \in [\min\{\mu, \nu\}, \max\{\mu, \nu\}]$; or*
  (*b2*) *$\beta \in (0, \min\{\mu, \nu\}) \cup (\max\{\mu, \nu\}, \infty)$, and*
    (P2) *$P(x) \geq P^\infty = P_\infty > 0$ for all $x \in \mathbb{R}^N$, and $|P_+| > 0$, where $P_+ = \{x \in \mathbb{R}^N : P(x) > P_\infty\}$.*

For the proof of our theorems, we shall consider an equivalent system to $(\mathscr{A}_\varepsilon)$. For this purpose, making a change of variable $\varepsilon y = x$, we can rewrite $(\mathscr{A}_\varepsilon)$ as the following equivalent equation

$$(\mathscr{P}_\varepsilon) \qquad \begin{cases} -\Delta u + P(\varepsilon x)u = \mu u^3 + \beta v^2 u & \text{in } \mathbb{R}^N, \\ -\Delta v + Q(\varepsilon x)v = \nu v^3 + \beta u^2 v & \text{in } \mathbb{R}^N, \\ u, v > 0, \quad u, v \in H^1(\mathbb{R}^N). \end{cases}$$

Thus, our theorems for $(\mathscr{A}_\varepsilon)$ are equivalent to the results for $(\mathscr{P}_\varepsilon)$. So, in the sequel we focus on the system $(\mathscr{P}_\varepsilon)$. Throughout this paper, we always assume that $\mu, \nu, \beta > 0$.

In Section 2, we provide basic variational setup of the problem and prove some preliminary estimates. We prove the existence, concentration and properties of positive ground state solutions (Theorem 1.2) in Section 3, and we prove the multiplicity result (Theorem 1.3) and nonexistence result (Theorem 1.4) in Sections 4 and 5 respectively.

2. Variational setting and preliminary results

Throughout the paper, we use the following notation:

- $(\cdot,\cdot)$ is the inner product of $H^1(\mathbb{R}^N)$ defined by $(u, v) = \int_{\mathbb{R}^N} (\nabla u \nabla v + uv)$, and the corresponding norm is $\|u\| = (u, u)^{\frac{1}{2}}$;
- $\|\cdot\|_M$ is an equivalent norm of $H^1(\mathbb{R}^N)$ defined by $\|u\|_M^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + M|u|^2)$, for a positive function or constant M;
- 2^* is a critical exponential defined by $2^* = 6$ when $N = 3$, and $2^* = \infty$ when $N = 1, 2$;
- $S_1 = \inf_{u \in H^1(\mathbb{R}^N)\setminus\{0\}} \dfrac{\int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2)}{\left(\int_{\mathbb{R}^N} u^4\right)^{1/2}}$.

- Let $w_1(x)$ be the unique positive solution of (1.5) with $\max_{x\in\mathbb{R}^N} w(x) = w(0)$. Then for $\lambda, \mu > 0$, $w_{\lambda,\mu}(x) = \sqrt{\frac{\lambda}{\mu}} w_1(\sqrt{\lambda}x)$ is the unique positive solution of

$$\Delta w - \lambda w + \mu w^3 = 0, \quad \text{in } \mathbb{R}^N, \quad w \in H^1(\mathbb{R}^N). \tag{2.1}$$

For Banach spaces X and Y with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, the norm of the product space $X \times Y$ is defined by $\|(x,y)\|_{X\times Y} := (\|x\|_X^2 + \|y\|_Y^2)^{\frac{1}{2}}$; and for Hilbert spaces X and Y with inner products $(\cdot,\cdot)_X$ and $(\cdot,\cdot)_Y$, the inner product of $X \times Y$ is defined by $((x,y),(w,z))_{X\times Y} = (x,w)_X + (y,z)_Y$.

For any $\varepsilon > 0$, let $H_{p,\varepsilon} = \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} P(\varepsilon x)u^2 < \infty\}$ denote the Hilbert space endowed with inner product

$$(u,v)_{p,\varepsilon} = \int_{\mathbb{R}^N} (\nabla u \nabla v + P(\varepsilon x)uv), \quad \text{for} \quad u, v \in H_{p,\varepsilon},$$

and the induced norm denoted by $\|u\|_{p,\varepsilon}^2 = (u,u)_{p,\varepsilon}$. Similarly, one can define Hilbert space $H_{q,\varepsilon} = \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} Q(\varepsilon x)u^2 < \infty\}$. Clearly, since $P(x)$ and $Q(x)$ are positive bounded continuous functions, it follows that $\|\cdot\|_{p,\varepsilon}$, $\|\cdot\|_{q,\varepsilon}$ and $\|\cdot\|$ are equivalent norms uniformly for $\varepsilon > 0$.

The positive solutions of (1.3) are limits of the solutions of $(\mathscr{P}_\varepsilon)$ in some sense. The existence of such solutions is stated in Theorem 1.1, and here we recall some basic properties of the positive solutions of (1.3). For the proof of these results one can refer to [40,41].

Lemma 2.1. *Suppose that $\beta > 0$. Let (u_β, v_β) be a positive solution of (1.3). Then*

1. *(u_β, v_β) satisfies*

$$\lim_{|x|\to\infty} |u_\beta(x)| = \lim_{|x|\to\infty} |v_\beta(x)| = 0, \quad \lim_{|x|\to\infty} |\nabla u_\beta(x)| = \lim_{|x|\to\infty} |\nabla v_\beta(x)| = 0, \tag{2.2}$$

and $u_\beta, v_\beta \in C_{loc}^{1,\sigma}(\mathbb{R}^N)$ with $\sigma \in (0,1)$. Furthermore there exist $C, c > 0$ such that $u_\beta(x) + v_\beta(x) \le Ce^{-c|x-x_\beta|}$, where $x_\beta \in \mathbb{R}^N$ such that $|u_\beta(x_\beta) + v_\beta(x_\beta)| = \max_{x\in\mathbb{R}^N} |u_\beta(x) + v_\beta(x)|$.
2. *Let $\mathscr{B}_\beta$ be the set of all ground state solutions of (1.3). Then $\mathscr{B}_\beta$ is compact in E.*

Now we give a variational framework for the solutions of $(\mathscr{P}_\varepsilon)$. Let $E_\varepsilon = H_{p,\varepsilon} \times H_{q,\varepsilon}$. Since P and Q are bounded positive functions, we know that for each $\varepsilon > 0$, $E_\varepsilon = E$. For $z \in E_\varepsilon$ we define a functional

$$\mathscr{L}_\varepsilon(z) = \mathscr{L}_\varepsilon(u,v) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + P(\varepsilon x)u^2 + |\nabla v|^2 + Q(\varepsilon x)v^2)$$
$$- \frac{1}{4}\int_{\mathbb{R}^N} (\mu u^4 + +2\beta u^2 v^2 + \nu v^4).$$

It is routine to verify that $\mathscr{L}_\varepsilon \in C^2(E_\varepsilon, \mathbb{R})$ and critical points of $\mathscr{L}_\varepsilon$ are weak solutions of $(\mathscr{P}_\varepsilon)$ (see [16,23]).

In order to find nontrivial critical points for $\mathscr{L}_\varepsilon$, we consider the following Nehari manifold for $(\mathscr{P}_\varepsilon)$:

$$\mathscr{N}_\varepsilon = \{z = (u, v) \in E_\varepsilon : z \neq 0, \mathscr{L}'_\varepsilon(z)z = 0\}. \tag{2.3}$$

That is, $z = (u, v) \in \mathscr{N}_\varepsilon$ satisfies

$$\int_{\mathbb{R}^N} (|\nabla u|^2 + P(\varepsilon x)u^2 + |\nabla v|^2 + Q(\varepsilon x)v^2) = \int_{\mathbb{R}^N} (\mu u^4 + 2\beta u^2 v^2 + \nu v^4). \tag{2.4}$$

This implies that for $(u, v) \in \mathscr{N}_\varepsilon$,

$$\begin{aligned} \mathscr{L}_\varepsilon|_{\mathscr{N}_\varepsilon}(u, v) &= \frac{1}{4} \int_{\mathbb{R}^N} (|\nabla u|^2 + P(\varepsilon x)u^2 + |\nabla v|^2 + Q(\varepsilon x)v^2) \\ &= \frac{1}{4} \int_{\mathbb{R}^N} (\mu u^4 + 2\beta u^2 v^2 + \nu v^4). \end{aligned} \tag{2.5}$$

Hence $\mathscr{L}_\varepsilon$ is bounded from below away from zero on $\mathscr{N}_\varepsilon$. And we define

$$A_\varepsilon = \inf_{z=(u,v)\in \mathscr{N}_\varepsilon} \mathscr{L}_\varepsilon(u, v), \tag{2.6}$$

then $A_\varepsilon > 0$ because of (2.4) and (2.5).

Similarly we define corresponding energy functional and Nehari manifold for the limiting equation (1.3) as follows:

$$\mathscr{L}_{\lambda_1\lambda_2}(u, v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda_1 u^2 + |\nabla v|^2 + \lambda_2 v^2) - \frac{1}{4} \int_{\mathbb{R}^N} (\mu u^4 + 2\beta u^2 v^2 + \nu v^4), \tag{2.7}$$

and

$$\mathscr{N}_{\lambda_1\lambda_2} = \{z = (u, v) \in E : z \neq 0, \mathscr{L}'_{\lambda_1\lambda_2}(z)z = 0\}.$$

We set

$$A_{\lambda_1\lambda_2} = \inf_{w\in \mathscr{N}_{\lambda_1\lambda_2}} \mathscr{L}_{\lambda_1\lambda_2}(w). \tag{2.8}$$

A particular case is that for some $y_0 \in \mathbb{R}^N$, $\lambda_1 = P(y_0)$ and $\lambda_2 = Q(y_0)$ in (1.3). We denote the corresponding notation by $\mathscr{L}_{P(y_0)Q(y_0)}$, $\mathscr{N}_{P(y_0)Q(y_0)}$ and $A_{P(y_0)Q(y_0)}$ respectively.

The following lemma shows the relation between the ground state energy level A_ε and the one for the limiting equation $A_{P(y_0)Q(y_0)}$. The proof is similar to [41, Lemma 4.1] (also see [42]), and we omit the details here.

Lemma 2.2. *Let $A_{P(y_0)Q(y_0)}$ and A_ε be defined as in (2.6) and (2.8). For $\beta > 0$, we have the following conclusions.*

1. *Suppose that for some $y_0 \in \mathbb{R}^N$, $A_{P(y_0)Q(y_0)}$ is attained by some $(u_0, v_0) \in \mathcal{N}_{P(y_0)Q(y_0)}$, then $\limsup_{\varepsilon\to 0} A_\varepsilon \le A_{P(y_0)Q(y_0)}$. Moreover if $y_0 \in \mathcal{V}$ and A_ε is attained by some $(u^\varepsilon, v^\varepsilon) \in \mathcal{N}_\varepsilon$ for all $\varepsilon > 0$ small, then $\lim_{\varepsilon\to 0} A_\varepsilon = A_{P(y_0)Q(y_0)}$.*
2. *Suppose that for $\lambda_1, \lambda_2 > 0$, $A_{\lambda_1\lambda_2}$ is attained by some $(u_0, v_0) \in \mathcal{N}_{\lambda_1\lambda_2}$, $P(\varepsilon x) \to \lambda_1$ and $Q(\varepsilon x) \to \lambda_2$ uniformly on any bounded subset of $\mathbb{R}^N$ as $\varepsilon \to 0$, then $\limsup_{\varepsilon\to 0} A_\varepsilon \le A_{\lambda_1\lambda_2}$.*

The next lemma states the monotonicity of the energy level $A_{\lambda_1\lambda_2}$ in the parameters λ_1 and λ_2.

Lemma 2.3. *Let $A_{\lambda_1\lambda_2}$ be defined as in (2.8). If $\lambda_1 \le \hat\lambda_1$ and $\lambda_2 \le \hat\lambda_2$, then $A_{\lambda_1\lambda_2} \le A_{\hat\lambda_1\hat\lambda_2}$. Moreover, if one of the inequalities is strict and $A_{\hat\lambda_1\hat\lambda_2}$ is attained by $z = (u, v) \in \mathcal{N}_{\hat\lambda_1\hat\lambda_2}$ $(u, v \neq 0)$, then $A_{\lambda_1\lambda_2} < A_{\hat\lambda_1\hat\lambda_2}$.*

Proof. The first part of the conclusion follows from the minimax characterization

$$0 < A_{\lambda_1\lambda_2} = \inf_{z\in\mathcal{N}_{\lambda_1\lambda_2}} \mathcal{L}_{\lambda_1\lambda_2}(z) = \inf_{w=(u,v)\in E, w\not\equiv 0} \max_{t>0} \mathcal{L}_{\lambda_1\lambda_2}(tw),$$

and the inequality $\mathcal{L}_{\hat\lambda_1\hat\lambda_2}(tw) \ge \mathcal{L}_{\lambda_1\lambda_2}(tw)$, for $t > 0$ and $w \in E$, when $\lambda_1 \le \hat\lambda_1$ and $\lambda_2 \le \hat\lambda_2$.

Now we give the proof of second part of the conclusion. Since $A_{\hat\lambda_1\hat\lambda_2}$ is attained, we can choose $z_1 = (u_1, v_1) \in \mathcal{N}_{\hat\lambda_1\hat\lambda_2}$ and $w = (u, v) \in E$ $(u, v \not\equiv 0)$ be such that $A_{\hat\lambda_1\hat\lambda_2} = \mathcal{L}_{\hat\lambda_1\hat\lambda_2}(z_1) = \max_{t>0} \mathcal{L}_{\hat\lambda_1\hat\lambda_2}(tw)$. On the other hand, let $z_2 = (u_2, v_2) \in E$ be such that $\mathcal{L}_{\lambda_1\lambda_2}(z_2) = \max_{t>0} \mathcal{L}_{\lambda_1\lambda_2}(tw)$. Therefore, one sees that

$$\begin{aligned} A_{\hat\lambda_1\hat\lambda_2} &\ge \mathcal{L}_{\hat\lambda_1\hat\lambda_2}(z_2) = \mathcal{L}_{\lambda_1\lambda_2}(z_2) + (\hat\lambda_1 - \lambda_1) \int_{\mathbb{R}^N} u_2^2 + (\hat\lambda_2 - \lambda_2) \int_{\mathbb{R}^N} v_2^2 \\ &\ge A_{\lambda_1\lambda_2} + (\hat\lambda_1 - \lambda_1) \int_{\mathbb{R}^N} u_2^2 + (\hat\lambda_2 - \lambda_2) \int_{\mathbb{R}^N} v_2^2, \end{aligned}$$

as required. □

The following Mountain-Pass geometry characterization of the functional $\mathcal{L}_\varepsilon$ can be shown by standard methods.

Lemma 2.4 *(Mountain-Pass geometry). For any $\beta > 0$, the functional $\mathcal{L}_\varepsilon$ satisfies the following conditions:*

(i) *There exist positive constants ϑ, α such that $\mathcal{L}_\varepsilon(z) \ge \vartheta$ for $\|z\| = \alpha$.*
(ii) *There exists $e \in E_\varepsilon$ with $\|e\| > \alpha$ such that $\mathcal{L}_\varepsilon(e) < 0$.*

From Lemma 2.4, one can apply the Ambrosetti–Rabinowitz Mountain-Pass Theorem without $(PS)_c$ condition (see [43]), and it follows that for any small $\varepsilon > 0$, there exists a $(PS)_c$-sequence $\{z_n\} \subset E_\varepsilon$ (with $c = A'_\varepsilon$ defined below) such that

$$\mathscr{L}_\varepsilon(z_n) \to A'_\varepsilon = \inf_{\gamma \in \Gamma} \max_{0 \le t \le 1} \mathscr{L}_\varepsilon(\gamma(t)) \quad \text{and} \quad \mathscr{L}'_\varepsilon(z_n) \to 0, \quad \text{as } n \to \infty, \tag{2.9}$$

where $\Gamma = \{\gamma \in C(E_\varepsilon, \mathbb{R}) : \mathscr{L}_\varepsilon(\gamma(0)) = 0,\ \mathscr{L}_\varepsilon(\gamma(1)) < 0\}$. As in [36, Proposition 3.11] (also see [44]), we shall use the following equivalent characterization of A'_ε, which is more appropriate to our purpose, given by

$$A'_\varepsilon = \inf_{z \in E_\varepsilon \setminus \{0\}} \max_{t > 0} \mathscr{L}_\varepsilon(tz) = A_\varepsilon. \tag{2.10}$$

3. Existence and concentration of positive solution

3.1. Compactness lemma for the functional $\mathscr{L}_\varepsilon$

In order to obtain the existence of positive solutions for $(\mathscr{P}_\varepsilon)$, we prove some compactness lemma for the functional $\mathscr{L}_\varepsilon$ to analyze the Palais–Smale sequence properties for the functional $\mathscr{L}_\varepsilon$. Throughout this subsection we assume that (**PQ0**), (**PQ1**) and $\beta > 0$ hold. From (**PQ1**) we know that $P_0 < P_\infty$ and $Q_0 < Q_\infty$, hence we can choose $\tau, \sigma > 0$ such that

$$P_0 \le \tau < P_\infty, \quad \text{and} \quad Q_0 \le \sigma < Q_\infty. \tag{3.1}$$

We assume that $\{z_n\} \subset E_\varepsilon$ is a sequence satisfying

$$\mathscr{L}_\varepsilon(z_n) \to A, \quad \mathscr{L}'_\varepsilon(z_n) \to 0, \quad \text{as } n \to \infty, \tag{3.2}$$

for some A satisfying

$$0 < A \le A_{\tau\sigma} < A_{P_\infty Q_\infty}, \tag{3.3}$$

where τ and σ are given in (3.1). Note that the inequalities in (3.3) follow from (3.1) and Lemma 2.3. First, we notice that the condition (3.2) implies that $\{z_n\}$ is bounded in E.

Lemma 3.1. *Assume that $\{z_n\}$ satisfies (3.2). Then $\{z_n\}$ is bounded in E.*

Proof. Let $\{z_n\} = \{(u_n, v_n)\} \subset E_\varepsilon$ such that (3.2) is satisfied. It follows from $P(\varepsilon x) \ge P_0$, $Q(\varepsilon x) \ge Q_0$, and $\mathscr{L}_\varepsilon(z_n) - \mathscr{L}'_\varepsilon(z_n)z_n = A + o(1)$ that

$$A + o(1) = \frac{1}{4} \int_{\mathbb{R}^N} (|\nabla u_n|^2 + P(\varepsilon x)u_n^2 + |\nabla v_n|^2 + Q(\varepsilon x)v_n^2) \ge c\|z_n\|^2. \tag{3.4}$$

That is, $\{z_n\}$ is bounded. □

From the Lions Concentration-Compactness Principle, we have the following lemma for any fixed $\varepsilon > 0$.

Lemma 3.2. *Assume that $\{z_n\}$ satisfies (3.2) and $z_n \rightharpoonup 0$ in E_ε, then one of the following conclusions holds:*

(*i*) $z_n \to 0$ *in* E_ε *as* $n \to \infty$; *or*
(*ii*) *there exist a sequence* $\{y_n\} \subset \mathbb{R}^N$ *and positive constants* $r, \delta > 0$ *such that* $\liminf_{n\to\infty} \int_{B_r(y_n)} |z_n|^2 \ge \delta > 0.$

Proof. Assume that $z_n = (u_n, v_n)$. Suppose that (*ii*) does not occur, *i.e.*, there exists $r > 0$ such that

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_r(y)} |z_n|^2 = \lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_r(y)} (u_n^2 + v_n^2) = 0.$$

Then by the Lions Concentration-Compactness Principle (see [45,46,43]), we deduce that $z_n \to 0$ in $L^t(\mathbb{R}^N) \times L^t(\mathbb{R}^N)$ for $t \in (2, 2^*)$. So one infers from $\mathscr{L}_\varepsilon'(z_n) \to 0$ that

$$\int_{\mathbb{R}^N} (|\nabla u_n|^2 + P(\varepsilon x)u_n^2) + \int_{\mathbb{R}^N} (|\nabla v_n|^2 + Q(\varepsilon x)v_n^2) = \int_{\mathbb{R}^N} (\mu u_n^4 + 2\beta u_n^2 v_n^2 + \nu v_n^4) + o(1) \to 0,$$

as $n \to \infty$. Since $P \ge P_0$ and $Q \ge Q_0$, then it follows that $z_n \to 0$ in E_ε as $n \to \infty$. □

Next we shall prove that indeed only the first alternative in Lemma 3.2 occurs if (3.3) is satisfied.

Lemma 3.3. *Assume that $\{z_n = (u_n, v_n)\}$ satisfies (3.2), (3.3) and $z_n \rightharpoonup 0$ in E_ε. Then for fixed $\varepsilon > 0$, $z_n \to 0$ in E_ε as $n \to \infty$.*

Proof. From the proof of Lemma 2.2, for any $z = (u, v) \in E \setminus \{(0, 0)\}$, the function $g(t) = \mathscr{L}_{P_\infty Q_\infty}(tz)$ has a unique global maximum point $t_z > 0$ and $t_z z \in \mathscr{N}_{P_\infty Q_\infty}$. Hence we can choose a positive sequence $\{t_n\}$ such that $\{t_n z_n\} \subset \mathscr{N}_{P_\infty M_\infty}$. We argue by contradiction. Suppose that $z_n = (u_n, v_n) \nrightarrow 0$ in E_ε, we first claim that

$$\limsup_{n\to\infty} t_n \le 1. \tag{3.5}$$

Assume by contradiction, there exist $\eta > 0$ and a subsequence (still denoted by $\{t_n\}$) such that $t_n \ge 1 + \eta$ for all $n \in \mathbb{N}$. From $\mathscr{L}_\varepsilon'(z_n)z_n = o(1)$ we have that

$$\int_{\mathbb{R}^N} (|\nabla u_n|^2 + P(\varepsilon x)u_n^2) + \int_{\mathbb{R}^N} (|\nabla v_n|^2 + Q(\varepsilon x)v_n^2) = \int_{\mathbb{R}^N} (\mu u_n^4 + 2\beta u_n^2 v_n^2 + \nu v_n^4) + o(1). \tag{3.6}$$

Moreover, since $\{t_n z_n\} \subset \mathscr{N}_{P_\infty Q_\infty}$, then we see that

$$\int_{\mathbb{R}^N} (|\nabla u_n|^2 + P_\infty u_n^2) + \int_{\mathbb{R}^N} (|\nabla v_n|^2 + Q_\infty v_n^2) = t_n^2 \int_{\mathbb{R}^N} (\mu u_n^4 + 2\beta u_n^2 v_n^2 + \nu v_n^4). \tag{3.7}$$

We deduce from (3.6) and (3.7) that

$$\int_{\mathbb{R}^N} (P_\infty - P(\varepsilon x))u_n^2 + \int_{\mathbb{R}^N} (Q_\infty - Q(\varepsilon x))u_n^2 = (t_n^2 - 1) \int_{\mathbb{R}^N} (\mu u_n^4 + 2\beta u_n^2 v_n^2 + \nu v_n^4) + o(1). \quad (3.8)$$

By the definition of P_∞ and Q_∞, for any $\epsilon > 0$, there exists $R = R(\epsilon) > 0$ such that

$$P(\varepsilon x) \geq P_\infty - \epsilon \quad \text{and} \quad Q(\varepsilon x) \geq Q_\infty - \epsilon \quad \text{for } |x| \geq R. \quad (3.9)$$

Since $\{z_n\}$ is a bounded sequence from Lemma 3.1, then we have that $z_n \to 0$ in $L^2(B_R(0)) \times L^2(B_R(0))$, and from (3.9), there exists $C > 0$ such that

$$(t_n^2 - 1) \int_{\mathbb{R}^N} (\mu u_n^4 + 2\beta u_n^2 v_n^2 + \nu v_n^4) \leq C\epsilon + o(1). \quad (3.10)$$

Since $z_n \nrightarrow 0$ in E_ε, it follows from Lemma 3.2 that there exist a sequence $\{y_n\} \in \mathbb{R}^N$ and positive constants $r, \delta > 0$ such that

$$\liminf_{n\to\infty} \int_{B_r(y_n)} (u_n^2 + v_n^2) \geq \delta. \quad (3.11)$$

If we set $w_n(x) = z(x + y_n) = (u(x + y_n), v(x + y_n))$, then there exists a function $w = (u, v)$, up to a subsequence, such that $w_n \rightharpoonup w$ in E, $w_n \to w$ in $[L^2_{loc}(\mathbb{R}^N)]^2$ and $w_n(x) \to w(x)$ *a.e.* in $\mathbb{R}^N$. Moreover, by (3.11), there exists a subset Ω in $\mathbb{R}^N$ with positive measure such that $w \neq 0$ *a.e.* in Ω. It follows from (3.10) and Fatou's lemma that

$$0 < [(1 + \eta)^2 - 1] \int_\Omega (\mu u^4 + 2\beta u^2 v^2 + \nu v^4) \leq C\epsilon, \quad (3.12)$$

for any $\epsilon > 0$, which yields a contradiction. Thus (3.5) holds.

Next we prove that indeed $\limsup_{n\to\infty} t_n \leq 1$ cannot happen. Then we obtain a contradiction and $z_n \to 0$ in E_ε. For this purpose, we distinguish the following two cases: (1) $\limsup\limits_{n\to\infty} t_n = 1$; (2) $\limsup\limits_{n\to\infty} t_n < 1$.

Case 1. $\limsup\limits_{n\to\infty} t_n = 1$. In this case, there exists a subsequence, still denoted by $\{t_n\}$ such that $t_n \to 1$ as $n \to \infty$. Hence, from (3.2) and (3.3),

$$o(1) + A_{\tau\sigma} \geq \mathscr{L}_\varepsilon(z_n) \geq \mathscr{L}_\varepsilon(z_n) + A_{P_\infty Q_\infty} - \mathscr{L}_{P_\infty Q_\infty}(t_n z_n). \quad (3.13)$$

We can estimate that

$$\begin{aligned}
&\mathscr{L}_\varepsilon(u_n, v_n) - \mathscr{L}_{P_\infty Q_\infty}(t_n u_n, t_n v_n) \\
&= \frac{1-t_n^2}{2} \int_{\mathbb{R}^N} (|\nabla u_n|^2 + |\nabla v_n|^2) + \frac{1}{2} \int_{\mathbb{R}^N} P(\varepsilon x) u_n^2 - \frac{t_n}{2} \int_{\mathbb{R}^N} P_\infty u_n^2 \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^N} Q(\varepsilon x) v_n^2 - \frac{t_n^2}{2} \int_{\mathbb{R}^N} Q_\infty v_n^2 + (1-t_n^4) \int_{\mathbb{R}^N} \left(\mu u_n^4 + 2\beta u_n^2 v_n^2 + \nu v_n^4\right).
\end{aligned} \tag{3.14}$$

From the boundedness of $\{z_n\}$, $t_n \to 1$, $z_n \rightharpoonup 0$ in E_ε as $n \to \infty$ and (3.9), we obtain that

$$\mathscr{L}_\varepsilon(z_n) - \mathscr{L}_{P_\infty Q_\infty}(t_n z_n) \geq o(1) - C\epsilon. \tag{3.15}$$

Taking the limit as $n \to \infty$ in (3.13), we have $A_{\tau\sigma} \geq A_{P_\infty Q_\infty}$. On the other hand, from (3.3), we have that $A_{\tau\sigma} < A_{P_\infty Q_\infty}$. This is a contradiction.

Case 2. $\limsup_{n\to\infty} t_n < 1$. Without loss of generality, we may suppose that $t_n < 1$ for all $n \in \mathbb{N}$. From (3.9), $\{t_n z_n\} \subset \mathscr{N}_{P_\infty Q_\infty}$, $z_n \to 0$ in $[L^2_{loc}(\mathbb{R}^N)]^2$ and $\|z_n\| \leq C$, we see that

$$\begin{aligned}
A_{P_\infty Q_\infty} &\leq \mathscr{L}_{P_\infty Q_\infty}(t_n z_n) = \mathscr{L}_\varepsilon(t_n z_n) + \frac{t_n^2}{2} \int_{\mathbb{R}^N} (P_\infty - P(\varepsilon x)) u_n^2 + \frac{t_n^2}{2} \int_{\mathbb{R}^N} (Q_\infty - Q(\varepsilon x)) v_n^2 \\
&\leq \mathscr{L}_\varepsilon(z_n) + C\epsilon + o(1) \leq A_{\tau\sigma} + C\epsilon + o(1).
\end{aligned} \tag{3.16}$$

Let $n \to \infty$ in (3.16), we get $A_{\tau\sigma} \geq A_{P_\infty Q_\infty}$, which again is in contradiction with $A_{\tau\sigma} < A_{P_\infty Q_\infty}$ from (3.3). □

The next lemma states that the functional $\mathscr{L}_\varepsilon$ satisfies $(PS)_A$-condition.

Lemma 3.4. *Assume that $\{z_n\} \subset E_\varepsilon$ satisfies (3.2) and (3.3). Then for fixed $\varepsilon > 0$, $\{z_n\}$ has a convergent subsequence in E_ε. That is, the functional $\mathscr{L}_\varepsilon$ satisfies the $(PS)_A$-condition.*

Proof. From Lemma 3.1, $\{z_n\}$ is bounded in E, hence there exists $z = (u, v) \in E_\varepsilon$ such that $z_n \rightharpoonup z$ in E_ε, and z is a critical point of $\mathscr{L}_\varepsilon$. Set $h_n = u_n - u$ and $k_n = v_n - v$. By Brezis–Lieb Lemma (see [43]), we have

$$\int_{\mathbb{R}^N} |\nabla h_n|^2 = \int_{\mathbb{R}^N} |\nabla u_n|^2 - \int_{\mathbb{R}^N} |\nabla u|^2 + o(1), \quad \int_{\mathbb{R}^N} |h_n|^4 = \int_{\mathbb{R}^N} |u_n|^4 - \int_{\mathbb{R}^N} |u|^4 + o(1),$$
$$\int_{\mathbb{R}^N} |\nabla k_n|^2 = \int_{\mathbb{R}^N} |\nabla v_n|^2 - \int_{\mathbb{R}^N} |\nabla v|^2 + o(1), \quad \int_{\mathbb{R}^N} |k_n|^4 = \int_{\mathbb{R}^N} |v_n|^4 - \int_{\mathbb{R}^N} |v|^4 + o(1).$$

Hence, as in [47], one can verify that $\mathscr{L}_\varepsilon(h_n, k_n) = \mathscr{L}_\varepsilon(u_n, v_n) - \mathscr{L}_\varepsilon(u, v) + o(1)$ and $\mathscr{L}'_\varepsilon(h_n, k_n) \to 0$ as $n \to \infty$. Thus, it follows from $\mathscr{L}'_\varepsilon(u, v) = 0$ that

$$\mathscr{L}_\varepsilon(z) = \mathscr{L}_\varepsilon(u, v) = \frac{1}{4}\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2 + P(\varepsilon x)u^2 + Q(\varepsilon x)v^2) \geq 0. \tag{3.17}$$

So from (3.17) we infer that $\mathscr{L}_\varepsilon(h_n, k_n) = \mathscr{L}_\varepsilon(u_n, v_n) - \mathscr{L}_\varepsilon(u, v) + o(1) \to A - d_1$ as $n \to \infty$, where $d_1 = \mathscr{L}_\varepsilon(z) \geq 0$. Hence $\{\phi_n = (h_n, k_n)\}$ is a $(PS)_c$ sequence with $c = A - d_1$. Moreover we have $A - d_1 \leq A \leq A_{\tau\sigma} < A_{P_\infty Q_\infty}$ thus (3.3) is satisfied, therefore Lemma 3.3 implies that $h_n = u_n - u \to 0$ and $k_n = v_n - v \to 0$ in $H^1(\mathbb{R}^N)$. □

Finally we prove that $\mathscr{L}_\varepsilon$ also satisfies $(PS)_A$-condition if it is restricted to the Nehari manifold $\mathscr{N}_\varepsilon$.

Lemma 3.5. *Assume that* $\{z_n\} \subset \mathscr{N}_\varepsilon$ *satisfies*

$$\mathscr{L}_\varepsilon(z_n) \to A, \quad \mathscr{L}'_\varepsilon(z_n)z_n = 0 \quad \text{and} \quad \mathscr{L}'_\varepsilon|_{\mathscr{N}_\varepsilon}(z_n) \to 0, \qquad \text{as} \quad n \to \infty \tag{3.18}$$

and (3.3)*. Then for fixed* $\varepsilon > 0$*,* $\{z_n\}$ *has a convergent subsequence in* $\mathscr{N}_\varepsilon$*.*

Proof. Suppose that $\{z_n\} = \{(u_n, v_n)\} \subset \mathscr{N}_\varepsilon$ satisfy (3.18) and (3.3). Then there exists a sequence $\{l_n\} \subset \mathbb{R}$ such that

$$o(1) = \mathscr{L}'_\varepsilon|_{\mathscr{N}_\varepsilon}(z_n) = \mathscr{L}'_\varepsilon(z_n) - l_n \mathscr{J}'_\varepsilon(z_n), \tag{3.19}$$

where

$$\mathscr{J}_\varepsilon(z_n) = \int_{\mathbb{R}^N} (|\nabla u_n|^2 + |\nabla v_n|^2 + P(\varepsilon x)u_n^2 + Q(\varepsilon x)v_n^2) - \int_{\mathbb{R}^N} (\mu u_n^4 + 2\beta u_n^2 v_n^2 + \nu v_n^4). \tag{3.20}$$

So, it follows from $z_n \in \mathscr{N}_\varepsilon$ that

$$\begin{aligned}\mathscr{J}'_\varepsilon(z_n)z_n &= 2\int_{\mathbb{R}^N} (|\nabla u_n|^2 + |\nabla v_n|^2 + P(\varepsilon x)u_n^2 + Q(\varepsilon x)v_n^2) - 4\int_{\mathbb{R}^N} (\mu u_n^4 + 2\beta u_n^2 v_n^2 + \nu v_n^4)\\ &= -2\int_{\mathbb{R}^N} (|\nabla u_n|^2 + |\nabla v_n|^2 + P(\varepsilon x)u_n^2 + Q(\varepsilon x)v_n^2) \leq 0.\end{aligned} \tag{3.21}$$

On the other hand, from Lemma 3.1, $\{z_n\}$ is bounded in E, hence there exists $C > 0$ such that $\mathscr{J}'_\varepsilon(z_n)z_n \geq -C$. Hence subject to a subsequence we may assume that $\mathscr{J}'_\varepsilon(z_n)z_n \to l \leq 0$. If $l = 0$, then we see that

$$|\mathscr{J}'_\varepsilon(z_n)z_n| \geq 2\int_{\mathbb{R}^N} (|\nabla u_n|^2 + |\nabla v_n|^2 + P(\varepsilon x)u_n^2 + Q(\varepsilon x)v_n^2), \tag{3.22}$$

which implies that $z_n \to 0$ in E thus contradicting with $\mathscr{L}_\varepsilon(z_n) \to A > 0$. So we must have $l < 0$. It follows that $l_n \to 0$ as $n \to \infty$, and therefore $\mathscr{L}'_\varepsilon(z_n) = o(1)$. So z_n is a $(PS)_A$ sequence for $\mathscr{L}_\varepsilon$ in E_ε, and the conclusion follows from Lemma 3.4. □

3.2. Existence of positive ground state solution

In this subsection we consider the existence of positive solution for large β. We assume that **(PQ0)** and **(PQ1)** are satisfied. We first recall results on the related scalar equations

$$-\Delta u + P(\varepsilon x)u = \mu u^3, \quad u \in H^1(\mathbb{R}^N), \tag{3.23}$$

and

$$-\Delta v + Q(\varepsilon x)v = \nu v^3, \quad v \in H^1(\mathbb{R}^N). \tag{3.24}$$

Following the idea of the proof of Theorem 1.1 in [41], one can prove the following results about the positive solutions of (3.23).

Lemma 3.6. *Suppose that* **(P1)** *is satisfied. Then for all sufficiently small $\varepsilon > 0$, we have that*

(*i*) (3.23) *has at least one positive ground state solution u_ε in $H^1(\mathbb{R}^N)$.*
(*ii*) *Let $\mathscr{M}'_\varepsilon$ be the set of all positive solutions of* (3.23)*. Then $\mathscr{M}'_\varepsilon$ is compact in $H^1(\mathbb{R}^N)$.*

Moreover, if P is uniformly continuous, then the following results hold.

(*iii*) *there exists a maximum point x_ε of u_ε such that $\lim_{\varepsilon\to 0} dist(\varepsilon x_\varepsilon, \mathscr{P}_0) = 0$, $\varepsilon x_\varepsilon \to y_0 \in \mathscr{P}_0$ and $h_\varepsilon(x) = u_\varepsilon(x + x_\varepsilon)$ converges in $H^1(\mathbb{R}^N)$ to $w_{P(y_0),\mu} = \sqrt{\frac{P(y_0)}{\mu}} w_1(\sqrt{P(y_0)}x)$, which is the unique positive solution of*

$$-\Delta u + P(y_0)u = \mu u^3, \quad u \in H^1(\mathbb{R}^N), \tag{3.25}$$

as $\varepsilon \to 0$. Moreover $u_\varepsilon \in C^{1,\sigma}_{loc}(\mathbb{R}^N)$ with $\sigma \in (0, 1)$, and there exist constants $C, c > 0$ such that

$$\lim_{|x|\to\infty} u_\varepsilon(x) = \lim_{|x|\to\infty} |\nabla u_\varepsilon(x)| = 0,\ |u_\varepsilon(x)| \le Ce^{-\frac{c}{\varepsilon}|x-x_\varepsilon|},\ x \in \mathbb{R}^N.$$

A parallel result as in Lemma 3.6 holds for (3.24). Let U_ε and V_ε denote a positive ground state solution of (3.23) and (3.24) respectively. To obtain a nontrivial solution for $(\mathscr{P}_\varepsilon)$, as in [23] we shall prove that for small enough $\varepsilon > 0$ we have

$$A_\varepsilon < \min\{\mathscr{L}_\varepsilon(U_\varepsilon, 0), \mathscr{L}_\varepsilon(0, V_\varepsilon)\}, \tag{3.26}$$

therefore the minimizer achieving A_ε is a nontrivial solution of $(\mathscr{P}_\varepsilon)$.

For some $y_0 \in \mathscr{V}$, we assume that $A_{P(y_0)Q(y_0)}$ is attained. Then we infer from Lemma 2.2 that for $\varepsilon > 0$ sufficiently small,

$$A_\varepsilon \le A_{P(y_0)Q(y_0)} + \delta_\varepsilon, \tag{3.27}$$

where $\delta_\varepsilon > 0$ and $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$. Moreover, from Lemma 3.6 we conclude that

$$\begin{aligned}
\mathscr{L}_\varepsilon(U_\varepsilon, 0) &= \frac{\mu}{4}\int_{\mathbb{R}^N} U_\varepsilon^4 = \frac{\mu}{4}\int_{\mathbb{R}^N} U_\varepsilon(x+x_\varepsilon)^4 = \frac{\mu}{4}\int_{\mathbb{R}^N} w_{P(y_0),\mu}^4 + o_\varepsilon(1)\\
&= \mathscr{L}_{P(y_0)Q(y_0)}(w_{P(y_0),\mu}, 0) + o_\varepsilon(1),\\
\mathscr{L}_\varepsilon(0, V_\varepsilon) &= \frac{\nu}{4}\int_{\mathbb{R}^N} V_\varepsilon^4 = \frac{\nu}{4}\int_{\mathbb{R}^N} V_\varepsilon(x+\tilde{x}_\varepsilon)^4 = \frac{\nu}{4}\int_{\mathbb{R}^N} w_{Q(y_0),\nu}^4 + o_\varepsilon(1)\\
&= \mathscr{L}_{P(y_0)Q(y_0)}(0, w_{Q(y_0),\nu}) + o_\varepsilon(1),
\end{aligned} \tag{3.28}$$

where $o_\varepsilon(1) \to 0$ as $\varepsilon \to 0$, and $x_\varepsilon, \tilde{x}_\varepsilon$ are the maximum points of U_ε and V_ε respectively. So in order to prove (3.26), it is sufficient to show that

$$A_{P(y_0)Q(y_0)} < \min\{\mathscr{L}_{P(y_0)Q(y_0)}(w_{P(y_0),\mu}, 0), \mathscr{L}_{P(y_0)Q(y_0)}(0, w_{Q(y_0),\nu})\}. \tag{3.29}$$

We shall find some conditions on P, Q, μ, ν to ensure (3.29) holds. First, as in Lemma 3.3 of [23], we know that for $\beta \ge 0$,

$$A_{P(y_0)Q(y_0)} = \inf_{z=(u,v)\in E\setminus\{(0,0)\}} \mathscr{J}(z), \tag{3.30}$$

where

$$\mathscr{J}(z) = \mathscr{J}(u,v) = \frac{(\|u\|_{P(y_0)}^2 + \|v\|_{Q(y_0)}^2)^2}{4\int_{\mathbb{R}^N}(\mu u^4 + 2\beta u^2 v^2 + \nu v^4)}. \tag{3.31}$$

For $(s,t) \in \Gamma = \{(s,t) : s \ge 0, t \ge 0, (s,t) \ne (0,0)\}$, we define a function

$$\begin{aligned}
g(s,t) &= \mathscr{J}(\sqrt{s}w_{P(y_0),\mu}, \sqrt{t}w_{Q(y_0),\nu})\\
&= \frac{(s\mu^{-1}S_1^2 P(y_0)^{2-\frac{N}{2}} + t\nu^{-1}S_1^2 Q(y_0)^{2-\frac{N}{2}})^2}{4S_1^2(s^2\mu^{-1}P(y_0)^{2-\frac{N}{2}} + t^2\nu^{-1}Q(y_0)^{2-\frac{N}{2}} + 2st\beta\mu^{-1}\nu^{-1}P(y_0)Q(y_0)\hat{h}_{P(y_0)Q(y_0)})},
\end{aligned} \tag{3.32}$$

where

$$\hat{h}_{P(y_0)Q(y_0)} = \frac{\int_{\mathbb{R}^N} w_1^2(\sqrt{P(y_0)}x)w_1^2(\sqrt{Q(y_0)}x)dx}{\int_{\mathbb{R}^N} w_1^4(x)dx}.$$

We claim that

$$\min\{P(y_0)^{-\frac{N}{2}}, Q(y_0)^{-\frac{N}{2}}\} \le \hat{h}_{P(y_0)Q(y_0)} \le P(y_0)^{-\frac{N}{4}} Q(y_0)^{-\frac{N}{4}}. \tag{3.33}$$

From Hölder inequality we have that

$$\begin{aligned}\int_{\mathbb{R}^N} w_1^2(\sqrt{P(y_0)}x) w_1^2(\sqrt{Q(y_0)}x)dx &\le (\int_{\mathbb{R}^N} w_1^2(\sqrt{P(y_0)}x)dx)^{\frac{1}{2}} (\int_{\mathbb{R}^N} w_1^2(\sqrt{Q(y_0)}x)dx)^{\frac{1}{2}} \\ &= P(y_0)^{-\frac{N}{4}} Q(y_0)^{-\frac{N}{4}} \int_{\mathbb{R}^N} w_1^4(x)dx,\end{aligned}$$

which gives the upper bound for $\hat{h}_{P(y_0)Q(y_0)}$ in (3.33). To prove the lower bound for $\hat{h}_{P(y_0)Q(y_0)}$, without loss of generality, we assume that $Q(y_0) \ge P(y_0)$. Since $w_1(x)$ is radially symmetric and strictly decreasing in $|x|$, it follows that for $\lambda \ge 1$ and $x \in \mathbb{R}^N$, $w_1(x) \ge w_1(\sqrt{\lambda}x)$. So a direct computation shows that

$$\begin{aligned}\int_{\mathbb{R}^N} w_1^2(\sqrt{P(y_0)}x) w_1^2(\sqrt{Q(y_0)}x)dx &= P(y_0)^{-\frac{N}{2}} \int_{\mathbb{R}^N} w_1^2(x) w_1^2(\sqrt{\frac{Q(y_0)}{P(y_0)}}x)dx \\ \ge P(y_0)^{-\frac{N}{2}} \int_{\mathbb{R}^N} w_1^4(\sqrt{\frac{Q(y_0)}{P(y_0)}}x)dx &= Q(y_0)^{-\frac{N}{2}} \int_{\mathbb{R}^N} w_1^4(x)dx.\end{aligned}$$

We claim that g attains its minimum over Γ in the interior, which implies that (3.29) holds. Clearly, along the boundary of Γ, we have that

$$\begin{aligned} g(s,0) &= \frac{P(y_0)^{2-\frac{N}{2}} S_1^2}{4\mu} = \mathscr{J}(w_{P(y_0),\mu}, 0) = \mathscr{L}_{P(y_0)Q(y_0)}(w_{P(y_0),\mu}, 0), \\ g(0,t) &= \frac{Q(y_0)^{2-\frac{N}{2}} S_1^2}{4\nu} = \mathscr{J}(0, w_{Q(y_0),\nu}) = \mathscr{L}_{P(y_0)Q(y_0)}(0, w_{Q(y_0),\nu}). \end{aligned} \tag{3.34}$$

Note that the function g is the ratio of two quadratic forms of s and t, and elementary analysis shows that the function

$$\tilde{g}(s,t) = \frac{(as+bt)^2}{cs^2 + 2dst + et^2}, \quad a,b,c,d,e > 0, \tag{3.35}$$

does not attain its minimum in Γ on the boundary if and only if

$$ad - bc > 0, \quad \text{and} \quad bd - ae > 0. \tag{3.36}$$

And the minimum is achieved when $t = ad - bc$ and $s = bd - ae$. For the function $g(s,t)$, the relations in (3.36) are equivalent to

$$\beta \hat{h}_{P(y_0)Q(y_0)} - \frac{Q(y_0)^{1-\frac{N}{2}}}{P(y_0)}\mu > 0, \quad \text{and} \quad \beta \hat{h}_{P(y_0)Q(y_0)} - \frac{P(y_0)^{1-\frac{N}{2}}}{Q(y_0)}\nu > 0, \tag{3.37}$$

or equivalently

$$\beta \hat{h}_{P(y_0)Q(y_0)} > \max\left\{ \frac{Q(y_0)^{1-\frac{N}{2}}}{P(y_0)}\mu, \frac{P(y_0)^{1-\frac{N}{2}}}{Q(y_0)}\nu \right\}. \tag{3.38}$$

Then one infers from (3.33) and (3.38) that if

$$\begin{aligned}\beta &> \max\left\{P(y_0)^{\frac{N}{2}}, Q(y_0)^{\frac{N}{2}}\right\} \cdot \max\left\{ \frac{Q(y_0)^{1-\frac{N}{2}}}{P(y_0)}\mu, \frac{P(y_0)^{1-\frac{N}{2}}}{Q(y_0)}\nu \right\} \\ &= \max\left\{ \mu\frac{Q_0}{P_0}, \nu\left(\frac{P_0}{Q_0}\right)^{1-\frac{N}{2}}, \nu\frac{P_0}{Q_0}, \mu\left(\frac{Q_0}{P_0}\right)^{1-\frac{N}{2}} \right\} := \beta_1\left(\frac{Q_0}{P_0}\right),\end{aligned} \tag{3.39}$$

where $\beta_1(z) = \beta_1(\mu, \nu, z)$ is defined in (1.6), then (3.29) holds. Summarizing the calculations above, we have the following lemma.

Lemma 3.7. *For $\beta > \beta_1(Q_0/P_0)$ defined as in (3.39) and sufficiently small $\varepsilon > 0$, any ground state solution of $(\mathscr{P}_\varepsilon)$ is a nontrivial one. That is, if $z = (u, v)$ is a ground state solution of $(\mathscr{P}_\varepsilon)$, then $u \not\equiv 0$ and $v \not\equiv 0$.*

Remark 3.8. As in [23], one can also find other conditions to guarantee (3.29) holds. We omit the details and leave it for interested readers. Also we note that if $P(x) \equiv Q(x)$, then $P_0 = Q_0$ and $\beta_1(1) = \max\{\mu, \nu\}$.

In order to prove the existence of a positive solution when β is large, we use a variational approach in which the compactness lemma (Lemma 3.4) is crucial. Let $\hat{\beta}_1$ be defined as in (1.9). Then for $\beta > \hat{\beta}_1$, each of $A_{P_\infty Q_\infty}$ and $A_{P_0Q_0}$ is attained by a respective positive ground state solution from part 2 of Theorem 1.1 and (1.6), and from Lemma 2.3 we have that $A_{P_0Q_0} < A_{P_\infty Q_\infty}$. Now we are in a position to prove the existence of a positive ground state solution of $(\mathscr{P}_\varepsilon)$, which implies the existence part (i) of Theorem 1.2.

Proposition 3.9. *Suppose that P, Q satisfy* (**PQ0**) *and* (**PQ1**). *Let $\hat{\beta}_1$ be defined as in (1.9). Then for $\beta > \hat{\beta}_1$ and $\varepsilon > 0$ sufficiently small, A_ε is attained by $z_\varepsilon = (u_\varepsilon, v_\varepsilon) \in E_\varepsilon$ such that $u_\varepsilon > 0$ and $v_\varepsilon > 0$.*

Proof. First, from Lemma 2.4, the functional $\mathscr{L}_\varepsilon$ satisfies a mountain pass geometry for $\beta > 0$. Then by using a version of the mountain pass theorem without $(PS)_c$ condition [43, Theorem 1.15], there exists a sequence $\{z_n\} \subset E_\varepsilon$ satisfying

$$\mathscr{L}_\varepsilon(z_n) \to A_\varepsilon, \quad \text{and} \quad \mathscr{L}'_\varepsilon(z_n) \to 0, \quad \text{as } n \to \infty. \tag{3.40}$$

From Lemma 3.1, $\{z_n\}$ is bounded in E_ε. Hence there exists $z = (u, v) \in E_\varepsilon$ such that $z_n \rightharpoonup z$ in E_ε.

Since $\beta > \hat{\beta}_1$, it follows that each of $A_{P_\infty Q_\infty}$ and $A_{P_0 Q_0}$ is attained by a respective positive ground state solution from part 2 of Theorem 1.1 and (1.6). Moreover from the condition **(PQ1)**, one can choose $\xi > 1$ such that $P_0 < \xi P_0 < P_\infty$ and $Q_0 < \xi Q_0 < Q_\infty$. Define $\tau = \xi P_0$ and $\sigma = \xi Q_0$. Since $\beta_1(\xi Q_0/\xi P_0) = \beta_1(Q_0/P_0)$, then $A_{\tau\sigma}$ is also attained when $\beta > \hat{\beta}_1$. Then from Lemma 2.3 we have $A_{P_0 Q_0} < A_{\tau\sigma} < A_{P_\infty Q_\infty}$. It follows from Lemma 2.2 that $A_\varepsilon \le A_{P_0 Q_0} + \eta_\varepsilon$, where $\eta_\varepsilon > 0$ and $\eta_\varepsilon \to 0$ as $\varepsilon \to 0$. On the other hand it follows from $A_{P_0 Q_0} < A_{\tau\sigma}$ that there exists $\eta > 0$ such that $A_{P_0 Q_0} < A_{P_0 Q_0} + \eta \le A_{\tau\sigma}$. So for $\varepsilon > 0$ small enough we deduce that $A_\varepsilon \le A_{P_0 Q_0} + \eta_\varepsilon \le A_{P_0 Q_0} + \eta \le A_{\tau\sigma} < A_{P_\infty Q_\infty}$. Now from (3.40) and Lemma 3.4, we obtain that $z_n \to z$ in E_ε. Moreover, since $\beta > \hat{\beta}_1 \ge \beta_1(Q_0/P_0)$, from Lemma 3.7, we know that for $z = (u, v)$, $u \not\equiv 0$ and $v \not\equiv 0$. Thus we prove that $z \in \mathscr{N}_\varepsilon$ such that $\mathscr{L}_\varepsilon(z) = A_\varepsilon$ and $\mathscr{L}'_\varepsilon(z) = 0$. Finally we prove that $u, v > 0$. In fact, since $(|u|, |v|) \in \mathscr{N}_\varepsilon$ and $A_\varepsilon = \mathscr{L}_\varepsilon(|u|, |v|)$, we conclude that $(|u|, |v|)$ is a nonnegative solution of $(\mathscr{P}_\varepsilon)$. Using the strong maximum principle we infer that $|u|, |v| > 0$. Thus A_ε is attained by a positive $z_\varepsilon = (u_\varepsilon, v_\varepsilon)$, where $u_\varepsilon = |u|$, $v_\varepsilon = |v|$. □

Remark 3.10. If $P(x) \equiv Q(x)$, then again $P_0 = Q_0$ and $P_\infty = Q_\infty$, so for $\beta > \hat{\beta}_1 = \max\{\mu, \nu\}$, the conclusion of Proposition 3.9 holds.

To conclude this subsection we prove part (*ii*) of Theorem 1.2.

Lemma 3.11. *Suppose that the assumptions of Theorem 1.2 are satisfied. Let $\mathscr{B}'_\varepsilon$ denote the set of all positive ground state solutions of $(\mathscr{P}_\varepsilon)$. Then $\mathscr{B}'_\varepsilon$ is compact in E for all small $\varepsilon > 0$.*

Proof. Let $\{z_n\} \subset \mathscr{B}'_\varepsilon \cap \mathscr{N}_\varepsilon$ be a bounded sequence satisfying $\mathscr{L}_\varepsilon(z_n) = A_\varepsilon$ and $\mathscr{L}'_\varepsilon(z_n) = 0$. Without loss of generality we assume that $z_n \rightharpoonup z \in E_\varepsilon$. Then it follows from the weak continuity of $\mathscr{L}'_\varepsilon$ that $\mathscr{L}'_\varepsilon(z) = 0$. Set $w_n = z_n - z$. As in Lemma 3.4, we can prove that $w_n \to 0$ in E. Hence $z \in \mathscr{B}'_\varepsilon$. □

3.3. Concentration of positive ground state solutions

In this subsection we study the concentration phenomenon of the positive ground state solutions obtained in Subsection 3.2. We begin with the following lemma which is needed in proving the concentration of positive ground state solutions.

Lemma 3.12. *Assume that $\beta > 0$ and $\lambda_1, \lambda_2 > 0$. Let $\{\tilde{z}_n = (\tilde{u}_n, \tilde{v}_n)\} \subset \mathscr{N}_{\lambda_1\lambda_2}$ be a sequence satisfying $\mathscr{L}_{\lambda_1\lambda_2}(\tilde{z}_n) \to A_{\lambda_1\lambda_2}$ as $n \to \infty$, where $\mathscr{L}_{\lambda_1\lambda_2}$ and $A_{\lambda_1\lambda_2}$ are defined in (2.7) and (2.8). Then either $\{\tilde{z}_n\}$ has a subsequence strongly convergent in E or there exists $\{y_n\} \subset \mathbb{R}^N$ such that the sequence $\tilde{w}_n(x) = \tilde{z}_n(x + y_n)$ converges strongly in E. In particular $A_{\lambda_1\lambda_2}$ is attained by some $z \in \mathscr{N}_{\lambda_1\lambda_2}$.*

Proof. First, it follows from the Ekeland's variational principle (see Theorem 8.5 of [43]) on $\mathscr{N}_{\lambda_1\lambda_2}$ that there exists a minimizing sequence $\{(u_n, v_n)\} \subset \mathscr{N}_{\lambda_1\lambda_2}$ such that

$$
\begin{aligned}
&(a_1) \quad \mathscr{L}_{\lambda_1\lambda_2}(u_n, v_n) \le A_{\lambda_1\lambda_2} + \frac{1}{n},\\
&(a_2) \quad \mathscr{L}_{\lambda_1\lambda_2}(u, v) \ge \mathscr{L}_{\lambda_1\lambda_2}(u_n, v_n) - \frac{1}{n}\|(u_n - u, v_n - v)\|_E, \quad \forall (u, v) \in \mathscr{N}_{\lambda_1\lambda_2},\\
&(a_3) \quad \|(\tilde{u}_n, \tilde{v}_n) - (u_n, v_n)\|_E \to 0.
\end{aligned} \tag{3.41}
$$

We claim that

$$\mathscr{L}'_{\lambda_1\lambda_2}(u_n, v_n) \to 0 \quad \text{as} \quad n \to \infty. \tag{3.42}$$

In fact, it is easy to check that $\{(u_n, v_n)\}$ is bounded and $\|(u_n, v_n)\|_E \geq \delta > 0$. Moreover, for a given $(\varphi, \phi) \in E$ such that $\|\varphi\|, \|\phi\| \leq 1$, we define

$$F_n(t, s) = \mathscr{L}_{\lambda_1\lambda_2}(u_n + su_n + t\varphi, v_n + sv_n + t\phi), \tag{3.43}$$

where $\mathscr{L}_{\lambda_1\lambda_2}$ is defined in (2.7). Obviously, $F_n(0, 0) = 0$ and $F \in C^1(\mathbb{R}^2, \mathbb{R})$. A direct computation shows that

$$\begin{aligned} \frac{\partial F_n}{\partial s}(0,0) &= 2(\|u_n\|^2_{\lambda_1} + \|v_n\|^2_{\lambda_2}) - 4\int\limits_{\mathbb{R}^N} (\mu u_n^4 + 2\beta u_n^2 v_n^2 + \nu v_n^4) \\ &= -2(\|u_n\|^2_{\lambda_1} + \|v_n\|^2_{\lambda_2}) \leq -c\delta < 0. \end{aligned} \tag{3.44}$$

By the implicit function theorem, there exists a C^1 function $s_n(t)$ defined on some interval $(-\tau_n, \tau_n)$ for $\tau_n > 0$, such that $s_n(0) = 0$ and

$$F_n(t, s_n(t)) = 0, \quad t \in (-\tau_n, \tau_n). \tag{3.45}$$

Differentiating (3.45), we have that

$$\frac{\partial F_n}{\partial t}(0,0) + \frac{\partial F_n}{\partial s}(0,0)s_n'(0) = 0. \tag{3.46}$$

Since

$$\begin{aligned} \left|\frac{\partial F_n}{\partial t}(0,0)\right| = \Bigg|2\int\limits_{\mathbb{R}^N} &(\nabla u_n \nabla\varphi + \lambda_1 u_n\varphi + \nabla v_n\nabla\phi + \lambda_2 v_n\phi) \\ &- 4\int\limits_{\mathbb{R}^N} (\mu u_n^3\varphi + \beta v_n^2 u_n\varphi + \beta u_n^2 v_n\phi + \nu v_n^3\phi)\Bigg| \leq c, \end{aligned} \tag{3.47}$$

then we obtain that

$$|s_n'(0)| \leq c. \tag{3.48}$$

Let $\varphi_{n,t} = u_n + s_n(t)u_n + t\varphi$ and $\phi_{n,t} = v_n + s_n(t)v_n + t\phi$. Then it follows from (3.45) that $(\varphi_{n,t}, \phi_{n,t}) \in \mathscr{N}_{\lambda_1\lambda_2}$ for $t \in (-\tau_n, \tau_n)$. Furthermore, we deduce from (a_2) of (3.41) that

$$\mathscr{L}_{\lambda_1\lambda_2}(\varphi_{n,t}, \phi_{n,t}) - \mathscr{L}_{\lambda_1\lambda_2}(u_n, v_n) \geq -\frac{1}{n}\|(s_n(t)u_n + t\varphi, s_n(t)v_n + t\phi)\|_E. \tag{3.49}$$

Note that $\mathscr{L}'_{\lambda_1\lambda_2}(u_n, v_n)(u_n, v_n) = 0$, hence by using Taylor expansion we have that

$$\begin{aligned}&\mathscr{L}_{\lambda_1\lambda_2}(\varphi_{n,t}, \phi_{n,t}) - \mathscr{L}_{\lambda_1\lambda_2}(u_n, v_n)\\ &= \mathscr{L}'_{\lambda_1\lambda_2}(u_n, v_n)(s_n(t)u_n + t\varphi, s_n(t)v_n + t\phi) + R(n,t) = 2t\mathscr{L}'_{\lambda_1\lambda_2}(u_n, v_n)(\varphi, \phi) + R(n,t),\end{aligned} \tag{3.50}$$

where $R(n,t) = o(\|(s_n(t)u_n + t\varphi, s_n(t)v_n + t\phi)\|)$ as $t \to 0$. It follows from (3.48) that

$$\limsup_{n\to\infty} \left\| \left(\frac{s_n(t)}{t} u_n + \varphi, \frac{s_n(t)}{t} v_n + \phi \right) \right\| \le c. \tag{3.51}$$

Thus, $R(n,t) = o(t)$ as $t \to 0$. One can deduce from (3.49)–(3.51) that

$$|\mathscr{L}'_{\lambda_1\lambda_2}(u_n, v_n)(\varphi, \phi)| \le \frac{c}{n}, \quad t \to 0. \tag{3.52}$$

That is, the claim (3.42) holds. So $\{z_n\}$ is a $(PS)_{A_{\lambda_1\lambda_2}}$-sequence of $\mathscr{L}_{\lambda_1\lambda_2}$. Similar to the proof of Lemma 3.1, one can verify that $\{z_n\}$ is a bounded sequence in E. Therefore, choosing a subsequence if necessary, we have that $z_n \rightharpoonup z = (u, v)$ in E and $\mathscr{L}'_{\lambda_1\lambda_2}(z) = 0$. In order to prove the conclusion of this lemma, we distinguish the following two cases:

Case 1. If $z \neq 0$, then it follows that $z \in \mathscr{N}_{\lambda_1\lambda_2}$. Moreover, one sees that

$$\begin{aligned}A_{\lambda_1\lambda_2} &\le \mathscr{L}_{\lambda_1\lambda_2}(z) = \mathscr{L}_{\lambda_1\lambda_2}(z) - \frac{1}{4}(\mathscr{L}'_{\lambda_1\lambda_2}(z), z) = \frac{1}{4}\int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda_1 u^2 + |\nabla v|^2 + \lambda_2 v^2)\\ &\le \liminf_{n\to\infty} \int_{\mathbb{R}^N} (|\nabla u_n|^2 + \lambda_1 u_n^2 + |\nabla v_n|^2 + \lambda_2 v_n^2) = \liminf_{n\to\infty} \left[\mathscr{L}_{\lambda_1\lambda_2}(z_n) - \frac{1}{4}\mathscr{L}'_{\lambda_1\lambda_2}(z_n)z_n \right]\\ &= A_{\lambda_1\lambda_2}.\end{aligned}$$

So we obtain that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (|\nabla u_n|^2 + \lambda_1 u_n^2) = \int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda_1 u^2), \quad \lim_{n\to\infty} \int_{\mathbb{R}^N} (|\nabla v_n|^2 + \lambda_2 v_n^2) = \int_{\mathbb{R}^N} (|\nabla v|^2 + \lambda_2 v^2).$$

Thus it follows from Brezis–Lieb lemma (see Lemma 1.32 of [43]) and Sobolev's inequality that $\|u_n - u\|$ and $\|v_n - v\| \to 0$ as $n \to \infty$.

Case 2. If $z = 0$, as in Lemma 3.2, we have that there exist $\{y_n\} \subset \mathbb{R}^N$, $r, \delta > 0$ such that

$$\liminf_{n\to\infty} \int_{B_r(y_n)} |z_n|^2 \ge \delta. \tag{3.53}$$

Set $w_n(x) = z_n(x + y_n) = (w_n^1, w_n^2)$, then $\|w_n^1\|_{\lambda_1} = \|u_n\|_{\lambda_1}$ and $\|w_n^2\|_{\lambda_2} = \|v_n\|_{\lambda_2}$, $\mathscr{L}_{\lambda_1\lambda_2}(w_n) \to A_{\lambda_1\lambda_2}$ and $\mathscr{L}'_{\lambda_1\lambda_2}(w_n) \to 0$. Then there exists $w \in E$ with $w \neq 0$ such that $w_n \rightharpoonup w$ in E. Then the conclusion follows from same arguments as in Case 1.

Finally it follows from (a_3) that the conclusions of this lemma hold. □

Now we are ready to prove the concentration of positive state solutions for small ε.

Lemma 3.13. *Suppose that P, Q satisfy* (**PQ0**) *and* (**PQ1**), *and P, Q are uniformly continuous. Assume that $\beta > \hat{\beta}_1$. Let $z_\varepsilon = (u_\varepsilon, v_\varepsilon)$ be a positive ground state solution of ($\mathscr{P}_\varepsilon$). Then there is a maximum point y_ε of $u_\varepsilon + v_\varepsilon$ and $y_0 \in \mathscr{V}$ such that dist$(\varepsilon y_\varepsilon, \mathscr{V}) \to 0$, $\varepsilon y_\varepsilon \to y_0$, $z_\varepsilon(x + y_\varepsilon)$ converges in E to a positive ground state solution of (1.3) with $\lambda_1 = P(y_0) = P_0$ and $\lambda_2 = Q(y_0) = Q_0$, as $\varepsilon \to 0$.*

Proof. Since $\beta > \hat{\beta}_1$, then $A_{P_0Q_0}$ is attained by some positive $z_0 \in \mathscr{N}_{P_0Q_0}$. From Proposition 3.9, for small $\varepsilon > 0$, A_ε is attained by some positive $z_\varepsilon \in \mathscr{N}_\varepsilon$. Therefore from Lemma 2.2, we have

$$\lim_{\varepsilon \to 0} A_\varepsilon = A_{P_0Q_0} < A_{P_\infty Q_\infty}. \tag{3.54}$$

Let $\{\varepsilon_j\}$ be a sequence of positive numbers converging to 0 as $j \to \infty$, and let $\{z_j = z_{\varepsilon_j}\} \subset \mathscr{N}_{\varepsilon_j}$ satisfying $\mathscr{L}_{\varepsilon_j}(z_j) = A_{\varepsilon_j}$ and $\mathscr{L}'_{\varepsilon_j}(z_j) = 0$. Then from Lemma 3.1, one sees that the sequence $\{z_j\}$ is bounded in E. So we can assume that $z_j \rightharpoonup z$ in E as $j \to \infty$. To prove the main results we divide the proof into the following three steps.

Step 1. $\{z_j\}$ is nonvanishing.

Similar to the proof of Lemma 3.2, one can verify that there exist $r, \delta > 0$ and a sequence $\{y'_j\} \subset \mathbb{R}^N$ such that

$$\liminf_{j \to \infty} \int_{B_r(y'_j)} |z_j|^2 \geq \delta > 0. \tag{3.55}$$

For $j \in \mathbb{N}$, define $y_j \in \mathbb{R}^N$ to be a maximum point of $u_j + v_j$ such that

$$u_j(y_j) + v_j(y_j) = \max_{y \in \mathbb{R}^N} \left[u_j(y) + v_j(y)\right]. \tag{3.56}$$

We claim that there exists $\varrho > 0$ (independent of j) such that $|z_j(y_j)| \geq \varrho > 0$ for all $j \in \mathbb{N}$. To the contrary, we assume that $|z_j(y_j)| \to 0$ as $j \to \infty$. We deduce from (3.55) that

$$0 < \delta \leq \int_{B_r(y'_j)} |z_j|^2 \leq c|z_j(y_j)|^2 \to 0 \quad \text{as } j \to \infty.$$

This is a contradiction. Furthermore, from (3.55) one deduces that there exist $R > 0$ and $\delta' > 0$ such that

$$\liminf_{j \to \infty} \int_{B_R(y_j)} |z_j|^4 \geq \delta' > 0. \tag{3.57}$$

Step 2. $\{\varepsilon_j y_j\}$ is bounded.

In order to prove this conclusion we set

$$w_j(x) = z_j(x + y_j) = (u_j^1(x), v_j^1(x)),$$
$$\hat{P}_{\varepsilon_j}(x) = P(\varepsilon_j(x + y_j)) \text{ and } \quad \hat{Q}_{\varepsilon_j}(x) = Q(\varepsilon_j(x + y_j)).$$

Then from (3.57) we have $w_j \rightharpoonup w = (u^1, v^1) \neq 0$ in E and $w_j \to w$ in $L^p_{loc}(\mathbb{R}^N) \times L^p_{loc}(\mathbb{R}^N)$ for $p \in (2, 2^*)$.

We prove that $w_j \to w$ in E. In fact, from the proof of Lemma 2.2, we can choose $t_j > 0$ such that $t_j w_j \in \mathcal{N}_{P_0Q_0}$. Set $\tilde{w}_j = t_j w_j = (\tilde{u}_j^1, \tilde{v}_j^1)$. It follows from (**PQ1**), $z_j \in \mathcal{N}_{\varepsilon_j}$ and Lemma 2.2 that

$$\begin{aligned}\mathcal{L}_{P_0Q_0}(\tilde{w}_j) &\le \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla \tilde{u}_j^1|^2 + \hat{P}_{\varepsilon_j}(x)(\tilde{u}_j^1)^2) + \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla \tilde{v}_j^1|^2 + \hat{Q}_{\varepsilon_j}(x)(\tilde{v}_j^1)^2)\\ &\quad - \frac{1}{4}\int_{\mathbb{R}^N} (\mu(\tilde{u}_j^1)^4 + 2\beta(\tilde{u}_j^1)^2(\tilde{v}_j^1)^2 + \nu(\tilde{v}_j^1)^4)\\ &= \mathcal{L}_{\varepsilon_j}(t_j z_j) \le \mathcal{L}_{\varepsilon_j}(z_j) = A_{\varepsilon_j} = A_{P_0Q_0} + o(1).\end{aligned}$$

On the other hand, $\mathcal{L}_{P_0Q_0}(\tilde{w}_j) \ge A_{P_0Q_0}$, thus $\lim\limits_{j\to\infty} \mathcal{L}_{P_0Q_0}(\tilde{w}_j) = A_{P_0Q_0}$. Since $w_j \rightharpoonup w$ in E, it is easy to check that $\{t_j\}$ is bounded. Without loss of generality we can assume that $t_j \to t \ge 0$ as $j \to \infty$. If $t = 0$, we have that $\tilde{w}_j = t_j w_j \to 0$ in view of the boundedness of w_j, and hence $\mathcal{L}_{P_0Q_0}(\tilde{w}_j) \to 0$ as $j \to \infty$, which contradicts with $A_{P_0Q_0} > 0$. So we must have $t > 0$ and the weak limit $\tilde{w} = (\tilde{u}^1, \tilde{v}^1)$ of $\tilde{w}_j$ is different from zero. Since $t_n \to t > 0$ and $w_n \rightharpoonup w$, we have from the uniqueness of the weak limit that $\tilde{w} = tw \neq 0$ and $\tilde{w} \in \mathcal{N}_{P_0Q_0}$. We choose $\lambda_1 = P_0$ and $\lambda_2 = Q_0$ in Lemma 3.12, then $\tilde{w}_j \to \tilde{w}$ in E, and consequently $w_j \to w$ in E. This proves the claim of $w_j \to w$ in E.

To further consider the limit w, notice that $w_j = (u_j^1, v_j^1)$ solves

$$\begin{cases} -\Delta u + P(\varepsilon_j x + \varepsilon_j y_j)u = \mu u^3 + \beta v^2 u & \text{in } \mathbb{R}^N,\\ -\Delta v + Q(\varepsilon_j x + \varepsilon_j y_j)v = \nu v^3 + \beta u^2 v & \text{in } \mathbb{R}^N,\\ u, v > 0, \quad u, v \in H^1(\mathbb{R}^N), \end{cases} \tag{3.58}$$

with corresponding energy functional

$$\tilde{\mathcal{L}}_{\varepsilon_j}(u, v) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + \hat{P}_{\varepsilon_j} u^2 + |\nabla v|^2 + \hat{Q}_{\varepsilon_j} v^2) - \frac{1}{4}\int_{\mathbb{R}^N} (\mu u^4 + 2\beta u^2 v^2 + \nu v^4).$$

We show that $\{\varepsilon_j y_j\}$ is bounded by using an idea of [42]. Assume by contradiction that $|\varepsilon_j y_j| \to \infty$ as $j \to \infty$. Without loss of generality we may assume that $P(\varepsilon_j y_j) \to \tilde{P}^\infty$ and $Q(\varepsilon_j y_j) \to \tilde{Q}^\infty$ as $j \to \infty$. It follows from the assumption (**PQ1**) that $P_0 < \tilde{P}^\infty$ and $Q_0 < \tilde{Q}^\infty$. Since P and Q are uniformly continuous functions, it follows that there exists $R > 0$ such that for all $|x| \le R$, as $j \to \infty$,

$$
\begin{aligned}
&|\hat{P}_{\varepsilon_j}(x) - \tilde{P}^{\infty}| \\
&\quad \le |P(\varepsilon_j(x+y_j)) - P(\varepsilon_j y_j)| + |P(\varepsilon_j y_j) - \tilde{P}^{\infty}| \le c\varepsilon_j |x| + |P(\varepsilon_j y_j) - \tilde{P}^{\infty}| \to 0, \\
&|\hat{Q}_{\varepsilon_j}(x) - \tilde{Q}^{\infty}| \\
&\quad \le |Q(\varepsilon_j(x+y_j)) - \tilde{Q}(\varepsilon_j y_j)| + |Q(\varepsilon_j y_j) - \tilde{Q}^{\infty}| \le c\varepsilon_j |x| + |Q(\varepsilon_j y_j) - \tilde{Q}^{\infty}| \to 0.
\end{aligned}
$$

In addition, for any $\phi = (\phi_1, \phi_2) \in C_0^{\infty}(\mathbb{R}^N) \times C_0^{\infty}(\mathbb{R}^N)$, we deduce from $w_j \to w$ in E that

$$
\begin{aligned}
\lim_{j\to\infty} \tilde{\mathscr{L}}'_{\varepsilon_j}(w_j)\phi &= \lim_{j\to\infty} \int_{\mathbb{R}^N} [(\nabla u_j^1 \nabla \phi_1 + \hat{P}_{\varepsilon_j}(x) u_j^1 \phi_1) + (\nabla v_j^1 \nabla \phi_2 + \hat{Q}_{\varepsilon_j}(x) v_j^1 \phi_2)] \\
&\quad - \lim_{j\to\infty} \int_{\mathbb{R}^N} [\mu (u_j^1)^3 \phi_1 + \nu (v_j^1)^3 \phi_2] - \beta \lim_{j\to\infty} \int_{\mathbb{R}^N} [(v_j^1)^2 u_j^1 \phi_1 + (u_j^1)^2 v_j^1 \phi_2] \\
&= \int_{\mathbb{R}^N} [(\nabla u^1 \nabla \phi_1 + \tilde{P}_{\infty} u^1 \phi_1) + (\nabla v^1 \nabla \phi_2 + \tilde{Q}_{\infty} v^1 \phi_2)] \\
&\quad - \int_{\mathbb{R}^N} [\mu (u^1)^3 \phi_1 + \nu (v^1)^3 \phi_2] - \beta \int_{\mathbb{R}^N} [(v^1)^2 u^1 \phi_1 + (u^1)^2 v^1 \phi_2] = 0.
\end{aligned}
$$

Thus, $w = (u^1, v^1)$ is a solution of (1.3) with $\lambda_1 = \tilde{P}_{\infty}$ and $\lambda_2 = \tilde{Q}_{\infty}$, and the corresponding energy functional satisfies

$$
\begin{aligned}
\mathscr{L}_{\tilde{P}_{\infty}\tilde{Q}_{\infty}}(w) &= \frac{1}{2}\left(\|u^1\|^2_{\tilde{P}^{\infty}} + \|v^1\|^2_{\tilde{Q}^{\infty}}\right) - \frac{1}{4}\int_{\mathbb{R}^N} [\mu (u^1)^4 + 2\beta (u^1)^2 (v^1)^2 + \nu (v^1)^4] \\
&\ge A_{\tilde{P}^{\infty}\tilde{Q}^{\infty}}.
\end{aligned} \tag{3.59}
$$

By using standard arguments one checks that $A_{\tilde{P}^{\infty}\tilde{Q}^{\infty}}$ is attained by $z = (u, v) \neq 0$, and z may be a semi-trivial solution of (1.3) with $\lambda_1 = \tilde{P}^{\infty}$ and $\lambda_2 = \tilde{Q}^{\infty}$. Furthermore, since $\beta > \hat{\beta}_1$, $P_0 < \tilde{P}^{\infty}$ and $Q_0 < \tilde{Q}^{\infty}$, then from Lemma 2.3 we have $A_{\tilde{P}^{\infty}\tilde{Q}^{\infty}} > A_{P_0 Q_0}$. On the other hand, one deduces from $\tilde{\mathscr{L}}'_{\varepsilon_j}(w_j)w_j = \mathscr{L}'_{\varepsilon_j}(z_j)z_j = 0$ that

$$
\begin{aligned}
A_{P_0 Q_0} &\ge \lim_{j\to\infty} A_{\varepsilon_j} = \lim_{j\to\infty} \tilde{\mathscr{L}}_{\varepsilon_j}(w_j) = \lim_{j\to\infty}\left[\tilde{\mathscr{L}}_{\varepsilon_j}(w_j) - \frac{1}{2}\tilde{\mathscr{L}}_{\varepsilon_j}(w_j)'(w_j)w_j\right] \\
&\ge \liminf_{j\to\infty}\left[\frac{1}{4}\int_{\mathbb{R}^N} (\mu (u_j^1)^4 + \nu (v_j^1)^4) + 2\beta (u_j^1)^2 (v_j^1)^2)\right] \\
&\ge \frac{1}{4}\int_{\mathbb{R}^N} \left[\mu (u^1)^4 + \nu (v^1)^4 + 2\beta (u^1)^2 (v^1)^2\right] = \mathscr{L}_{\tilde{P}_{\infty}\tilde{Q}_{\infty}}(w) \ge A_{\tilde{P}^{\infty}\tilde{Q}^{\infty}}.
\end{aligned} \tag{3.60}
$$

That is a contradiction. Therefore the sequence $\{\varepsilon_j y_j\}$ is bounded. Without loss of generality, we assume that $\varepsilon_j y_j \to y_0$, and consequently $w = (u^1, v^1)$ is a solution of (1.3) with $\lambda_1 = P(y_0)$ and $\lambda_2 = Q(y_0)$.

Step 3. We prove that $y_0 \in \mathscr{V}$.

We argue by contradiction. Assume that $y_0 \notin \mathscr{V}$. Again it follows from Lemma 2.3 that

$$A_{P_0 Q_0} < A_{P(y_0)Q(y_0)}. \tag{3.61}$$

By using the same arguments as in the proof of Step 2 (with $\tilde{P}^\infty$ and $\tilde{Q}^\infty$ replaced by $P(y_0)$ and $Q(y_0)$), we derive a contradiction by using (3.59), (3.60) and (3.61). So $y_0 \in \mathscr{V}$. Finally we prove that $w = (u^1, v^1)$ is a positive ground state solution of (1.3) with $\lambda_1 = P_0 = P(y_0)$ and $\lambda_2 = Q_0 = Q(y_0)$. Indeed from (3.54) and (3.60) (with $\tilde{P}^\infty$ and $\tilde{Q}^\infty$ replaced by P_0 and Q_0), we obtain that

$$\lim_{j\to\infty} A_{\varepsilon_j} = A_{P_0 Q_0} \geq \mathscr{L}_{P_0 Q_0}(w) \geq A_{P_0 Q_0},$$

which implies that $\mathscr{L}_{P_0 Q_0}(w) = A_{P_0 Q_0}$ and w is a ground state solution of (1.3). Since $\beta > \hat{\beta}_1$, then w must be positive from Lemma 3.7. $\square$

In order to obtain exponential decay of positive solutions of $(\mathscr{P}_\varepsilon)$, we need the following regularity results, which can be found in, for example, [44, Proposition 2-3].

Lemma 3.14. *Let $z \in H^1(\mathbb{R}^N)$ satisfying*

$$-\Delta z + (Q(x) + H(x))z = f(x, z), \quad z \in H^1(\mathbb{R}^N),$$

where $Q(x) \geq 0$ in $\mathbb{R}^N$, $Q \in L^\infty_{loc}(\mathbb{R}^N, \mathbb{R}^+)$, and $H \in L^{\frac{N}{2}}(\mathbb{R}^N)$, f is a Caratheodory function such that

$$0 \leq f(x, s) \leq C_f(s + s^{r-1}), \quad x \in \mathbb{R}^N, \ s \geq 0,$$

where $2 < r < 2^*$ if $N = 3$, $2 < r < \infty$ if $N = 1, 2$. Then $z \in L^p(\mathbb{R}^N)$ for all $2 \leq p < \infty$. Furthermore, there is a positive constant C_p depending on p, C_f and Q such that $|z|_{L^p(\mathbb{R}^N)} \leq C_p \|z\|_{H^1(\mathbb{R}^N)}$. Moreover, the dependence on Q of C_p can be given uniformly on Cauchy sequences Q_k in $L^{\frac{N}{2}}(\mathbb{R}^N)$.

Lemma 3.15. *Suppose that $t > N$, $k \in L^{\frac{t}{2}}(\Lambda)$ and $z \in H^1(\Lambda)$ satisfies in the weak sense*

$$-\Delta z \leq k(x),$$

where Λ is an open subset of $\mathbb{R}^N$. Then for any ball $B_{2R}(y) \subset \Lambda$, one has that

$$\sup_{B_R(y)} z \leq C(|z^+|_{L^2(B_{2R}(y))} + |k|_{L^{\frac{t}{2}}(B_{2R}(y))})$$

where C depends on N, t and R.

Now we are ready to prove the following exponential decay results for the positive ground state solution of $(\mathscr{P}_\varepsilon)$.

Lemma 3.16. *Under the assumptions of Theorem 1.2, if $z_\varepsilon = (u_\varepsilon, v_\varepsilon)$ is a positive ground state solution of $(\mathscr{P}_\varepsilon)$, one has that when $\varepsilon > 0$ is small, $u_\varepsilon, v_\varepsilon \in C^{1,\sigma}_{loc}(\mathbb{R}^N)$ for $\sigma \in (0, 1)$, and*

$$\lim_{|x|\to\infty} u_\varepsilon(x) = \lim_{|x|\to\infty} v_\varepsilon(x) = \lim_{|x|\to\infty} |\nabla u_\varepsilon(x)| = \lim_{|x|\to\infty} |\nabla v_\varepsilon(x)| = 0, \tag{3.62}$$

and there exist $C, c > 0$ such that

$$u_\varepsilon(x) + v_\varepsilon(x) \le C e^{-c|x-y_\varepsilon|}, \tag{3.63}$$

where y_ε satisfies $|u_\varepsilon(y_\varepsilon) + v_\varepsilon(y_\varepsilon)| = \max_{x\in\mathbb{R}^N} |u_\varepsilon(x) + v_\varepsilon(x)|$.

Proof. The proof of $u_\varepsilon, v_\varepsilon \in C^{1,\sigma}_{loc}(\mathbb{R}^N)$ for $\sigma \in (0, 1)$ and (3.62) can be proved the same way as in (i) of Lemma 2.1. In the following we prove the exponential decay of $w_\varepsilon = u_\varepsilon + v_\varepsilon$. Let $\varepsilon_j \to 0$ be a positive sequence, let $\{z_j = (u_j, v_j)\}$ be a sequence of positive ground state solutions such that $\mathscr{L}_{\varepsilon_j}(z_j) = A_{\varepsilon_j}$ and $\mathscr{L}'_{\varepsilon_j}(z_j) = 0$, and let y_j be the maximum point as defined in (3.56). As in the proof of Lemma 3.13, we have that $(u^1_j(x), v^1_j(x)) = z_j(x + y_j) = (u_j(x + y_j), v_j(x + y_j))$ satisfies (3.58). Let $w_j = u^1_j + v^1_j$. Then from (3.58), w_j satisfies

$$-\Delta w_j + (\hat{P}_{\varepsilon_j}(x) + \hat{Q}_{\varepsilon_j}(x))w_j = g_j(x), \quad \text{in } \mathbb{R}^N, \tag{3.64}$$

where

$$g_j(x) = \hat{P}_{\varepsilon_j}(x)v^1_j + \hat{Q}_{\varepsilon_j}(x)u^1_j + \mu(u^1_j)^3 + \nu(v^1_j)^3 + \beta(u^1_j(v^2_j)^2 + v^1_j(u^1_j)^2). \tag{3.65}$$

So we deduce from Lemma 3.14 that $w_j \in L^t(\mathbb{R}^N)$ for all $t \ge 2$ and

$$|w_j|_{L^t(\mathbb{R}^N)} \le N_t \|w_j\|_{H^1(\mathbb{R}^N)}, \tag{3.66}$$

for some N_t not depending on j. As in the proof of Lemma 3.13, we may assume that $u^1_j \to u^1$ and $v^1_j \to v^1$ in $H^1(\mathbb{R}^N)$, hence for any $l \in (2, 2^*]$,

$$\lim_{R\to\infty} \int\limits_{|x|\ge R} [(u^1_j)^2 + (u^1_j)^l + (v^1_j)^2 + (v^1_j)^l] = 0, \quad \text{uniformly for } j \in \mathbb{N}. \tag{3.67}$$

From (3.64) it follows that

$$-\Delta w_j \le g_j(x) \quad \text{in } \mathbb{R}^N, \tag{3.68}$$

and the estimate (3.66) implies that for all $t \ge 2$, there exists $C > 0$ such that $|g_j|_{L^t(\mathbb{R}^N)} \le C$ for $j \in \mathbb{N}$. Thus by Lemma 3.15 we infer that for all $y \in \mathbb{R}^N$,

$$\sup_{B_1(y)} w_j \le c(|w_j|_{L^2(B_2(y))} + |g_j|_{L^t(B_2(y))}). \tag{3.69}$$

This implies that $|w_j|_\infty$ is uniformly bounded. Now combining the limit (3.67) with the inequality (3.69), we reach that

$$\lim_{|x|\to\infty} w_j(x) = 0 \quad \text{uniformly for all } j \in \mathbb{N}.$$

Form this we deduce that there is $\varepsilon_0 > 0$ such that

$$\lim_{|x|\to\infty} w_\varepsilon(x) = \lim_{|x|\to\infty} [u_\varepsilon(x) + v_\varepsilon(x)] = 0 \quad \text{uniformly for all } \varepsilon \in (0, \varepsilon_0].$$

So by using the same arguments as in the proof of [41, Theorem 3.8], we can show (3.63) holds. □

Now we complete the proof of Theorem 1.2.

Proof of Theorem 1.2. From Lemmas 3.9–3.11, one sees that the conclusions (i)–(ii) of Theorem 1.2 hold. The conclusions (iii)–(iv) of Theorem 1.2 follow from Lemmas 3.13 and 3.16. The results for $(\mathscr{P}_\varepsilon)$ go back to $(\mathscr{A}_\varepsilon)$ with the change of variables: $x \mapsto x/\varepsilon$. □

4. Multiple positive solutions for $(\mathscr{P}_\varepsilon)$

In this section we prove the existence of multiple positive solutions of $(\mathscr{P}_\varepsilon)$ by using Ljusternik–Schnirelmann category theory. Our methods are inspired by the work of [38] for scalar nonlinear elliptic equations. Later, some authors use this methods for other problems, for instance, see [48,41,49] and the references therein. Since $\mathscr{V} = \mathscr{P}_0 \cap \mathscr{Q}_0 \neq \emptyset$, we shall make good use of the ground state solution of (1.3) with $(\lambda_1, \lambda_2) = (P_0, Q_0) = (P(y_0), Q(y_0))$ for $y_0 \in \mathscr{P}_0 \cap \mathscr{Q}_0$. Fix a $\delta > 0$ and we define a nonincreasing cutoff function $J \in C_0^\infty(\mathbb{R}^+, [0,1])$ by

$$J(s) = \begin{cases} 1, & 0 \le s \le \frac{\delta}{2}, \\ \text{smooth function}, & \frac{\delta}{2} \le s \le \delta, \\ 0, & s \ge \delta. \end{cases}$$

Since $\mathscr{V} = \mathscr{P}_0 \cap \mathscr{Q}_0 \neq \emptyset$, then we assume that $z_0 = (u_0, v_0)$ is a positive ground state solution of (1.3) with $(\lambda_1, \lambda_2) = (P_0, Q_0)$. For any $y \in \mathscr{V}$, we define

$$(U_{\varepsilon,y}(x), V_{\varepsilon,y}(x)) = J(|\varepsilon x - y|)\left(u_0(\frac{\varepsilon x - y}{\varepsilon}), v_0(\frac{\varepsilon x - y}{\varepsilon})\right). \tag{4.1}$$

Then $(U_{\varepsilon,y}, V_{\varepsilon,y}) \in E_\varepsilon$, and there exists a unique $t_{1,\varepsilon}(y) > 0$ such that $t_{1,\varepsilon}(U_{\varepsilon,y}, V_{\varepsilon,y}) \in \mathscr{N}_\varepsilon$. So we define a mapping $\Psi_\varepsilon : \mathscr{V} \to \mathscr{N}_\varepsilon$ by

$$\Psi_\varepsilon(y) = t_{1,\varepsilon}(y)(U_{\varepsilon,y}, V_{\varepsilon,y}). \tag{4.2}$$

Now we study the behavior of $\mathscr{L}_\varepsilon(\Psi_\varepsilon(y))$ as $\varepsilon \to 0$. That is, the following results holds.

Lemma 4.1. *Assume that* (**PQ0**) *and* (**PQ1**) *hold. For* $\beta > 0$, *one has that* $\lim_{\varepsilon\to 0} \mathscr{L}_\varepsilon(\Psi_\varepsilon(y)) = A_{P_0Q_0}$ *uniformly for* $y \in \mathscr{V}$.

The proof of this lemma is similar to [48, Lemma 11] and we omit the details. Moreover, by using similar arguments as in Lemma 3.13 one can prove the following result.

Lemma 4.2. *Suppose that the assumptions of Theorem 1.3 are satisfied. Let* $\varepsilon_n \to 0$ *and* $\{z_n\} \subset \mathscr{N}_{\varepsilon_n}$ *such that* $\mathscr{L}_{\varepsilon_n}(z_n) \to A_{P_0Q_0}$, *then there exists a sequence* $\{y_n\} \subset \mathbb{R}^N$ *such that* $z_n(x + y_n)$ *has a convergent subsequence in* E, *and* $\varepsilon_n y_n \to y \in \mathscr{V}$.

From the condition **(PQ1)**, $\mathscr{V}$ is a compact set in $\mathbb{R}^N$, so $\mathscr{V}_\delta = \{x \in \mathbb{R}^N : dist(x, \mathscr{V}) \le \delta\}$. For $\delta > 0$, let $\varrho = \varrho(\delta)$ be a positive number so that $\mathscr{V}_\delta \subset B_\varrho(0)$. Let $\gamma : \mathbb{R}^N \to \mathbb{R}^N$ be a function defined by

$$\gamma(x) = \begin{cases} x, & |x| \le \varrho, \\ \dfrac{\varrho x}{|x|}, & |x| \ge \varrho. \end{cases}$$

Then we define a barycenter type map $\mathcal{K}_\varepsilon : \mathscr{N}_\varepsilon \to \mathbb{R}^N$ by

$$\mathcal{K}_\varepsilon(z) = \mathcal{K}_\varepsilon(u, v) = \frac{\int_{\mathbb{R}^N} \gamma(\varepsilon x) u^2}{2\int_{\mathbb{R}^N} u^2} + \frac{\int_{\mathbb{R}^N} \gamma(\varepsilon x) v^2}{2\int_{\mathbb{R}^N} v^2}. \tag{4.3}$$

Similar to the proof of Lemma 4.1, by using Lebesgue's Theorem it is easy to verify that $\lim_{\varepsilon\to 0} \mathcal{K}_\varepsilon(\Psi_\varepsilon(y)) = y$ uniformly for $y \in \mathscr{V}$.

Let $\kappa(\varepsilon)$ be any positive function such that $\kappa(\varepsilon) \to 0$ as $\varepsilon \to 0$. We define the following set:

$$\mathscr{O}_\varepsilon = \{z \in \mathscr{N}_\varepsilon : \mathscr{L}_\varepsilon(z) \le A_{P_0Q_0} + \kappa(\varepsilon)\}. \tag{4.4}$$

In fact, for any $y \in \mathscr{V}$, we deduce from Lemma 4.1 that $\kappa(\varepsilon) = |\mathscr{L}_\varepsilon(\Psi_\varepsilon(y)) - A_{P_0Q_0}| \to 0$ as $\varepsilon \to 0^+$. That is, $\Psi_\varepsilon(y) \in \mathscr{O}_\varepsilon$ and $\mathscr{O}_\varepsilon \ne \emptyset$ for $\varepsilon > 0$. The following lemma provides a relation between the image of $\mathscr{O}_\varepsilon$ under $\mathcal{K}_\varepsilon$ and the neighborhood $\mathscr{V}_\delta$ of $\mathscr{V}$.

Lemma 4.3. *Suppose that the assumptions of Theorem 1.3 are satisfied. Then for* $\delta > 0$ *satisfying* $\mathscr{V}_\delta \subset B_\varrho(0)$, *and* $\mathcal{K}_\varepsilon, \mathscr{O}_\varepsilon$ *defined as in (4.3) and (4.4) respectively, we have that* $\lim_{\varepsilon\to 0} \sup_{z\in\mathscr{O}_\varepsilon} dist(\mathcal{K}_\varepsilon(z), \mathscr{V}_\delta) = 0$.

Proof. Let $\{\varepsilon_n\}$ be a positive sequence such that $\varepsilon_n \to 0$. By definition, there exists $\{z_n\} \subset \mathscr{O}_{\varepsilon_n}$ such that $dist(\mathcal{K}_{\varepsilon_n}(z_n), \mathscr{V}_\delta) = \sup_{z\in\mathscr{O}_{\varepsilon_n}} dist(\mathcal{K}_{\varepsilon_n}(z), \mathscr{V}_\delta) + o(1)$ as $n \to \infty$. Thus it is sufficient to find a sequence $\{\tilde{y}_n\} \subset \mathscr{V}_\delta$ satisfying $|\mathcal{K}_{\varepsilon_n}(z_n) - \tilde{y}_n| = o(1)$ as $n \to \infty$. From $\mathscr{L}_{P_0Q_0}(tz_n) \le \mathscr{L}_\varepsilon(tz_n)$ for $t \ge 0$ and $\{z_n\} \subset \mathscr{O}_{\varepsilon_n} \subset \mathscr{N}_{\varepsilon_n}$, we obtain that $A_{P_0Q_0} \le A_{\varepsilon_n} \le \mathscr{L}_{\varepsilon_n}(z_n) \le A_{P_0Q_0} + \kappa(\varepsilon_n)$. This leads to $\mathcal{L}_{\varepsilon_n}(z_n) \to A_{P_0Q_0}$ as $n \to \infty$. By Lemma 4.2 one sees that there exists a sequence $\{y_n\} \subset \mathbb{R}^N$ such that $\tilde{y}_n = \varepsilon_n y_n \in \mathscr{V}_\delta$ for n sufficiently large. Hence

$$\mathcal{K}_{\varepsilon_n}(z_n) = \tilde{y}_n + \frac{\int_{\mathbb{R}^N} (\gamma(\varepsilon_n\xi + \tilde{y}_n) - \tilde{y}_n) u_n^2(\xi + \tilde{y}_n)}{2\int_{\mathbb{R}^N} u_n^2(\xi + \tilde{y}_n)} + \frac{\int_{\mathbb{R}^N} (\gamma(\varepsilon_n\xi + \tilde{y}_n) - \tilde{y}_n) v_n^2(\xi + \tilde{y}_n)}{2\int_{\mathbb{R}^N} v_n^2(\xi + \tilde{y}_n)}.$$

Since $\varepsilon_n\xi + \tilde{y}_n \to y \in \mathscr{V}$ uniformly for ξ in any compact subset of $\mathbb{R}^N$, we have that $\mathcal{K}_{\varepsilon_n}(z_n) = \tilde{y}_n + o(1)$ as $n \to \infty$, which implies the desired convergence. □

We now prove the existence of multiple positive solutions.

Lemma 4.4. *Suppose that the assumptions of Theorem 1.3 are satisfied. Then $(\mathscr{P}_\varepsilon)$ has at least $cat_{\mathscr{V}_\delta}(\mathscr{V})$ distinct nontrivial solutions for $\varepsilon > 0$ small.*

Proof. For a fixed small $\delta > 0$, by using Lemmas 4.1 and 4.3, there exists $\varepsilon_\delta > 0$ small enough such that, the diagram

$$\mathscr{V} \xrightarrow{\Psi_\varepsilon} \mathscr{O}_\varepsilon \xrightarrow{\mathcal{K}_\varepsilon} \mathscr{V}_\delta \tag{4.5}$$

is well defined for $\varepsilon \in (0, \varepsilon_\delta)$. It is known that

$$\lim_{\varepsilon \to 0} \mathcal{K}_\varepsilon(\Psi_\varepsilon(y)) = y \quad \text{uniformly for } y \in \mathscr{V}. \tag{4.6}$$

For $\varepsilon > 0$ small enough, we denote $\mathcal{K}_\varepsilon(\Psi_\varepsilon(y)) = y + \zeta(y)$ for $y \in \mathscr{V}$, where $|\zeta(y)| < \delta/2$ uniformly for $y \in \mathscr{V}$. Define $\eta(t, y) = y + (1-t)\zeta(y)$. Then $\eta : [0, 1] \times \mathscr{V} \to \mathscr{V}_\delta$ is continuous. Obviously, $\eta(0, y) = \mathcal{K}_\varepsilon(\Psi_\varepsilon(y))$, $\eta(1, y) = y$ for all $y \in \mathscr{V}$. Thus, we obtain that the composite mapping $\mathcal{K}_\varepsilon \circ \Psi_\varepsilon$ is homotopic to the inclusion mapping $id : \mathscr{V} \to \mathscr{V}_\delta$. So it follows from Lemma 2.2 of [38] that

$$cat_{\mathscr{O}_\varepsilon}(\mathscr{O}_\varepsilon) \geq cat_{\mathscr{V}_\delta}(\mathscr{V}). \tag{4.7}$$

Next, let us choose a function $\kappa(\varepsilon) > 0$ such that $\kappa(\varepsilon) \to 0$ as $\varepsilon \to 0$, and $A_{P_0Q_0} + \kappa(\varepsilon) < A_{P_\infty Q_\infty}$ for $\varepsilon > 0$ sufficiently small. We deduce from Lemma 3.5 that $\mathcal{L}_\varepsilon$ satisfies the Palais–Smale condition on $\mathscr{O}_\varepsilon$ for small $\varepsilon > 0$. Hence, by the Ljusternik–Schnirelmann theory of critical points (see Theorem 2.1 of [38] or [43]), it follows that $\mathcal{L}_\varepsilon$ has at least $cat_{\mathscr{O}_\varepsilon}(\mathscr{O}_\varepsilon)$ distinct critical points in $\mathscr{O}_\varepsilon$. Furthermore, since $\beta > \hat{\beta}_1$, and all the critical values are less than $A_{P_0Q_0} + \kappa(\varepsilon)$, it follows from the proof of Lemma 3.7 that these critical points are nontrivial for $\varepsilon > 0$ sufficiently small. □

Next to prove the properties of positive solutions obtained in Lemma 4.4, we state the following lemma which can be proved with similar arguments as in the proof of Lemma 3.13.

Lemma 4.5. *Suppose that the assumptions of Theorem 1.3 are satisfied. If for $n \in \mathbb{N}$, $z_n = (u_n, v_n)$ is a solution of $(\mathscr{P}_\varepsilon)$ satisfying $\mathscr{L}'_{\varepsilon_n}(z_n) = 0$, $\mathscr{L}_{\varepsilon_n}(z_n) \to A_{P_0 Q_0}$, and there exist $r, \delta > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^N$ (as in Lemma 4.2) such that $\liminf_{n\to\infty} \int_{B_r(y_n)} |z_n|^2 \geq \delta > 0$, $w_n(x) = (\tilde{u}_n, \tilde{v}_n) = z_n(x + y_n) \to w = (\tilde{u}, \tilde{v})$ in E as $n \to \infty$ with $\tilde{u}, \tilde{v} \neq 0$, then we have that $\tilde{u}_n, \tilde{v}_n \in L^\infty(\mathbb{R}^N)$ and $\|\tilde{u}_n\|_{L^\infty(\mathbb{R}^N)} + \|\tilde{v}_n\|_{L^\infty(\mathbb{R}^N)} \leq C$ for $n \in \mathbb{N}$, and $\lim_{|x|\to\infty} \tilde{u}_n(x) = \lim_{|x|\to\infty} \tilde{v}_n(x) = 0$ uniformly for $n \in \mathbb{N}$ and $|\tilde{u}_n(x)| + |\tilde{v}_n(x)| \leq ce^{-c|x-y_n|}$ for $n \in \mathbb{N}$ and $x \in \mathbb{R}^N$.*

Now we prove the following lemma which shows the concentration phenomenon and exponential decay for the positive solutions of $(\mathscr{P}_\varepsilon)$.

Lemma 4.6. *Suppose that the assumptions of Theorem 1.3 hold. If $z_\varepsilon = (u_\varepsilon, v_\varepsilon)$ is a solution of $(\mathscr{P}_\varepsilon)$ and x_ε is a maximum point of $u_\varepsilon + v_\varepsilon$, then we have that $u_\varepsilon, v_\varepsilon \in C^{1,\sigma}_{loc}(\mathbb{R}^N)$ with $\sigma \in (0, 1)$, and*

$$\lim_{\varepsilon\to 0} P(\varepsilon x_\varepsilon) = P_0, \quad \lim_{\varepsilon\to 0} Q(\varepsilon x_\varepsilon) = Q_0, \tag{4.8}$$

$$\lim_{|x|\to\infty} u_\varepsilon(x) = \lim_{|x|\to\infty} v_\varepsilon(x) = 0, \quad \lim_{|x|\to\infty} |\nabla u_\varepsilon(x)| = \lim_{|x|\to\infty} |\nabla v_\varepsilon(x)| = 0. \tag{4.9}$$

Furthermore there exist constants $C, c > 0$ (independent of ε) such that $|u_\varepsilon(x)| + |v_\varepsilon(x)| \leq Ce^{-c|x-x_\varepsilon|}$ for all $x \in \mathbb{R}^N$.

Proof. Let $\{\varepsilon_n\}$ be a positive sequence converging to 0, and let z_n be a nontrivial solution of $(\mathscr{P}_{\varepsilon_n})$. Let $\{y_n\}$ be the sequence in $\mathbb{R}^N$ given by Lemma 4.2 and let $w_n(x) = z_n(x + y_n) = (\tilde{u}_n(x), \tilde{v}_n(x))$. Then, up to a subsequence, it follows from Lemmas 4.2 and 4.5 that $w_n \to w = (\tilde{u}, \tilde{v})$, $\tilde{u}, \tilde{v} \not\equiv 0$, and $\varepsilon_n y_n \to y \in \mathscr{V}$. Then $u_\varepsilon, v_\varepsilon \in C^{1,\sigma}_{loc}(\mathbb{R}^N)$ with $\sigma \in (0, 1)$ follows from Lemma 3.16. As in [50,41], we shall prove that there exists a $\delta > 0$ such that $\|w_n\|_{L^\infty(\mathbb{R}^N)} = \||\tilde{u}_n| + |\tilde{v}_n|\|_{L^\infty(\mathbb{R}^N)} \geq \delta > 0$. Argue by contradiction, if $\|w_n\|_{L^\infty(\mathbb{R}^N)} \to 0$ as $n \to \infty$, one infers from z_n is a solution of $(\mathscr{P}_\varepsilon)$ that

$$\begin{aligned}
&\int_{\mathbb{R}^N} (|\nabla \tilde{u}_n|^2 + P_{\varepsilon_n}(y_n + \varepsilon_n x)\tilde{u}_n^2) + \int_{\mathbb{R}^N} (|\nabla \tilde{v}_n|^2 + Q_{\varepsilon_n}(y_n + \varepsilon_n x)\tilde{v}_n^2) \\
&\leq \mu \|\tilde{u}_n\|^2_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N} \tilde{u}_n^2 + \nu \|\tilde{v}_n\|^2_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N} \tilde{v}_n^2 + 2\beta \|\tilde{v}_n\|^2_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N} \tilde{u}_n^2 \\
&\to 0, \quad \text{as} \quad n \to \infty.
\end{aligned}$$

This implies that $\|\tilde{u}_n\|^2 + \|\tilde{v}_n\|^2 \to 0$ as $n \to \infty$. However, $\tilde{u}_n \to \tilde{u} \neq 0$ and $\tilde{v}_n \to \tilde{v} \neq 0$ as $n \to \infty$, which is a contradiction. Thus there exists a $\delta > 0$ such that $\|w_n\|_{L^\infty(\mathbb{R}^N)} \geq \delta > 0$. Let $\tilde{x}_n$ be the global maximum point of $|\tilde{u}_n| + |\tilde{v}_n|$. Then we infer from Lemma 4.5 and the claim above, that $\{\tilde{x}_n\} \subset B_R(0)$ for some $R > 0$. Thus, the global maximum of $|u_n| + |v_n|$ given by $x_n = y_n + \tilde{x}_n$ which gives $\varepsilon_n x_n = \varepsilon_n y_n + \varepsilon_n \tilde{x}_n$. Since $\{\tilde{x}_n\}$ is bounded, it follows that $\varepsilon_n x_n \to y \in \mathscr{V}$.

So we obtain (4.8). Finally, we infer from the above arguments, Lemma 4.4 and the boundedness of $\{\tilde{x}_n\}$ that $|u_n(x)| + |v_n(x)| \le ce^{-c|x-x_n+\tilde{x}_n|} \le ce^{-c|x-x_n|}$, and the estimates in (4.9). □

Proof of Theorem 1.3. The results for $(\mathscr{A}_\varepsilon)$ can be obtained from the ones for $(\mathscr{P}_\varepsilon)$ via change of variable $x \mapsto x/\varepsilon$. Therefore the results in Theorem 1.3 follow from Lemma 4.5 and Lemma 4.6. □

5. Nonexistence of positive solution

In this section, we prove the nonexistence results in Theorem 1.4. Part (i) is quite straightforward, and it has been used in previous work as well (see for example [10]). Assume that $(\mathscr{P}_\varepsilon)$ has a nontrivial solution (u, v). Multiplying the equation of u in $(\mathscr{P}_\varepsilon)$ by v, the equation of v by u, and integrating over $\mathbb{R}^N$, we obtain

$$\int_{\mathbb{R}^N} uv[(P(\varepsilon x) - Q(\varepsilon x)) + (\beta - \mu)u^2 + (\nu - \beta)v^2] = 0. \tag{5.1}$$

Then the conclusion (i) of Theorem 1.4 follows from (5.1). So in the rest of this section we prove the conclusion (ii) of Theorem 1.4 under the assumption **(PQ2)**. To do this, we assume that $A_{P_\infty Q_\infty}$ is attained by a nontrivial positive ground state solution. According to Theorem 1.1, we know that there exist $0 < \hat{\beta}_2 < \hat{\beta}_3$ such that for $\beta \in (0, \hat{\beta}_2) \cup (\hat{\beta}_3, \infty)$, the limiting equation (1.3) with $(\lambda_1, \lambda_2) = (P_\infty, Q_\infty)$ has a positive ground state solution. We prove the following result.

Proposition 5.1. *Assume that* **(PQ0)** *and* **(PQ2)** *are satisfied, and* (1.3) *with* $(\lambda_1, \lambda_2) = (P_\infty, Q_\infty)$ *has a nontrivial positive ground state solution for* $\beta \in (0, \hat{\beta}_2) \cup (\hat{\beta}_3, \infty)$*, then* $(\mathscr{P}_\varepsilon)$ *has no positive ground state solution for* $\beta \in (0, \hat{\beta}_2) \cup (\hat{\beta}_3, \infty)$.

Proof. We first claim that $A_\varepsilon = A_{P_\infty Q_\infty}$ for any $\varepsilon > 0$. In fact, since $P(x) \ge P_\infty$ and $Q(x) \ge Q_\infty$ for all $x \in \mathbb{R}^N$, then $A_\varepsilon \ge A_{P_\infty Q_\infty}$ from the proof of Lemma 2.3. Next we show that $A_{P_\infty Q_\infty} \ge A_\varepsilon$ for any fixed $\varepsilon > 0$. Since $\beta \in (0, \hat{\beta}_2) \cup (\hat{\beta}_3, \infty)$, we know that (1.3) with $(\lambda_1, \lambda_2) = (P_\infty, Q_\infty)$ has a nontrivial positive ground state solution (u^∞, v^∞). Moreover, (u^∞, v^∞) is the unique global maximum of $\mathscr{L}_{P_\infty Q_\infty}(tu^\infty, tv^\infty)$. Set $w_n = (u^\infty(\cdot - y_n), v^\infty(\cdot - y_n))$, where $\{y_n \in \mathbb{R}^N\}$ is a sequence satisfying $|y_n| \to \infty$ as $n \to \infty$. As in Lemma 2.2, it follows that there exists $t_n > 0$ such that $t_n w_n \in \mathscr{N}_\varepsilon$ is the unique global maximum of $\mathscr{L}_\varepsilon(tw_n)$ for each n. Then we can calculate that

$$\begin{aligned} &A_\varepsilon \le \mathscr{L}_\varepsilon(t_n w_n) \\ &= \mathscr{L}_{P_\infty Q_\infty}(t_n u^\infty, t_n v^\infty) + \frac{t_n^2}{2} \int_{\mathbb{R}^N} [(P(\varepsilon x + \varepsilon y_n) - P_\infty)(u^\infty)^2 + (Q(\varepsilon x + \varepsilon y_n) - Q_\infty)(v^\infty)^2] \\ &\le A_{P_\infty Q_\infty} + \frac{t_n^2}{2} \int_{\mathbb{R}^N} [(P(\varepsilon x + \varepsilon y_n) - P_\infty)(u^\infty)^2 + (Q(\varepsilon x + \varepsilon y_n) - Q_\infty)(v^\infty)^2]. \end{aligned} \tag{5.2}$$

It is clear that for any $\epsilon > 0$, there exists $R > 0$ such that

$$\int_{|x|\geq R} (P(\varepsilon x+\varepsilon y_n)-P_\infty)(u^\infty)^2 \leq c\epsilon. \tag{5.3}$$

On the other hand, from Lebesgue's dominated convergence theorem we have

$$\lim_{n\to\infty}\int_{|x|<R} (P(\varepsilon x+\varepsilon y_n)-P_\infty)(u^\infty)^2 = \int_{|x|<R} (\lim_{n\to\infty} P(\varepsilon x+\varepsilon y_n)-P_\infty)(u^\infty)^2 = 0. \tag{5.4}$$

Combining (5.3) and (5.4), we have

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N} (P(\varepsilon x+\varepsilon y_n)-P_\infty)(u^\infty)^2 = 0. \tag{5.5}$$

A similar estimate holds for $\int_{\mathbb{R}^N}(Q(\varepsilon x+\varepsilon y_n)-Q_\infty)(v^\infty)^2$. So it follows from (5.2) that $A_{P^\infty Q^\infty} = A_\varepsilon$ for any fixed $\varepsilon > 0$.

We complete the proof by using a contradiction argument. Assume that for some $\varepsilon_0 > 0$ that there exists a positive $\hat{z}$ such that $\hat{z} = (\hat{u}, \hat{v}) \in \mathcal{N}_{\varepsilon_0}$ and $A_{\varepsilon_0} = \mathcal{L}_{\varepsilon_0}(\hat{z})$. We know that $\hat{z}$ is the unique global maximum of $\mathcal{L}_{\varepsilon_0}(t\hat{z})$. Hence

$$A_{P_\infty Q_\infty} = A_{\varepsilon_0} \leq \mathcal{L}_{P_\infty Q_\infty}(t^\infty \hat{z}) = \max_{t>0} \mathcal{L}_{P_\infty Q_\infty}(t\hat{z}). \tag{5.6}$$

On the other hand, by using (**PQ2**), we have $\mathcal{L}_{P_\infty Q_\infty}(z) \leq \mathcal{L}_{\varepsilon_0}(z)$ for any $z \in E$. Thus combing with (5.6), we have

$$A_{P_\infty Q_\infty} \leq \mathcal{L}_{P_\infty Q_\infty}(t^\infty \hat{z}) \leq \mathcal{L}_{\varepsilon_0}(t_\infty \hat{z}) \leq \mathcal{L}_{\varepsilon_0}(\hat{z}) = A_{\varepsilon_0} = A_{P_\infty Q_\infty}. \tag{5.7}$$

This implies $A_{P_\infty Q_\infty} = \mathcal{L}_{P_\infty Q_\infty}(t^\infty \hat{z}) = \mathcal{L}_{\varepsilon_0}(t^\infty \hat{z})$. Moreover, $z^\infty = t^\infty \hat{z} = (u^\infty, v^\infty)$ satisfies (1.3) with $\lambda_1 = P_\infty$ and $\lambda_2 = Q_\infty$. So one has

$$\mathcal{L}_{P_\infty Q_\infty}(z^\infty) = \mathcal{L}_{\varepsilon_0}(z^\infty) + \frac{1}{2}\int_{\mathbb{R}^N}\Big[(P_\infty - P(\varepsilon_0 x))(u^\infty)^2 + (Q_\infty - Q(\varepsilon_0 x))(v^\infty)^2\Big]. \tag{5.8}$$

On the other hand we deduce from (**PQ2**) that

$$\frac{1}{2}\int_{\mathbb{R}^N}\Big[(P_\infty - P(\varepsilon_0 x))(u^\infty)^2 + (Q_\infty - Q(\varepsilon_0 x))(v^\infty)^2\Big] < 0. \tag{5.9}$$

Equations (5.8) and (5.9) together imply that $\mathcal{L}_{P_\infty Q_\infty}(z^\infty) < \mathcal{L}_{\varepsilon_0}(z^\infty)$. This is a contradiction. □

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