DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **34**, Number **9**, September **2014**

doi:10.3934/dcds.2014.34.3875

pp. **3875–3899**

POSITIVE STEADY STATE SOLUTIONS OF A DIFFUSIVE LESLIE-GOWER PREDATOR-PREY MODEL WITH HOLLING TYPE II FUNCTIONAL RESPONSE AND CROSS-DIFFUSION

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(Communicated by Hirokazu Ninomiya)

Abstract. In this paper we consider a diffusive Leslie-Gower predator-prey model with Holling type II functional response and cross-diffusion under zero Dirichlet boundary condition. By using topological degree theory, bifurcation theory, energy estimates and asymptotic behavior analysis, we prove the existence, uniqueness and multiplicity of positive steady states solutions under certain conditions on the parameters.

1. **Introduction.** Consider the following steady state prey-predator model with nonlinear diffusions:

$$\begin{cases} -\Delta[(d_1+\tilde{\alpha}\tilde{v})\tilde{w}] = \tilde{w}\left(\tilde{a} - e\tilde{w} - \dfrac{\tilde{c}_1\tilde{v}}{\tilde{w}+\tilde{k}_1}\right), & x\in\Omega, \\ -d_2\Delta\tilde{v} = \tilde{v}\left(\tilde{b} - \dfrac{\tilde{c}_2\tilde{v}}{\tilde{w}+\tilde{k}_2}\right), & x\in\Omega, \\ \tilde{w}=\tilde{v}=0, & x\in\partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^N$, $N\geq 1$, is a bounded open domain with smooth boundary $\partial\Omega$; $\tilde{a},\tilde{b},\tilde{e},\tilde{c}_1,\tilde{c}_2,\tilde{k}_1,\tilde{k}_2$ are positive constants; $\tilde{\alpha}$ is a nonnegative constant. Problem (1.1) models the interactions between a predator, with population density $\tilde{v}(x)$, and a prey, with population density $\tilde{w}(x)$, inhabiting a spatial region Ω. In reaction

2010 *Mathematics Subject Classification.* 35J57, 35K55, 92C15, 92C40.

Key words and phrases. Leslie-Gower predator-prey model, Holling type II functional response, positive steady state solutions, multiplicity, uniqueness, stability.

The first author is supported by NSFC grant 11126141 and 11201380, China Scholarship Council, the Fundamental Research Funds for the Central Universities grant XDJK2012B007, Doctor Fund of Southwest University grant SWU111021 and Educational Fund of Southwest University grant 2010JY053, the second author is supported by National Research Foundation of Korea Grant funded by the Korean Government (Ministry of Education, Science and Technology, NRF-2011-357-C00006), and the third author is supported by NSF grants DMS-1022648, DMS-1313243 and Shanxi 100-talent program.

terms, $\tilde{a}$ and $\tilde{b}$ are the growth rate of prey $\tilde{w}$ and predator $\tilde{v}$, respectively; e measures the strength of competition among individuals of prey species; $\tilde{c}_1$ is the maximum value of the per capita reduction rate of $\tilde{w}$ due to $\tilde{v}$; $\tilde{k}_1$ and $\tilde{k}_2$ measure the extent to which environment provides protection to prey $\tilde{w}$ and to predator $\tilde{v}$, respectively; $\tilde{c}_2$ has a similar meaning as $\tilde{c}_1$ (see [1, 12, 23]). In diffusion terms, positive constants d_1 and d_2 represent natural dispersive forces of movements of prey and predator, respectively. The nonlinear diffusion $\Delta[\tilde{\alpha}\tilde{v}\tilde{w}] = \tilde{\alpha}\nabla[\tilde{w}\nabla(\tilde{v}) + \tilde{v}\nabla\tilde{w}]$ produces the most characteristic term in (1.1), which models a tendency that prey escapes to region with lower predator density (see the monograph of Okubo and Levin [36] for a further ecological background).

By rescaling as follows

$$w = \frac{e}{d_1}\tilde{w},\ v = d_2\tilde{v},\ \alpha = \frac{\tilde{\alpha}}{d_1 d_2},\ a = \frac{\tilde{a}}{d_1},\ b = \tilde{b},$$
$$c_1 = \frac{e\tilde{c}_1}{d_1^2 d_2},\ c_2 = \frac{e\tilde{c}_2}{d_1 d_2},\ k_1 = \frac{e\tilde{k}_1}{d_1},\ k_2 = \frac{e\tilde{k}_2}{d_1},$$

(1.1) is equivalently rewritten as

$$\begin{cases} -\Delta[(1+\alpha v)w] = w\left(a - w - \dfrac{c_1 v}{w+k_1}\right), & x \in \Omega, \\ -\Delta v = v\left(b - \dfrac{c_2 v}{w+k_2}\right), & x \in \Omega, \\ w = v = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where α is a nonnegative constant, and a, b, c_1, c_2, k_1, k_2 are positive constants.

The system (1.2) is based on a classical predator-prey model of Leslie and Gower [23] with more reasonable Holling type II functional responses [16] in both prey and predator interaction terms (see [49] for more detailed explanation), and the corresponding ODE system is regarded as one of prototypical predator-prey systems in the ecological studies. The kinetic model of (1.2) was proposed based on the biological fact that if the predator v is more capable of switching from its favorite food, say the prey u, to other food options, then it has better ability to survive when the prey population is low.

On the other hand the spatial component of ecological interactions has been identified as an important factor in how ecological communities shaped, and understanding the role of space is challenging both theoretically and empirically [34]. Empirical evidence suggests that the spatial scale and the structure of environment can influence population interactions [2]. The reaction-diffusion system with cross-diffusion was proposed by Shigesada *et al.* in [45] to investigate the habitat segregation phenomena between two species. Since then, strongly coupled parabolic and elliptic equations have received considerable attention in recent years, and various forms of the systems have been considered in the literature (see [8, 22, 26, 29, 30, 31, 32, 39, 40, 41, 47] for competition models and see [9, 10, 11, 13, 14, 17, 19, 20, 21, 33, 35, 37, 42] for prey-predator models).

In this paper we consider the positive solutions of (1.2), which incorporates the cross-diffusion, Holling type II functional response (see the equation of w), and modified Leslie-Gower functional response (see the equation of v). The main concern here is the structure of the set of positive solutions of (1.2) under the combined effect of cross-diffusion, Holling type II functional response, and modified Leslie-Gower functional response. Positive solutions of (1.2) with $\alpha = 0$ have been considered in [48, 49]. In this paper, for the case of $\alpha \geq 0$, we prove some results on the existence, multiplicity, uniqueness and bifurcation structure of positive solutions to (1.2).

The organization of the remaining part of the paper is as follows. In Section 2, we give some preliminaries, which are essential tools in our later study. In Section 3, we consider the stability results about the trivial and semi-trivial solutions. In Section 4, we study the existence of positive solutions by using degree theory. In Section 5, the multiplicity of positive solutions is investigated. Finally the uniqueness of the positive solution when $N = 1$ is studied in Section 6.

2. **Preliminaries.**

In this section we list some notation, definitions and well-known facts which will be used in the sequel. We use $||\cdot||_X$ as the norm of Banach space X, $\langle\cdot,\cdot\rangle$ as the duality pair of a Banach space X and its dual space X^*. For a linear operator L, we use $\mathcal{N}(L)$ as the null space of L and $\mathcal{R}(L)$ as the range space of L, and we use $L[w]$ to denote the image of w under the linear mapping L. For a multilinear operator L, we use $L[w_1, w_2, \cdots, w_k]$ to denote the image of $(w_1, w_2, \cdots, w_k)$ under L, and when $w_1 = w_2 = \cdots = w_k$, we use $L[w_1]^k$ instead of $L[w_1, w_1, \cdots, w_1]$. For a nonlinear operator F, we use F_u as the partial derivative of F with respect to argument u.

First we recall some well-known abstract bifurcation theorems. Consider an abstract equation

$$F(\lambda, u) = 0,$$

where $F : \mathbb{R} \times X \to Y$ is a nonlinear differential mapping, and X, Y are Banach spaces such that X is continuously embedding in Y. The following bifurcation and stability theorems were obtained in [4, 5, 38] (see also [43, 44]).

Theorem 2.1. *Let U be a neighborhood of (λ_0, u_0) in $\mathbb{R} \times X$, and let $F : U \to Y$ be a twice continuously differentiable mapping. Assume that $F(\lambda, u_0) = 0$ for all $(\lambda, u_0) \in U$. At (λ_0, u_0), F satisfies*

$$\dim\mathcal{N}(F_u(\lambda_0, u_0)) = \mathrm{codim}\mathcal{R}(F_u(\lambda_0, u_0)) = 1.$$

and

$$F_{\lambda u}(\lambda_0, u_0)[w_0] \notin \mathcal{R}(F_u(\lambda_0, u_0)).$$

Here $\mathcal{N}(F_u(\lambda_0, u_0)) = \mathrm{span}\{w_0\}$. Let Z be the complement of span$\{w_0\}$ in X. Then the solution set of $F(\lambda, u) = 0$ near (λ_0, u_0) consists precisely of the curves $u = u_0$ and $\Gamma := \{(\lambda(s), u(s)) : s \in I = (-\epsilon, \epsilon)\}$, where $\lambda : I \to \mathbb{R}$, $z : I \to Z$ are C^1 functions such that $u(s) = u_0 + sw_0 + sz(s)$, $\lambda(0) = \lambda_0$, $z(0) = 0$, and

$$\lambda'(0) = -\frac{\langle \ell, F_{uu}(\lambda_0, u_0)[w_0, w_0]\rangle}{2\langle \ell, F_{\lambda u}(\lambda_0, u_0)[w_0]\rangle},$$

where $\ell \in Y^*$ satisfies $\mathcal{R}(F_u(\lambda_0, u_0)) = \{\phi \in Y : \langle \ell, \phi\rangle = 0\}$. Moreover if in addition, $F_u(\lambda, u)$ is a Fredholm operator for all $(\lambda, u) \in U$, then the bifurcation curve Γ is contained in Σ, which is a connected component of $\overline{S}$, where $\overline{S} := \{(\lambda, u) \in U : F(\lambda, u) = 0, u \neq u_0\}$; and either Σ is not compact in U, or Σ contains a point (λ_*, u_0) with $\lambda_* \neq \lambda_0$.

Theorem 2.2. *Assume that all assumptions in Theorem 2.1 are satisfied, and let $\{\lambda(t), u(t)\}$ be the solution curve in Theorem 2.1. Then there exists C^2 functions*

$m : (\lambda_0-\epsilon, \lambda_0+\epsilon) \to \mathbb{R}$, $z : (\lambda_0-\epsilon, \lambda_0+\epsilon) \to X$, $\mu : (-\delta, \delta) \to \mathbb{R}$, and $w : (-\delta, \delta) \to X$ such that

$$F_u(\lambda, u_0)z(\lambda) = m(\lambda)z(\lambda), \quad \lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon),$$
$$F_u(\lambda(t), u(t))w(t) = \mu(t)w(t), \quad t \in (-\delta, \delta),$$

where $m(\lambda_0) = \mu(0) = 0$, $z(\lambda_0) = w(0) = w_0$. Moreover, near $t = 0$ the functions $\mu(t)$ and $-t\lambda'(t)m'(\lambda_0)$ have the same zeros and, whenever $\mu(t) \neq 0$, the same sign. More precisely,

$$\lim_{t\to 0} \frac{-t\lambda'(t)m'(\lambda_0)}{\mu(t)} = 1.$$

Next we recall some well-known facts about linear elliptic equations and diffusive logistic equation. For each $q \in C(\overline{\Omega})$, let $\lambda_1(q)$ be the principal eigenvalue of

$$\begin{cases} -\Delta u + q(x)u = \lambda u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{2.1}$$

As is well known, the principal eigenvalue $\lambda_1(q)$ is given by the following variational characterization:

$$\lambda_1(q) = \inf_{\phi \in H_0^1(\Omega), \|\phi\|_{L^2(\Omega)}=1} \int_\Omega (|\nabla\phi|^2 + q(x)\phi^2)dx.$$

We denote $\lambda_1(0)$ by λ_1 and let $\phi_1(x)$ be the positive eigenfunction corresponding to λ_1 with $\|\phi_1\|_{L^2(\Omega)} = 1$. Furthermore, the principal eigenvalue $\lambda_1(q)$ has some useful properties as follows (see [22, Proposition A.1] or [46, Proposition 1.1]).

Theorem 2.3. (i)**:** *If $q_i \in C(\overline{\Omega})$ $(i = 1, 2)$ satisfy $q_1 \geq q_2$ in $\overline{\Omega}$ and $q_1 \not\equiv q_2$, then $\lambda_1(q_1) > \lambda_1(q_2)$.*

(ii)**:** *For $q_n \in C(\overline{\Omega})$ and $q \in C(\overline{\Omega})$, let $\phi_n \in H_0^1(\Omega)$ and $\phi \in H_0^1(\Omega)$ be the corresponding eigenfunctions of (2.1) satisfying $\|\phi_n\|_{L^2(\Omega)} = \|\phi\|_{L^2(\Omega)} = 1$, where $n \in \mathbb{N}$. If $\lim_{n\to\infty} \|q_n - q\|_{L^\infty(\Omega)} = 0$, then $\lim_{n\to\infty} \lambda_1(q_n) = \lambda_1(q)$ and $\lim_{n\to\infty} \phi_n = \phi$ strongly in $H_0^1(\Omega)$.*

(iii)**:** *Let (c, d) be an open interval and assume that a mapping $\beta \mapsto q_\beta$ is continuously differentiable from (c, d) to $C(\overline{\Omega})$ with respect to supremum norm. If $\phi_\beta \in H_0^1(\Omega)$ with $\|\phi_\beta\|_{L^2(\Omega)} = 1$ is the unique positive eigenfunction corresponding to $\lambda_1(q_\beta)$, then $\beta \mapsto \lambda_1(q_\beta)$ is continuously differentiable from (c, d) to $\mathbb{R}$ and*

$$\frac{d}{d\beta}\lambda_1(q_\beta) = \int_\Omega \frac{\partial q_\beta}{\partial\beta}\phi_\beta^2 dx.$$

For $q \in C(\overline{\Omega})$, let p be a sufficiently large constant such that $p - q(x) > 0$ for any $x \in \overline{\Omega}$. Define a bounded linear operator $T : C(\overline{\Omega}) \to C(\overline{\Omega})$ by $u = Tv = (-\Delta + pI)^{-1}(p - q(x))v$, where $u \in C(\overline{\Omega})$ is the unique solution of the following problem

$$\begin{cases} -\Delta u + pu = (p - q(x))v, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{2.2}$$

Denote $r(T)$ be the spectral radius of T. Then the relationship between $\lambda_1(q)$ and $r(T)$ can be given as follows (see [7, Proposition 1] or [25, Lemmas 2.1 and 2.3]).

Theorem 2.4. *Let $q \in C(\overline{\Omega})$ and let p be a sufficiently large number such that $p > q(x)$ for any $x \in \overline{\Omega}$. Then we have*

(i): $\lambda_1(q) > 0$ *if and only if* $r((-\Delta + pI)^{-1}(p - q(x))) < 1$;
(ii): $\lambda_1(q) < 0$ *if and only if* $r((-\Delta + pI)^{-1}(p - q(x))) > 1$;
(iii): $\lambda_1(q) = 0$ *if and only if* $r((-\Delta + pI)^{-1}(p - q(x))) = 1$.

Consider the following steady state problem for logistic equation with linear diffusion

$$\begin{cases} -\Delta u = u(l-u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{2.3}$$

where l is a positive constant and $\Omega \subset \mathbb{R}^N$ is a bounded open set with smooth boundary $\partial\Omega$. Then the following results are well known (see [7, Lemma 1] and [15, Propositions 6.1-6.4]).

Theorem 2.5. *(i)*: *If* $l \le \lambda_1$, *then* (2.3) *has no nontrivial solutions.*

(ii): *If* $l > \lambda_1$, *then there exists a unique positive solution* $\theta_l(x)$ *of* (2.3) *satisfying* $0 < \theta_l(x) < l$ *for all* $x \in \Omega$.

(iii): $\lim_{l \to \lambda_1^+} \theta_l(x) = 0$ *uniformly in* Ω. *More precisely,*

$$\theta_l = \left(\int_\Omega \phi_1^3 dx\right)^{-1} (l - \lambda_1)\phi_1 + o(l - \lambda_1) \quad \text{as } l \to \lambda_1^+.$$

(iv): $\lim_{l\to\infty} \theta_l(x) = \infty$ *and* $\lim_{l\to\infty} \theta_l(x)/l = 1$ *uniformly in* K, *where* K *is any compact subset of* Ω.

(v): *The mapping* $l \mapsto \theta_l$ *is* C^1 *from* (λ_1, ∞) *to* $C(\overline{\Omega})$ *and* $\theta_l(x)$ *is strictly increasing with respect to* l. *More precisely,*

$$\frac{\partial \theta_l}{\partial l} = (-\Delta + (2\theta_l - l)I)^{-1}\theta_l,$$

where $(-\Delta + (2\theta_l - l)I)^{-1}$ *is the inverse operator of* $-\Delta + (2\theta_l - l)I$ *with zero Dirichlet boundary condition.*

Finally we introduce some concepts of fixed point index theory in a cone [6]. Let E be a Banach space and $W \subset E$ be a closed convex set. W is called a total wedge in E if $\gamma W \subset W$ for all $\gamma \ge 0$ and $\overline{W - W} = E$. For $y \in W$, define $W_y = \{x \in E : y + \gamma x \in W \text{ for some } \gamma > 0\}$ and $S_y = \{x \in \overline{W}_y : -x \in \overline{W}_y\}$. Then $\overline{W}_y$ is a wedge containing W, y, $-y$, while S_y is a closed subset of E containing y. Let T be a compact linear operator on E which satisfies $T(\overline{W}_y) \subset \overline{W}_y$. We say that T has property $\mathfrak{a}$ on $\overline{W}_y$ if there exist $t \in (0,1)$ and $\omega \in \overline{W}_y \setminus S_y$ such that $(I - tT)\omega \in S_y$. Let $A : W \to W$ be a compact operator with a fixed point $y \in W$, and let D be a relatively open subset of W such that A has no fixed point on the boundary of D. We denote by $\deg_W(I - A, D)$ the degree of $I - A$ in D relative to W, and by $\text{index}_W(A, y)$ the fixed point index of A at y relative to W. The following result is well-known: (see [6], [25, Theorem D] or [27, Lemma 4.1]).

Theorem 2.6. *Assume that* W *is a total wedge, and let* $A : W \to W$ *be a compact operator with a fixed point* $y \in W$ *and it is Fréchet differentiable at* y. *Let* $L = A'(y)$ *be the Fréchet derivative of* A *at* y. *Then* L *maps* $\overline{W}_y$ *into itself. Moreover, if* $I - L$ *is invertible on* $\overline{W}_y$, *then the following results hold.*

(i): *If* L *has property* $\mathfrak{a}$ *on* $\overline{W}_y$, *then* $\text{index}_W(A, y) = 0$;
(ii): *If* L *does not have property* $\mathfrak{a}$ *on* $\overline{W}_y$, *then* $\text{index}_W(A, y) = (-1)^\sigma$, *where* σ *is the sum of multiplicities of all eigenvalues of* L *which is greater than* 1.

3. **Analysis of the trivial and semi-trivial solutions of (1.2).** In this section we analyze the trivial and semi-trivial solutions of (1.2). It is obvious that the trivial solution of (1.2) is $(0,0)$ and semi-trivial solutions of (1.2) are $(\theta_a, 0)$ (if $a > \lambda_1$) and $(0, k_2\theta_b/c_2)$ (if $b > \lambda_1$). Here θ_a and θ_b are the unique positive solutions of (2.3) with $l = a$ or $l = b$, respectively. The main result of this section is the following theorem.

Theorem 3.1. *Consider the system* (1.2).

(i)**:** *The trivial steady state* $(0,0)$ *is locally asymptotically stable if* $a < \lambda_1$ *and* $b < \lambda_1$*, while it is unstable if* $a > \lambda_1$ *or* $b > \lambda_1$*;*

(ii)**:** *Assume that* $a > \lambda_1$*. Then the semi-trivial steady state* $(\theta_a, 0)$ *is locally asymptotically stable if* $b < \lambda_1$*, while it is unstable if* $b > \lambda_1$*;*

(iii)**:** *Assume that* $b > \lambda_1$*. Then the semi-trivial steady state* $(0, k_2\theta_b/c_2)$ *is locally asymptotically stable if* $\lambda_1\left(\frac{ck_2\theta_b - ak_1c_2}{k_1(c_2+k_2\alpha\theta_b)}\right) > 0$*, while it is unstable if* $\lambda_1\left(\frac{ck_2\theta_b - ak_1c_2}{k_1(c_2+k_2\alpha\theta_b)}\right) < 0$.

Proof. We only prove the case (iii) since the proofs of other two cases are similar. From the linearization principle, the stability of $(0, k_2\theta_b/c_2)$ is determined by the following eigenvalue problem

$$\begin{cases} -\Delta\left[\left(1+\dfrac{k_2\alpha\theta_b}{c_2}\right)\phi\right] + \left(\dfrac{c_1k_2\theta_b}{k_1c_2} - a\right)\phi = \lambda\phi, & x \in \Omega, \\ -\Delta\psi - \dfrac{\theta_b^2}{c_2}\phi + (2\theta_b - b)\psi = \lambda\psi, & x \in \Omega, \\ \phi = \psi = 0, & x \in \partial\Omega. \end{cases} \tag{3.1}$$

Since (3.1) is not completely coupled, we only need to consider the following two eigenvalue problems

$$\begin{cases} -\Delta\psi + (2\theta_b - b)\psi = \lambda\psi, & x \in \Omega, \\ \psi = 0, & x \in \partial\Omega, \end{cases} \tag{3.2}$$

and

$$\begin{cases} -\Delta\left[\left(1+\dfrac{k_2\alpha\theta_b}{c_2}\right)\phi\right] + \left(\dfrac{c_1k_2\theta_b}{k_1c_2} - a\right)\phi = \lambda\phi, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega. \end{cases} \tag{3.3}$$

Then it follows from [24, page 76] that the eigenvalues of (3.1) are the union of the eigenvalues of (3.2) and (3.3). Denote the principal eigenvalue of (3.2) and (3.3) by λ_* and λ^*, respectively. Then

$$\lambda_* = \lambda_1(2\theta_b - b) > \lambda_1(\theta_b - b) = 0.$$

In order to determine the sign of λ^*, letting $\varphi = (1 + \frac{k_2\alpha\theta_b}{c_2})\phi$, (3.3) is equivalent to

$$\begin{cases} -\Delta\varphi + \dfrac{c_1k_2\theta_b - ak_1c_2}{k_1(c_2+k_2\alpha\theta_b)}\varphi = \lambda\dfrac{c_2}{c_2+k_2\alpha\theta_b}\varphi, & x \in \Omega, \\ \varphi = 0, & x \in \partial\Omega. \end{cases}$$

By the variational characterization of principal eigenvalue, we have

$$\lambda^* = \inf_{\varphi\in H_0^1(\Omega),\ \varphi\not\equiv 0}\left\{\frac{\displaystyle\int_\Omega |\nabla\varphi|^2dx + \int_\Omega \frac{c_1k_2\theta_b - ak_1c_2}{k_1(c_2+k_2\alpha\theta_b)}\varphi^2dx}{\displaystyle\int_\Omega \frac{c_2}{c_2+k_2\alpha\theta_b}\varphi^2dx}\right\}.$$

Since $0 < \dfrac{c_2}{c_2 + k_2\alpha\theta_b} < 1$ for $x \in \Omega$,

$$\lambda^* \begin{cases} > \lambda_1\left(\dfrac{ck_2\theta_b - ak_1c_2}{k_1(c_2+k_2\alpha\theta_b)}\right), & \text{if } \lambda_1\left(\dfrac{ck_2\theta_b - ak_1c_2}{k_1(c_2+k_2\alpha\theta_b)}\right) > 0, \\ < \lambda_1\left(\dfrac{ck_2\theta_b - ak_1c_2}{k_1(c_2+k_2\alpha\theta_b)}\right), & \text{if } \lambda_1\left(\dfrac{ck_2\theta_b - ak_1c_2}{k_1(c_2+k_2\alpha\theta_b)}\right) < 0. \end{cases}$$

Combining the above results, one can see that if $\lambda_1\left(\frac{ck_2\theta_b - ak_1c_2}{k_1(c_2+k_2\alpha\theta_b)}\right) > 0$, then all eigenvalues of (3.1) are positive, and thus $(0, k_2\theta_b/c_2)$ is locally asymptotically stable. On the other hand, if $\lambda_1\left(\frac{ck_2\theta_b - ak_1c_2}{k_1(c_2+k_2\alpha\theta_b)}\right) < 0$, then (3.1) has a negative eigenvalue, which implies the instability of $(0, k_2\theta_b/c_2)$. $\square$

Next we make some explanations to Theorem 3.1 (*iii*). To this end we define a curve $\mathcal{C}$ on (b, a)-plane by

$$\mathcal{C} = \left\{(b,a) \in \mathbb{R}^2 : \lambda_1\left(\frac{c_1k_2\theta_b - ak_1c_2}{k_1(c_2+k_2\alpha\theta_b)}\right) = 0,\ a \geq \lambda_1,\ b \geq \lambda_1\right\}, \tag{3.4}$$

where θ_b is the unique positive solution of (2.3) with $l = b$ if $b > \lambda_1$ and $\theta_b = 0$ if $b = \lambda_1$. Then we have the following lemma, which describes the profile of $\mathcal{C}$.

Lemma 3.2. *The curve $\mathcal{C}$ defined by (3.4) can be expressed as*

$$\mathcal{C} = \{(b,a) \in \mathbb{R}^2 : a = \chi(b),\ b \geq \lambda_1\}.$$

Here $\chi(b)$ is a strictly increasing C^1 function. Furthermore, it satisfies the following properties:

$$\chi(\lambda_1) = \lambda_1,\ \chi'(\lambda_1) = \frac{k_2(c_1c_2 + \lambda_1k_1\alpha)}{k_1} \text{ and } \lim_{b\to\infty}\chi(b) = \infty.$$

Proof. We only prove the case $\alpha > 0$. For the case $\alpha = 0$,

$$a = \chi(b) = \lambda_1\left(\frac{c_1k_2\theta_b}{k_1c_2}\right)$$

and one can show the conclusion of this lemma in a similar manner. Set

$$S(a,b) = \lambda_1(\varphi(a,\theta_b)),\ (a,b) \in [\lambda_1,\infty) \times [\lambda_1,\infty),$$

where

$$\varphi(a,z) = \frac{c_1k_2z - ak_1c_2}{k_1(c_2+k_2\alpha z)}.$$

By Theorem 2.5, θ_b is a continuous and strictly increasing function with respect to b such that $\lim_{b\to\lambda_1^+}\theta_b(x) = 0$ uniformly in Ω and $\lim_{b\to\infty}\theta_b(x) = \infty$ uniformly in any compact subsets of Ω. Since $\varphi(a,z)$ is strictly decreasing with respect to a and is strictly increasing with respect to z, it follows from Theorem 2.3 that $S(a,b)$ is strictly decreasing with respect to a, and it is strictly increasing with respect to b.

Since $\lim_{b\to\lambda_1^+}\varphi(a,\theta_b) = -a$ uniformly in Ω, it follows from Theorem 2.3 (*ii*) that

$$S(a,\lambda_1) = \lim_{b\to\lambda_1^+}\lambda_1(\varphi(a,\theta_b)) = \lambda_1 - a. \tag{3.5}$$

By the variational characterization of principal eigenvalue,

$$S(a,b) = \inf_{\phi\in H_0^1(\Omega),\|\phi\|_{L^2(\Omega)}=1}\left\{\int_\Omega|\nabla\phi|^2dx + \int_\Omega\varphi(a,\theta_b)\phi^2dx\right\}.$$

Recall ϕ_1 is the positive eigenfunction of λ_1 with $\|\phi_1\|_{L^2(\Omega)} = 1$. Then

$$S(a,b) \le \int_\Omega |\nabla\phi_1|^2 dx + \int_\Omega \varphi(a,\theta_b)\phi_1^2 dx.$$

Since $\lim_{b\to\infty} \varphi(a,\theta_b) = c_1/(k_1\alpha)$ for any $x \in \Omega$, by Lebesgue's dominate convergence theorem,

$$\begin{aligned}\lim_{b\to\infty} S(a,b) &\le \lim_{b\to\infty} \left(\int_\Omega |\nabla\phi_1|^2 dx + \int_\Omega \varphi(a,\theta_b)\phi_1^2 dx\right) \\ &= \|\nabla\phi_1\|^2_{L^2(\Omega)} + \frac{c}{k_1\alpha}\int_\Omega \phi_1^2 dx = \lambda_1 + \frac{c}{k_1\alpha}.\end{aligned} \tag{3.6}$$

On the other hand,

$$S(a,b) = \|\nabla\phi_{a,b}\|^2_{L^2(\Omega)} + \int_\Omega \varphi(a,\theta_b)\phi_{a,b}^2 dx, \tag{3.7}$$

where $\phi_{a,b}(x)$ is the positive eigenfunction corresponding to $\lambda_1(\varphi(a,\theta_b))$ with $\|\phi_{a,b}\|_{L^2(\Omega)} = 1$. Then

$$\|\nabla\phi_{a,b}\|^2_{L^2(\Omega)} = S(a,b) - \int_\Omega \varphi(a,\theta_b)\phi_{a,b}^2 dx \le \left(\lambda_1 + \frac{c_1}{k_1\alpha}\right) + \max\left\{a, \frac{c_1}{k_1\alpha}\right\}.$$

By the reflexive property of $H_0^1(\Omega)$, there exists a function $\phi_{a,\infty} \in H_0^1(\Omega)$ with $\|\phi_{a,\infty}\|_{L^2(\Omega)} = 1$ and a subsequence of $\{\phi_{a,b}\}_b$, denoted by $\{\phi_{a,b}\}_b$ again, such that

1. $\phi_{a,b} \rightharpoonup \phi_{a,\infty}$ weakly in $H_0^1(\Omega)$ as $b \to \infty$,
2. $\phi_{a,b} \to \phi_{a,\infty}$ strongly in $L^2(\Omega)$ as $b \to \infty$.

Then, by (3.7),

$$\lim_{b\to\infty} S(a,b) \ge \|\phi_{a,\infty}\|^2_{L^2(\Omega)} + \frac{c_1}{k_1\alpha}\int_\Omega \phi_{a,\infty}^2 dx \ge \lambda_1 + \frac{c_1}{k_1\alpha}. \tag{3.8}$$

Thus, by (3.6) and (3.8),

$$\lim_{b\to\infty} S(a,b) = \lambda_1 + \frac{c_1}{k_1\alpha}. \tag{3.9}$$

It follows from (3.5) and (3.9) that for each $a \in [\lambda_1,\infty)$, there exists a unique $b_a \in [\lambda_1,\infty)$ such that $S(a,b_a) = 0$. Define a function $\zeta(a)$ by

$$\zeta(a) = b_a,\ \ a \in [\lambda_1,\infty).$$

Clearly $\zeta(a)$ is a continuous function for $a \in [\lambda_1,\infty)$, and it satisfies $\zeta(\lambda_1) = \lambda_1$ by the fact that $S(\lambda_1,\lambda_1) = 0$. By Theorem 2.3 and Theorem 2.5, $S(a,b)$ satisfies that

$$\frac{\partial S}{\partial a}(a,b) = \int_\Omega \frac{\partial\varphi(a,\theta_b)}{\partial a}\phi_{a,b}^2 dx = -c_2\int_\Omega \frac{\phi_{a,b}^2}{c_2 + k_2\alpha\theta_b} dx < 0, \tag{3.10}$$

$$\begin{aligned}\frac{\partial S}{\partial b}(a,b) &= \int_\Omega \frac{\partial\varphi(a,\theta_b)}{\partial b}\phi_{a,b}^2 dx = \int_\Omega \left.\frac{\varphi(a,z)}{\partial z}\right|_{z=\theta_b} \frac{\partial\theta_b}{\partial b}\phi_{a,b}^2 dx \\ &= \int_\Omega \frac{k_2(c_1c_2 + ac_2k_1\alpha)}{k_1(c_2 + k_2\alpha\theta_b)^2}\frac{\partial\theta_b}{\partial b}\phi_{a,b}^2 dx > 0.\end{aligned} \tag{3.11}$$

By the implicit function theorem, $\zeta(a)$ is a C^1-function for $a \in (\lambda_1,\infty)$ and

$$\zeta'(a) = -\frac{\dfrac{\partial S}{\partial a}(a,\zeta(a))}{\dfrac{\partial S}{\partial b}(a,\zeta(a))} > 0.$$

Next we prove $\lim_{a\to\infty}\zeta(a)=\infty$. Assume on the contrary that $\lim_{a\to\infty}\zeta(a)=b_\infty<\infty$. Since $\zeta(a)$ is strictly increasing, it follows from Theorem 2.3 (i) and the monotone property of φ that

$$0=S(a,\zeta(a))<S(a,b_\infty)=\lambda_1(\varphi(a,\theta_{b_\infty})). \tag{3.12}$$

Since $\theta_{b_\infty}<b_\infty$ for all $x\in\Omega$, then

$$\varphi(a,\theta_{b_\infty})<\frac{c_1k_2\theta_{b_\infty}}{k_1(c_2+k_2\alpha\theta_{b_\infty})}-\frac{ac_2}{c_2+k_2\alpha b_\infty}.$$

Consequently

$$\lambda_1(\varphi(a,\theta_{b_\infty}))\le\lambda_1\left(\frac{c_1k_2\theta_{b_\infty}}{k_1(c_2+k_2\alpha\theta_{b_\infty})}\right)-\frac{ac_2}{c_2+k_2\alpha b_\infty}\to-\infty\quad\text{as}\quad a\to\infty,$$

which contradicts with (3.12).

Define $a=\chi(b)$ be the inverse function of $b=\zeta(a)$. Then all the conclusions of the lemma hold except $\chi'(\lambda_1)$. Thus we only need to compute $\chi'(\lambda_1)$ to complete the proof. By Theorem 2.5 and Theorem 2.3, the following results hold true (see [46, page 432]):

1. $\lim\limits_{b\to\lambda_1^+}\theta_b(x)=0$ uniformly in Ω;
2. $\lim\limits_{b\to\lambda_1^+}\phi_{\chi(b),b}=\phi_1$ strongly in $H_0^1(\Omega)$;
3. $\lim\limits_{b\to\lambda_1^+}\dfrac{\partial\theta_b}{\partial b}=\left(\displaystyle\int_\Omega\phi_1^3dx\right)^{-1}\phi_1$ uniformly in Ω.

From (3.10) and (3.11), it follows that

$$\frac{\partial S}{\partial a}(\lambda_1,\lambda_1)=-1,\ \frac{\partial S}{\partial b}(\lambda_1,\lambda_1)=\frac{k_2(c_1+\lambda_1k_1\alpha)}{k_1},$$

and thus

$$\chi'(\lambda_1)=-\frac{\dfrac{\partial S}{\partial b}(\lambda_1,\lambda_1)}{\dfrac{\partial S}{\partial a}(\lambda_1,\lambda_1)}=\frac{k_2(c_1+\lambda_1k_1\alpha)}{k_1}.$$

□

By virtue of Lemma 3.2, the stability result for the semi-trivial steady state $(0,k_2\theta_b/c_2)$ reads as follows (see Fig 1):

Corollary 1. *The semi-trivial steady state* $(0,\theta_b)$ *is asymptotically stable in a region* $\mathcal{A}$, *while it is unstable in a region* $\mathcal{B}$, *where*

$$\mathcal{A}=\{(b,a)\in\mathbb{R}^2:a<\chi(b),\ b>\lambda_1\}$$

and

$$\mathcal{B}=\{(b,a)\in\mathbb{R}^2:a>\chi(b),\ b>\lambda_1\}.$$

4. **Existence of positive solutions.** By using the transformation $u=(1+\alpha v)w$, (1.2) can be rewritten as follows:

$$\begin{cases}-\Delta u=f(u,v):=u\rho_1(u,v), & x\in\Omega,\\ -\Delta v=g(u,v):=v\rho_2(u,v), & x\in\Omega,\\ u=v=0, & x\in\partial\Omega,\end{cases} \tag{4.1}$$

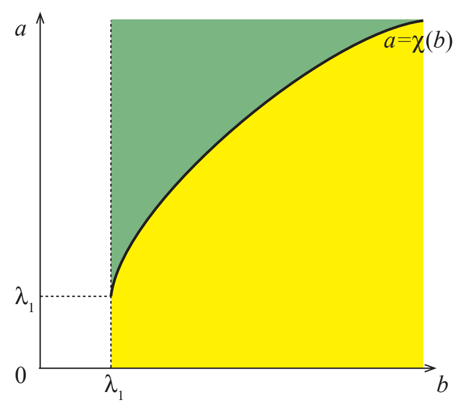

FIGURE 1. The stable region (yellow) and unstable region (green) for $(0, k_2\theta_b/c_2)$.

where

$$\rho_1(u,v) := \frac{a}{1+\alpha v} - \frac{u}{(1+\alpha v)^2} - \frac{c_1 v}{k_1(1+\alpha v)+u},$$
$$\rho_2(u,v) := b - \frac{c_2 v(1+\alpha v)}{k_2(1+\alpha v)+u}.$$

It is easy to see that (4.1) (or equivalently (1.2)) has no positive solutions if $a \le \lambda_1$ or $b \le \lambda_1$. Indeed let (u,v) be a positive solution of (4.1), then

$$\begin{cases} -\Delta u < au, & x \in \Omega, \\ -\Delta v < bv, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases}$$

and it follows from the property of the principal eigenvalue that $a > \lambda_1$ and $b > \lambda_1$. Since we are interested in positive solutions, throughout this section, we assume that $a > \lambda_1$ and $b > \lambda_1$ hold.

Next we derive an *a priori* estimate for nonnegative solutions of (4.1).

Lemma 4.1. *Let (u,v) be a nonnegative solution of (4.1). Then*

$$u(x) \le M_1(a) := \begin{cases} a\left(1 + \dfrac{\alpha(a+k_1)^2}{4c_1}\right) & \text{if } a > k_1, \\ a\left(1 + \dfrac{\alpha a k_1}{c_1}\right) & \text{if } a \le k_1, \end{cases} \tag{4.2}$$

and

$$v(x) \le M_2(a,b) := \frac{b}{c_2}(k_2 + M_1(a)). \tag{4.3}$$

Proof. We only prove that (4.2) holds since the other case (4.3) can be proved in a similar manner. Assume that $u(x)$ attains its maximum at $x_0 \in \Omega$. Thus

$$0 \le -\Delta u(x_0) = u(x_0)\rho_1(u(x_0), v(x_0)). \tag{4.4}$$

If $u(x_0) = 0$, it is obvious that $u(x) \equiv 0$, and (4.2) holds. On the other hand, if $u(x_0) > 0$, (4.4) implies that

$$\tau + \frac{c_1 v(x_0)}{k_1 + \tau} \leq a,$$

where $\tau = u(x_0)/(1 + \alpha v(x_0))$. Thus $\tau \leq a$ and

$$c_1 v(x_0) \leq (a - \tau)(k_1 + \tau) \leq \begin{cases} \dfrac{(a + k_1)^2}{4} & \text{if} \quad a > k_1, \\ a k_1 & \text{if} \quad a \leq k_1, \end{cases}$$

which imply that (4.2) holds. □

Now we introduce the following notations:

- $E = C(\overline{\Omega}) \times C(\overline{\Omega})$. It is obvious that E is a Banach space with the norm
$$\|(u, v)\|_E = \max_{x \in \overline{\Omega}} |u(x)| + \max_{x \in \overline{\Omega}} |v(x)|.$$
- $W = K \times K$, where $K = \{u \in C(\overline{\Omega}) : u(x) \geq 0 \text{ for } x \in \overline{\Omega}\}$.
- $D = \{(u, v) \in W : u(x) < M_1(a) + 1,\ v(x) < M_2(a, b) + 1 \text{ for } x \in \overline{\Omega}\}$, where $M_1(a)$ and $M_2(a, b)$ are defined in Lemma 4.1.

From Lemma 4.1, nonnegative solutions of (4.1) must be in D. Define a positive and compact operator $A : \overline{D} \to E$ by

$$A(u, v) := (-\Delta + pI)^{-1} \begin{pmatrix} f(u, v) + pu \\ g(u, v) + pv \end{pmatrix},$$

where p is a sufficiently large number such that

$$p + \rho_1(u, v) > 0 \text{ and } p + \rho_2(u, v) > 0 \text{ for } (u, v) \in \overline{D}.$$

Note that (4.1) is equivalent to $(u, v) = A(u, v)$ by the regularity of elliptic equations, and therefore it suffices to prove that A has a nontrivial fixed point in D in order to show the existence of positive solutions of (4.1). To this end we need to compute the fixed point index of the trivial and semi-trivial solutions of (4.1). It is easy to see that (4.1) has a trivial solution $(u, v) = (0, 0)$ and two semi-trivial solutions $(\theta_a, 0)$ and $(0, k_2\theta_b/c_2)$ since $a > \lambda_1$ and $b > \lambda_1$. Moreover the following lemma holds.

Lemma 4.2. *Assume that $a > \lambda_1$ and $b > \lambda_1$. Then*

(*i*): $\deg_W(I - A, D) = 1$;
(*ii*): $\mathrm{index}_W(A, (0, 0)) = 0$;
(*iii*): $\mathrm{index}_W(A, (\theta_a, 0)) = 0$;
(*iv*): $\mathrm{index}_W(A, (0, k_2\theta_b/c_2)) = 0$ *if* $\lambda_1\left(\frac{c_1k_2\theta_b - ak_1c_2}{k_1(c_2 + k_2\alpha\theta_b)}\right) < 0$;
and $\mathrm{index}_W(A, (0, k_2\theta_b/c_2)) = 1$ *if* $\lambda_1\left(\frac{c_1k_2\theta_b - ak_1c_2}{k_1(c_2 + k_2\alpha\theta_b)}\right) > 0$.

Proof. (*i*) For each $t \in [0, 1]$, we define a positive and compact operator $A_t : \overline{D} \to E$ by

$$A_t(u, v) = (-\Delta + pI)^{-1} \begin{pmatrix} tf(u, v) + pu \\ tg(u, v) + pv \end{pmatrix}.$$

Then $A_1 = A$, A_t has no fixed point on ∂D, and $A_t(\overline{D}) \subset W$. Thus $\deg_W(I - A_t, D)$ is well defined for all $t \in [0, 1]$. By the homotopy invariance of Leray-Schauder degree and $(0, 0)$ is the only fixed point of A_0 in D, we obtain that

$$\deg_W(I - A, D) = \deg_W(I - A_0, D) = \mathrm{index}_W(A_0, (0, 0)).$$

Set

$$L_0 = A_{0(u,v)}(0,0) = (-\Delta + pI)^{-1}\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}.$$

It is easy to see that $I - L_0$ is invertible on $\overline{W}_{(0,0)} = K \times K$ and $r(L_0) < 1$ by Theorem 2.4 (i). Since $r(L_0) < 1$, then L_0 does not have property $\mathfrak{a}$ on $\overline{W}_{(0,0)}$. Thus $\mathrm{index}_W(A_0,(0,0)) = 1$ by Theorem 2.6 (ii).

(ii) Let $L = A_{(u,v)}(0,0)$, then

$$L = (-\Delta + pI)^{-1}\begin{pmatrix} p+f_u & f_v \\ g_u & p+g_v \end{pmatrix}_{(u,v)=(0,0)} = (-\Delta + pI)^{-1}\begin{pmatrix} p+a & 0 \\ 0 & p+b \end{pmatrix}.$$

Assume that $L(\xi,\eta) = (\xi,\eta)$ for some $(\xi,\eta) \in \overline{W}_{(0,0)} = K \times K$. Then it is easy to verify that $\xi = \eta \equiv 0$ since $a \neq \lambda_1$ and $b \neq \lambda_1$. Thus $I - L$ is invertible on $\overline{W}_{(0,0)}$. Since $a > \lambda_1$, by Theorem 2.4 (ii), we see that $r_a := r((-\Delta + pI)^{-1}(p+a)) > 1$ and r_a is the principal eigenvalue of the operator $(-\Delta + pI)^{-1}(p+a)$ with a corresponding eigenfunction $\phi_a(x) > 0$ in Ω and $\phi_a|_{\partial\Omega} = 0$. Set $t_a = 1/r_a \in (0,1)$, then $(\phi_a, 0) \notin S_{(0,0)} = \{(0,0)\}$, but $(I - t_a L)(\phi_a, 0) = (0,0) \in S_{(0,0)}$. This shows that L has property $\mathfrak{a}$, and thus $\mathrm{index}_W(A,(0,0)) = 0$ by Theorem 2.6 (i).

(iii) Let $L = A_{(u,v)}(\theta_a, 0)$, then

$$\begin{aligned} L &= (-\Delta + pI)^{-1}\begin{pmatrix} p+f_u & f_v \\ g_u & p+g_v \end{pmatrix}_{(u,v)=(\theta_a,0)} \\ &= (-\Delta + pI)^{-1}\begin{pmatrix} p-(2\theta_a - a) & -\alpha(a-2\theta_a)\theta_a - \dfrac{c_1\theta_a}{\theta_a + k_1} \\ 0 & p+b \end{pmatrix}. \end{aligned}$$

Assume that $L(\xi,\eta) = (\xi,\eta)$ for some $(\xi,\eta) \in \overline{W}_{(\theta_a,0)} = C(\overline{\Omega}) \times K$, *i.e.* (ξ,η) satisfies

$$\begin{cases} -\Delta\xi + (2\theta_a - a)\xi = \left(-\alpha(a-2\theta_a)\theta_a - \dfrac{c_1\theta_a}{\theta_a + k_1}\right)\eta, & x \in \Omega, \\ -\Delta\eta = b\eta, & x \in \Omega, \\ \xi = \eta = 0, & x \in \partial\Omega. \end{cases}$$

Since $b > \lambda_1$, then $\eta \equiv 0$, and thus the conclusion follows by similar arguments as in the proof of (ii).

(iv) Let $L = A_{(u,v)}(0, k_2\theta_b/c_2)$, then

$$\begin{aligned} L &= (-\Delta + pI)^{-1}\begin{pmatrix} p+f_u & f_v \\ g_u & p+g_v \end{pmatrix}_{(u,v)=(0,k_2\theta_b/c_2)} \\ &= (-\Delta + pI)^{-1}\begin{pmatrix} p - \dfrac{c_1k_2\theta_b - ak_1c_2}{k_1(c_2 + k_2\alpha\theta_b)} & 0 \\ \dfrac{\theta_b^2}{c_2 + k_2\alpha\theta_b} & p-(2\theta_b - b) \end{pmatrix}. \end{aligned}$$

If $\lambda_1\left(\frac{c_1k_2\theta_b - ak_1c_2}{k_1(c_2+k_2\alpha\theta_b)}\right) < 0$, then by the similar argument as in the proof of (ii), we have $r(L) < 1$ and $\mathrm{index}_W(A,(0,k_2\theta_b/c_2)) = 0$; if $\lambda_1\left(\frac{c_1k_2\theta_b - ak_1c_2}{k_1(c_2+k_2\alpha\theta_b)}\right) > 0$, then, by the similar argument as in the proof of (i), $r(L) < 1$ and $\mathrm{index}_W(A,(0,k_2\theta_b/c_2)) = 1$. $\square$

The following existence theorem is a consequence of Lemma 4.2.

Theorem 4.3. *Assume that $a > \lambda_1$ and $b > \lambda_1$. Then (4.1) admits a positive solution if*

$$\lambda_1 \left(\frac{c_1 k_2 \theta_b - a k_1 c_2}{k_1(c_2 + k_2 \alpha \theta_b)} \right) < 0.$$

Proof. Assume on the contrary that (4.1) has no positive solution. Since $a > \lambda_1$ and $b > \lambda_1$, then (4.1) admits a trivial solution $(0,0)$ and two semi-trivial solutions $(\theta_a, 0)$, $(0, k_2\theta_b/c_2)$. Hence

$$\deg_W(I - A, D) = \mathrm{index}_W(A, (0,0)) + \mathrm{index}_W(A, (\theta_a, 0)) + \mathrm{index}_W(A, (0, k_2\theta_b/c_2)). \tag{4.5}$$

By Lemma 4.1, the left hand side of (4.5) is 1, but the right hand side is 0, which is a contradiction. □

By virtue of Lemma 3.2, the coexistence result (Theorem 4.1) reads as follows (see the green region in Figure 1).

Corollary 2. *Let $a = \chi(b)$ be the function defined in Lemma 3.2. Then (4.1) has a positive solution if $a > \chi(b)$ and $b > \lambda_1$.*

Remark 1. By Theorem 3.1 and Theorem 4.1, (1.2) has a positive solution when the trivial steady state $(0,0)$ and two semi-trivial steady states $(\theta_a, 0)$, $(0, k_2\theta_b/c_2)$ are all unstable. Moreover if we consider the evolution equation corresponding to (1.2), *i.e.*,

$$\begin{cases} w_t - \Delta[(1+\alpha v)w] = w\left(a - w - \dfrac{c_1 v}{w + k_1}\right), & x \in \Omega,\ t > 0, \\ v_t - \Delta v = v\left(b - \dfrac{c_2 v}{w + k_2}\right), & x \in \Omega,\ t > 0, \\ w = v = 0, & x \in \partial\Omega,\ t > 0, \\ w(x,0) = w_0(x), \quad v(x,0) = v_0(x). & \end{cases} \tag{4.6}$$

Then the system (4.6) is persistent if $a > \chi(b)$, $b > \lambda_1$, and $(w_0, v_0) \in \mathcal{O}$ (see [2] for details), where

$$\mathcal{O} = \{(\phi, \psi) \in E : \phi(x), \psi(x) \geq 0,\ x \in \Omega\} \setminus \{(0,0), (\theta_a, 0), (0, k_2\theta_b/c_2)\}.$$

5. **Multiplicity of positive solutions.** In this section we use bifurcation theory to show that (4.1) may have multiple positive solutions for certain parameters. For fixed $b > \lambda_1$, we rewrite (4.1) as the following form with parameter a:

$$\begin{cases} -\Delta u = f(a, u, v), & x \in \Omega, \\ -\Delta v = g(u, v), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{5.1}$$

where $a > 0$ is a positive parameter,

$$f(a, u, v) := \frac{au}{1 + \alpha v} - \frac{u^2}{(1+\alpha v)^2} - \frac{c_1 u v}{k_1 + u + k_1 \alpha v},$$

$$g(u, v) := bv - \frac{c_2 \alpha v^3 + c_2 v^2}{k_2 + u + k_2 \alpha v}.$$

Recall $a = \chi(b)$ be the solution of

$$\lambda_1 \left(\frac{c_1 k_2 \theta_b - a k_1 c_2}{k_1(c_2 + k_2 \alpha \theta_b)} \right) = 0.$$

For a fixed $b > \lambda_1$, it was shown in Corollary 2 that (5.1) has a positive solution if $a > \chi(b)$. Since the necessary condition for the existence of positive solutions of (5.1) is $a > \lambda_1$, then a natural question is whether positive solution exist when $a \in (\lambda_1, \chi(b))$. In this section we show that it is possible that there exists a positive constant $\epsilon_b \in (0, \lambda_1 - \chi(b))$ such that (5.1) has at least two positive solutions when $\chi(b) - \epsilon_b < a < \chi(b)$, and it has at least one positive solutions when $a \geq \chi(b) - \epsilon_b$.

Recall that (5.1) has a semi-trivial nonnegative solution $(u, v) = (0, k_2\theta_b/c_2)$ for any $a > 0$ as long as $b > \lambda_1$. Here we use a as a bifurcation parameter, and consider the bifurcation of positive solutions from the branch of semi-trivial solutions $\{(a, 0, k_2\theta_b/c_2)\}$. By linearizing (5.1) at $(0, k_2\theta_b/c_2)$, we obtain the following eigenvalue problem:

$$\begin{cases} \Delta\phi + \dfrac{ak_1c_2 - c_1k_2\theta_b}{k_1(c_2 + k_2\alpha\theta_b)}\phi = \lambda\phi, & x \in \Omega, \\ \Delta\psi + \dfrac{\theta_b^2}{c_2 + k_2\alpha\theta_b}\phi + (b - 2\theta_b)\psi = \lambda\psi, & x \in \Omega, \\ \phi = \psi = 0, & x \in \partial\Omega. \end{cases} \tag{5.2}$$

A necessary condition for bifurcation is that the principal eigenvalue of (5.2) is zero, which occurs when $a = \chi(b)$.

Let Φ be the positive eigenfunction corresponding to $a = \chi(b)$, *i.e.*, $(\chi(b), \Phi)$ satisfies

$$\begin{cases} \Delta\Phi + \dfrac{\chi(b)k_1c_2 - c_1k_2\theta_b}{k_1(c_2 + k_2\alpha\theta_b)}\Phi = 0, & x \in \Omega, \\ \Phi = 0, & x \in \partial\Omega. \end{cases} \tag{5.3}$$

We assume that Φ is normalized so that $\|\Phi\|_{L^\infty(\Omega)} = 1$. Since $\lambda_1(2\theta_b - b) > \lambda_1(\theta_b - b) = 0$, then $-\Delta + 2\theta_b - b$ is invertible, and $(-\Delta + 2\theta_b - b)^{-1}$ maps positive functions to positive functions by the maximum principle. Define

$$\Psi = (-\Delta + 2\theta_b - b)^{-1}\left(\frac{\theta_b^2}{c_2 + k_2\alpha\theta_b}\Phi\right),$$

then both Φ and Ψ are positive in Ω.

With the functions defined above, we have the following result regarding the bifurcation of positive solutions of (5.1) from $(a, 0, k_2\theta_b/d)$ at $a = \chi(b)$.

Theorem 5.1. *Let $b > \lambda_1$ be fixed. Then $a = \chi(b)$ is a bifurcation value of (5.1) where positive solutions bifurcate from the line of semi-trivial solutions $\Gamma_0 = \{(a, 0, k_2\theta_b/c_2) : a > 0\}$; near $(\chi(b), 0, k_2\theta_b/c_2)$, there exists $\delta > 0$ such that all the positive solutions of (5.1) lie on a smooth curve*

$$\Gamma_1 = \{(a(s), u(s), v(s)) : s \in (0, \delta)\}$$

and

$$\begin{cases} a(s) = \chi(b) + s\hat{a}_1 + sa_2(s), \\ u(s) = s\Phi + su_1(s, x), \\ v(s) = \dfrac{k_2}{c_2}\theta_b + s\Psi + sv_1(s, x). \end{cases}$$

Here $s \mapsto (a_2(s), u_1(s, x), v_1(s, x))$ is a smooth function from $(0, \delta)$ to $\mathbb{R} \times X$ for $X = C_0^{2+\sigma}(\overline{\Omega}) \times C_0^{2+\sigma}(\overline{\Omega})$ with $\sigma \in (0, 1)$ such that $a_2(0) = 0$, $u_1(0, x) = v_1(0, x) = 0$

and

$$\hat{a}_1 = \frac{\displaystyle\int_\Omega \frac{(c_2k_1^2 - c_1k_2\theta_b)\Phi^3 + k_1c_2(\chi(b)k_1\alpha + c_1)\Phi^2\Psi}{k_1^2(c_2 + k_2\alpha\theta_b)^2}dx}{\displaystyle\int_\Omega \frac{\Phi^2}{c_2 + k_2\alpha\theta_b}dx}. \tag{5.4}$$

Moreover $a = \chi(b)$ is the unique bifurcation value for which positive solutions bifurcate from Γ_0.

Proof. Let $X = C_0^{2+\sigma}(\overline{\Omega}) \times C_0^{2+\sigma}(\overline{\Omega})$ and $Y = C^\sigma(\overline{\Omega}) \times C^\sigma(\overline{\Omega})$, where $\sigma \in (0,1)$ is a constant. Define a nonlinear mapping $F : \mathbb{R} \times X \to Y$ by

$$F(a,u,v) = \begin{pmatrix} \Delta u + f(a,u,v) \\ \Delta v + g(u,v) \end{pmatrix}.$$

We consider the bifurcation at $(a,u,v) = (\chi(b), 0, k_2\theta_b/c_2)$. From straightforward calculations, we get

$$F_{(u,v)}(a,u,v)[\xi,\eta] = \begin{pmatrix} \Delta\xi + f_u\xi + f_v\eta \\ \Delta\eta + g_u\xi + g_v\eta \end{pmatrix},$$

$$F_a(a,u,v) = \begin{pmatrix} \dfrac{u}{1+\alpha v} \\ 0 \end{pmatrix},$$

$$F_{a(u,v)}(a,u,v)[\xi,\eta] = \begin{pmatrix} \dfrac{\xi}{1+\alpha v} - \dfrac{\alpha u\eta}{(1+\alpha v)^2} \\ 0 \end{pmatrix},$$

$$F_{(u,v)(u,v)}(a,u,v)[\xi,\eta]^2 = \begin{pmatrix} f_{uu}\xi^2 + 2f_{uv}\xi\eta + f_{vv}\eta^2 \\ g_{uu}\xi^2 + 2g_{uv}\xi\eta + g_{vv}\eta^2 \end{pmatrix}.$$

At $(a,u,v) = (\chi(b), 0, k_2\theta_b/c_2)$, it is easy to see that the kernel space is

$$\mathcal{N}(F_{(u,v)}(\chi(b), 0, k_2\theta_b/c_2)) = \mathrm{span}\{(\Phi, \Psi)\}$$

and the range space is

$$\mathcal{R}(F_{(u,v)}(\chi(b), 0, k_2\theta_b/c_2)) = \left\{(\hbar_1, \hbar_2) \in Y : \int_\Omega \hbar_1(x)\Phi(x)dx = 0\right\}.$$

Then

$$F_{a(u,v)}(\chi(b), 0, k_2\theta_b/c_2)[\Phi, \Psi] = \left(\frac{c_2\Phi}{c_2 + k_2\alpha\theta_b}, 0\right) \notin \mathcal{R}(F_{(u,v)}(\chi(b), 0, k_2\theta_b/c_2))$$

since

$$\int_\Omega \frac{c_2\Phi^2}{c_2 + k_2\alpha\theta_b}dx \neq 0.$$

Thus we can apply Theorem 2.1 to conclude that the set of positive solutions to (5.1) near $(\chi(b), 0, k_2\theta_b/c_2)$ is a smooth curve $\Gamma_1 = \{(a(s), u(s), v(s)) : s \in (0,\delta)\}$, such that $a(0) = \chi(b)$, $u(s) = s\Phi + o(s)$, $v(s) = k_2\theta_b/d + s\Psi + o(s)$. Moreover, by Theorem 2.1,

$$\hat{a}_1 = a'(0) = -\frac{\langle \ell, F_{(u,v)(u,v)}(\chi(b), 0, k_2\theta_b/c_2)[\Phi, \Psi]^2\rangle}{2\langle \ell, F_{a(u,v)}(\chi(b), 0, k_2\theta_b/c_2)[\Phi, \Psi]\rangle},$$

where ℓ is a linear functional on Y defined as $\langle \ell, (\hbar_1, \hbar_2)\rangle = \displaystyle\int_\Omega \hbar_1(x)\Phi(x)dx$. Thus $\hat{a}_1$ is given by (5.4).

Finally we prove $a = \chi(b)$ is the unique bifurcation point where positive solutions bifurcate from $(0, k_2\theta_b/c_2)$. Suppose that there is a sequence $\{(a_n, u_n, v_n)\}_{n\geq 1}$ of positive solutions of (5.1) such that

$$\lim_{n\to\infty}(a_n, u_n, v_n) = \left(\overline{a}, 0, \frac{k_2\theta_b}{c_2}\right) \in \mathbb{R}\times X.$$

Let $\phi_n = u_n/\|u_n\|_{L^\infty(\Omega)}$. From the first equation of (5.1) with $a = a_n$,

$$\begin{cases} -\Delta\phi_n = \dfrac{a_n}{1+\alpha v_n}\phi_n - \dfrac{u_n}{(1+\alpha v_n)^2}\phi_n - \dfrac{c_1 v_n}{k_1 + u_n + k_1\alpha v_n}\phi_n, & x\in\Omega, \\ \phi_n = 0, & x\in\partial\Omega. \end{cases} \tag{5.5}$$

By Lemma 4.1 and the regularity theory of elliptic equations, there exists a subsequence of $\{\phi_n\}_{n\geq 1}$ such that it converges uniformly in $C_0^{2+\sigma}(\overline{\Omega})$ to some nonnegative function $\phi\in C_0^{2+\sigma}(\overline{\Omega})$ with $\|\phi\|_{L^\infty(\Omega)} = 1$. Letting $n\to\infty$ in (5.5), $(\overline{a}, \phi)$ satisfies

$$\begin{cases} \Delta\phi + \dfrac{\overline{a}k_1c_2 - c_1k_2\theta_b}{k_1(c_2 + k_2\alpha\theta_b)}\phi = 0, & x\in\Omega, \\ \phi = 0, & x\in\partial\Omega. \end{cases}$$

By Krein-Rutman Theorem, $\phi = \Phi$ and $\overline{a} = \chi(b)$. This completes the proof. □

Next we discuss the stability of the positive solutions obtained from Theorem 5.1.

Theorem 5.2. *Let $b > \lambda_1$ be fixed and let $\hat{a}_1$ be defined as in (5.4). If $\hat{a}_1 \neq 0$, then there exists $\tilde{\delta}\in(0,\delta]$ such that for $s\in(0,\tilde{\delta})$, the positive solution $(a(s), u(s), v(s))$ bifurcating from $(\chi(b), 0, k_2\theta_b/d)$ is not degenerate, where δ is the constant in Theorem 5.1. Moreover $(u(s), v(s))$ is unstable if $\hat{a}_1 < 0$, and it is stable if $\hat{a}_1 > 0$.*

Proof. In order to study the stability of the bifurcating positive solution $(u(s), v(s))$ of (5.1), we consider the following eigenvalue problem

$$\begin{cases} L(s)\left[\begin{pmatrix}\xi(s)\\ \eta(s)\end{pmatrix}\right] = \mu(s)\begin{pmatrix}\xi(s)\\ \eta(s)\end{pmatrix}, & x\in\Omega, \\ \xi(s) = \eta(s) = 0, & x\in\partial\Omega, \end{cases}$$

where

$$\begin{aligned} L(s) :=& -F_{(u,v)}(a(s), u(s), v(s)) \\ =& \begin{pmatrix} -\Delta - f_u(a(s), u(s), v(s)) & -f_v(a(s), u(s), v(s)) \\ -g_u(u(s), v(s)) & -\Delta - g_v(u(s), v(s)) \end{pmatrix}. \end{aligned}$$

Furthermore

$$\lim_{s\to 0^+} L(s) = L_0 := \begin{pmatrix} -\Delta + \dfrac{c_1k_2\theta_b - k_1c_2\chi(b)}{k_1(c_2 + k_2\alpha\theta_b)} & 0 \\ -\dfrac{\theta_b^2}{c_2 + k_2\alpha\theta_b} & -\Delta - b + 2\theta_b \end{pmatrix}.$$

Since $\lambda_1\left(\dfrac{c_1k_2\theta_b - k_1c_2\chi(b)}{k_1(c_2 + k_2\alpha\theta_b)}\right) = 0$ and $\lambda_1(2\theta_b - b) > \lambda_1(\theta_b - b) = 0$, then 0 is the first eigenvalue of L_0 with the corresponding eigenfunction (Φ, Ψ). Moreover the real part of all other eigenvalues of L_0 are positive and are apart from 0. By the perturbation theory of linear operators [18], we know that for $s > 0$ small, $L(s)$ has a unique eigenvalue $\mu(s)$ such that $\lim_{s\to 0^+}\mu(s)=0$ and all other eigenvalues of $L(s)$ have positive real parts and are apart from 0.

Now we determine the sign of $\mu(s)$ for small $s>0$ by using Theorem 2.2. Consider the following eigenvalue problem

$$\begin{cases} -F_{(u,v)}(a,0,k_2\theta_b/c_2)\left[\begin{pmatrix}\phi(a)\\ \psi(a)\end{pmatrix}\right]=\gamma(a)\begin{pmatrix}\phi(a)\\ \psi(a)\end{pmatrix}, & x\in\Omega,\\ \phi(a)=\psi(a)=0, & x\in\partial\Omega.\end{cases}$$

Then $\phi(a)$ satisfies

$$\begin{cases} -\Delta\phi(a)+\dfrac{c_1k_2\theta_b-ak_1c_2}{k_1(c_2+k_2\alpha\theta_b)}\phi(a)=\gamma(a)\phi(a), & x\in\Omega,\\ \phi(a)=0, & x\in\partial\Omega.\end{cases}\tag{5.6}$$

Since $\gamma(\chi(b))=0$ and $\phi(\chi(b))=\Phi$, then by differentiating (5.6) with respect to a at $a=\chi(b)$, we obtain that

$$\begin{cases} -\Delta\varphi-\dfrac{c_2}{c_2+k_2\alpha\theta_b}\Phi+\dfrac{c_1k_2\theta_b-\chi(b)k_1c_2}{k_1(c_2+k_2\alpha\theta_b)}\varphi=\gamma'(\chi(b))\Phi, & x\in\Omega,\\ \varphi=0, & x\in\partial\Omega,\end{cases}\tag{5.7}$$

where $\varphi=\phi'(\chi(b))$. Multiplying both sides of (5.7) with Φ and integrating it over Ω, we obtain from (5.3) that

$$\gamma'(\chi(b))\int_\Omega\Phi^2dx=-\int_\Omega\frac{c_2}{c_2+k_2\alpha\theta_b}\Phi^2dx,$$

and

$$\gamma'(\chi(b))=-\frac{\displaystyle\int_\Omega\frac{c_2}{c_2+k_2\alpha\theta_b}\Phi^2dx}{\displaystyle\int_\Omega\Phi^2dx}.\tag{5.8}$$

Since $\hat{a}_1\neq 0$, then it follows from Theorem 2.2 and (5.8) that $\mu(s)\neq 0$ for $s>0$ small and

$$\lim_{s\to 0^+}\frac{\mu(s)}{s}=-\gamma'(\chi(b))a'(0)=\hat{a}_1\frac{\displaystyle\int_\Omega\frac{c_2}{c_2+k_2\alpha\theta_b}\Phi^2dx}{\displaystyle\int_\Omega\Phi^2dx}.\tag{5.9}$$

Since all the other eigenvalues of $L(s)$ have positive real parts and are apart from 0, then the conclusions follow from (5.9). $\square$

Based on the above preparations, we give the multiplicity result on positive solutions of (5.1) as follows:

Theorem 5.3. *Assume the conditions of Theorem 5.1 are satisfied, and let $\hat{a}_1$ be defined as in (5.4). If $\hat{a}_1<0$, then there exists a positive constant $\epsilon_b\in(0,\chi(b)-\lambda_1)$ such that problem (5.1) has at least two positive solutions if $\chi(b)-\epsilon_b<a<\chi(b)$, and it has at least one positive solutions if $a\geq\chi(b)-\epsilon_0$.*

Proof. From Theorem 5.1, (5.1) has a curve $\Gamma_1=\{(a(s),u(s),v(s)):s\in(0,\delta)\}$ of positive solutions near $(\chi(b),0,k_2\theta_b/c_2)$. Since $a_1<0$, $a(s)<\chi(b)$ for $s>0$ small. Assume on the contrary that (5.1) has a unique positive solution $(\hat{u},\hat{v})$ when $a<\chi(b)$ but near $\chi(b)$. Then it is obvious that $(\hat{u},\hat{v})$ must be the positive solution bifurcating from $(\chi(b),0,k_2\theta_b/c_2)$, which was obtained from Theorem 5.1. That is

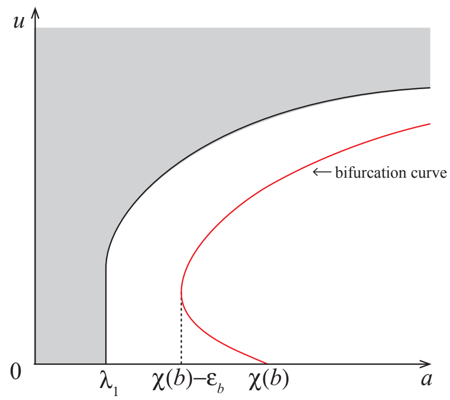

FIGURE 2. Possible bifurcation diagram of u when $a_1 < 0$.

$(\hat{u}, \hat{v}) = (u(s), v(s))$, which is not degenerate by Theorem 5.2. Thus $I - A_{(u,v)}(\hat{u}, \hat{v}) : \overline{W}_{(\hat{u},\hat{v})} \to \overline{W}_{(\hat{u},\hat{v})}$ is invertible. Recall that A is the operator defined in (4). Since $\overline{W}_{(\hat{u},\hat{v})} - S_{(\hat{u},\hat{v})} = \emptyset$, $A_{(u,v)}(\hat{u}, \hat{v})$ does not have property property $\mathfrak{a}$. Consequently

$$\mathrm{index}_W(A, (\hat{u}, \hat{v})) = \pm 1.$$

Notice that $\lambda_1 < a < \chi(b)$ for $s > 0$ small and $b > \lambda_1$. It follows from Lemma 4.2 (iv) and Lemma 3.2 that

$$\begin{aligned} 1 &= \deg_W(I - A, D) \\ &= \mathrm{index}_W(A, (0,0)) + \mathrm{index}_W(A, (\theta_a, 0)) \\ &\quad + \mathrm{index}_W(A, (0, k_2\theta_b/c_2)) + \mathrm{index}_W(A, (\hat{u}, \hat{v})) \\ &= 0 + 0 + 1 \pm 1, \end{aligned}$$

which is a contradiction. Thus if $a < \chi(b)$ and near $\chi(b)$, then there exist at least two positive solutions of (5.1).

By Theorem 2.1, the curve Γ_1 of bifurcating positive solutions is contained in a connected component Σ of the set of positive solutions of (5.1). Moreover the closure of Σ contains another semi-trivial solution on $\{(a, 0, k_2\theta_b/c_2) : a > 0\}$, or the closure of Σ contains semi-trivial solution $\{(a, \theta_a, 0) : a > \lambda_1\}$ or Σ is unbounded. By Theorem 5.1, $a = \chi(b)$ is the unique bifurcation value for positive solutions of (5.1) from the line of semi-trivial solution $\{(a, 0, k_2\theta_b/c_2) : a > 0\}$, so the first alternative is not possible. By Theorem 3.1, $(\theta_a, 0)$ is not degenerate for all $a > \lambda_1$ since $b > \lambda_1$, so the second alternative is not possible. Thus Σ must be unbounded. Furthermore, by Lemma 4.1 and regularity theory of elliptic equations, for each $a > 0$ there exists $C(a) > 0$ such that $0 < \|(u, v)\|_X \leq C(a)$, and there is no positive solutions when $a \leq \lambda_1$. Thus there exists $\epsilon_b \in (0, \chi(b) - \lambda_1)$ such that the projection of Σ on the a-axis contains an interval $[\chi(b) - \epsilon_b, \infty)$. In particular (5.1) has at least two positive solutions if $\chi(b) - \epsilon_b < a < \chi(b)$, and it has at least one positive solutions if $a \geq \chi(b) - \epsilon_b$ (see Figure 2). $\square$

Remark 2. We remark that $\hat{a}_1 < 0$ can be achieved by fixing $\alpha, c_2, k_2 > 0$, $b > \lambda_1$ and letting $c_1 = k_1 = \varepsilon > 0$ in (5.4). Then Φ and Ψ are all independent of ε, while

$$\begin{aligned}\hat{a}_1 \int_\Omega \frac{\Phi^2}{c_2 + k_2\alpha\theta_b} dx &= c_2 \int_\Omega \frac{\Phi^3}{(c_2 + k_2\alpha\theta_b)^2} dx - \frac{k_2}{\varepsilon} \int_\Omega \frac{\theta_b\Phi^3}{(c_2 + k_2\alpha\theta_b)^2} dx \\ &\quad + c_2(\chi(b)\alpha + 1) \int_\Omega \frac{\Phi^2\Psi}{(c_2 + k_2\alpha\theta_b)^2} dx \to -\infty \text{ as } \varepsilon \to 0^+.\end{aligned}$$

Thus $\hat{a}_1 < 0$ if $\varepsilon > 0$ is small enough.

6. **Uniqueness of positive solutions.** In this section we study the uniqueness of positive solutions to problem (4.1) (or equivalently (1.2)) when $N = 1$. Consider the following system

$$\begin{cases} -u'' = \dfrac{au}{1+\alpha v} - \dfrac{u^2}{(1+\alpha v)^2} - \dfrac{c_1 uv}{k_1 + u + k_1\alpha v}, & x \in (0, L), \\ -v'' = bv - \dfrac{c_2\alpha v^3 + c_2 v^2}{k_2 + u + k_2\alpha v}, & x \in (0, L), \\ u(0) = u(L) = v(0) = v(L) = 0, \end{cases} \tag{P_α}$$

where L is a positive constant.

For the uniqueness result, we first assume the following condition holds

(H)**:** $b > \lambda_1$ and $a > \lambda_1\left(\frac{c_1k_2}{c_2k_1}\theta_b\right)$.

Then there exists a positive solution of (P_α) with $\alpha = 0$ by Theorem 4.3 (or [48, Theorem 1.1]). Setting

$$\hbar_1(\alpha) = \lambda_1\left(\frac{c_1k_2\theta_b - ak_1c_2}{k_1(c_2 + k_2\alpha\theta_b)}\right),$$

then

$$\hbar_1(0) = \lambda_1\left(\frac{c_1k_2}{c_2k_1}\theta_b\right) - a < 0 \quad \text{and} \quad \lim_{\alpha\to\infty} \hbar_1(\alpha) = \lambda_1 > 0.$$

By the continuity of $\hbar_1(\alpha)$, there exists the smallest root $\overline{\alpha}_1 = \overline{\alpha}_1(a, b, c_1, c_2, k_1, k_2)$ of $\hbar_1(\alpha) = 0$, and it satisfies that $\hbar_1(\alpha) < 0$ for $\alpha \in [0, \overline{\alpha}_1)$. It follows from Theorem 4.3 that (P_α) has a positive solution if $\alpha \in [0, \overline{\alpha}_1)$ and (H) holds.

By Lemma 4.1, all positive solutions (u, v) of (P_α) satisfy

$$u(x) \le a + B\alpha \quad \text{and} \quad v(x) \le \frac{b}{c_2}(k_2 + a + B\alpha), \quad x \in \overline{\Omega}, \tag{6.1}$$

where

$$B = \begin{cases} \dfrac{a(a+k_1)^2}{4c_1} & if\ a > k_1, \\ \dfrac{a^2k_1}{c_1} & if\ a \le k_1. \end{cases}$$

Consider the quartic polynomial

$$\begin{aligned}\hbar_2(\alpha) \quad = \quad & -2B^3\alpha^4 - 2B^2(3a + 2k_1)\alpha^3 - 2B(3a^2 + 4k_1a + k_1^2)\alpha^2 \\ & + (-2a^3 - 5k_1a^2 + Bc_1)\alpha + ac_1(1 + k_1),\end{aligned} \tag{6.2}$$

where B is defined in (6.1). Since $\hbar_2(0) = ac_1(1 + k_1) > 0$ and $\lim_{\alpha\to\infty} \hbar_2(\alpha) = -\infty$, there exists the smallest positive root $\overline{\alpha}_2 = \overline{\alpha}_2(a, c_1, k_1)$ of $\hbar_2(\alpha) = 0$. Thus $\hbar_2(\alpha) < 0$ for $\alpha \in [0, \overline{\alpha}_2)$ and $\hbar_2(\overline{\alpha}_2) = 0$.

Now we give the uniqueness result of positive solution to problem (P_α) as follows.

Theorem 6.1. *Assume*

$$\frac{c_2k_1^2}{\lambda_1 c_1} - k_2 > \lambda_1, \tag{6.3}$$

and

$$a + k_2 < \frac{c_2k_1^2}{bc_1}. \tag{6.4}$$

Then problem (P_α) *has exactly one positive solution if* (H) *holds and* $\alpha \in [0, \overline{\alpha})$, *where*

$$\overline{\alpha} := \min\left\{\overline{\alpha}_1,\ \overline{\alpha}_2,\ \frac{1}{B}\left(\frac{c_2k_1^2}{bc_1} - k_2 - a\right)\right\}.$$

Remark 3. The condition (6.3) is needed so that the set of values of (a, b) satisfying (6.4) and (H) is not an empty set (see Figure 3).

In order to prove Theorem 6.1, we first consider the uniqueness of positive solution to the problem (P_0).

Lemma 6.2. *Under the assumptions of Theorem 6.1, problem* (P_0) *has exactly one positive solution.*

Proof. The proof follows from the methods developed by López-Gomez and Pardo [28] (or see Casal et al. [3]). Since $b > \lambda_1$ and $a > \lambda_1\left(c_1k_2\theta_b/(c_2k_1)\right)$, there exists a positive solution of (P_0) by Theorem 4.3 (or [48, Theorem 1.1]). Thus we only need to prove the uniqueness. Let (u_0, v_0) be a positive solution of (P_0). Then the linearized system of (P_0) is

$$\begin{cases} -\phi'' + L_1^0\phi = -\dfrac{c_1u_0}{u_0 + k_1}\psi, & x \in (0, L), \\ -\psi'' + L_2^0\psi = \dfrac{c_2v_0^2}{(u_0 + k_2)^2}\phi, & x \in (0, L), \\ \phi(0) = \phi(L) = \psi(0) = \psi(L) = 0, \end{cases}$$

where

$$L_1^0\phi = \left(2u_0 + \frac{c_1k_1v_0}{(u_0 + k_1)^2} - a\right)\phi,\ \ L_2^0\psi = \left(\frac{2c_2v_0}{u_0 + k_2} - b\right)\psi.$$

Since (u_0, v_0) is a positive solution of (P_0), it follows from the Krein-Rutman Theorem that

$$\lambda_1\left(u_0 + \frac{c_1v_0}{u_0 + k_1} - a\right) = 0 \text{ and } \lambda_1\left(\frac{c_2v_0}{u_0 + k_2} - b\right) = 0.$$

Clearly

$$\lambda_1(L_2^0) > \lambda_1\left(\frac{c_2v_0}{u_0 + k_2} - b\right) = 0.$$

Since $u_0 \le \theta_a < a$, one can see that $v_0 < b(a + k_2)/c_2$. It follows from $a + k_2 \le c_2k_1^2/(bc_1)$ that $v_0 < k_1^2/c_1$, and thus

$$\lambda_1(L_1^0) > \lambda_1\left(u_0 + \frac{c_1v_0}{u_0 + k_1} - a\right) = 0.$$

Then by similar analysis as [3, Lemma 5.2], $(\phi, \psi) = (0, 0)$. Finally the conclusion of this theorem follows from the proof of Lemma 5.4 and Theorem 5.1 in [3] with obvious modifications. ☐

Now we consider the problem (P_α). Let (u_0, v_0) be any positive solution of (P_α). Then the linearized system of (P_α) at (u_0, v_0) is

$$\begin{cases} -\phi'' + L_1^\alpha \phi = -M_1(u_0, v_0; \alpha)\psi, & x \in (0, L), \\ -\psi'' + L_2^\alpha \psi = M_2(u_0, v_0; \alpha)\phi, & x \in (0, L), \\ \phi(0) = \phi(L) = \psi(0) = \psi(L) = 0, \end{cases} \tag{6.5}$$

where

$$\begin{aligned} L_1^\alpha \phi &= \left[-\frac{a}{1+\alpha v_0} + \frac{2u_0}{(1+\alpha v_0)^2} + \frac{c_1 k_1 v_0 + c_1 k_1 \alpha v_0^2}{(k_1 + u_0 + k_1 \alpha v_0)^2} \right] \phi, \\ L_2^\alpha \psi &= \left[-b + \frac{2c_2 k_2 \alpha^2 v_0^3 + 3c_2 \alpha u_0 v_0^2 + 4c_2 k_2 \alpha v_0^2 + 2c_2 u_0 v_0 + 2c_2 k_2 v_0}{(k_2 + u_0 + k_2 \alpha v_0)^2} \right] \psi, \\ M_1(u_0, v_0; \alpha) &= \frac{a \alpha u_0}{(1+\alpha v_0)^2} - \frac{2\alpha u_0^2}{(1+\alpha v_0)^3} + \frac{c_1 k_1 u_0 + c_1 u_0^2}{(k_1 + u_0 + k_1 \alpha v_0)^2}, \\ M_2(u_0, v_0; \alpha) &= \frac{c_2 \alpha v_0^3 + c_2 v_0^2}{(k_2 + u_0 + k_2 \alpha v_0)^2}. \end{aligned}$$

Lemma 6.3. *Under the assumptions of Theorem 6.1, the linearized system* (6.5) *with $\alpha \in [0, \overline{\alpha})$ has only trivial solution $(\phi, \psi) = (0, 0)$. In other words, the positive solutions of* (P_α) *are never degenerate in this case.*

Proof. Since the conclusion for $\alpha = 0$ has been proved in Lemma 6.2, we assume that $\alpha \in (0, \overline{\alpha})$. If we show that $\lambda_1(L_1^\alpha) > 0$, $\lambda_1(L_2^\alpha) > 0$, $M_1(u_0, v_0; \alpha) > 0$ and $M_2(u_0, v_0; \alpha) > 0$, then the conclusion follows from similar arguments as in the proof of [3, Lemma 5.2].

Since (u_0, v_0) is a positive solution of (P_α), it follows from the Krein-Rutman Theorem that

$$\lambda_1 \left(-\frac{a}{1+\alpha v_0} + \frac{u_0}{(1+\alpha v_0)^2} + \frac{c_1 v_0}{k_1 + u_0 + k_1 \alpha v_0} \right) = 0,$$

and

$$\lambda_1 \left(-b + \frac{c_2 \alpha v_0^2 + c_2 v_0}{k_2 + u_0 + k_2 \alpha v_0} \right) = 0. \tag{6.6}$$

Since

$$\alpha < \frac{1}{B} \left(\frac{c_2 k_1^2}{b c_1} - k_2 - a \right),$$

it follows from (6.1) that $v_0 < k_1^2 / c_1$, and thus

$$\lambda_1(L_1^\alpha) > \lambda_1 \left(-\frac{a}{1+\alpha v_0} + \frac{u_0}{(1+\alpha v_0)^2} + \frac{c_1 v_0}{k_1 + u_0 + k_1 \alpha v_0} \right) = 0.$$

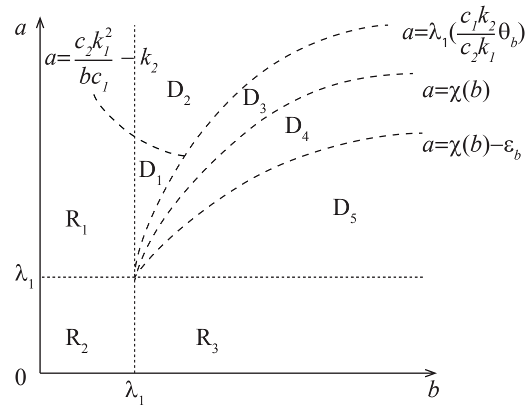

FIGURE 3. Illustration of the parameter regions of (a, b) in the main results

Similarly $\lambda_1(L_2^\alpha) > 0$ by (6.6). Clearly $M_2(u_0, v_0; \alpha) > 0$, and we only need to prove that $M_1(u_0, v_0; \alpha) > 0$ to complete the proof.

$$\begin{aligned}
&M_1(u_0, v_0; \alpha)\\
=&\frac{u_0}{(k_1+u_0+k_1\alpha v_0)^2}\left[\frac{a\alpha(k_1(1+\alpha v_0)+u_0)^2}{(1+\alpha v_0)^2}+c_1k_1+c_1u_0\right.\\
&\left.-2\alpha u_0\left(\frac{u_0^2}{(1+\alpha v_0)^3}+\frac{2k_1u_0}{(1+\alpha v_0)^2}+\frac{k_1^2}{1+\alpha v_0}\right)\right]\\
>&\frac{u_0}{(k_1+u_0+k_1\alpha v_0)^2}(a\alpha k_1^2+c_1k_1+c_1u_0-2\alpha u_0(u_0^2+2k_1u_0+k_1^2))\\
=&\frac{u_0}{(k_1+u_0+k_1\alpha v_0)^2}(-2\alpha u_0^3-4k_1\alpha u_0^2+(-2k_1^2\alpha+c_1)u_0+a\alpha k_1^2+c_1k_1)\\
=&\frac{u_0}{(k_1+u_0+k_1\alpha v_0)^2}\hbar(u_0;\alpha).
\end{aligned}$$

Here $\hbar(t;\alpha) = -2\alpha t^3 - 4k_1\alpha t^2 + (-2k_1^2\alpha + c_1)t + a\alpha k_1^2 + c_1k_1$ for $t \geq 0$. Then

$$\frac{d}{dt}\hbar(t;\alpha) = -6\alpha t^2 - 8k_1\alpha t - 2k_1^2\alpha + c_1,$$

and $\hbar(t;\alpha)$ is either decreasing for all $t \in [0, \infty)$ or, for some $\delta > 0$, increasing for $t \in (0, \delta)$ and decreasing for $t \in (\delta, \infty)$. Since $\hbar(0;\alpha) = a\alpha k_1^2 + c_1k_1 > 0$ and $0 \leq u_0 \leq a + B\alpha$, if $\hbar(a + B\alpha;\alpha) > 0$ then $\hbar(u_0;\alpha) > 0$ and consequently $M_1(u_0, v_0;\alpha) > 0$. By direct calculation, $\hbar(a + B\alpha;\alpha) = \hbar_2(\alpha)$, where $\hbar_2(\alpha)$ is the quartic polynomial defined in (6.2). Since $\alpha \in [0, \overline{\alpha})$, $\hbar_2(\alpha) > 0$, and thus the proof is complete. □

Based on Lemma 6.3, the conclusion of Theorem 6.1 follows from the proof of Lemma 5.4 and Theorem 5.1 in [3] with obvious modifications.

7. **Conclusions.** In this paper, we study the stability of trivial and semi-trivial solutions, and the existence, multiplicity and uniqueness of solution to problem (1.2). Our main results can be summarized as follows (see Figure 3):

1. the trivial solution $(0, 0)$ is globally asymptotically stable if $(a, b) \in R_2 = \{0 < a < \lambda_1, 0 < b < \lambda_1\}$, and it is unstable if $(a, b) \notin R_2$ (see Theorem 3.1);

2. the semi-trivial solution $(\theta_a, 0)$ exists if $a > \lambda_1$; it is locally asymptotically stable if $(a,b) \in R_1 = \{a > \lambda_1, 0 < b < \lambda_1\}$, and it is unstable otherwise (see Theorem 3.1);
3. the semi-trivial solution $(0, k_2\theta_b/c_2)$ exists if $b > \lambda_1$; it is locally asymptotically stable if (a,b) satisfies $a < \chi(b)$ and $b > \lambda_1$, which includes the regions $R_3 = \{0 < a < \lambda_1, b > \lambda_1\}$, D_4 and D_5 in Figure 3; here $int(\overline{D_4 \cup D_5}) = \{\chi(b) > a > \lambda_1, b > \lambda_1\}$ (see Theorem 3.1);
4. problem (1.2) possesses at least one positive solution if $(a,b) \in int(\overline{D_1 \cup D_2 \cup D_3})$; here D_1 and D_2 are separated by $a = (c_2k_1^2)/(bc_1) - k_2$, $D_1 \cup D_2$ and D_3 are separated by $a = \lambda_1(c_1k_2\theta_b/(c_2k_1))$, while D_3 and D_4 share the boundary $a = \chi(b)$ (see Theorem 4.3);
5. problem (1.2) possesses at least two positive solutions if $(a,b) \in D_4$ and $\hat{a}_1 < 0$, where $\hat{a}_1$ is defined in (5.4); here D_4 is a narrow region just below the curve $a = \chi(b)$ (see Theorem 5.3);
6. problem (1.2) possesses exactly one positive solution if $(a,b) \in D_1$ if $N = 1$ and α is small enough (see Theorem 6.1).

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Received August 2013; revised November 2013.

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