ON THE EXISTENCE AND UNIQUENESS OF A LIMIT CYCLE FOR A LIÉNARD SYSTEM WITH A DISCONTINUITY LINE

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Abstract. In this paper, we investigate the existence and uniqueness of crossing limit cycle for a planar nonlinear Liénard system which is discontinuous along a straight line (called a discontinuity line). By using the Poincaré mapping method and some analysis techniques, a criterion for the existence, uniqueness and stability of a crossing limit cycle in the discontinuous differential system is established. An application to Schnakenberg model of an autocatalytic chemical reaction is given to illustrate the effectiveness of our result. We also consider a class of discontinuous piecewise linear differential systems and give a necessary condition of the existence of crossing limit cycle, which can be used to prove the non-existence of crossing limit cycle.

1. **Introduction.** A great variety of mathematical models with practical applications stemming from engineering devices and electrical circuits are discontinuous in nature. These modelings can be described by differential equations with discontinuous right-hand sides (called discontinuous differential systems), which switch between different modes. Compared with continuous differential systems, the discontinuous differential systems sometimes provide more realistic descriptions to the real world processes. Hence the discontinuous differential systems deserve considerable attention from both application and theoretical point of view. For more details on the theory, we refer to the books [5, 23] and references therein.

Qualitative theory of planar systems of ordinary differential equations, and analysis of existence and uniqueness of limit cycle are important problems which have attracted great interests of many researchers. For the continuous or smooth differential systems, there have been many achievements, see for example [8, 12, 6, 21, 20, 27, 29, 17, 1] and references therein. In recent years, much progress has been made in studying relevant problems for the discontinuous differential systems, see for example [2, 3, 4, 13, 14, 15, 5, 7, 9, 10, 19, 22] and references therein. But most

of these existing papers focus on the existence, uniqueness or multiplicity of limit cycle for the piecewise linear discontinuous differential systems [2, 7, 9, 10].

Liénard equation $x'' + f(x)x' + g(x) = 0$, or equivalently

$$x' = y - F(x), \qquad y' = -g(x), \tag{1}$$

where $F(x) = \int_0^x f(s)ds$, originating from physics is an important class of nonlinear systems of differential equations. The Liénard equation is a suitable model for many practical problems in engineering, and it has been the focus of many recent studies. The existence of limit cycle of continuous Liénard systems can be proved by various methods based on the well known Poincaré-Bendixson theorem by constructing a trapping zone where the limit cycle is located. For example, the following result was proved in [29, Page 161, Theorem 1.2].

Theorem 1.1. *Assume that the system* (1) *satisfies the following conditions:*

(i) *when $|x| < A$ with A large enough, $F(x)$ and $g(x)$ are Lipschitz continuous;*

(ii) $xg(x) > 0$ *for* $x \neq 0$, *and* $G(x) = \int_0^x g(s)ds$ *with* $G(\pm\infty) = +\infty$;

(iii) $F(x) < 0$ *for* $0 < x < x_1$, *and* $F(x) > 0$ *for* $x_2 < x < 0$;

(iv) *there exist $M > \max(x_1, |x_2|)$ and $k_2 < k_1$ such that $F(x) \geq k_1$ for $x > M$ and $F(x) \leq k_2$ for $x < -M$.*

Then there exists at least a closed orbit for system (1).

We note that in Theorem 1.1, the functions F and g are continuous, *i.e.*, the vector field of system (1) is continuous. However when the vector field is discontinuous, the classical Poincaré-Bendixson theorem cannot be directly applied. In this paper we investigate the existence and uniqueness of limit cycle for a class of Liénard systems with discontinuity occurring along a straight line by using Poincaré mapping method, thus our result generalizes Theorem 1.1 to the setting of discontinuous differential system with a line discontinuity.

For the nonlinear discontinuous planar Liénard systems, there have been several recent studies, for example [14, 15, 19, 22]. In [22], the uniqueness of crossing limit cycle for a nonlinear Liénard system with a discontinuity line was shown, provided that periodic orbit exists. In this paper, we investigate the existence, uniqueness and stability of crossing limit cycle for the same nonlinear Liénard system with a discontinuity line. This partially generalizes some existing results in [7, 21] (see Remarks 1 and 2). First we provide a new criterion for the existence and uniqueness of a crossing limit cycle in the discontinuous differential system with a discontinuity line. Second we consider a class of discontinuous piecewise linear differential systems and give a necessary condition of the existence of crossing limit cycle, which can be used to show the nonexistence of crossing limit cycle. Here a crossing limit cycle is defined as a limit cycle with trajectory intersecting with the discontinuity line. Such a limit cycle usually is a continuous orbit with discontinuous derivative on the discontinuity line.

Several examples are included to illustrate our main theoretical result. It is known that many planar ecological models and chemical reaction models can be transformed into the Liénard systems, and the uniqueness of limit cycle in the original system can be proved via the uniqueness of limit cycle for the Liénard systems [11, 16, 25, 26]. The uniqueness of limit cycle for the continuous Liénard systems has been proved by many authors including [26, 28], and our result here can be regarded as partial extension of these results to discontinuous Liénard systems. We

apply our result for the discontinuous Liénard systems to prove the uniqueness of crossing limit cycle of a discontinuous Schnakenberg type chemical reaction system, which is a prototypical autocatalytic chemical reaction model [18, 24]. The uniqueness of continuous Schnakenberg systems has been proved in [11]. A discontinuous Schnakenberg system naturally appears if the reaction is controlled by the concentration of reacting chemicals, and we show that the limit cycle is preserved under the discontinuity perturbation but it is non-smooth at the intersection points of the limit cycle with the discontinuity line (see Section 4 for the result and Matlab numerical simulation). Another example of piecewise linear discontinuous Liénard system is also given, and some non-existence of crossing limit cycle result is also proved.

The paper is organized as follows. In the next section, we present some preliminaries for the discontinuous planar nonlinear Liénard system. In Section 3, we first present some geometrical behaviors and qualitative properties for the discontinuous differential system, and then a new criterion concerning the existence, uniqueness and stability of crossing limit cycle is stated. In Section 4, examples including the discontinuous Schnakenberg model are given to illustrate the effectiveness of our theoretical result. In Section 5, we prove a non-existence of crossing limit cycle result. Concluding remarks are given in Section 6.

2. **Preliminaries.** Consider the following nonlinear Liénard system with a discontinuity line

$$
\begin{cases}
\dfrac{dx}{dt} = F(x) - y, \\
\dfrac{dy}{dt} = g(x),
\end{cases}
\tag{2}
$$

where the function g is defined by

$$
g(x) = \begin{cases}
g_-(x), & x < 0, \\
g_+(x), & x > 0.
\end{cases}
\tag{3}
$$

Let Σ_0 be the discontinuity line of $g(x)$:

$$
\Sigma_0 = \{(x, y) \in \mathbb{R}^2 : x = 0\},
$$

and let $\Sigma_\pm$ be two subregions of $\mathbb{R}^2$ separated by Σ_0:

$$
\Sigma_+ = \{(x, y) \in \mathbb{R}^2 : x > 0\}, \quad \Sigma_- = \{(x, y) \in \mathbb{R}^2 : x < 0\}.
$$

Then $\mathbb{R}^2 = \Sigma_+ \cup \Sigma_0 \cup \Sigma_-$ and the normal vector to Σ_0 is $\mathbf{n}^T = (1, 0)$.

For system (2) with (3), the corresponding vector field is denoted by

$$
V(x, y) = \begin{cases}
V_-(x, y), & (x, y) \in \Sigma_- \cup \Sigma_0, \\
V_+(x, y), & (x, y) \in \Sigma_+ \cup \Sigma_0,
\end{cases}
\tag{4}
$$

where $V_\pm(x, y) = (F(x) - y, \ g_\pm(x))^T$.

Inspired by the papers [3] and [7], we present the following several definitions.

Definition 2.1. For any $(0, y) \in \Sigma_0$, if

$$
[\mathbf{n}^T V_-(0, y)] \cdot [\mathbf{n}^T V_+(0, y)] \leq 0,
$$

then we say the point $(0, y)$ is a *sliding point*. A set of sliding points is called to be a *sliding set*.

Definition 2.2. A periodic orbit Γ is called to be a *crossing limit cycle*, if it is an isolated periodic orbit which does not share points with the sliding set, *i.e.*, for any $(0, y) \in \Sigma_0 \cap \Gamma$, one has that

$$[\mathbf{n}^T V_-(0, y)] \cdot [\mathbf{n}^T V_+(0, y)] > 0.$$

Definition 2.3. A periodic orbit Γ is called to be a *sliding limit cycle*, if it is an isolated periodic orbit which has some points in the sliding set.

From now on we impose the following hypotheses for system (2):

$(H0)$ $g_+ \in C^1([0, +\infty), \mathbb{R})$ and $g_- \in C^1((-\infty, 0], \mathbb{R})$ with $g_-(0) \leq 0 \leq g_+(0)$;
$(H1)$ $xg(x) > 0$ for $x \neq 0$;
$(H2)$ $F \in C(\mathbb{R}, \mathbb{R})$ and $F \in C^1(\mathbb{R} \setminus \{0\}, \mathbb{R})$, and there exist $x_0 > 0$ and $x_1 < 0$ such that $F(0) = F(x_0) = F(x_1) = 0$, $F(x) > 0$ for $x \in (0, x_0)$, $F(x) < 0$ for $x \in (x_1, 0)$ and $F'(x) < 0$ for $x \in (-\infty, x_1) \cup (x_0, +\infty)$.

It follows from $(H0)$ and $(H2)$ that the first and second components of the vector field (4) are continuous and discontinuous on the plane, respectively. Due to $(H1)$ the singular points of system (2) must be on the y-axis, and since $x' = -y$ when $x = 0$ it follows that the unique possible singular point is the origin $O(0, 0)$. If the system (2) has a periodic orbit L in $\mathbb{R}^2$, by Poincaré-Bedixson theorem for phase portraits in $\mathbb{R}^2$ then the interior of the bounded region limited by L must contain a singular point and such a point in (2) must be the origin. So the periodic orbit L surrounds the unique singular point O, and all orbits other than the origin cross Σ_0 transversally in a counterclockwise fashion. Moreover, by the Filippov theory (see [5, 7]), the origin $O(0, 0)$ is the unique sliding point on Σ_0. Therefore there exists no sliding limit cycle for system (2), and consequently we focus our attention on the crossing limit cycle in the following.

3. **Existence and uniqueness of a crossing limit cycle.** In this section, we use the left and right Poincaré mappings P_L, P_R to investigate the existence, uniqueness and stability of crossing limit cycle of system (2).

From $(H0)$, $(H2)$ and the Filippov theory, then for any initial point $P(x_0, y_0) \in \mathbb{R}^2 \setminus \{O\}$, there exists a unique solution $\widetilde{\varphi}(P, t)$ satisfying $\widetilde{\varphi}(P, 0) = P$ of system (2) for $t \in (-T_1, T_2)$, where $(-T_1, T_2)$ is the maximum existence interval of the solution. The corresponding orbit is denoted by $\widetilde{L}_P = \{\widetilde{\varphi}(P, t) : -T_1 < t < T_2\}$. Similarly, we also denote by $\widetilde{L}_P^+ = \{\widetilde{\varphi}(P, t) : 0 \leq t < T_2\}$ the positive orbit, and $\widetilde{L}_P^- = \{\widetilde{\varphi}(P, t) : -T_1 < t \leq 0\}$ the negative orbit. For convenience, we also define the following regions:

$\Sigma_0^+ = \{(x, y) : x = 0, y > 0\}, \qquad \Sigma_0^- = \{(x, y) : x = 0, y < 0\},$

$\Sigma_F^0 = \{(x, y) \in \mathbb{R}^2 : F(x) = y\},$

$\Sigma_F^+ = \{(x, y) \in \mathbb{R}^2 : x > 0, F(x) > y\}, \quad \Sigma_F^- = \{(x, y) \in \mathbb{R}^2 : x > 0, F(x) < y\},$

$\widetilde{\Sigma}_F^+ = \{(x, y) \in \mathbb{R}^2 : x < 0, F(x) > y\}, \quad \widetilde{\Sigma}_F^- = \{(x, y) \in \mathbb{R}^2 : x < 0, F(x) < y\}.$

We first prove the following lemma concerning the geometric properties of solutions to the discontinuous Liénard system (2). In particular we show that all solutions exist globally in time.

Lemma 3.1. *Suppose that $(H0)$-$(H2)$ hold for system (2), and let $\widetilde{\varphi}(P, t)$ be the unique solution of system (2) with $\widetilde{\varphi}(P, 0) = P$ for $t \in (-T_1, T_2)$, where $(-T_1, T_2)$ is the maximum existence interval of the solution.*

1. If $P = (x_P, F(x_P)) \in \Sigma_F^0$ and $x_P \neq 0$, then there exists $T_+ > 0$ (resp. $T_- < 0$) such that $\widetilde{L}_P^+$ (resp. $\widetilde{L}_P^-$) intersects with $\Sigma_0\backslash\{O\}$ the first time when $t = T_+$ (resp. $t = T_-$).

2. If $P = (0, y_P) \in \Sigma_0$ and $y_P \neq 0$, then there exists $T_+ > 0$ (resp. $T_- < 0$) such that $\widetilde{L}_P^+$ (resp. $\widetilde{L}_P^-$) intersects with $\Sigma_F^0\backslash\{O\}$ the first time when $t = T_+$ (resp. $t = T_-$).

3. For any $P \in \mathbb{R}^2\backslash\{O\}$, the solution $\widetilde{\varphi}(P, t)$ exists for $t \in (-\infty, +\infty)$ so $T_1 = T_2 = +\infty$.

Proof. 1. We only consider the positive orbit $\widetilde{L}_P^+$ starting from the point $P(x_P, F(x_P))$ with $x_P > 0$ as other cases can be proved similarly. From the phase portrait of system (2), the positive orbit $\widetilde{L}_P^+$ enters the region Σ_F^- for $t > 0$ small. From the direction of the vector field $V_+(x, y)$, either $\widetilde{L}_P^+$ remains in Σ_F^- for all $t \in (0, T_2)$, or it intersects with $\Sigma_0\backslash\{O\}$ the first time when $t = T_+$. If the former case occurs, then it follows from $(H1)$ that

$$\frac{dx}{dt} = F(x) - y < 0, \quad \frac{dy}{dt} = g_+(x) > 0,$$

in Σ_F^-. Hence $x(t)$ is strictly decreasing and $y(t)$ is strictly increasing as long as the orbit stays in Σ_F^-. Since there is no singular point in Σ_F^-, then $\widetilde{L}_P^+$ is unbounded in Σ_F^-. Hence there exists a vertical asymptotic line $x = a \geq 0$ such that the slope of the orbit $\widetilde{L}_P^+$ approaches to it as $t \to T_2^-$, which contradicts with

$$\frac{dy}{dx} = \frac{g_+(x)}{F(x) - y} \to 0 \quad \text{as} \quad x \to a^+, y \to +\infty.$$

So the positive orbit $\widetilde{L}_P^+$ must intersect with the positive y-axis at some $t = T_+$.

2. We only consider the positive orbit $\widetilde{L}_P^+$ starting from the point $P(0, y_P)$ with $y_P < 0$. From the phase portrait, the positive orbit $\widetilde{L}_P^+$ enters the region Σ_F^+ and it is counterclockwise. It follows from $(H1)$ that

$$\frac{dx}{dt} = F(x) - y > 0, \quad \frac{dy}{dt} = g_+(x) > 0,$$

in Σ_F^+, so $x(t)$ and $y(t)$ are strictly increasing as long as the orbit stays in Σ_F^+. Since the orbit cannot approach any singular point, the positive orbit $\widetilde{L}_P^+$ must intersect with the curve Σ_F^0.

3. From the parts 1 and 2, any orbit $\widetilde{L}_P$ starting from $P \in \mathbb{R}^2\backslash\{O\}$ is counterclockwisely rotating around the origin O infinitely many times as t increases. In particular, each solution exists for all time $t \in \mathbb{R}$ thus $T_1 = T_2 = +\infty$.   □

The results in Lemma 3.1 enable us to make the following definition similar to the one in [21].

Definition 3.2. For any $(0, y) \in \Sigma_0^+$, let $(0, -p_L(y)) \in \Sigma_0^-$ be the first intersection point of the orbit of system (2) starting from $(0, y)$ with Σ_0 in a counterclockwise fashion. Then the planar mapping $P_L : (0, y) \to (0, -p_L(y))$ is called to be the *left Poincaré mapping* of system (2). Similarly, for any $(0, -y) \in \Sigma_0^-$, let $(0, p_R(y)) \in \Sigma_0^+$ be the first intersection point of the orbit starting from $(0, -y)$ with Σ_0 in a counterclockwise fashion. Then the planar mapping $P_R : (0, -y) \to (0, p_R(y))$ is called to be the *right Poincaré mapping* of system (2).

By the continuous dependence of solutions of system (2) with respect to the initial values, the mappings P_L and P_R are continuous and consequently the component functions p_L and p_R are also continuous. Clearly the results in Lemma 3.1 show that an orbit starting from $(0, -y)$ with $y > 0$ enters the region Σ_+, and it is counterclockwise around the origin in the half plane. The slope of the orbit is positive in Σ_F^+, is vertical on Σ_F^0, and it becomes negative in Σ_F^-. Then it reaches $(0, p_R(y)) \in \Sigma_0^+$. Similarly continuing the orbit, it runs around in the left half plane Σ_- to reach $(0, -p_L(p_R(y))) \in \Sigma_0^-$. If $p_L(p_R(y)) = y$ for some $y > 0$, then the orbit corresponding to $(0, -y)$ is a periodic orbit of system (2).

The following lemma is a key step to establish the existence of a periodic orbit.

Lemma 3.3. *Suppose that* (H0)-(H2) *hold for system* (2). *Then functions* $q_z(y) = p_z(y) - y$, *where* $z \in \{L, R\}$, *satisfy*

$$q_z(y) > 0, \ \ y \in (0, \delta_z), \quad q_z(y) < 0, \ \ y \in (M_z, +\infty), \quad and \quad \lim_{y \to +\infty} q_z(y) = -\infty,$$
(5)

where $0 < \delta_z < M_z < +\infty$.

Proof. We only prove the statement for the function $q_R(y)$ with $y > 0$, as the proof for $q_L(y)$ is symmetrical. Consider a trajectory arc starting from the point $(0, -y)$ with $y > 0$. Then from the above discussion, the orbit starting from $(0, -y)$ enters Σ_+ to make a half-turn around the origin in Σ_+, and it eventually comes back to Σ_0^+ at some point $(0, p_R(y))$ with $p_R(y) > 0$. Let

$$\lambda(x, y) = \frac{1}{2} y^2 + G(x),$$
(6)

where $G(x) = \int_0^x g(s) ds$ satisfying $G(0) = 0$. Then along the orbit with $x(t) > 0$, one has that

$$\frac{d\lambda(x(t), y(t))}{dt} = y(t) g_+(x(t)) + g_+(x(t))[F(x(t)) - y(t)] = g_+(x(t)) F(x(t)). \quad (7)$$

Consider any trajectory arc $\widehat{A_1 C_1 B_1}$ which is completely contained in the strip $0 < x < x_0$ (see Figure 1), and the orbit starts from $A_1(0, -y_{A_1})$ with $y_{A_1} > 0$ and comes back to Σ_0^+ at point $B_1(0, y_{B_1})$, $y_{B_1} > 0$. Since $F(x) > 0$ for $0 < x < x_0$, it follows from (7) that along such an arc we have

$$\lambda(B_1) - \lambda(A_1) = \int_{\lambda(A_1)}^{\lambda(B_1)} d\lambda = \int_{\widehat{A_1 B_1}} F(x) dy = \int_{-y_{A_1}}^{y_{B_1}} F(x(y)) dy > 0,$$

which implies that $y_{B_1} - y_{A_1} > 0$ and it is equivalent to

$$p_R(y_{A_1}) - y_{A_1} > 0.$$
(8)

Thus $q_R(y) > 0$ for small $y > 0$.

Next we consider any two trajectory arcs $\widehat{A_2 C_2 D_2 B_2}$ and $\widehat{A_3 C_3 D_3 B_3}$ which are not completely contained in the strip $0 < x < x_0$, and these two orbits start from $A_2(0, -y_{A_2})$ and $A_3(0, -y_{A_3})$ with $y_{A_3} > y_{A_2} > 0$, cross the curve Σ_F^0 for $x > x_0$ and come back to Σ_0^+ at points $B_2(0, y_{B_2})$ and $B_3(0, y_{B_3})$ respectively (see Figure 1). By the properties of planar autonomous systems, we have $y_{B_3} > y_{B_2} > 0$. Here C_2 and D_2 are where $\widehat{A_2 C_2 D_2 B_2}$ intersects with the straight line $l_{x_0} = \{(x, y) \in \mathbb{R}^2 : x = x_0\}$, E_3 and F_3 are where $\widehat{A_3 C_3 D_3 B_3}$ intersects with l_{x_0}, C_2 and C_3, and D_2 and D_3 have the same y-coordinate.
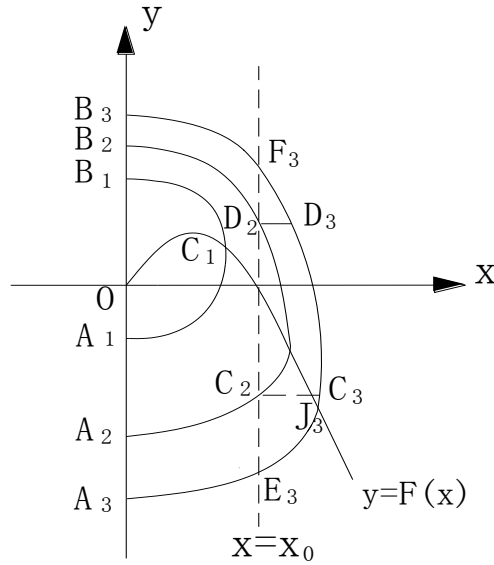
FIGURE 1. Trajectory arcs of system (2) in the region Σ_+

On the arcs $\widehat{A_3E_3}$ and $\widehat{A_2C_2}$, for the same value x, the positive value of $F(x) - y$ along $\widehat{A_3E_3}$ is greater than the one along $\widehat{A_2C_2}$, it follows that

$$
\begin{aligned}
0 < \lambda(E_3) - \lambda(A_3) = \int_{\lambda(A_3)}^{\lambda(E_3)} d\lambda &= \int_{\widehat{A_3E_3}} \frac{F(x)g_+(x)}{F(x) - y(x)} dx \\
&< \int_{\widehat{A_2C_2}} \frac{F(x)g_+(x)}{F(x) - y(x)} dx = \int_{\lambda(A_2)}^{\lambda(C_2)} d\lambda = \lambda(C_2) - \lambda(A_2).
\end{aligned}
\tag{9}
$$

Similarly

$$
0 < \lambda(B_3) - \lambda(F_3) < \lambda(B_2) - \lambda(D_2). \tag{10}
$$

On the other hand, along the arcs $\widehat{C_2D_2}$ and $\widehat{C_3D_3}$, since $F'(x) < 0$ for $x > x_0$, then it follows that for the same value y, the negative value $F(x)$ along $\widehat{C_3D_3}$ is smaller than the one along $\widehat{C_2D_2}$. From that one has that

$$
\begin{aligned}
\lambda(D_3) - \lambda(C_3) = \int_{\lambda(C_3)}^{\lambda(D_3)} d\lambda &= \int_{\widehat{C_3D_3}} F(x)dy \\
&< \int_{\widehat{C_2D_2}} F(x)dy = \int_{\lambda(C_2)}^{\lambda(D_2)} d\lambda = \lambda(D_2) - \lambda(C_2) < 0.
\end{aligned}
\tag{11}
$$

Moreover, due to $F(x) < 0$ for $x > x_0$, then

$$
\begin{aligned}
\lambda(C_3) - \lambda(E_3) = \int_{\lambda(E_3)}^{\lambda(C_3)} d\lambda = \int_{\widehat{E_3C_3}} F(x)dy = \int_{y_{E_3}}^{y_{C_3}} F(x(y))dy < 0, \\
\lambda(F_3) - \lambda(D_3) = \int_{\lambda(D_3)}^{\lambda(F_3)} d\lambda = \int_{\widehat{D_3F_3}} F(x)dy = \int_{y_{D_3}}^{y_{F_3}} F(x(y))dy < 0,
\end{aligned}
\tag{12}
$$

where $E_3(x_0, y_{E_3})$, $C_3(x_{C_3}, y_{C_3})$, $D_3(x_{D_3}, y_{D_3})$ and $F_3(x_0, y_{F_3})$.

Adding from (9) to (12), we obtain that

$$\lambda(B_3) - \lambda(A_3) < \lambda(B_2) - \lambda(A_2). \tag{13}$$

Denote by $\lambda(x, y) = H(y) + G(x)$ with $H(y) = y^2/2$, then $H(y) = H(-y)$ and $H'(y) > 0$ for $y > 0$. So (13) is equivalent to

$$H(p_R(y_{A_3})) - H(y_{A_3}) < H(p_R(y_{A_2})) - H(y_{A_2}). \tag{14}$$

Therefore for trajectory arcs starting from $(0, -y)$ $(y > 0)$ which cross the curve Σ_F^0 for $x > x_0$, then the function $y \mapsto H(p_R(y)) - H(y)$ is strictly decreasing.

Finally we verify that $q_R(y) = p_R(y) - y \to -\infty$ as $y \to +\infty$. If $p_R(y)$ is bounded then the conclusion is obvious. If $p_R(y)$ is unbounded $(p_R(y) \to +\infty$ as $y \to +\infty)$, then one has that

$$\lambda(E_3) - \lambda(A_3) = \int_{\lambda(A_3)}^{\lambda(E_3)} d\lambda = \int_{\widehat{A_3 E_3}} \frac{F(x)g_+(x)}{F(x) - y} dx \to 0,$$

as the value of y along the arc $\widehat{A_3 E_3}$ tends to $-\infty$ uniformly. Similarly, we have that $\lambda(B_3) - \lambda(F_3) \to 0$. On the other hand, from (11)-(12) we see that $\lambda(F_3) - \lambda(E_3) < 0$. Hence as the y-coordinate of E_3 $y_{E_3} \to -\infty$ and the y-coordinate of F_3 $y_{F_3} \to +\infty$ along the arc $\widehat{E_3 F_3}$, one has that

$$\lambda(F_3) - \lambda(E_3) = \int_{\lambda(E_3)}^{\lambda(F_3)} d\lambda = \int_{\widehat{E_3 F_3}} F(x) dy$$
$$= \int_{y_{E_3}}^{y_{F_3}} F(x(y)) dy < -\delta_0(y_{F_3} - y_{E_3}) \to -\infty,$$

where $\delta_0 > 0$ is a constant. Therefore, we obtain that $\lambda(B_3) - \lambda(A_3)$ tends to $-\infty$ as $y_{A_3} \to +\infty$. So the function $q_R(y) = p_R(y) - y$ tends to $-\infty$ as $y \to +\infty$. The proof of Lemma 3.3 is complete. $\qquad\square$

Now we are ready to state our main existence and uniqueness result for the crossing limit cycle in the discontinuous Liénard system (2).

Theorem 3.4. *Assume that* $(H0)$-$(H2)$ *hold, then there exists a unique stable crossing limit cycle surrounding the origin for system* (2).

Proof. The existence of a periodic orbit of system (2) is equivalent to the existence of two positive values y_L and y_R such that

$$p_R(y_R) = y_L, \quad p_L(y_L) = y_R. \tag{15}$$

By adding and subtracting the two equations in (15) we get

$$p_R(y_R) + y_R = p_L(y_L) + y_L, \quad p_R(y_R) - y_R = y_L - p_L(y_L), \tag{16}$$

which are sufficient and necessary conditions of the existence of a periodic orbit of system (2).

By the properties of planar autonomous systems, the functions $p_z(y)$ and $p_z(y)+y$ (where $z \in \{L, R\}$) are strictly increasing for $y > 0$. Hence we can define two functions

$$M_R(Y) = p_R(y) - y, \quad M_L(Y) = p_L(y) - y,$$

for $Y = p_z(y) + y > 0$ with $y > 0$ (where $z \in \{L, R\}$). Then (16) is transformed into the form

$$M_R(Y) + M_L(Y) = 0, \tag{17}$$

where Y is an equivalent variable as y. That is, the existence of two values $y_L > 0$ and $y_R > 0$ is transformed into the existence of a value $Y > 0$ such that (17) holds.

It follows from Lemma 3.3 that $p_z(y) - y > 0$ (where $z \in \{L, R\}$) for small $y > 0$ and $p_z(y) - y < 0$ when $y > 0$ is sufficiently large. Furthermore, the function $M_R(Y) + M_L(Y)$ is positive for some value $Y > 0$, but it is eventually negative tending to $-\infty$ as $Y \to +\infty$. Therefore, by the intermediate value theorem, there exists at least one value $Y > 0$ such that (17) holds. Consequently, there exist at least two values $y_L > 0$ and $y_R > 0$ such that (15) holds. Together with the Poincaré-Bendixson theorem of the planar phase portrait (*i.e.*, the interior of the bounded region enclosed by the periodic orbit must contain a singular point), we conclude that the system (2) possesses at least one periodic orbit surrounding the origin.

Let L be a periodic orbit of system (2). In the following, we show the uniqueness and stability of the periodic orbit L. Consider the function (6), when $x(t) \in (0, x_0) \cup (x_1, 0)$, it follows from $(H1)$-$(H2)$ that

$$\frac{d\lambda(x(t), y(t))}{dt} = y(t)g(x(t)) + g(x(t))[F(x(t)) - y(t)] = g(x(t))F(x(t)) > 0. \quad (18)$$

So the unique singular point O is unstable and all periodic orbits are not completely contained in the strip $x_1 < x < x_0$, they must encircle two points $(x_1, 0)$ and $(x_0, 0)$.

Suppose on the contrary that the system (2) has two periodic orbits $\widetilde{L}, L$ with $\widetilde{L}$ is in the interior of L, then the periodic orbit $\widetilde{L}$ encircles the points $(x_1, 0)$ and $(x_0, 0)$ as interior points. Let $(0, \widetilde{y}_L)$ and $(0, -\widetilde{y}_R)$ be the intersection points of $\widetilde{L}$ with Σ_0^+ and Σ_0^- respectively, and let $(0, y_L)$ and $(0, -y_R)$ be the intersection points of L with Σ_0^+ and Σ_0^- respectively. By the properties of planar autonomous systems, one has that $0 < \widetilde{y}_L < y_L$ and $0 < \widetilde{y}_R < y_R$.

It is easy to see that (15) is equivalent to

$$H(p_R(y_R)) = H(y_L), \quad H(p_L(y_L)) = H(y_R).$$

And from (16), one has that

$$\begin{aligned} H(p_R(y_R)) - H(y_R) &= H(y_L) - H(p_L(y_L)), \\ H(p_R(\widetilde{y}_R)) - H(\widetilde{y}_R) &= H(\widetilde{y}_L) - H(p_L(\widetilde{y}_L)). \end{aligned} \quad (19)$$

From (14) in Lemma 3.3, then the functions $H(p_z(y)) - H(y)$ (where $z \in \{L, R\}$) are strictly decreasing for all orbits crossing the curve Σ_F^0 for $x > x_0$ and $x < x_1$. So one has that

$$H(p_R(y_R)) - H(y_R) < H(p_R(\widetilde{y}_R)) - H(\widetilde{y}_R). \quad (20)$$

This together with (19) mean that $H(p_L(\widetilde{y}_L)) - H(\widetilde{y}_L) < H(p_L(y_L)) - H(y_L)$, which is a contradiction. Therefore, the periodic orbit L is the unique limit cycle.

Now we show the stability. It follows from (18) that the limit cycle L is stable from interior. Let $\overline{L}$ be any orbit of system (2) located in the exterior of L, then the orbit $\overline{L}$ must cross the curve Σ_F^0 for $x > x_0$ and $x < x_1$. Assume that the orbit $\overline{L}$ starts from the point $(0, -\overline{y}_R), \overline{y}_R > 0$ and the right Poincaré mapping P_R maps it to $(0, \overline{y}_L) \in \Sigma_0^+$. Then $y_L = p_R(y_R)$ and $\overline{y}_L = p_R(\overline{y}_R)$. Moreover, by the properties of planar autonomous systems one has that $0 < y_R < \overline{y}_R$ and $0 < y_L < \overline{y}_L$. We claim that

$$p_L(\overline{y}_L) < \overline{y}_R. \quad (21)$$

Since $\overline{y}_R > y_R$, $\overline{y}_L > y_L$ and $H'(y) > 0$ for $y > 0$, it follows that

$$
\begin{aligned}
H(p_R(\overline{y}_R)) - H(\overline{y}_R) \quad &< \quad H(p_R(y_R)) - H(y_R) \\
&= \quad H(y_L) - H(p_L(y_L)) < H(\overline{y}_L) - H(p_L(\overline{y}_L)),
\end{aligned}
$$

i.e., $H(\overline{y}_R) > H(p_L(\overline{y}_L))$ and then the claim (21) holds. Hence the limit cycle is also stable from the exterior.

In conclusion, the system (2) possesses a unique stable crossing limit cycle L surrounding the origin. The proof of Theorem 3.4 is complete. $\square$

In [22], the authors studied the uniqueness of crossing limit cycle for a discontinuous nonlinear Liénard system, provided that the periodic orbit exists (see Theorems 2-3 in [22]). In this paper we investigate the same discontinuous nonlinear Liénard system, but we give a new criterion concerning the existence, uniqueness and stability of crossing limit cycle under other conditions (see Theorem 3.4). Compared with the hypothesis $(H2)$ in [22], we note that the sufficient condition for the existence and uniqueness of crossing limit cycle in this paper is different, and these two conditions do not imply each other.

4. **Applications.** In this section, two examples illustrating our theoretical result are discussed.

Example 4.1. We first consider a celebrated nonlinear differential equation model of autocatalytic chemical reaction. Schnakenberg [24] proposed a continuous planar system as follows:

$$
\begin{cases}
\dfrac{du}{d\tau} = a - u + u^2 v, \\
\dfrac{dv}{d\tau} = b - u^2 v,
\end{cases}
\tag{22}
$$

where $u(\tau)$ and $v(\tau)$ are concentrations of two chemical products, a and b are the concentrations of two chemical source. It is assumed that two chemical source are in abundance, so a and b are two positive constants. It is known that $u = a+b, v = \frac{b}{(a+b)^2}$ is the unique equilibrium point of system (22) and the system has at most one limit cycle (see [11]). Indeed when $b - a > (b+a)^3$, (22) possesses a unique limit cycle (see Figure 2 left panel) and otherwise the unique equilibrium point is globally asymptotically stable [18].

Corresponding to the continuous system (22), we consider a Schnakenberg system with discontinuous vector field of the form

$$
\begin{cases}
\begin{cases}
\dfrac{du}{d\tau} = a - u + u^2 v, \\
\dfrac{dv}{d\tau} = b - u^2 v,
\end{cases} & \text{for} \quad u \geq a+b, \\[6mm]
\begin{cases}
\dfrac{du}{d\tau} = a - 2u + (v + \dfrac{1}{a+b})u^2, \\
\dfrac{dv}{d\tau} = b + 1 + u - (v + \dfrac{1}{a+b})u^2,
\end{cases} & \text{for} \quad 0 < u < a+b,
\end{cases}
\tag{23}
$$

where a, b satisfy

$$
4a(a+b)^3 < \min\{4ab - 4a^2, b^2, a^2 + 4b^2 - 4ab, 8ab\}.
\tag{24}
$$

It is easy to verify that the u-component of the vector field of system (23) is continuous at $u = a + b$, but the v-component is discontinuous. So the discontinuity line is of the form $\widetilde{\Sigma}_0 = \{(u, v) : u = a + b, -\infty < v < +\infty\}$.
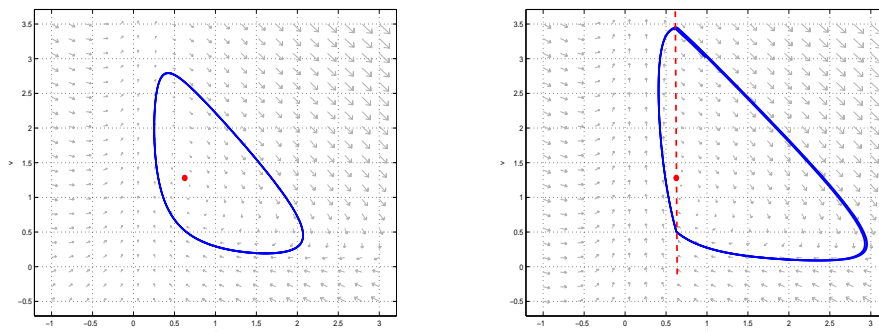
FIGURE 2. Limit cycles of continuous and discontinuous Schnaken-berg systems (22) and (23). Here $a = 1/8$ and $b = 1/2$ in both plots. Left: a unique limit cycle of continuous system (22); Right: a unique crossing limit cycle of discontinuous system (23). Red dash line is the discontinuity line $\widetilde{\Sigma}_0$.

To convert (23) into a Liénard form, we make a variable transformation as follows

$$t = -\tau, \ \ x = -\frac{1}{u} + \beta, \ \ y = (u + v) - \frac{1}{\beta} - b\beta^2, \tag{25}$$

where $\beta = 1/(a + b) > 0$. Then the system (23) is transformed into the Liénard system form with the discontinuity line $\Sigma_0 = \{(x, y) \in \mathbb{R}^2 : x = 0\}$ as follows

$$\begin{cases} \begin{cases} \dfrac{dx}{dt} = -y - \dfrac{1}{\beta} - b\beta^2 - a(x - \beta)^2 - (x - \beta) - \dfrac{1}{x - \beta}, \\ \dfrac{dy}{dt} = -\dfrac{1}{\beta} - \dfrac{1}{x - \beta}, \end{cases} & \text{for} \ \ \beta > x \geq 0, \\[4ex] \begin{cases} \dfrac{dx}{dt} = -y - \dfrac{1}{\beta} - b\beta^2 - a(x - \beta)^2 - (x - \beta) - \dfrac{1}{x - \beta} - x, \\ \dfrac{dy}{dt} = -1 - \dfrac{1}{\beta} - \dfrac{1}{x - \beta}, \end{cases} & \text{for} \ \ x < 0. \end{cases} \tag{26}$$

We can apply our theory for $\Sigma_+ = \{(x, y) \in \mathbb{R}^2 : 0 < x < \beta\}$ and $\Sigma_- = \{(x, y) \in \mathbb{R}^2 : x < 0\}$. From (26) we observe that $g^- \in C^1((-\infty, 0], \mathbb{R}), g^+ \in C^1([0, \beta), \mathbb{R})$ satisfying $g(0) = 0 = g(0^+), g(0^-) = -1$ and $xg(x) > 0$ for $x \in (-\infty, 0) \cup (0, \beta)$, so $(H0)$-$(H1)$ hold. Moreover, it is easy to see that $F \in C((-\infty, \beta), \mathbb{R})$ satisfying $F(0) = 0$ and $F \in C^1((-\infty, \beta) \setminus \{0\}, \mathbb{R})$ due to

$$f(x) = F'(x) = \begin{cases} -2a(x - \beta) - 1 + \dfrac{1}{(x - \beta)^2}, & \text{for } \beta > x \geq 0, \\[3ex] -2a(x - \beta) - 2 + \dfrac{1}{(x - \beta)^2}, & \text{for } x < 0, \end{cases}$$

so $f(x)$ is discontinuous at $x = 0$. For $x \in (-\infty, \beta)$, the function $F(x)$ can be rewritten as the form

$$(x - \beta)F(x) = \begin{cases} -x[ax^2 + (1 - 3a\beta)x + (\beta^{-1} + (3a + b)\beta^2 - 2\beta)], & \text{for } \beta > x \geq 0, \\ -x[ax^2 + (2 - 3a\beta)x + (\beta^{-1} + (3a + b)\beta^2 - 3\beta)], & \text{for } x < 0. \end{cases}$$

By (24) and some simple computations, there exist $x_0 > 0$ and $x_1 < 0$ with

$$x_0 = \frac{(3a\beta - 1) + \sqrt{(3a\beta - 1)^2 - 4a[\beta^{-1} - 2\beta + (3a + b)\beta^2]}}{2a},$$

$$x_1 = \frac{(3a\beta - 2) + \sqrt{(3a\beta - 2)^2 - 4a[\beta^{-1} - 3\beta + (3a + b)\beta^2]}}{2a},$$

such that $(H2)$ is satisfied with $-\infty$ being replaced by β. Hence by Theorem 3.4 the discontinuous system (23) possesses a unique stable crossing limit cycle surrounding the origin (See Figure 2 right panel). We observe that the crossing limit cycle is non-smooth but continuous at the intersection points of the limit cycle with the discontinuity line $\widetilde{\Sigma}_0$.
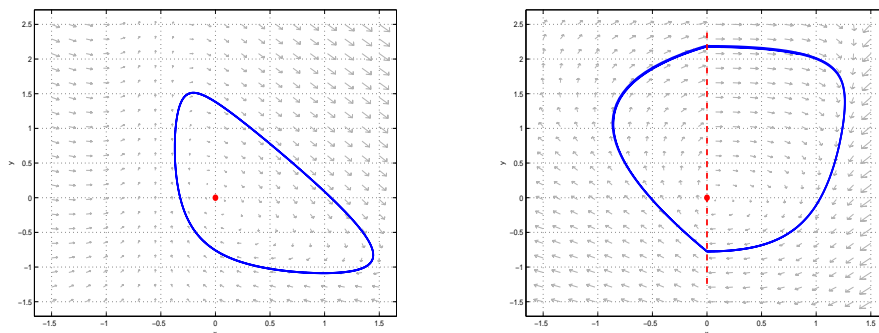

FIGURE 3. Limit cycles of continuous and discontinuous transformational Schnakenberg systems. Here $a = 1/8$ and $b = 1/2$ in both plots. Left: a unique limit cycle of the transformational continuous system of (22) with $(0, 0)$ being the equilibrium point; Right: a unique crossing limit cycle of transformational discontinuous system (26). Red dash line is the discontinuity line Σ_0.

From Example 4.1, the unique limit cycle of continuous Schnakenberg system (22) is preserved under the discontinuity perturbation into (23). Compared with the limit cycle of continuous system (22) (Figure 2 left panel), the limit cycle of the discontinuous system (23) has larger oscillation amplitude. This indicates that the oscillatory nature of the chemical reaction is preserved, and the reaction-switch increases the oscillation. In Figure 3 we also plot the limit cycles for the transformational systems of (22) and (23) under the transformation (25).

Example 4.2. Consider the Liénard system with a discontinuity line $\Sigma_0 = \{(x, y) \in \mathbb{R}^2 : x = 0\}$ as follows

$$\begin{cases} \dfrac{dx}{dt} = F(x) - y, \\ \dfrac{dy}{dt} = g(x), \end{cases} \tag{27}$$

where $F(x) = \int_0^x f(s)ds$ and functions f, g are given by

$$f(x) = \begin{cases} 2x + 2, & x < 0, \\ -2x + 1, & x \geq 0, \end{cases} \qquad g(x) = \begin{cases} 2x - 1, & x < 0, \\ x, & x \geq 0. \end{cases} \tag{28}$$

It is easy to see that $(H0)$ and $(H1)$ are satisfied. It follows that $F(x) = x^2 + 2x$ for $x < 0$, $F(x) = -x^2 + x$ for $x \geq 0$ with $F(0) = 0$ and so $F \in C(\mathbb{R}, \mathbb{R})$ but $F \in C^1(\mathbb{R} \setminus \{0\}, \mathbb{R})$ due to $f(0^+) = 1 = f(0), f(0^-) = 2$. Moreover, it is easy to verify that $F(-2) = F(1) = 0$ satisfying $F(x) > 0$ for $x \in (-\infty, -2) \cup (0, 1)$, $F(x) < 0$ for $x \in (-2, 0) \cup (1, +\infty)$ and $f(x) < 0$ for $x \in (1/2, +\infty) \cup (-\infty, -1)$, so $(H2)$ holds. Therefore, by Theorem 3.4 the discontinuous Liénard system (27) with (28) has a unique stable crossing limit cycle surrounding the origin. Indeed Matlab numerical simulation shows the result shown in the right panel of Figure 4.
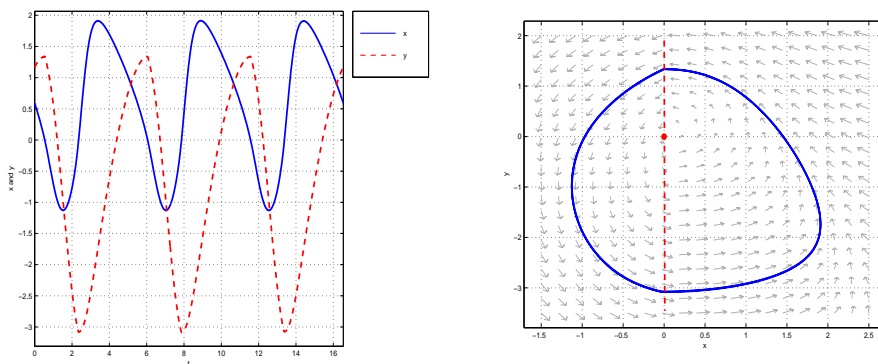

FIGURE 4. Crossing limit cycle of discontinuous Liénard system (27) with (28). Left: time series $(t, x(t))$ and $(t, y(t))$; Right: phase portrait. Red dash line in the right panel is Σ_0.

5. **Non-existence results.** In this section, we prove a non-existence result for the discontinuous Liénard system. Here we assume that $F(x)$ and $g(x)$ in the discontinuous Liénard system (2) are some piecewise linear functions as follows

$$F(x) = \begin{cases} a_R(x - x_R) + a_C x_R, & x \geq x_R, \\ a_C x, & 0 < x \leq x_R, \\ d_C x, & x_L \leq x \leq 0, \\ a_L(x - x_L) + d_C x_L, & x \leq x_L, \end{cases} \tag{29}$$

$$g(x) = \begin{cases} b_R(x - x_R) + b_C x_R + h_1, & x \geq x_R, \\ b_C x + h_1, & 0 < x \leq x_R, \\ e_C x + h_2, & x_L \leq x < 0, \\ b_L(x - x_L) + e_C x_L + h_2, & x \leq x_L. \end{cases} \tag{30}$$

One can first obtain an existence and uniqueness result from Theorem 3.4 for this specific case.

Corollary 1. *Consider the system* (2) *satisfying* (29)-(30). *Assume that* $a_R < 0$, $a_C > 0$, $d_C > 0$, $a_L < 0$, $b_R > 0$, $b_C > 0$, $e_C > 0$, $b_L > 0$, $h_1 \geq 0$ *and* $h_2 \leq 0$, *then the system* (2) *has a unique stable crossing limit cycle surrounding the origin.*

Proof. It is easily verify that the hypotheses $(H0)$-$(H2)$ are satisfied and so the conclusion holds.   □

When $b_C = b_R$ and $e_C = b_L$ in (30) then $g(x)$ can be rewritten as the form

$$g(x) = \begin{cases} b_R x + h_1, & x > 0, \\ b_L x + h_2, & x < 0. \end{cases} \tag{31}$$

Corollary 2. *Consider the system* (2) *satisfying* (29) *and* (31), *and assume that* $b_R > 0$, $b_L > 0$, $h_1 \geq 0$ *and* $h_2 \leq 0$, *then the system* (2) *has a unique stable crossing limit cycle surrounding the origin.*

Remark 1. In (29)-(30), if $a_C = d_C, b_C = e_C$, $h_1 = h_2 = 0$ and $g(0) = 0$ then $F(x)$ and $g(x)$ are continuous piecewise linear functions with three linear zones. In this case, when $b_C = 1$ Corollary 1 in this paper becomes Theorem 1 in [21].

If the discontinuous Liénard system (2) has a crossing periodic orbit L surrounding the origin, by the Poincaré-Bendixson theorem we conclude that the crossing periodic orbit L intersects with the discontinuity line Σ_0 at exactly two different points denoted by $M(0, y_M)$ and $N(0, y_N)$ satisfying $y_M < 0 < y_N$ (see Figure 5). In the following, we consider the system (2) with the functions $F(x)$ and $g(x)$ being as in (29)-(30), and give a necessary condition of the existence of crossing periodic orbit. For convenience, let L intersect with the lines $l_{x_L} = \{(x, y) \in \mathbb{R}^2 : x = x_L\}$ and $l_{x_R} = \{(x, y) \in \mathbb{R}^2 : x = x_R\}$ at points $D(x_L, y_D), C(x_L, y_C)$ and $A(x_R, y_A), B(x_R, y_B)$ respectively, satisfying $y_C < 0 < y_D$ and $y_B < 0 < y_A$ (see Figure 5). Since $\widehat{DC} \cup \overline{CD}$ is a closed Jordan curve, it follows that the interior $\Delta_2^- = \text{int}\{\widehat{DC} \cup \overline{CD}\}$ and the value $\sigma_2^- = \text{area}(\Delta_2^-)$ are well defined, where $\widehat{DC}$ denotes the directed arc from the point D to C and $\overline{CD}$ denotes the directed line segment from the point C to D. Similarly, denote by $\Delta_1^- = \text{int}\{\widehat{ND} \cup \overline{DC} \cup \widehat{CM} \cup \overline{MN}\}$, $\Delta_1^+ = \text{int}\{\widehat{AN} \cup \overline{NM} \cup \widehat{MB} \cup \overline{BA}\}$, $\Delta_2^+ = \text{int}\{\widehat{BA} \cup \overline{AB}\}$ and $\sigma_1^- = \text{area}(\Delta_1^-)$, $\sigma_1^+ = \text{area}(\Delta_1^+)$, $\sigma_2^+ = \text{area}(\Delta_2^+)$.

We have the following necessary condition for the existence of crossing periodic orbit.

Theorem 5.1. *Consider the discontinuous piecewise linear differential system* (2) *with* (29)-(30). *If the system has a crossing periodic orbit L intersecting with Σ_0 at two points $M(0, y_M)$ and $N(0, y_N)$ with $y_M < 0 < y_N$, then*

$$a_C \sigma_1^+ + a_R \sigma_2^+ + d_C \sigma_1^- + a_L \sigma_2^- = 0. \tag{32}$$

Proof. Define $V(x, y)$ as the vector field of system (2) satisfying (29)-(30) as follows

$$V(x, y) = \begin{cases} V_1^-(x, y), & x_L \leq x \leq 0, y \in \mathbb{R}, \\ V_2^-(x, y), & x \leq x_L, y \in \mathbb{R}, \\ V_1^+(x, y), & 0 \leq x \leq x_R, y \in \mathbb{R}, \\ V_2^+(x, y), & x \geq x_R, y \in \mathbb{R}. \end{cases}$$
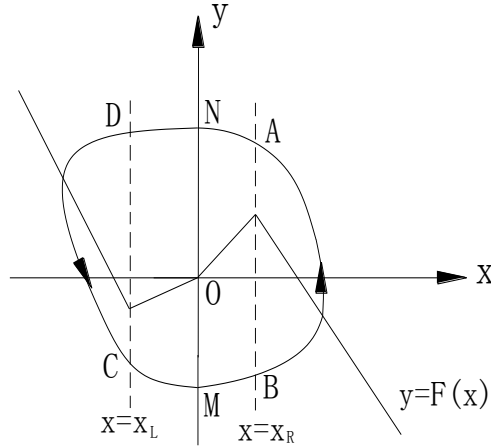
FIGURE 5. The crossing periodic orbit L surrounding the origin

Then it follows from (29)-(30) that the orthogonal vector field of V is of the form

$$\begin{cases} V_1^{-\perp}(x,y) = (-e_C x - h_2, d_C x - y), & x_L \leq x \leq 0, y \in \mathbb{R}, \\ V_2^{-\perp}(x,y) = (-b_L(x-x_L) - e_C x_L - h_2, a_L(x-x_L) + d_C x_L - y), x \leq x_L, y \in \mathbb{R}, \\ V_1^{+\perp}(x,y) = (-b_C x - h_1, a_C x - y), & 0 \leq x \leq x_R, y \in \mathbb{R}, \\ V_2^{+\perp}(x,y) = (-b_R(x-x_R) - b_C x_R - h_1, a_R(x-x_R) + a_C x_R - y), x \geq x_R, y \in \mathbb{R}. \end{cases}$$

By Green's formula one has that

$$\begin{aligned}
&\oint_{\widehat{ND} \cup \overline{DC} \cup \widehat{CM} \cup \overline{MN}} V_1^{-\perp} d\mathbf{r} = \iint_{\Delta_1^-} d_C d\sigma = d_C \sigma_1^-, \\
&\oint_{\widehat{DC} \cup \overline{CD}} V_2^{-\perp} d\mathbf{r} = \iint_{\Delta_2^-} a_L d\sigma = a_L \sigma_2^-, \\
&\oint_{\widehat{AN} \cup \overline{NM} \cup \widehat{MB} \cup \overline{BA}} V_1^{+\perp} d\mathbf{r} = \iint_{\Delta_1^+} a_C d\sigma = a_C \sigma_1^+, \\
&\oint_{\widehat{BA} \cup \overline{AB}} V_2^{+\perp} d\mathbf{r} = \iint_{\Delta_2^+} a_R d\sigma = a_R \sigma_2^+.
\end{aligned} \tag{33}$$

On the other hand, it follows from the orthogonality that

$$\begin{aligned}
&\oint_{\widehat{ND} \cup \overline{DC} \cup \widehat{CM} \cup \overline{MN}} V_1^{-\perp} d\mathbf{r} = \int_{\overline{DC} \cup \overline{MN}} V_1^{-\perp} d\mathbf{r} = \int_{y_D}^{y_C} (d_C x_L - y) dy + \int_{y_M}^{y_N} -y\, dy, \\
&\oint_{\widehat{DC} \cup \overline{CD}} V_2^{-\perp} d\mathbf{r} = \int_{\overline{CD}} V_2^{-\perp} d\mathbf{r} = \int_{y_C}^{y_D} (d_C x_L - y) dy, \\
&\oint_{\widehat{AN} \cup \overline{NM} \cup \widehat{MB} \cup \overline{BA}} V_1^{+\perp} d\mathbf{r} = \oint_{\overline{NM} \cup \overline{BA}} V_1^{+\perp} d\mathbf{r} = \int_{y_B}^{y_A} (a_C x_R - y) dy + \int_{y_N}^{y_M} -y\, dy, \\
&\oint_{\widehat{BA} \cup \overline{AB}} V_2^{+\perp} d\mathbf{r} = \oint_{\overline{AB}} V_2^{+\perp} d\mathbf{r} = \int_{y_A}^{y_B} (a_C x_R - y) dy.
\end{aligned} \tag{34}$$

Combining (33)-(34), we obtain (32).   □

Now we can use Theorem 5.1 to prove the following nonexistence result for the crossing limit cycle.

Theorem 5.2. *Consider the discontinuous piecewise linear differential system* (2) *with* (29)-(30). *If* a_R, a_C, d_C, a_L *have the same sign, then there exists no crossing limit cycle for system* (2).

Proof. If a_R, a_C, d_C, a_L have the same sign, then the equality (32) cannot be satisfied hence there exists no crossing limit cycle. □

When $a_C = a_R, d_C = a_L$ and $b_C = b_R, e_C = b_L$ in (29)-(30), we consider the functions $F(x)$ and $g(x)$ of the form

$$F(x) = \begin{cases} a_R x, & x > 0, \\ a_L x, & x < 0, \end{cases} \quad g(x) = \begin{cases} b_R x + h_1, & x > 0, \\ b_L x + h_2, & x < 0. \end{cases} \tag{35}$$

Then Theorem 5.1 implies the following result:

Corollary 3. *If the discontinuous system* (2) *satisfying* (35) *has a crossing periodic orbit L intersecting with Σ_0 at two points $M(0, y_M)$ and $N(0, y_N)$ with $y_M < 0 < y_N$, then one has that $a_R \sigma^+ + a_L \sigma^- = 0$, where $\sigma^+ = \text{area}(\Delta^+), \sigma^- = \text{area}(\Delta^-)$ with $\Delta^- = \text{int}\{L^- \cup \overline{MN}\}, \Delta^+ = \text{int}\{L^+ \cup \overline{NM}\}$, and L^-, L^+ denote the parts of L contained in Σ_-, Σ_+ respectively.*

Remark 2. If $a_R a_L \geq 0$ and $a_R + a_L \neq 0$, there exists no crossing limit cycle of the system. In this case, Corollary 3 in this paper becomes the conclusion (b) of Theorem 4.3 in [7].

6. **Concluding remarks.** In this paper, we investigate the existence, uniqueness and stability of crossing limit cycle for a planar nonlinear Liénard system with a discontinuity line. The discontinuity line is the y-axis Σ_0, and in each subregion separated by Σ_0 the system is smooth. Since the vector field of the system is discontinuous, we adopt the Filippov theory to define orbits of the system when they intersect with Σ_0 such that the orbits can be concatenated in a natural way. By using the left and right Poincaré mappings, we provide a new criterion concerning the existence, uniqueness and stability of a crossing limit cycle for such discontinuous system. An application to the discontinuous Schnakenberg model of autocatalytic chemical reaction is given to illustrate the effectiveness of the obtained theoretical result. We also consider a class of discontinuous piecewise linear differential systems and prove a necessary condition of the existence of crossing limit cycle. This necessary condition can be used to prove the non-existence of crossing limit cycle.

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