Z. Angew. Math. Phys. 66 (2015), 3061–3083
© 2015 Springer Basel
0044-2275/15/063061-23
published online September 9, 2015
DOI 10.1007/s00033-015-0578-y

Zeitschrift für angewandte Mathematik und Physik ZAMP

Existence of positive solutions to a Laplace equation with nonlinear boundary condition

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Abstract. The positive solutions of a Laplace equation with a superlinear nonlinear boundary condition on a bounded domain are studied. For higher-dimensional domains, it is shown that non-constant positive solutions bifurcate from a branch of trivial solutions at a sequence of bifurcation points, and under additional conditions on nonlinearity, the existence of a non-constant positive solution for any sufficiently large parameter value is proved by using variational approach. It is also proved that for one-dimensional domain, there is only one bifurcation point, all non-constant positive solutions lie on the bifurcating curve, and for large parameter values, there exist at least two non-constant positive solutions. For a special case, there are exactly two non-constant positive solutions.

Mathematics Subject Classfication. 35J65, 35J25, 35J20.

Keywords. Laplace equation · Nonlinear boundary condition · Bifurcation · Variational method · Multiple solutions.

1. Introduction

Reaction–diffusion equations are mathematical models for describing various physical and biological phenomena. For a well-posed reaction–diffusion problem, boundary conditions are required to obtain proper solutions. Normally boundary conditions are linear functions of the values or normal derivatives of the solutions on the boundary, but in recent studies, an increasing number of models require nonlinear boundary conditions [3,4,8,9,11,12,23,24,41,48].

In this article, we consider a Laplace equation with a nonlinear boundary condition as follows:

$$\begin{cases} -\Delta u = 0, & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = \lambda r(x) f(u), & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 1$, n is the unit outer normal to $\partial\Omega$, and λ is a nonnegative parameter. The weight function $r(x)$ satisfies

(r) $r : \partial\Omega \to \mathbb{R}$ is of class $C^{1,\theta}(\partial\Omega)$ for $\theta \in (0,1)$;

and the growth function $f(u)$ satisfies

(f) $f : \mathbb{R} \to \mathbb{R}$ is a smooth function satisfying $f > 0$ in $(0,1)$, $f < 0$ in $(-\infty, 0) \cup (1, \infty)$, $f(0) = f(1) = 0$, $f'(0) > 0$ and $f'(1) < 0$.

C.-G. Kim is Partially supported by National Research Foundation of Korea Grant funded by the Korean Government (Ministry of Education, Science and Technology, NRF-2011-357-C00006). Z.-P. Liang is Partially supported by National Natural Science Foundation of China (11571209, 11301313), Science Council of Shanxi Province (2015021007, 2013021001-4) and Shanxi 100 Talent program. J.-P. Shi is Partially supported by NSF grant DMS-1313243 and Shanxi 100 Talent program.

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The equation (1.1) is the steady state equation for the diffusive boundary reaction equation:

$$\begin{cases} u_t - D\Delta u = 0, & t > 0,\ x \in \Omega, \\ \dfrac{\partial u}{\partial n} = \lambda r(x) f(u), & t > 0,\ x \in \partial\Omega, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{1.2}$$

The system (1.2) is a variation of the classical reaction–diffusion model in which the reaction occurs in the interior of the reactor Ω. In the system (1.2), the function $u(x,t)$ is the concentration of a chemical of interest, and the chemical molecules make random walk in the reactor; hence its movement is governed by a diffusion equation. On the other hand, a chemical reaction involving this chemical occurs on the boundary of the reactor, and it generates a location-dependent flux $r(x)f(u)$ as a boundary condition.

The nonlinearity $f(u)$ satisfying (f) is usually called logistic type function as the prototypical example $f(u) = au - bu^2$ $(a, b > 0)$ appears in logistic growth model or Fisher-KPP model in genetics studies. The weight function $r(x)$ plays an important role in the structure of the solutions to (1.1). Previous work (see [32]) shows that (1) when $r(x)$ is positive, then for all $\lambda > 0$ the only nonnegative solutions of (1.1) are the constant ones $u = 0$ and $u = 1$; (2) when $r(x)$ is sign-changing, $\int_{\partial\Omega} r(x)\mathrm{d}s < 0$ and $f''(u) \le 0$, then there exists a critical value $\lambda_1 > 0$ such that only when $\lambda > \lambda_1$, (1.1) has a unique non-constant solution u in $\mathcal{H} = \{u \in H^1(\Omega) : 0 \le u \le 1 \text{ a.e. } x \in \Omega\}$, and all non-constant solutions in $\mathcal{H}$ for $\lambda > \lambda_1$ are on a curve bifurcating from $(\lambda, u) = (\lambda_1, 0)$.

We study (1.1) for the case of negative $r(x)$ in this paper. Our main results for spatial dimension $N \ge 2$ can be summarized as follows:

1. there are a sequence of bifurcation points $\lambda_k \to \infty$ such that non-constant positive solutions of (1.1) bifurcate from the branch of trivial solution $u = 1$ at $\lambda = \lambda_k$;
2. with some more conditions on $f(u)$, (1.1) possesses a non-constant positive solution for any sufficiently large $\lambda > 0$.

The first result is established by using bifurcation theory, and the second one is proved via variational method (see Sect. 3). It is a bit surprising that the result for $N = 1$ is different. Indeed, we also prove that when $N = 1$, there is only one bifurcation point $\lambda_1 > 0$ for the positive solutions from the trivial branch $u = 1$, and all non-constant positive solutions lie on the bifurcating curve. Moreover, we show that for $\lambda > \lambda_1$, there exist at least two non-constant positive solutions, and with more restrictive $f(u)$, we show that there are exactly two non-constant positive solutions for each $\lambda > \lambda_*$ and $\lambda \ne \lambda_1$, where λ_* is a saddle-node bifurcation point satisfying $\lambda_* \le \lambda_1$ (see Sect. 4).

It is interesting to compare equation (1.1) with its more well-known counterpart with reaction occurring in the interior with zero flux boundary condition:

$$\begin{cases} -\Delta u = \lambda r(x) f(u), & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{1.3}$$

Here f again satisfies (f). The structure of the nonnegative solutions of (1.3) is

1. If $r(x)$ is positive, then the only nonnegative solutions of (1.3) are $u = 0$ and $u = 1$ from the maximum principle.
2. If $r(x)$ is sign-changing, $\int_\Omega r(x)\mathrm{d}x < 0$ and $f''(u) \le 0$, then there exists $\tilde{\lambda_1} > 0$ such that only when $\lambda > \tilde{\lambda_1}$, (1.3) has a unique non-constant positive solution, and all non-constant positive solutions $\{(\lambda, u) : \lambda > \tilde{\lambda_1}\}$ are on a curve bifurcating from $(\lambda, u) = (\tilde{\lambda_1}, 0)$ (see [20]).
3. If $r(x)$ is negative, then there are a sequence of bifurcation points $\tilde{\lambda_k} \to \infty$ such that non-constant positive solutions of (1.3) bifurcate from the branch of trivial solution $u = 1$ at $\lambda = \tilde{\lambda_k}$, and when $\lambda \to \infty$, there are solutions exhibiting spiky pattern (see [25,26,42,49]).

Hence the results for the boundary reaction equation (1.1) and interior reaction equation (1.3) are very similar. But note that the results above for (1.3) and negative $r(x)$ also hold for $N = 1$, which is different from the one for (1.1). This subtle difference can be attributed to the fact that any positive solution of (1.1) achieves its any local maximum/minimum on the boundary (see Lemma 2.2), while the solutions of (1.3) can have "interior peak" solutions [18,19]. Note that (1.3) with positive or sign-changing $r(x)$ appears in the studies of migration–selection genetics models [28–30,34,35], while (1.3) with negative $r(x)$ appears in the studies of pattern formation PDEs and chemotaxis systems [6,7,26,36,37].

In recent years, the existence, multiplicity, and uniqueness of positive solutions of nonlinear elliptic equations with nonlinear boundary conditions have been considered by many authors. For example, the bifurcation of positive solutions of diffusive logistic equation with nonlinear boundary condition has been studied in [8–10,17,45,47], and other types of nonlinear boundary conditions have been also considered in [14,16,44]. On the other hand, nonlinear elliptic equations with nonlinear boundary condition defined in half space have been considered in [12,22,31,38,50].

We review some preliminaries of the linear eigenvalue problem, results for positive and sign-changing $r(x)$ and bifurcation theory in Sect. 2. The main results for dimension $N \geq 2$ are stated and proved in Sect. 3, while the results for $N = 1$ are proved in Sect. 4. The proof of Lemma 3.4 is given in Sect. 5.

2. Preliminaries

2.1. Linear eigenvalue problem

First we recall some results for the following eigenvalue problem

$$\begin{cases} \Delta \phi = 0, & x \in \Omega, \\ \dfrac{\partial \phi}{\partial n} = \lambda s(x) \phi, & x \in \partial\Omega, \end{cases} \tag{2.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 1$, λ is a nonnegative parameter. For the higher-dimensional domain Ω, the following basic result is well known (see, e.g. [5,46]).

Proposition 2.1. *Suppose that Ω is a smooth bounded domain in $\mathbb{R}^N$ with $N \geq 2$, and $s : \partial\Omega \to \mathbb{R}$ is of class $C^{1,\theta}(\partial\Omega)$ for $\theta \in (0, 1)$. If there exists a measurable subset Ω_0 of $\partial\Omega$ such that $|\Omega_0| > 0$ and $s(x) > 0$ for $x \in \Omega_0$, then there exists a sequence of eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ of (2.1) such that $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$ and $\lambda_n \to \infty$ as $n \to \infty$. Moreover,*

1. *If ϕ_i and ϕ_j are eigenfunctions corresponding to eigenvalues λ_i and λ_j, respectively, and $\lambda_i \neq \lambda_j$, then*
$$\int_{\Omega} \nabla\phi_i(x) \cdot \nabla\phi_j(x) \mathrm{d}x = \int_{\partial\Omega} s(x)\phi_i(x)\phi_j(x) \mathrm{d}S = 0.$$
2. *If $s(x)$ is a sign-changing function satisfying*
$$\int_{\partial\Omega} s(x) \mathrm{d}S < 0, \tag{2.2}$$
then the eigenfunction ϕ_1 corresponding to λ_1 can be chosen as positive; if $s(x)$ is positive for all $x \in \partial\Omega$ or $s(x)$ is sign-changing but does not satisfy (2.2), then all eigenfunctions $\phi_i(x)$ $(i \geq 1)$ are sign-changing in Ω.

It is clear that the eigenvalue $\lambda_0 = 0$ corresponds to the eigenfunction $\phi_0(x) = 1$. The result for the principal eigenvalue was proved in [46, Theorem 2.2]. We also remark that for the case that $s(x) < 0$ for all $x \in \partial\Omega$ and $N \geq 1$, 0 is the only nonnegative eigenvalue of (2.1).

On the other hand when $N = 1$, equation (2.1) becomes the following two-point boundary value problem

$$\begin{cases} \phi''(x) = 0, & x \in (0,1), \\ -\phi'(0) = \lambda s_0 \phi(0), & \phi'(1) = \lambda s_1 \phi(1), \end{cases} \tag{2.3}$$

where s_0 and s_1 are nonzero constants. Then, by direct calculation, the problem (2.3) has only two eigenvalues $\lambda_0 = 0$ and $\lambda_1 = \dfrac{s_0 + s_1}{s_0 s_1}$, and the eigenfunction associated with λ_1 is $\phi_1(x) = x - \dfrac{s_1}{s_0 + s_1}$. When s_0 and s_1 are both positive, $\lambda_1 > 0$ and ϕ_1 is sign-changing, and when $s_0 s_1 < 0$ but $s_0 + s_1 < 0$, $\lambda_1 > 0$ and ϕ_1 can be chosen as positive.

2.2. Results for positive and sign-changing potential functions

In this paper, we consider (1.1) for the case that the potential function $r(x)$ is negative. The cases of $r(x)$ is positive or sign-changing have been considered previously, and in this subsection, we will review these results. First we prove a maximum principle for a Laplace equation with a general nonlinear boundary condition as follows:

$$\begin{cases} -\Delta u = 0, & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = g(x, u), & x \in \partial\Omega, \end{cases} \tag{2.4}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ with $N \geq 1$ and $g \in C(\partial\Omega \times \mathbb{R})$. From the strong maximum principle and Hopf's lemma for the elliptic equations, we have the following lemma.

Lemma 2.2. *Suppose that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a non-constant solution of (2.4). If u achieves a local maximum at $x = x_0 \in \overline{\Omega}$, then $x_0 \in \partial\Omega$, and $g(x_0, u(x_0)) > 0$. Similarly, if u achieves a local minimum at $x = x_0 \in \overline{\Omega}$, then $x_0 \in \partial\Omega$, and $g(x_0, u(x_0)) < 0$.*

Proof. Assume on the contrary that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a non-constant solution of (2.4), and it achieves a local maximum at $x = x_0 \in \Omega$. Then there exists an open ball $B_\delta(x_0) \subset \Omega$ with radius $\delta > 0$ and center x_0 such that $u(x_0) \geq u(x)$ for all $x \in \overline{B}_\delta(x_0)$. From the strong maximum principle, we have $u(x) \equiv u(x_0)$ in $\overline{B}_\delta(x_0)$. We can proceed to prove that u is constant in Ω, but that is a contradiction to the fact u is a non-constant solution. Thus $x_0 \in \partial\Omega$, and there exists an open ball B containing x_0 such that $u(x_0) > u(x)$ for all $x \in B \cap \Omega$. It follows from Hopf's lemma that

$$\frac{\partial u}{\partial n}(x_0) > 0,$$

which implies that $g(x_0, u(x_0)) > 0$. In the same way, if u achieves a local minimum at $x = x_0 \in \overline{\Omega}$, then $x_0 \in \partial\Omega$, and $g(x_0, u(x_0)) < 0$. □

From Lemma 2.2, we have the following result directly.

Theorem 2.3. *Suppose that $r(x)$ satisfies (r), and $f(u)$ satisfies (f). Assume in addition that $r(x) > 0$ for all $x \in \partial\Omega$. Then for any $\lambda > 0$, the only nonnegative solutions of (1.1) are $u = 0$ or 1.*

On the other hand, Madeira and do Nascimento [32] studied the problem (1.1) with an indefinite weight $r(x)$ and the results are as follows.

Theorem 2.4. *Suppose that $r(x)$ satisfies (r), and $f(u)$ satisfies (f). Assume in addition that $r(x)$ is a sign-changing function with $\int_{\partial\Omega} r(x)\mathrm{d}S < 0$ and $f''(u) < 0$ for $u \in [0,1]$. Then (1.1) has only the constant solutions $u = 0$ and $u = 1$ when $\lambda \leq \lambda_1$, and (1.1) has a unique non-constant solution $u_\lambda \in \mathcal{H}$ for each $\lambda > \lambda_1$. Here $\mathcal{H} = \{u \in H^1(\Omega) : 0 \leq u \leq 1$ a.e. $x \in \Omega\}$, and λ_1 is the positive principal eigenvalue of (2.1) with $s(x) = r(x)f'(0)$.*

We comment that the above results hold for both the cases of $N \geq 2$ and $N = 1$. Apparently Theorem 2.3 classifies all nonnegative solutions of (1.1) when $r(x)$ is positive and Theorem 2.4 classifies all nonnegative solutions in $\mathcal{H}$ when $r(x)$ is sign-changing. We shall consider the case when $r(x)$ is negative in this paper.

2.3. Bifurcation theory

Our main analytic tool in this paper is the bifurcation theory, and in this subsection, we review some abstract bifurcation theorems which will be used. Nonlinear problem can often be formulated in the form of an abstract equation

$$F(\lambda, u) = 0,$$

where $F : \mathbb{R} \times X \to Y$ is a nonlinear differentiable mapping and X, Y are Banach spaces. In the following, we use F_u as the partial derivative of F with respect to argument u, and we use $\langle \cdot, \cdot \rangle$ as the duality pair of a Banach space X and its dual space X^*. We say that 0 is a simple eigenvalue of $F_u(\lambda_0, u_0)$ if the following assumption is satisfied:

$(F1)$ $dim N(F_u(\lambda_0, u_0)) = codim R(F_u(\lambda_0, u_0)) = 1$, and $N(F_u(\lambda_0, u_0)) = span\{\phi_1\}$,

where $N(T)$ and $R(T)$ are the null space and the range space of linear operator T, respectively. Crandall and Rabinowitz [13] proved the following celebrated local bifurcation theorem from a simple eigenvalue.

Theorem 2.5. (Transcritical and pitchfork bifurcations, [13, Theorem 1.7]). *Let U be a neighborhood of (λ_0, u_0) in $\mathbb{R} \times X$, and let $F : U \to Y$ be a twice continuously differentiable mapping. Assume that $F(\lambda, u_0) = 0$ for $(\lambda, u_0) \in U$. At (λ_0, u_0), F satisfies $(F1)$ and*

$(F2)$ $F_{\lambda u}(\lambda_0, u_0)[\phi_1] \notin R(F_u(\lambda_0, u_0))$.

Let Z be any complement of $span\{\phi_1\}$ in X. Then the solutions of $F(\lambda, u) = 0$ near (λ_0, u_0) different from (λ, u_0) form a curve $\{(\lambda(s), u(s)) : s \in I = (-\epsilon, \epsilon)\}$, where $\lambda : I \to \mathbb{R}$, $z : I \to Z$ are C^1 functions such that $u(s) = u_0 + s\phi_1 + sz(s)$, $\lambda(0) = \lambda_0$, $z(0) = 0$, and

$$\lambda'(0) = -\frac{\langle l, F_{uu}(\lambda_0, u_0)[\phi_1, \phi_1] \rangle}{2\langle l, F_{\lambda u}(\lambda_0, u_0)[\phi_1] \rangle}, \tag{2.5}$$

where $l \in Y^*$ satisfying $N(l) = R(F_u(\lambda_0, u_0))$. If F satisfies

$(F3)$ $F_{uu}(\lambda_0, u_0)[\phi_1, \phi_1] \notin R(F_u(\lambda_0, u_0))$,

then $\lambda'(0) \neq 0$, and a transcritical bifurcation occurs. If F satisfies

$(F3')$ $F_{uu}(\lambda_0, u_0)[\phi_1, \phi_1] \in R(F_u(\lambda_0, u_0))$,

and in addition $F \in C^3$, then $\lambda'(0) = 0$ and

$$\lambda''(0) = -\frac{\langle l, F_{uuu}(\lambda_0, u_0)[\phi_1, \phi_1, \phi_1] \rangle + 3\langle l, F_{uu}(\lambda_0, u_0)[\phi_1, \theta] \rangle}{3\langle l, F_{\lambda u}(\lambda_0, u_0)[\phi_1] \rangle},$$

where θ satisfies $F_{uu}(\lambda_0, u_0)[\phi_1, \phi_1] + F_u(\lambda_0, u_0)[\theta] = 0$. A pitchfork bifurcation typically satisfies $\lambda''(0) \neq 0$.

We will also use a secondary bifurcation result which was first proved in [13, Theorem 1], and Liu, Shi and Wang [27] extended it as follows.

Theorem 2.6. (Secondary Bifurcation Theorem [27, Theorem 2.7]). *Let W and Y be Banach spaces, let Ω be an open subset of W and let $G : \Omega \to Y$ be twice differentiable. Suppose that*

$$G(w_0) = 0, \; dim N(G'(w_0)) = 2, \; codim R(G'(w_0)) = 1.$$

Then

1. *if for any* $\phi(\neq 0) \in N(G'(w_0))$, $G''(w_0)[\phi]^2 \notin R(G'(w_0))$, *then the set of solutions to* $G(w) = 0$ *near* $w = w_0$ *is the singleton* $\{w_0\}$.
2. *if there exists* $\phi_1(\neq 0) \in N(G'(w_0))$ *such that* $G''(w_0)[\phi_1]^2 \in R(G'(w_0))$, *and there exists* $\phi_2 \in N(G'(w_0))$ *such that* $G''(w_0)[\phi_1, \phi_2] \notin R(G'(w_0))$, *then* w_0 *is a bifurcation point of* $G(w) = 0$ *and in some neighborhood of* w_0, *the totality of solutions of* $G(w) = 0$ *form two continuous curves intersecting only at* w_0. *Moreover, the solution curves are in form of* $w_0 + s\psi_i + s\theta_i(s)$, $s \in (-\delta, \delta)$, $\theta_i(0) = \theta_i'(0) = 0$, *where* $\psi_i\,(i = 1, 2)$ *are the two linear independent solutions of the equation* $\langle l_1, G''(w_0)[\psi, \psi]\rangle = 0$ *and* $l_1 \in Y^*$ *satisfying* $N(l_1) = R(G'(w_0))$.

Finally we recall the following global bifurcation theorem due to Shi and Wang [43] which is essentially based on almost the same conditions of Theorem 2.5, and it is also a generalization of the classical Rabinowitz global bifurcation theorem [39].

Theorem 2.7. *Let* V *be an open connected subset of* $\mathbb{R} \times X$ *and* $(\lambda_0, u_0) \in V$, *and let* F *be a continuously differentiable mapping from* V *into* Y. *Suppose that*

1. $F(\lambda, u_0) = 0$ *for* $(\lambda, u_0) \in V$,
2. *the partial derivative* $F_{\lambda u}(\lambda, u)$ *exists and is continuous in* (λ, u) *near* (λ_0, u_0),
3. $F_u(\lambda_0, u_0)$ *is a Fredholm operator with index* 0, *and* $dim N(F_u(\lambda_0, u_0)) = 1$,
4. $F_{\lambda u}[w_0] \notin R(F_u(\lambda_0, u_0))$, *where* $w_0 \in X$ *spans* $N(F_u(\lambda_0, u_0))$.

Let Z *be any complement of* $span\{w_0\}$ *in* X. *Then there exist an open interval* $I_1 = (-\epsilon, \epsilon)$ *and continuous functions* $\lambda : I_1 \to \mathbb{R}$, $\psi : I_1 \to Z$ *such that* $\lambda(0) = \lambda_0$, $\psi(0) = 0$, *and if* $u(s) = u_0 + sw_0 + s\psi(s)$ *for* $s \in I_1$, *then* $F(\lambda(s), u(s)) = 0$. *Moreover,* $F^{-1}(\{0\})$ *near* (λ_0, u_0) *consists precisely of the curves* $u = u_0$ *and* $\Gamma = \{(\lambda(s), u(s)) : s \in I_1\}$. *If in addition* $F_u(\lambda, u)$ *is a Fredholm operator for all* $(\lambda, u) \in V$, *then the curve* Γ *is contained in* Σ, *which is a connected component of* $\overline{S}$, *where* $S := \{(\lambda, u) \in V : F(\lambda, u) = 0, u \neq u_0\}$, *and either* Σ *is not compact in* V *or* Σ *contains a point* (λ_*, u_0) *with* $\lambda_* \neq \lambda_0$.

3. Existence for higher-dimensional domains

In this section, we consider the existence of positive solutions to (1.1) for a bounded domain $\Omega \subset \mathbb{R}^N$ with $N \geq 2$ and under the condition

$$r(x) < 0, \qquad \text{for all} \quad x \in \partial\Omega. \tag{3.1}$$

Clearly (1.1) has two lines of trivial solutions:

$$\Gamma_0 := \{(\lambda, 0) : \lambda \geq 0\} \quad \text{and} \quad \Gamma_1 := \{(\lambda, 1) : \lambda \geq 0\}, \tag{3.2}$$

and also

$$\Gamma_{00} := \{(0, u) : u \in \mathbb{R},\ u \geq 0\}. \tag{3.3}$$

If (3.1) is satisfied, and $u(x)$ is a non-constant solution of (1.1), then by Lemma 2.2, u is a positive solution of (1.1) such that

$$\max_{x\in\overline{\Omega}} u(x) = \max_{x\in\partial\Omega} u(x) > 1$$

and

$$0 < \min_{x\in\overline{\Omega}} u(x) = \min_{x\in\partial\Omega} u(x) < 1.$$

To consider the solutions of (1.1) in a functional setting, we define $X = W^{2,p}(\Omega)$ and $Y = L^p(\Omega) \times W^{1-\frac{1}{p},p}(\partial\Omega)$, where $p > N$. Define a nonlinear mapping $F : \mathbb{R} \times X \to Y$ by

$$F(\lambda, u) = \left(\Delta u, \frac{\partial u}{\partial n} - \lambda r(x) f(u)\right). \tag{3.4}$$

We prove the existence of positive solutions of (1.1) by using bifurcation theory for the bifurcation of positive solutions from the line of trivial solutions Γ_1. We first determine possible bifurcation points along the lines of trivial solutions Γ_0, Γ_1 and Γ_{00}. We say that $(\lambda_*, 1)$ is a bifurcation point on the line of trivial solutions $\Gamma_1 = \{(\lambda, 1) : \lambda > 0\}$ if there exists a sequence (λ^n, u^n) of solutions to (1.1) such that $u^n \neq 1$, $\lambda^n \to \lambda_*$ and $||u^n - 1||_X \to 0$ as $n \to \infty$. And a bifurcation point on the line Γ_0 or Γ_{00} can be defined similarly.

Lemma 3.1. *Suppose that $r(x)$ satisfies (r) and (3.1), and $f(u)$ satisfies (f).*

1. *If $(\lambda, 1)$ with $\lambda > 0$ is a bifurcation point of (1.1) on the trivial branch Γ_1, then λ is an eigenvalue of (2.1) with $s(x) = f'(1)r(x)$.*
2. *There is no bifurcation point of (1.1) on the trivial branch Γ_0 for $\lambda > 0$.*
3. *If $(0, u)$ is a bifurcation point of (1.1) on the trivial branch Γ_{00}, then $u = 0$ or 1.*

Proof. 1. Suppose that $(\lambda, 1)$ is a bifurcation point on Γ_1, then there exists a sequence $\{(\lambda^n, u^n)\}$ such that $u^n (\not\equiv 1)$ is a solution of (1.1) with $\lambda = \lambda^n$ and

$$(\lambda^n, u^n) \to (\lambda, 1) \quad \text{in} \quad \mathbb{R} \times W^{2,p}(\Omega) \quad \text{as} \quad n \to \infty.$$

Thus $u^n \to 1$ in $H^1(\Omega)$ as $n \to \infty$. Setting

$$v^n := \frac{u^n - 1}{\|u^n - 1\|_{H^1(\Omega)}},$$

there exists a subsequence of $\{v^n\}$, still denoted by $\{v^n\}$, and $v_\lambda \in H^1(\Omega) \setminus \{0\}$ such that as $n \to \infty$,

$$\begin{aligned} v^n &\rightharpoonup v_\lambda \quad \text{in} \quad H^1(\Omega), \\ v^n &\to v_\lambda \quad \text{in} \quad L^2(\partial\Omega), \\ v^n &\to v_\lambda \quad \text{a.e. in} \quad \partial\Omega. \end{aligned}$$

On the other hand, v^n satisfies

$$\begin{cases} \Delta v^n = 0, & x \in \Omega, \\ \dfrac{\partial v^n}{\partial n} = \lambda^n r(x) \dfrac{f(\|u^n - 1\|_{H^1(\Omega)} v^n + 1)}{\|u^n - 1\|_{H^1(\Omega)}}, & x \in \partial\Omega, \end{cases}$$

and thus, for all $\phi \in H^1(\Omega)$,

$$\begin{aligned} \int_\Omega \nabla v^n \cdot \nabla \phi \mathrm{d}x &= \lambda^n \int_{\partial\Omega} r(x) \frac{f(\|u^n - 1\|_{H^1(\Omega)} v^n + 1)}{\|u^n - 1\|_{H^1(\Omega)}} \phi \mathrm{d}S \\ &= \lambda^n \int_{\partial\Omega} r(x) \left(f'(1) v^n + \frac{o(\|u^n - 1\|_{H^1(\Omega)} v^n)}{\|u^n - 1\|_{H^1(\Omega)}} \right) \phi \mathrm{d}S \\ &= \lambda^n \int_{\partial\Omega} r(x) \left(f'(1) + \frac{o(u^n - 1)}{u^n - 1} \right) v^n \phi \mathrm{d}S. \end{aligned}$$

Here $o(s)$ means that $o(s)/s \to 0$ as $s \to 0$. Consequently,

$$\int_\Omega \nabla v_\lambda \cdot \nabla \phi \mathrm{d}x = \lambda \int_{\partial\Omega} r(x) f'(1) v_\lambda \phi \mathrm{d}S$$

for all $\phi \in H^1(\Omega)$, which implies that λ is an eigenvalue of (2.1) with $s(x) = r(x)f'(1)$.

2. Suppose that $(\lambda, 0)$ is a bifurcation point on Γ_0, then the same arguments as in part 1 show that λ is an eigenvalue of (2.1) with $s(x) = r(x)f'(0)$. From (3.1), we have $s(x) < 0$ for all $x \in \partial\Omega$, then from the remark after Proposition 2.1, (2.1) has no positive eigenvalue, and hence such bifurcation point does not exist on Γ_0.

3. Suppose that $(0,u)$ is a bifurcation point on Γ_{00}. By integration of the equation (1.1), we obtain

$$\lambda^n \int_{\partial\Omega} r(x) f(u^n) \mathrm{d}S = \int_{\Omega} \Delta u^n \mathrm{d}x = 0,$$

which implies that $f(u)=0$; thus we must have $u=0$ or $u=1$ from the condition (f).

□

From the part 3 of Lemma 3.1, we have two possible bifurcation points $(0,0)$ and $(0,1)$ along Γ_{00}. But indeed only the trivial solutions on Γ_0 and Γ_1 bifurcate from these two points. We make this fact clear by using the secondary bifurcation theorem (Theorem 2.6) as follows.

Lemma 3.2. *Suppose that $r(x)$ satisfies (r) and (3.1), and $f(u)$ satisfies (f). Then,*

1. $(\lambda,u)=(0,1)$ *is a bifurcation point of* (1.1) *such that totality of the solutions of* (1.1) *near* $(0,1)$ *consists precisely of the curves* $\mathcal{C}_1=\{(\lambda,u)=(0,c): c\in(1-\delta,1+\delta)\}$ *and* $\mathcal{C}_2=\{(\lambda,u)=(\lambda,1):\lambda\in[0,\delta)\}$ *for sufficiently small* $\delta>0$.
2. $(\lambda,u)=(0,0)$ *is a bifurcation point of* (1.1) *such that totality of the solutions of* (1.1) *near* $(0,0)$ *consists precisely of the curves* $\mathcal{C}_1=\{(\lambda,u)=(0,c): c\in(-\delta,\delta)\}$ *and* $\mathcal{C}_2=\{(\lambda,u)=(\lambda,0):\lambda\in[0,\delta)\}$ *for sufficiently small* $\delta>0$.

Proof. Define a nonlinear mapping $G:\mathbb{R}\times X\to Y$ by

$$G(w)=\left(\Delta u, \frac{\partial u}{\partial n}-\lambda r(x) f(u)\right), \quad w=(\lambda,u)\in\mathbb{R}\times X.$$

and let $w_0=(0,1)$. Then $N(G'(w_0))=span\{(0,1),(1,0)\}$ and $R(G'(w_0))=N(l_1)$, where

$$\langle l_1,(h_1,h_2)\rangle=\int_{\Omega} h_1 \mathrm{d}x-\int_{\partial\Omega} h_2 \mathrm{d}S.$$

Since

$$G''(w_0)[(0,1),(0,1)])=(0,0)\in R(G'(w_0))$$

and

$$G''(w_0)[(0,1),(1,0)])=(0,-r(x)f'(1))\notin R(G'(w_0)),$$

then applying Theorem 2.6, we obtain that $(0,1)$ is a bifurcation point of (1.1) and totality of the solutions of (1.1) near $(0,1)$ forms two continuous curves intersecting only at $(0,1)$. The case $(0,0)$ can be proved in a similar manner. □

The results in Lemma 3.1 and Lemma 3.2 show that the only non-trivial bifurcation points from the set of trivial solutions are the eigenvalues of (2.1) with $s(x)=f'(1)r(x)$. From Proposition 2.1, let $\{\lambda_n\}$ be the sequence of eigenvalues of (2.1) with $s(x)=f'(1)r(x)$ such that $0=\lambda_0<\lambda_1\le\lambda_2\le\cdots\le\lambda_n\le\cdots$, and $\lambda_n\to\infty$ as $n\to\infty$. The following local bifurcation result can be proved by using Theorem 2.5.

Theorem 3.3. *Suppose that $r(x)$ satisfies (r) and (3.1), and $f(u)$ satisfies (f). Assume that for some $k\in\mathbb{N}$, the eigenvalue λ_k of (2.1) with $s(x)=f'(1)r(x)$ is simple with an associative eigenfunction ϕ_k. Then the solution set of (1.1) near $(\lambda,u)=(\lambda_k,1)$ consists precisely of the curves Γ_1 and*

$$\mathcal{S}_k=\{(\lambda_k(t),u_k(t)):t\in I=(-\eta_k,\eta_k)\subset\mathbb{R}\},$$

where $\lambda_k(t) = \lambda_k + t\lambda_k'(0) + tz_1^k(t)$ *and* $u_k(t) = 1 + t\phi_k + tz_2^k(t)$ *are continuous functions such that* $z_i^k(0) = 0$, $i = 1, 2$. *Moreover, if* f *is* C^2 *near* $u = 1$, *then the curve* $\mathcal{S}_k$ *is* C^1, *and*

$$\lambda_k'(0) = -\frac{\int_{\partial\Omega} \lambda_k r(x) f''(1)\phi_k^3 \mathrm{d}S}{2\int_{\partial\Omega} r(x) f'(1)\phi_k^2 \mathrm{d}S} = -\frac{\lambda_k f''(1)\int_\Omega \phi_k |\nabla\phi_k|^2 \mathrm{d}x}{f'(1)\int_\Omega |\nabla\phi_k|^2 \mathrm{d}x}. \tag{3.5}$$

Proof. We verify all the assumptions in Theorem 2.5. We prove it in several steps:

1. Since λ_k is assumed to be simple, then $dim N(F_u(\lambda_k, 1)) = 1$ and $N(F_u(\lambda_k, 1)) = span\{\phi_k\}$.
2. Let $(h_1, h_2) \in R(F_u(\lambda_k, 1))$ and let $w \in X$ satisfy

$$\begin{cases} \Delta w = h_1, & x \in \Omega, \\ \dfrac{\partial w}{\partial n} - \lambda_k r(x) f'(1) w = h_2, & x \in \partial\Omega. \end{cases} \tag{3.6}$$

Multiplying the equation in (3.6) by ϕ_k and integrating on Ω, we obtain

$$\begin{aligned} \int\limits_\Omega \phi_k h_1 \mathrm{d}x &= \int\limits_\Omega w\Delta\phi_k \mathrm{d}x + \int\limits_{\partial\Omega} \left(\phi_k \frac{\partial w}{\partial n} - w\frac{\partial \phi_k}{\partial n}\right) \mathrm{d}S \\ &= \int\limits_{\partial\Omega} \phi_k h_2 \mathrm{d}S, \end{aligned}$$

which shows that $(h_1, h_2) \in R(F_u(\lambda_k, 1))$ if and only if

$$\int\limits_\Omega \phi_k h_1 \mathrm{d}x - \int\limits_{\partial\Omega} \phi_k h_2 \mathrm{d}S = 0.$$

In the following, we define $l \in X^*$ by

$$\langle l, (h_1, h_2)\rangle = \int\limits_\Omega \phi_k h_1 \mathrm{d}x - \int\limits_{\partial\Omega} \phi_k h_2 \mathrm{d}S.$$

Consequently, $R(F_u(\lambda_k, 1)) = N(l)$, and $codim R(F_u(\lambda_k, 1)) = 1$.
3. Since

$$F_{\lambda u}(\lambda_k, 1)[\phi_k] = (0, -r(x) f'(1)\phi_k),$$

then we have

$$\langle l, F_{\lambda u}(\lambda_k, 1)[\phi_k]\rangle = \int\limits_{\partial\Omega} r(x) f'(1)\phi_k^2 \mathrm{d}S > 0,$$

and $F_{\lambda u}(\lambda_k, 1)[\phi_k] \notin R(F_u(\lambda_k, 1))$. Thus the proof is complete in view of Theorem 2.5, and (3.5) can be obtained by using (2.5).

□

We remark that the simplicity assumption on the eigenvalues of (2.1) is not restrictive, as the simplicity is generically true with respect to perturbation of the boundary, see, for example, Henry [21] Chapter 6. On the other hand, in the case of a higher multiplicity eigenvalue $\lambda = \lambda_k$, the bifurcation of non-constant solutions still occurs due to the variational structure of (1.1) so a bifurcation theorem of variational problem (see Theorem 11.4 of Rabinowitz [40]). Here we will not give the details of that approach, and in the last part of this section, we use variational method directly to prove the existence of positive solutions.

Theorem 3.3 shows that non-constant positive solutions bifurcate from the line of trivial solutions $u = 1$ in the (λ, u) space. The global bifurcation theorem (Theorem 2.7) can be applied to obtain a global picture of the bifurcation diagram, but we will need a critical *a priori* estimate for the positive solutions of (1.1). To prove the boundedness of solutions of (1.1), we make use of the blow-up method (see, e.g. [15,26]). The following lemma can be proved in a similar way as the proof of [26, Theorem 3]. For the sake of completeness, we present its proof in Sect. 5.

Lemma 3.4. *Suppose that $r(x)$ satisfies (r) and (3.1), and $f(u)$ satisfies (f). In addition, we assume that $f(u)$ satisfies that*

($f1$) *Let $f(u) = u - g(u)$. Then $g(u)$ satisfies*

$$\lim_{u\to 0}\frac{g(u)}{u}=0,\quad \lim_{u\to\infty}\frac{g(u)}{u^p}=A_1, \tag{3.7}$$

for positive constants A_1 and $p \in (1, p^*)$. Here, $p^* = N/(N-2)$ if $N \geq 3$, and $p^* = \infty$ if $N = 2$.

Then there exists $M > 0$ independent of λ such that if $u(x)$ is a positive solution to (1.1) with $\lambda \in (0,\infty)$, then $u(x) < M$ for all $x \in \overline{\Omega}$.

Now we give a main result of global bifurcation of positive solutions of (1.1) with negative $r(x)$.

Theorem 3.5. *Suppose that all conditions in Theorem 3.3 are satisfied, and $f(u)$ also satisfies ($f1$). Let $V := \{(\lambda, u) \in (0,\infty)\times X : u(x) > 0,\ x \in \overline{\Omega}\}$. Then the curve $\mathcal{S}_k$ in Theorem 3.3 is contained in Λ_k, which is a connected component of $\overline{S}$, where $S := \{(\lambda, u) \in V : F(\lambda, u) = 0, u \neq 1\}$, and either Λ_k is unbounded in the λ-direction or Λ_k contains a point $(\lambda_*, 1)$ with $\lambda_* \neq \lambda_k$. Here λ_* is another eigenvalue of (2.1) with $s(x) = f'(1)r(x)$.*

Proof. Since $F_u(\lambda, u)$ is a Fredholm operator for all $(\lambda, u) \in V$, it follows from Theorem 2.7 that the curve $\mathcal{S}_k$ in Theorem 3.3 is contained in Λ_k, and either Λ_k is not compact in V or Λ_k contains a point $(\lambda_*, 1)$ with $\lambda_* \neq \lambda_k$. If Λ_k contains a point $(\lambda_*, 1)$ with $\lambda_* \neq \lambda_k$, by Lemma 3.1, λ^* is an eigenvalue of (2.1) with $s(x) = f'(1)r(x)$. On the other hand, if Λ_k does not contain a point $(\lambda_*, 1)$ with $\lambda_* \neq \lambda_k$, it follows from Lemma 2.2, Lemma 3.1, and Lemma 3.2 that $\Lambda_k \cap \partial V$ is an empty set, and thus Λ_k is unbounded in the λ-direction by Lemma 3.4. □

The global bifurcation result in Theorem 3.5 shows the existence of non-constant positive solutions for λ-values at least near the bifurcation points. But it is possible that $\Lambda_i = \Lambda_j$ for $i, j \in \mathbb{N}$ and $i \neq j$. Hence Theorem 3.5 cannot guarantee the existence of non-constant positive solutions for all large λ. In the last part of this section, we prove the existence of a non-constant positive solution of (1.1) for large $\lambda > 0$ by using the mountain pass theorem of Ambrosetti and Rabinowitz [2].

Similar to the setting in ($f1$), let $f(u) = u - g(u)$ for $u \in \mathbb{R}$, and we make the following hypotheses on $g(u)$ following [26]:

($g1$) $g : \mathbb{R} \to \mathbb{R}$ is locally Hölder continuous, $g(u) = 0$ for all $u < 0$, and $g(u) > 0$ for all $u > 0$.

($g2$) $g(u) = o(u)$ as $u \to 0$ and $\dfrac{g(u)}{u} \to \infty$ as $u \to \infty$.

($g3$) There exist positive constants c_1, c_2, and $p \in (1, p^*)$ such that

$$g(u) \leq c_1 + c_2u^p \quad \text{for } u > 0,$$

where p^* is the constant defined in ($f1$).

($g4$) There exist $\mu > 2$ and $\epsilon > 0$ such that

$$0 < \mu G(u) \leq ug(u) \quad \text{for } u \geq \epsilon,$$

where $G(s) = \int_0^s g(t)\mathrm{d}t$.

($g5$) $\inf_{u\in Z}\{2^{-1}u^2 - G(u)\} > 0$, where $Z = \{u > 0 : g(u) = u\}$.

Note that $Z \neq \emptyset$ by $(g1)$ and $(g2)$.

In the remaining part of this section, let X denote the Sobolev space $W^{1,2}(\Omega)$ with norm

$$\|u\|_X = \left(\int_\Omega |\nabla u|^2 \mathrm{d}x - \lambda \int_{\partial\Omega} r(x) u^2 \mathrm{d}S \right)^{\frac{1}{2}},$$

which is equivalent to the usual norm in $W^{1,2}(\Omega)$ because of (3.1). We define a functional $E : X \to \mathbb{R}$ by

$$E(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \mathrm{d}x - \lambda \int_{\partial\Omega} r(x) \left(\frac{1}{2} u^2 - G(u) \right) \mathrm{d}S, \quad u \in X.$$

Since the embedding $X \hookrightarrow L^k(\partial\Omega)$ is compact if $k \in [1, p^* + 1)$, then by standard arguments, we have the following lemma (see, e.g. [33, Lemma 4.2]).

Lemma 3.6. *Suppose that $r(x)$ satisfies (r), and $(g1)$ and $(g3)$ hold. Then E is well defined on X, and $E \in C^1(X, \mathbb{R})$ with*

$$E'(u)\phi = \int_\Omega \nabla u \cdot \nabla \phi \mathrm{d}x - \lambda \int_{\partial\Omega} r(x)(u - g(u))\phi \mathrm{d}S \quad \text{for all} \quad u, \phi \in X.$$

Now we verify that the conditions in the mountain pass theorem are satisfied.

Lemma 3.7. *Assume that $r(x)$ satisfies (r) and (3.1), and $(g1) - (g4)$ hold. Then*

(1) *$u = 0$ is a strict local minimum of E;*
(2) *given $v \in X$ with $v \neq 0$ on $\partial\Omega$, there exists $\rho_0 > 0$ such that $E(\rho_0 v) \leq 0$;*
(3) *E satisfies the Palais–Smale condition, i.e., let $\{u_n\}$ be any sequence in X such that $|E(u_n)|$ is uniformly bounded and $E'(u_n) \to 0$ as $n \to \infty$, then $\{u_n\}$ has a convergent subsequence.*

Proof. (1) In view of $(g2)$ and $(g3)$, given $\delta > 0$, there exists $C_\delta > 0$ such that

$$G(s) \leq \frac{1}{2}\delta |s|^2 + C_\delta |s|^{p+1}, \quad s \in \mathbb{R},$$

which implies that

$$E(u) \geq \frac{1}{2}\|u\|_X^2 - C_1 \left(\frac{\delta}{2} \|u\|^2_{L^2(\partial\Omega)} + C_\delta \|u\|^{p+1}_{L^{p+1}(\partial\Omega)} \right)$$

for some $C_1 > 0$. Since the embedding $X \hookrightarrow L^k(\partial\Omega)$ is compact if $k \in [1, p^* + 1)$, one can choose δ so small that

$$E(u) > 0 = E(0)$$

for all u with $0 < \|u\|_X \leq \epsilon_1$ and for some sufficiently small $\epsilon_1 > 0$.

(2) By $(g1), (g3)$ and $(g4)$, there exist positive constants c and d such that

$$G(s) \geq c|s|^\mu - d, \quad s \in \mathbb{R},$$

which implies

$$E(u) \leq \frac{1}{2}\|u\|_X^2 - C_2 \|u\|^\mu_{L^\mu(\partial\Omega)} + C_3$$

for some $C_2, C_3 > 0$. Given $v \in W^{1,2}(\Omega)$ with $\|v\|_{L^\mu(\partial\Omega)} > 0$,

$$E(\rho v) \leq \frac{1}{2}\|v\|_X^2 \rho^2 - C_2 \|v\|^\mu_{L^\mu(\partial\Omega)} \rho^\mu + C_3 \to -\infty \quad \text{as} \quad \rho \to \infty.$$

Thus there exists $\rho_0 > 0$ such that $E(\rho_0 v) \leq 0$.

(3) Let $\{u_n\}$ be a sequence such that $|E(u_n)| \le C_4$ for all $n \in \mathbb{N}$ and for some constant $C_4 > 0$, and $E'(u_n) \to 0$ as $n \to \infty$. Then, for all n sufficiently large, one has

$$|E'(u_n)u_n| \le \|u_n\|_X,$$

which implies

$$E(u_n) - \frac{1}{\mu}E'(u_n)u_n \le C_4 + \frac{1}{\mu}\|u_n\|_X.$$

Consequently, by $(g4)$,

$$\left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|_X^2 - C_5 \le C_4 + \frac{1}{\mu}\|u_n\|_X$$

for some constant $C_5 > 0$, and $\|u_n\|_X$ is bounded. By standard arguments, E satisfies the Palais–Smale condition (see, e.g. [33, Proposition 4.3]).

□

Now we are able to prove the following existence result for non-constant positive solutions of (1.1) for all large λ. Let $B(p,\delta) := \{x \in \mathbb{R}^N : |x-p| < \delta\}$ and $B(\delta) := B(0,\delta)$.

Theorem 3.8. *Suppose that $N \ge 2$, $r(x)$ satisfies (r) and (3.1), and $(g1)-(g5)$ hold. Then (1.1) has a non-constant positive solution for all sufficiently large $\lambda > 0$.*

Proof. Let $x_0 \in \partial\Omega$. Assume that there exist an open neighborhood U of x_0, $B(\delta_1)$ and a diffeomorphism $\Psi : U \to B(\delta_1)$ such that

(1) $\Psi(x_0) = 0$ and $D\Psi(x_0) = I$,
(2) $\Psi(U \cap \Omega) = \mathbb{R}^N_+ \cap B(\delta_1)$ and $\Psi(\partial\Omega \cap \overline{U}) = \partial\mathbb{R}^N_+ \cap \overline{B(\delta_1)}$.

Let $\lambda > 1/\delta_1^2$ and $V = \Psi^{-1}(B(\lambda^{-\frac{1}{2}}))$. Define a test function

$$e_\lambda(y) = \begin{cases} \lambda^{\frac{N-1}{2}}(1 - \lambda^{\frac{1}{2}}|y|), & |y| < \lambda^{-\frac{1}{2}}, \\ 0, & |y| \ge \lambda^{-\frac{1}{2}}. \end{cases}$$

Define $\tilde{e}_\lambda(x) = e_\lambda(\Psi^{-1}(y))$. Then, $\tilde{e}_\lambda \in W_0^{1,2}(\mathbb{R}^N)$. By straightforward computation, we have

$$\int_\Omega |\nabla \tilde{e}_\lambda|^2 \mathrm{d}x \le \int_V |\nabla \tilde{e}_\lambda|^2 \mathrm{d}x \le C_1 \int_{B(\lambda^{-\frac{1}{2}})} |\nabla e_\lambda|^2 \mathrm{d}y \le c_1 \lambda^{\frac{N}{2}}, \tag{3.8}$$

where $C_1, c_1 > 0$ are constants independent of λ. Furthermore, there exist C_2, c_2 also independent of λ such that

$$\int_{\partial\Omega} \tilde{e}_\lambda^2 \mathrm{d}S = \int_{\partial\Omega\cap V} \tilde{e}_\lambda^2 \mathrm{d}S \le C_2 \int_{\partial\mathbb{R}^N_+ \cap \overline{B(\delta_1)} \cap B(\lambda^{-\frac{1}{2}})} e_\lambda^2 \, \mathrm{d}s = c_2 \lambda^{\frac{N-1}{2}}. \tag{3.9}$$

Set $h(t) := E(t\tilde{e}_\lambda)$ for $t \in [0,\infty)$. By Lemma 3.7 (1) and (2), there exists $t_0 > 0$ such that $h(t_0) = 0$ and $h(t) > 0$ for all $t \in (0, t_0)$. Let $\Gamma = \{l \in C([0,1], X) : l(0) = 0,\ l(1) = t_0\tilde{e}_\lambda\}$. Then

$$c_\lambda := \inf_{l\in\Gamma} \max_{s\in[0,1]} E(l(s)) > 0$$

is a critical value of E in view of the mountain pass theorem of Ambrosetti and Rabinowitz [2]. Thus E has a critical point $u_\lambda \in X \setminus \{0\}$ with $E(u_\lambda) = c_\lambda > 0$.

On the other hand, we show that $c_\lambda = O(\lambda^{\frac{3-N}{2}})$ as $\lambda \to \infty$. By [26], there is a unique $\sigma \in (0,1)$ depending only on N such that

$$\int_{S_\sigma} e_\lambda^2 \mathrm{d}x = \frac{1}{2} \int_{\partial\mathbb{R}^N_+ \cap \overline{B(\delta_1)}} e_\lambda^2 \mathrm{d}x, \tag{3.10}$$

where $S_\sigma = \{x \in \partial\mathbb{R}^N_+ \cap \overline{B(\delta_1)} : e_\lambda > \sigma\lambda^{\frac{N-1}{2}}\}$. Set $\tilde{S_\sigma} = \Psi^{-1}(S_\sigma)$. From $(g2)$, it follows that for any $R > 0$, there exists $M_R > 0$ such that $g(s) > Rs$ for all $s \geq M_R$. Given $t > M_R\sigma^{-1}\lambda^{-\frac{N-1}{2}}$, let

$$S := \left\{ y \in \partial\mathbb{R}^N_+ \cap \overline{B(\delta_1)} : e_\lambda(y) > \frac{M_R}{t} \right\}, \quad \tilde{S} = \Psi^{-1}(S).$$

Then $\tilde{S}_\sigma \subset \tilde{S}$. Put $M = \max_{\partial\Omega}(-r(x))$ and $m = \min_{\partial\Omega}(-r(x))$. For $\lambda > 1/\delta_1^2$, by (3.8),(3.9) and (3.10) and noting that $r(x) < 0$ for $x \in \partial\Omega$, we have

$$\begin{aligned}
h'(t) &= t\left(\int_\Omega |\nabla\tilde{e}_\lambda|^2 \mathrm{d}x - \lambda \int_{\partial\Omega} r(x)\tilde{e}_\lambda^2 \mathrm{d}S \right) + \lambda \int_{\partial\Omega} r(x) g(t\tilde{e}_\lambda)\tilde{e}_\lambda \mathrm{d}S \\
&\leq c_1 t\lambda^{\frac{N}{2}} + Mt\lambda \int_{\partial\Omega} \tilde{e}_\lambda^2 \mathrm{d}S - m\lambda \int_{\partial\Omega\cap\tilde{S}} g(t\tilde{e}_\lambda)\tilde{e}_\lambda \mathrm{d}S \\
&\leq c_1 t\lambda^{\frac{N}{2}} + Mtc_2\lambda^{\frac{N+1}{2}} - C_3 m\lambda \int_{S_\sigma} g(te_\lambda)e_\lambda \mathrm{d}S \\
&\leq c_1 t\lambda^{\frac{N}{2}} + Mtc_2\lambda^{\frac{N+1}{2}} - RtC_3 m\lambda \int_{S_\sigma} e_\lambda^2 \mathrm{d}S \\
&= c_1 t\lambda^{\frac{N}{2}} + Mtc_2\lambda^{\frac{N+1}{2}} - \frac{1}{2}RtC_3 m\lambda \int_{\partial\mathbb{R}^N_+\cap\overline{B(\delta_1)}} e_\lambda^2 \mathrm{d}S \\
&= c_1 t\lambda^{\frac{N}{2}} + Mtc_2\lambda^{\frac{N+1}{2}} - \frac{1}{2}Rtc_3 m\lambda^{\frac{N+1}{2}} \\
&= t\lambda^{\frac{N+1}{2}}(c_1\lambda^{-\frac{1}{2}} + Mc_2 - \frac{1}{2}Rc_3 m) \\
&\leq t\lambda^{\frac{N+1}{2}}\left(c_1\delta_1 + Mc_2 - \frac{1}{2}Rc_3 m \right),
\end{aligned}$$

where $c_3, C_3 > 0$ are constants independent of λ. Choosing $R = R_1$ large enough such that $c_1\delta_1 + Mc_2 - \frac{1}{2}R_1 c_3 m < 0$, we see that $h'(t) < 0$ provided $t > t_1 := M_{R_1}\sigma^{-1}\lambda^{-\frac{N-1}{2}}$. Since, for any $t \in [0, \infty)$, $G(t\tilde{e}_\lambda(x)) \geq 0$ for all $x \in \partial\Omega$, and it follows from (3.8) and (3.9) that

$$h(t) \leq \frac{t^2}{2}\lambda^{\frac{N+1}{2}}\left(c_1\lambda^{-\frac{1}{2}} + Mc_2 \right) \leq \frac{t^2}{2}\lambda^{\frac{N+1}{2}}(c_1\delta_1 + Mc_2),$$

which implies that

$$\begin{aligned}
c_\lambda &\leq \max_{t\in[0,t_0]} E(t\tilde{e}_\lambda) \leq \max_{t\in[0,t_1]} h(t) \leq \frac{t_1^2}{2}\lambda^{\frac{N+1}{2}}(c_1\delta_1 + Mc_2) \\
&= \frac{1}{2}M_{R_1}^2\sigma^{-2}\lambda^{\frac{3-N}{2}}(c_1\delta_1 + Mc_2).
\end{aligned}$$

If w is a constant positive solution of (1.1), then it follows from $(g5)$ that $E(w) \geq c\lambda$, where $c = -\inf_{u\in Z}\{2^{-1}u^2 - G(u)\}\int_{\partial\Omega} r(x)\mathrm{d}S > 0$. Since $N \geq 2$ and $c_\lambda = O(\lambda^{\frac{3-N}{2}})$ as $\lambda \to \infty$, then we can conclude that u_λ is not a constant positive solution or zero solution but a non-constant positive solution of (1.1) for sufficiently large $\lambda > 0$. □

Define a function $\hat{f}(u) = u - \hat{g}(u)$ for $u \in \mathbb{R}$, where $\hat{g}(u) = 0$ for $u < 0$ and $\hat{g}(u) = g(u)$ for $u \geq 0$. If $f(u) = u - g(u)$ satisfies (f) and $(f1)$, then $\hat{g}(u)$ satisfies $(g1) - (g3)$ and $(g5)$. Hence, by Theorem 3.8, we have the following corollary:

Corollary 3.9. *Suppose that* (r), (f), $(f1)$, *and* $(g4)$ *hold. Then* (1.1) *has a non-constant positive solution for sufficiently large* $\lambda > 0$.

We comment that if, in addition, $g(u)/u$ is strictly increasing, then we can show that a least energy positive solution of (1.1) exists under the conditions of Theorem 3.8 or Corollary 3.9, following similar arguments in [37].

4. Existence and exact multiplicity for one-dimensional domain

When $N = 1$, (1.1) becomes the following two-point boundary value problem

$$\begin{cases} u''(x) = 0, \quad x \in (0,1), \\ -u'(0) = \lambda r_0 f(u(0)), \quad u'(1) = \lambda r_1 f(u(1)), \end{cases} \tag{4.1}$$

where λ is a nonnegative parameter, and $f(u)$ satisfies (f). Here we assume that $r_0 < 0$ and $r_1 < 0$. In this section, we also assume that $f(u)$ satisfies

$(f2)$ There exists a unique $u_1 \in (0,1)$ such that $f'(u) > 0$ for $u \in [0, u_1)$, $f'(u_1) = 0$ and $f'(u) < 0$ for $u \in (u_1, \infty)$, and $\lim_{u \to \infty} f(u) = -\infty$.

If u is a solution of (4.1), u is a linear function, i.e., $u(x) = Ax + B$, for some $A, B \in \mathbb{R}$. We can still use the bifurcation approach in Sect. 3 to consider the solutions of (4.1), which we briefly discuss without detailed proof. Define $X_1 := W^{2,p}(0,1)$ and $Y_1 := L^p(0,1) \times \mathbb{R} \times \mathbb{R}$, where $p > 1$, and define a nonlinear mapping $H : \mathbb{R} \times X_1 \to Y_1$ by

$$H(\lambda, u) = (u'', -u'(0) - \lambda r_0 f(u(0)), u'(1) - \lambda r_1 f(u(1))). \tag{4.2}$$

Then similar to Lemma 3.1, the only possible bifurcation points from the lines of trivial solutions are $(\lambda, 1)$ where λ are the eigenvalues of the following eigenvalue problem:

$$\begin{cases} \phi''(x) = 0, \quad x \in (0,1), \\ -\phi'(0) = \lambda r_0 f'(1)\phi(0), \quad \phi'(1) = \lambda r_1 f'(1)\phi(1), \end{cases} \tag{4.3}$$

From the results in Sect. 2.1, we have only one positive eigenvalue $\lambda_1 = \dfrac{r_0 + r_1}{r_0 r_1 f'(1)} > 0$ with a corresponding eigenfunction $\phi_1(x) = x - \dfrac{r_1}{r_0 + r_1}$. Similar to Theorem 3.3, we can show a local bifurcation from Γ_1 occurs at $(\lambda, u) = (\lambda_1, 1)$.

Theorem 4.1. *Assume that* $f(u)$ *satisfies* (f), *and* $r_i < 0$ *for* $i = 1, 2$. *Then the solution set of* (4.1) *near* $(\lambda_1, 1)$ *consists precisely of the curves* Γ_1 *and*

$$\Sigma = \{(\lambda(t), u(t)) : t \in I = (-\eta, \eta) \subset \mathbb{R}\},$$

where $\lambda(t) = \lambda_1 + z_1(t)$ *and* $u(t) = 1 + t\phi_1 + tz_2(t)$ *are continuous functions such that* $z_i(0) = 0$, $i = 1, 2$. *Moreover, suppose that* f *is sufficiently smooth near* $u = 1$,

1. *if* $r_0 < r_1$ *and* $f''(1) < 0$, *then* $\lambda'(0) < 0$;
2. *if* $r_0 > r_1$ *and* $f''(1) < 0$, *then* $\lambda'(0) > 0$;
3. *if* $r_0 = r_1$ *and* $-f'(1)f'''(1) + 3(f''(1))^2 > 0$, *then* $\lambda'(0) = 0$ *and* $\lambda''(0) > 0$.

The proof of Theorem 4.1 is similar to that of Theorem 3.3. We only point out that if $f \in C^2$ near $u = 1$,

$$\lambda'(0) = \frac{(r_1 - r_0)f''(1)}{2r_0r_1(f'(1))^2},$$

and if $r_0 = r_1$, then $\lambda'(0) = 0$ and if $f \in C^3$ near $u = 1$, then

$$\lambda''(0) = \frac{-f'(1)f'''(1) + 3(f''(1))^2}{6r_0(f'(1))^3}.$$

For the $N = 1$ case, by using the fact that any solution $u(x)$ must be a linear function, we can obtain a more precise global bifurcation diagram. For that purpose, we set a solution $u(x)$ of (4.1) to be

$$u(x) = (C - B)x + B = Cx + B(1 - x), \tag{4.4}$$

where $B = u(0)$ and $C = u(1)$. Then the boundary conditions become

$$B - C = \lambda r_0 f(B), \quad C - B = \lambda r_1 f(C). \tag{4.5}$$

Hence a solution (λ, u) of (4.1) is equivalent to a solution (λ, B, C) of (4.5). Any non-constant solution $u(x)$ of (4.1) satisfies $C \neq B$, while $B = C = 0$ and $B = C = 1$ give the two trivial solutions $u = 0$ and $u = 1$ for any $\lambda > 0$, and $B = C > 0$ gives the trivial solution for $\lambda = 0$.

Adding the two equations in (4.5) implies that B and C must satisfy a relation

$$r_0 f(B) + r_1 f(C) = 0. \tag{4.6}$$

Since $f(u)$ satisfies (f) and $(f2)$, then the relation of B and C can be further determined as follows:

Lemma 4.2. *Suppose that $f(u)$ satisfies (f) and $(f2)$, $r_0 < 0$ and $r_1 < 0$. Then*

1. *For any fixed $0 < B < 1$, there exists a unique $C = C_1(B) > 1$ such that (4.6) holds; moreover, the function $C_1 : (0, 1) \to (1, \infty)$ is smooth such that $C_1'(B) > 0$ for $0 < B < u_1$, $C_1'(B) < 0$ for $u_1 < B < 1$, and*
$$\lim_{B\to 0^+} C_1(B) = \lim_{B\to 1^-} C_1(B) = 1. \tag{4.7}$$
2. *There exists $B_* > 1$ such that for any $B > B_*$, there is no $C > 0$ such that (4.6) holds; for any fixed $1 < B < B_*$, there exist exactly two $C = C_2(B), C_3(B) \in (0, 1)$ such that $C_2(B) > u_1 > C_3(B)$, and (4.6) holds for $(B, C_2(B))$ and $(B, C_3(B))$; moreover, the functions $C_i : (1, B_*) \to (0, 1)$ $(i = 2, 3)$ are smooth such that $C_2'(B) < 0$ and $C_3'(B) > 0$ for $1 < B < B_*$, and*
$$\lim_{B\to 1^+} C_2(B) = 1, \ \lim_{B\to 1^+} C_3(B) = 0, \ \lim_{B\to B_*^-} C_2(B) = \lim_{B\to B_*^-} C_3(B) = u_1. \tag{4.8}$$

Proof. The relation (4.6) implies that

$$f(C) = -\frac{r_0}{r_1} f(B). \tag{4.9}$$

Since $B, C > 0$, then $B \in (0, 1)$ implies that $C > 1$, and $B > 1$ implies that $C \in (0, 1)$. We first assume that $B \in (0, 1)$, then there exists $C > 1$ such that (4.9) holds since $f(1) = 0$ and $f(u) \to -\infty$ as $u \to \infty$ from $(f2)$, and such C is unique since $f'(u) < 0$ for $u > 1$. We denote this C by $C_1(B)$, and we have

$$f(C_1(B)) = -\frac{r_0}{r_1} f(B). \tag{4.10}$$

By differentiating (4.10) in B, we obtain that

$$f'(C_1(B))C_1'(B) = -\frac{r_0}{r_1} f'(B), \tag{4.11}$$

which implies that $C_1'(B) > 0$ for $0 < B < u_1$, $C_1'(B) < 0$ for $u_1 < B < 1$. The limits in (4.7) is clear from (4.9) and the fact that $f(B) \to 0$ as $B \to 0^+$ or $B \to 1^-$. The case of $B > 1$ can be proved similarly, by observing that the graphs of $(B, C_1(B))$ and the inverse function of $(B, C_2(B))$ and $(B, C_3(B))$ have

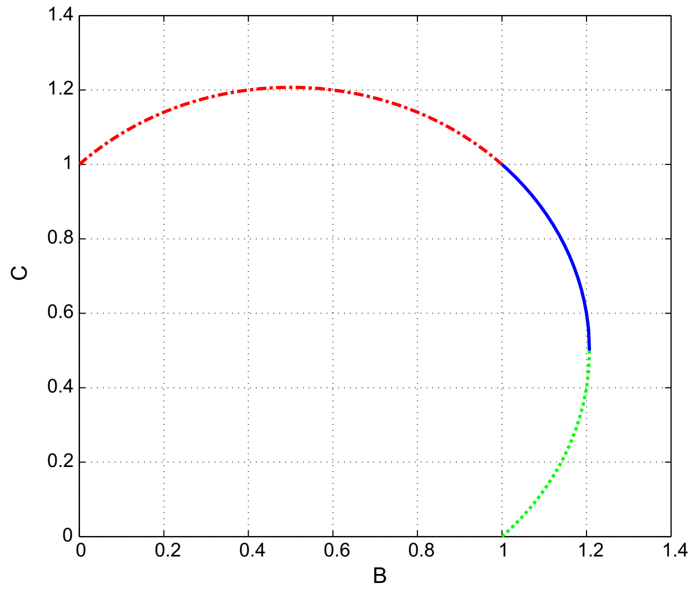

FIG. 1. The graphs of $C = C_i(B)\,(i = 1, 2, 3)$ when $f(u) = u - u^2$ and $r_0 = r_1 = -1$. Here the arc between $(0, 1)$ and $(1, 1)$ is $C_1(B)$, the one between $(1, 1)$ and $(B_*, 0.5)$ is $C_2(B)$, and the one between $(B_*, 0.5)$ and $(1, 0)$ is $C_3(B)$, where $B_* = (1 + \sqrt{2})/2$

the same structure (see, e.g. Fig. 1). Here $B_* > 1$ can be determined uniquely by $r_0 f(B_*) + r_1 f(u_1) = 0$ since $f'(u) < 0$ for $u > u_1$. □

The structure of $C_i(B)$ $(i = 1, 2, 3)$ given in Lemma 4.2 indicates that the solutions of (4.1) can be classified as follows:

Corollary 4.3. *Suppose that $f(u)$ satisfies (f) and $(f2)$, $r_0 < 0$ and $r_1 < 0$. Then any non-constant solution $u(x)$ of (4.1) is a linear function in form of (4.4), with either*

1. $0 < B < 1$, $C = C_1(B) > 1$, *and the corresponding $u(x)$ is increasing; or*
2. $1 < B \leq B_*$, $C = C_2(B) \in (0, 1)$ *or* $C = C_3(B) \in (0, 1)$, *and the corresponding $u(x)$ is decreasing.*

It remains to determine the parameter λ from B and C. From (4.5), we obtain that

$$\lambda = \lambda_i(B) = \frac{B - C_i(B)}{r_0 f(B)}, \quad i = 1, 2, 3. \tag{4.12}$$

Here the domain of $\lambda_i(B)$ is same as the one for $C_i(B)$, $(i = 1, 2, 3)$.

Now we are ready to state the global bifurcation result for (4.1).

Theorem 4.4. *Suppose that $f(u)$ satisfies (f) and $(f2)$, $r_0 < 0$ and $r_1 < 0$. Define*

$$\Sigma = \bigcup_{i=1}^{3} \Sigma_i, \quad \text{where} \quad \Sigma_i = \{(\lambda_i(B), B) : B \in I_i\}, \tag{4.13}$$

$I_1 = (0, 1)$, and $I_2 = I_3 = (1, B_*]$ where B_* is defined in Lemma 4.2. Then

1. *If (λ, u) is a positive solution of (4.1), then there exists $i \in \{1, 2, 3\}$ and $B \in I_i$ such that $u(x) = C_i(B)x + B(1 - x)$ and $\lambda = \lambda_i(B)$ which is defined in (4.12).*

2. $\overline{\Sigma}$ *is a smooth curve in* $\mathbb{R}^2_+$ *satisfying*

$$\begin{aligned} &\lambda_1(1^-) = \lambda_2(1^+) = \frac{r_0 + r_1}{r_0 r_1 f'(1)} \equiv \lambda_1, \ \ \lambda_1'(1^-) = \lambda_2'(1^+), \\ &\lambda_2(B_*^-) = \lambda_3(B_*^-), \ \ \lambda_2'(B_*^-) = \lambda_3'(B_*^-), \\ &\lim_{B\to 0^+} \lambda_1(B) = \lim_{B\to 1^+} \lambda_3(B) = \infty. \end{aligned} \tag{4.14}$$

3. *Let* $\lambda_* = \min_i \inf_{B\in I_i} \lambda_i(B)$. *Then* $0 < \lambda_* \le \lambda_1$. *For each* $\lambda > \lambda_*$ *and* $\lambda \ne \lambda_1$, (4.1) *possesses at least two non-constant positive solutions, and when* $\lambda > \lambda_1$, (4.1) *possesses at least one increasing positive solution and one decreasing positive solution.*

Proof. If (λ, u) is a positive solution of (4.1), then from arguments given above, $u(x) = Cx + B(1 - x)$, (C, B) satisfies $C = C_i(B)$ and $B \in I_i$ from Lemma 4.2. Hence the set of positive solutions of (4.1) is equivalent to Σ. The continuity of $\lambda_i(B)$ and $\lambda_i'(B)$ at $B = 1$ and $B = B_*$ can be easily established from the smoothness properties of $C_i(B)$ and $f(B)$. The limits of $\lambda_i(B)$ can also be easily shown from properties of $C_i(B)$ in Lemma 4.2.

From (4.14) (especially the infinite limits), one can see that $0 < \lambda_* \le \lambda_1$, and for each $\lambda > \lambda_*$, $\lambda = \lambda_i(B)$ is achieved at least twice on Σ except when $B = 1$ and $\lambda = \lambda_1$, and thus (4.1) possesses at least two non-constant positive solutions for each $\lambda > \lambda_*$ and $\lambda \ne \lambda_1$. For $\lambda > \lambda_1$, (4.1) has at least one positive solution on Σ_1 (which consists of increasing solutions), and another on $\Sigma_2 \cup \Sigma_3$ (which consists of decreasing solutions). □

As $\lambda \to \infty$, (4.1) has two positive solutions with (B, C) approaching to $(1, 0)$ or $(0, 1)$, which implies that the two solutions with patterns $u_1^\infty(x) = x$ and $u_2^\infty(x) = 1 - x$ respectively. In general, $\lambda = \lambda_*$ is a saddle-node bifurcation point, while $\lambda = \lambda_1$ is a transcritical bifurcation point. From Theorem 4.1, $\lambda_* < \lambda_1$ when $r_0 \ne r_1$. When $r_0 = r_1$, it is likely $\lambda_* = \lambda_1$ and two bifurcation points merge to create a pitchfork bifurcation. For f satisfying more restrictive convexity condition, it is possible to show that for each $\lambda > \lambda_*$ and $\lambda \ne \lambda_1$, (4.1) possesses exactly two non-constant positive solutions. Here we only point out that for the prototypical $f(u) = u - u^2$, the exact multiplicity results holds. Indeed, for $f(u) = u - u^2$, $C_i(B)$ and $\lambda_i(B)$ can be explicitly solved as

$$C_1(B) = \frac{1 + \sqrt{1 + 4r_2(B - B^2)}}{2}, \ \ \lambda_1(B) = \frac{B - C_1(B)}{r_0(B - B^2)}, \ \ B \in (0, 1], \tag{4.15}$$

$$C_2(B) = \frac{1 + \sqrt{1 + 4r_2(B - B^2)}}{2}, \ \ \lambda_2(B) = \frac{B - C_2(B)}{r_0(B - B^2)}, \ \ B \in (1, B_*], \tag{4.16}$$

$$C_3(B) = \frac{1 - \sqrt{1 + 4r_2(B - B^2)}}{2}, \ \ \lambda_3(B) = \frac{B - C_3(B)}{r_0(B - B^2)}, \ \ B \in (1, B_*], \tag{4.17}$$

where

$$r_2 = \frac{r_0}{r_1}, \ \ B_* = \frac{1 + \sqrt{1 + r_2^{-1}}}{2}. \tag{4.18}$$

The exact multiplicity of solutions for this case can be easily deduced from the explicit form above. Figure 2 shows the bifurcation diagrams of (4.1) with $f(u) = u - u^2$. One can see that a saddle-node bifurcation occurs in the portion Σ_1 when $r_0 < r_1 < 0$, and it occurs in the portion Σ_2 when $r_1 < r_0 < 0$. In all three diagrams in Fig. 2, the bifurcation point is at $\lambda_1 = 1.5$, but when $r_1 \ne r_2$, there is a saddle-node bifurcation point $\lambda_* < \lambda_1$ such that two non-constant positive solutions also exist for $\lambda \in (\lambda_*, \lambda_1)$.

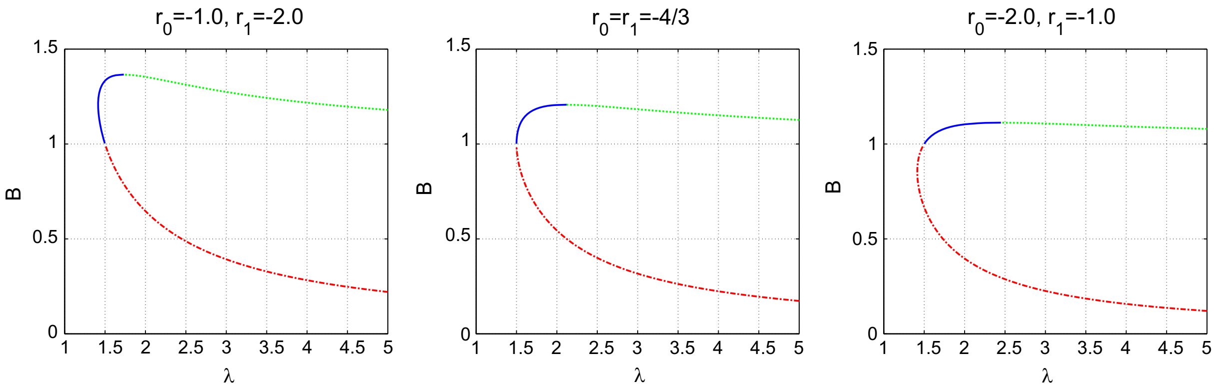

FIG. 2. The bifurcation diagrams for (4.1) when $f(u) = u - u^2$. The *horizontal axis* is λ, and the *vertical axis* is B. *Left*: $r_0 = -1$ and $r_1 = -2$; *middle*: $r_0 = r_1 = -4/3$; *right*: $r_0 = -2$ and $r_1 = -1$

5. Proof of Lemma 3.4

Proof of Lemma 3.4. Fix $\epsilon > 0$, we first prove that positive solutions of (1.1) with $\lambda \in [\epsilon, \infty)$ are uniformly bounded. Note that

$$\|u\|_{L^\infty(\Omega)} = \|u\|_{L^\infty(\partial\Omega)}$$

by Lemma 2.2. Assume on the contrary that there exist a sequence $\{\lambda^k\}$ with $\lambda^k \in [\epsilon, \infty)$, a sequence of non-constant solutions $\{u_k\}$ of (1.1) for $\lambda = \lambda^k$, and a sequence of points $\{P_k\}$ on $\partial\Omega$ such that

$$M_k := \max_{x\in\overline{\Omega}} u_k(x) = u_k(P_k) \to \infty, \quad P_k \to P \in \partial\Omega,$$

as $k \to \infty$.

Without loss of generality, we may assume that P is the origin and the x_N-axis is normal to $\partial\Omega$ at P. Then there exists a smooth function $\psi(x')$, $x' = (x_1, \cdots, x_{N-1})$, defined for $|x'| < \delta_0$ satisfying $\psi(0) = 0$, $(\partial\psi/\partial x_j)(0) = 0$ for $j = 1, \cdots, N-1$, $\Omega \cap \mathcal{O} = \{(x', x_N) : x_N > \psi(x')\}$, and $\partial\Omega \cap \mathcal{O} = \{(x', x_N) : x_N = \psi(x')\}$ in a neighborhood of $\mathcal{O}$ of P. For $y \in \mathbb{R}^N$ with $|y|$ sufficiently small, we define a mapping $x = \Phi(y) = (\Phi_1(y), \cdots, \Phi_N(y))$ by $\Phi_j(y) = y_j - y_N(\partial\psi/\partial x_j)(y')$ for $j = 1, \cdots, N-1$ and $\Phi_N(y) = y_N + \psi(y')$. Since $\Phi'(0) = I$, Φ has the inverse mapping $y = \Psi(x) := \Phi^{-1}(x)$ in the neighborhood of $x = 0$. We write $\Psi(x) = (\Psi_1(x), \cdots, \Psi_N(x))$, and put

$$a_{ij}(y) := \sum_{l=1}^{N} \frac{\partial\Psi_i}{\partial x_l}(\Phi(y))\frac{\partial\Psi_j}{\partial x_l}(\Phi(y)),$$
$$b_j(y) := (\Delta\Psi_j)(\Phi(y)),$$

where $1 \le i, j \le N$. Defining $v_k(y) = u_k(x)$, then v_k satisfies

$$\begin{cases} \displaystyle\sum_{i,j=1}^{N} a_{ij}(y)\frac{\partial^2 v_k}{\partial y_i \partial y_j} + \sum_{j=1}^{N} b_j(y)\frac{\partial v_k}{\partial y_j} = 0, & y \in B_{2\delta}^+, \\ \displaystyle\frac{\partial v_k}{\partial y_N} = -\lambda^k r(\Phi(y)) f(v_k), & y \in \{y_N = 0\} \cap B_{2\delta}, \end{cases} \tag{5.1}$$

where $B_{2\delta} = \{y \in \mathbb{R}^N : |y| < 2\delta\}$, $B_{2\delta}^+ = B_{2\delta} \cap \mathbb{R}_+^N$, and $\delta > 0$ is sufficiently small. Moreover, we put $Q_k = \Psi(P_k)$ and also write $Q_k = (q_k', 0)$. Since $Q_k \to 0$ as $k \to \infty$, we may assume that $|Q_k| < \delta$ for all

k. Let $d_k = (\lambda^k)^{-1} M_k^{1-p}$. Then $d_k \to 0$ as $k \to \infty$. We define a scaled function by

$$w_k(z) = M_k^{-1} v_k(d_k z' + q_k', d_k z_N). \tag{5.2}$$

Note that w_k is well defined in the half ball B^+_{δ/d_k} and that $0 < w_k(z) \le 1$ for all k. By (5.1), w_k satisfies

$$\begin{cases} \displaystyle\sum_{i,j=1}^{N} a_{ij}^k(z) \frac{\partial^2 w_k}{\partial z_i \partial z_j} + d_k \sum_{j=1}^{N} b_j^k(z) \frac{\partial w_k}{\partial z_j} = 0, & z \in B^+_{\delta/d_k}, \\ \displaystyle\frac{\partial w_k}{\partial z_N} = -r_k(z)(M_k^{1-p} w_k - M_k^{-p} g(M_k w_k)), & z \in \{z_N = 0\} \cap B_{\delta/d_k}, \end{cases} \tag{5.3}$$

where $a_{ij}^k(z) = a_{ij}(d_k z' + q_k', d_k z_N)$, $b_j^k(z) = b_j(d_k z' + q_k', d_k z_N)$, and $r_k(z) = r(\Phi(d_k z' + q_k', d_k z_N))$. Choose a sequence $\{R_n\}$ such that $R_n \to \infty$ as $n \to \infty$. For fixed n, $B^+_{4R_n} \subset B^+_{\delta/d_k}$ provided k is sufficiently large. Note that $a_{ij}^k(z)$ and $b_j^k(z)$ are uniformly bounded in k with respect to $C^2(\overline{B}_{\delta/d_k})$-norm, and $r_k(z)$ is uniformly bounded in k with respect to $C^2(\{z_N = 0\} \cap B_{\delta/d_k})$-norm. By $(f1)$,

$$\lim_{k\to\infty} |M_k^{-p} g(M_k w_k(z)) - A_1 w_k^p(z)| = 0,$$

and $M_k^{-p} f(M_k w_k(z))$ remains uniformly bounded in $\{z_N = 0\} \cap B_{\delta/d_k}$. Applying the elliptic L^r-estimates to (5.3) in the domain $\overline{B}^+_{2R_n}$, $\{w_k\}$ is uniformly bounded in $W^{2,r}(B^+_{2R_n})$ for each $r > 1$. Choosing $r > N$, $\{w_k\}$ is uniformly bounded in $C^{1,\beta}(\overline{B}^+_{2R_n})$, where $\beta \in (0,1)$. By the Schauder estimates for elliptic equations, on each $D \subsetneq B^+_{R_n}$, $\{w_k\}$ is uniformly bounded in $C^{2,\beta}(\overline{D})$ with $\beta \in (0,1)$. By standard arguments, there exists a subsequence, still denoted by $\{w_k\}$, such that w_k converges uniformly to $w \in C^{2,\beta'}(\mathbb{R}^N_+) \cap C^{1,\beta'}(\overline{\mathbb{R}}^N_+)$, for $\beta' \in (0, \beta)$, on any compact subset of $\mathbb{R}^N_+$. It follows from $\Psi'(0) = I$ that $a_{ij}(0) = \delta_{ij}$. Since $a_{ij}^k(z) \to a_{ij}(0)$ and $d_k \to 0$ as $k \to \infty$, w is a nonnegative solution of

$$\begin{cases} -\Delta w = 0, & \text{in} \quad \mathbb{R}^N_+, \\ \displaystyle\frac{\partial w}{\partial z_N} = A_1 r(0) w^p, & \text{on} \quad \{z_N = 0\}. \end{cases} \tag{5.4}$$

Since $A_1 r(0) < 0$, $w \equiv 0$ by [22, Theorem 1.1 and Theorem 1.2] (or see [41, Sect. 4]), which is a contradiction to the fact that

$$w(0) = \lim_{k\to\infty} w_k(0) = \lim_{k\to\infty} M_k^{-1} v_k(Q_k) = 1.$$

Thus positive solutions of (1.1) with $\lambda \in [\epsilon, \infty)$ are uniformly bounded.

By the same argument as above, we can prove that there exists $C_0 > 0$ such that for all solutions u_λ with $\lambda \in (0, \epsilon]$,

$$\max_{x\in\overline{\Omega}} u_\lambda(x) \le C_0 \lambda^{\frac{-1}{p-1}}. \tag{5.5}$$

Here C_0 is independent of $\lambda \in (0, \epsilon]$. Indeed, if we assume that (5.5) does not hold, and again let $d_k = (\lambda^k)^{-1} M_k^{1-p}$ and $M_k = \max_{x\in\overline{\Omega}} u_k(x)$, then $d_k \to 0$ as $k \to \infty$, and we can proceed to a contradiction as above.

Let u be a non-constant solution of (1.1) with $\lambda \in (0, \epsilon]$. By $(f1)$, there exists $A_2 > 0$ such that

$$g(z) \le \frac{1}{2}(z + A_2 z^p) \quad \text{for} \quad z \ge 0. \tag{5.6}$$

Multiplying the equation in (1.1) by u^{2s-1} $(s \geq 1)$ and integrating it over Ω, by (5.6), we have

$$\frac{2s-1}{s^2}\int_{\Omega}|\nabla(u^s)|^2\mathrm{d}x - \lambda\int_{\partial\Omega} r(x)u^{2s}\mathrm{d}S$$
$$= -\lambda\int_{\partial\Omega} r(x)u^{2s-1}g(u)\mathrm{d}S \leq -\lambda\int_{\partial\Omega} r(x)\frac{1}{2}(u^{2s}+A_2u^{2s-1+p})\mathrm{d}S,$$

which implies that, by (5.5),

$$\frac{2s-1}{s^2}\int_{\Omega}|\nabla(u^s)|^2\mathrm{d}x \leq -A_2C_0^{p-1}\int_{\partial\Omega} r(x)u^{2s}\mathrm{d}S.$$

For $s \geq 1$, we have $\dfrac{s^2}{2s-1} \leq s$, and thus

$$\int_{\Omega}|\nabla(u^s)|^2\mathrm{d}x \leq sA_2C_0^{p-1}\max_{x\in\partial\Omega}(-r(x))\int_{\partial\Omega} u^{2s}\mathrm{d}S. \tag{5.7}$$

Note that the norm

$$\|w\|_1 = \left(\int_{\Omega}|\nabla w|^2\mathrm{d}x + \int_{\partial\Omega} w^2\mathrm{d}S\right)^{\frac{1}{2}}$$

is equivalent to the usual norm in $H^1(\Omega)$. By a boundary trace imbedding theorem [1, Theorem 5.36], there exists a constant $\gamma > 0$ such that for all $w \in H^1(\Omega)$,

$$\left(\int_{\partial\Omega} w^{\nu}\mathrm{d}S\right)^{\frac{1}{\nu}} \leq \gamma\left(\int_{\Omega}|\nabla w|^2\mathrm{d}x + \int_{\partial\Omega} w^2\mathrm{d}S\right)^{\frac{1}{2}},$$

where $\nu = 2(N-1)/(N-2)$ if $N \geq 3$, and ν is fixed such that $\nu > 2$ if $N = 2$. It follows from (5.7) that, for all $s \geq 1$,

$$\left(\int_{\partial\Omega} u^{s\nu}\mathrm{d}S\right)^{\frac{2}{\nu}} \leq C_1 s\int_{\partial\Omega} u^{2s}\mathrm{d}S, \tag{5.8}$$

where

$$C_1 = \gamma^2\left(A_2C_0^{p-1}\max_{x\in\partial\Omega}(-r(x)) + 1\right).$$

Let

$$r_j = p(2^{-1}\nu)^{j-1}, \quad \alpha_j = \int_{\partial\Omega} u^{r_j}\mathrm{d}S, \tag{5.9}$$

for $j \geq 1$. Then, by (5.8), we have

$$\alpha_{j+1} \leq (C_2r_j)^{\frac{\nu}{2}}\alpha_j^{\frac{\nu}{2}} \quad \text{for} \quad j \geq 1, \tag{5.10}$$

where $C_2 = C_1/2$. Let $\mu_j = \log\alpha_j$ for $j \geq 1$. By (5.9) and (5.10), there exists $C^* > 0$ such that

$$\mu_{j+1} \leq \frac{\nu}{2}\mu_j + C^*(j+1) \quad \text{for} \quad j \geq 1.$$

Define $\{\tau_j\}$ by $\tau_1 = \mu_1$ and $\tau_{j+1} = \frac{\nu}{2}\tau_j + C^*(j+1)$ for $j \geq 1$. Then $\mu_j \leq \tau_j$ for all $j \geq 1$. By the same arguments as in the proof of [26, Corollary 2.1 and Theorem 3],

$$\|u\|_{L^\infty(\partial\Omega)} \leq C_3 \alpha_1^{\frac{1}{p}}, \tag{5.11}$$

for some constant $C_3 > 0$.

On the other hand, integrating the equation in (1.1) over Ω, we obtain that

$$-\int_{\partial\Omega} r(x) u \mathrm{d}S = -\int_{\partial\Omega} r(x) g(u) \mathrm{d}S,$$

so that

$$\int_{\partial\Omega} g(u)\mathrm{d}S \leq \left(\max_{x\in\partial\Omega}(-r(x))\right)\left(\min_{x\in\partial\Omega}(-r(x))\right)^{-1}\int_{\partial\Omega} u\mathrm{d}S.$$

It follows from (f_1) that there exist positive constants b_1, b_2 such that

$$g(z) \geq b_1 z^p - b_2, \quad z \geq 0,$$

and using Hölder inequality, we have

$$\int_{\partial\Omega} u^p \mathrm{d}S \leq b_3,$$

where b_3 is a positive constant depending only on g and $|\partial\Omega|$. Thus the proof is complete by (5.11). □

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(Received: November 25, 2014; revised: July 18, 2015)