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Profile of the unique limit cycle in a class of general predator–prey systems

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ABSTRACT

Many predator–prey systems with oscillatory behavior possess a unique limit cycle which is globally asymptotically stable. For a class of general predator–prey system, we show that the solution orbit of the limit cycle exhibits the temporal pattern of a relaxation oscillator, when a certain parameter is small.

1. Introduction

With the wide applications in the natural world, the predator–prey systems has been one of the important topics in ecology and mathematical biology. Along with the development of modern mathematics, the predator–prey systems have been by using qualitative analysis and stability theory. For the research of limit cycles, in 1975 Freedman and Waltman [4,5] used various techniques for establishing the existence of limit cycles. In 1981 Cheng [3] (see also Liou and Cheng [14]) published a result giving a criterion for the uniqueness of limit cycles for a special class of predator–prey models. In 1950's, Zhang proved a uniqueness theorem of limit cycles of generalized Liénard equations, which was later recorded in [25] in 1986. Zhang's result was used by Kuang and Freedman [13] to consider a Gause type predator–prey system:

$$\begin{cases} \dot{x} = xg(x) - \xi(y)p(x), \\ \dot{y} = \eta(y)(-\gamma + q(x)). \end{cases}$$

in [13], they converted this predator–prey model to a Liénard equation, then showed that the new model satisfies the conditions in [25], consequently proved the uniqueness of limit cycle of this predator–prey system. Models of this type were introduced by Gause et al. [6], and since then, variations of this model have been utilized in Armstrong [1], Hassell [7], Hassell and May [10], and Rosenzweig [19], Alberecht et al. [2], May [16], Rosenzweig [18].

In 2009, Hsu and Shi [11] studied a predator–prey system in the particular form:

$$\begin{cases} \frac{du}{dt} = u(1-u) - \frac{muv}{a+u}, \\ \frac{dv}{dt} = -dv + \frac{muv}{a+u}, \end{cases} \tag{1.1}$$

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where $a,\ m,\ d > 0$. The system (1.1) is often called Rosenzweig–MacArthur predator–prey system from the seminar work of Rosenzweig and MacArthur [20]. Hsu and Shi [11] considered the relaxation oscillator profile of the limit cycle of (1.1) by a careful phase portrait analysis and estimates. It is important to know whether such profile is special only to (1.1) or it holds for a more general class of predator–prey systems. Hence the objective of this paper is to study the dynamical properties of a general predator–prey systems, in particular, the asymptotic behavior of the limit cycle.

In this paper, we consider a class of more general predator–prey systems in the form

$$\begin{cases} \frac{du}{dt} = ug(u) - vp(u), \\ \frac{dv}{dt} = v(-d + p(u)), \\ u(0) \geq 0, \quad v(0) \geq 0, \end{cases} \tag{1.2}$$

where $d > 0$, the functions $g(u)$ and $p(u)$ are sufficiently smooth so that the existence, uniqueness, and continuous dependence on parameters of solutions to the initial-value problem are satisfied. The functions $u(t)$ and $v(t)$ represent the prey and predator populations, respectively, at a given time $t \geqslant 0$. In this paper, we assume that the functions $g(u)$ and $p(u)$ in (1.2) satisfy

(H1): $g \in C^2(\mathbf{R}^+),\ g(0) > 0$, there exists $k > 0$, such that for any $u > 0,\ u \neq k,\ g(u)(u-k) < 0$ and $g(k) = 0$.
(H2): $p \in C^2(\mathbf{R}^+),\ p(0) = 0,\ p'(u) > 0$ for any $u \geqslant 0$, and there exists $a \in (0, k)$ such that $p(a) = d$.
(H3): Define $F(u) = \frac{ug(u)}{p(u)}$ if $u > 0$ and $F(0) = \frac{g(0)}{p'(0)}$. Then $F \in C^2(\mathbf{R}^+)$. We assume there exists $a_* \in (0, k)$, such that for any $u > 0,\ u \neq a_*,\ F'(u)(u - a_*) < 0$ and $F'(a_*) = 0$.

It is known that (see Hsu [8]) if (H1)–(H3) are satisfied, then (1.2) possesses a unique coexistence equilibrium point $(a, F(a))$. The local stability of $(a, F(a))$ depends on the sign of $F'(a)$: when $a_* < a < k$, then $F'(a) < 0$ and $(a, F(a))$ is locally asymptotically stable; and when $0 < a < a_*$, then $F'(a) > 0$ and $(a, F(a))$ is unstable. Moreover the global stability of $(a, F(a))$ when $a_* < a < k$ can be established through a Lyapunov functional or Dulac criterion under some extra conditions (see [8,9]). On the other hand, when $0 < a < a_*$, the instability of $(a, F(a))$ implies the existence of a periodic orbit from the Poincaré–Bendixon theory. The uniqueness of the periodic orbit will make the periodic orbit a limit cycle—the attractor for the predator–prey system. Since the work of Cheng [3], the uniqueness of the limit cycle in (1.2) has been proved under some extra conditions [12,24]. Here we site a result of Kuang and Freedman [13]: if (H1)–(H3) are satisfied, and also

(H4): for all $0 \leqslant u \leqslant k,\ u \neq a$, we have $\frac{d}{du}\left(\frac{p(u)F'(u)}{-d+p(u)}\right) \leqslant 0$,

then the limit cycle of (1.2) is unique and is global asymptotically orbital stable. Moreover, we can verify the uniqueness of limit cycle holds if (H1)–(H3) are satisfied, and also

(H4)′: $F \in C^3(\mathbf{R}^+)$, and $uF'''(u) + 2F''(u) \leqslant 0$ for $0 \leqslant u \leqslant k$,

which can be obtained from results in [22,23].

We also recall that the growth rate of the prey is of logistic type if $g(u)$ is strictly decreasing, and it is of weak Allee effect type if $g(u)$ is increasing for $0 < u < c$ and is decreasing for $c < u < k$. Conditions (H1) and (H3) allow for either type of growth. For example, $g(u) = k - u$ is a logistic growth; for $g(u) = (k-u)(u+a)$, it is weak Allee effect type when $0 \leqslant a < k$, and it is logistic type when $a > k$. Some examples of $p(u)$ are Holling type II functional response $p(u) = mu/(b+u)$, or Ivlev type as $p(u) = m(1 - e^{-bu})$.

Our result here generalizes the one in [11], in which the relaxation oscillation profile of the limit cycle in a predator–prey model was first studied. An earlier work for relaxation oscillator in predator–prey model appeared in [15]. For many other mathematical models with limit cycle behavior and small parameters, such relaxation oscillation have been well-documented in, for example, [17,21]. For such relaxation oscillation profile, the prey population $u(t)$ is near zero for a very long period when d is small (see Figs. 2 and 3 for illustration). Biologically this means the prey population is vulnerable to extinction even with small stochastic perturbations.

We prove our main results in Section 2 for the case $d \to 0$. We will use δ_i and C_i, ($i \in \mathbf{N}$), to denote various positive constants. These constants are independent of d in Section 2. We give an example and some numerical simulations to illustrate our results in Section 3.

2. Asymptotic behavior of the limit cycle for *d* small

In this section, we consider the asymptotical profile of the limit cycle of (1.2). We assume that $0 < a < a_*$ and the conditions (H1)–(H4) (or (H1)–(H4)′) hold. We define

$$f(u, v) = ug(u) - vp(u), \quad g(u, v) = v(-d + p(u)). \tag{2.1}$$

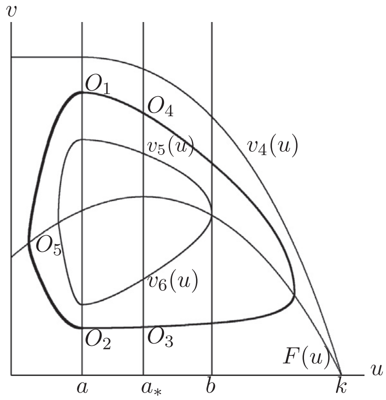

Fig. 1. Illustration of the phase portrait (not up to scale) and the limit cycle in the proof. The isoclines are the thin solid curves: $u = 0,\ v = 0,\ u = a$ and the parabola $v = F(u)$; the limit cycle is the thick solid curve $O_1O_2O_3O_4$; the boundary of the invariant region $R_3 : v = v_4(u)$ is the outer boundary (together with $u = 0$ and $v = 0$); $v = v_5(u)$ and $v = v_6(u)$ are the upper and lower portions of inner boundary respectively; the line $u = b$ satisfies $F(b) = F(a)$. (This graph is essentially from [11].)

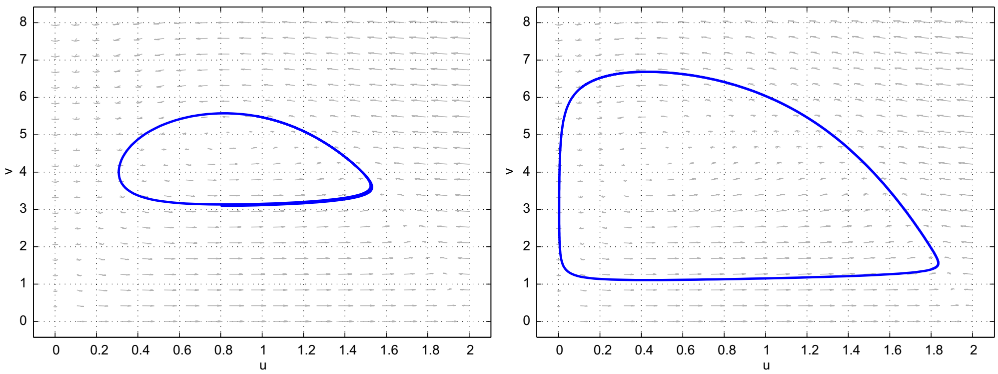

Fig. 2. Phase portraits of (3.1). Left: $d = 0.45$ with initial value $(u, v) = (0.8, 3.1)$; Right: $d = 0.3$ with initial value $(u, v) = (1.8, 1.4)$.

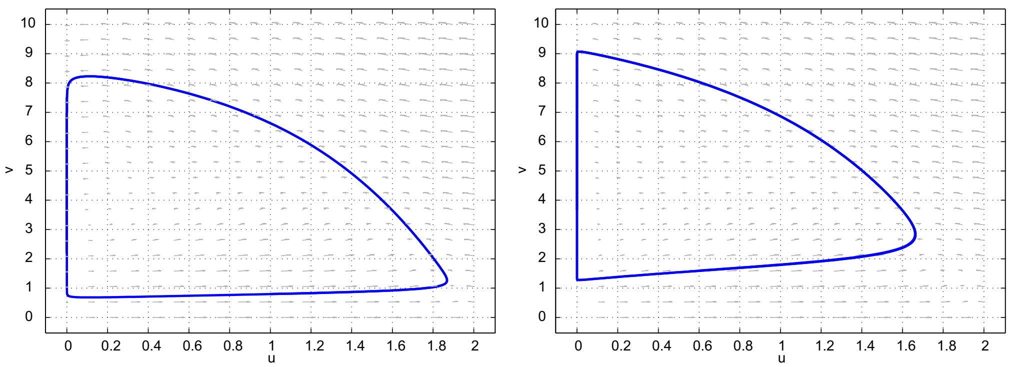

Fig. 3. Phase portraits of (3.1). Left: $d = 0.1$ with initial value $(u, v) = (1, 0.8)$; Right: $d = 0.01$ with initial value $(u, v) = (0.6, 1.6)$.

First we construct an invariant region in which the limit cycle is located. To achieve that, we give an estimate of the unstable manifold $U = \{(u_1(t), v_1(t)) : t \in \mathbf{R}\}$ of the saddle point $(k, 0)$. From the phase portrait, it satisfies $0 < u_1(t) < k$ for all $t \in \mathbf{R}$; U is above the isocline $v = F(u)$ when $a < u < k$. Since it is monotone for $a < u < k$, we denote this portion by $\{(u, v_1(u)) : a \leqslant u \leqslant k\}$ with $v_1(k) = 0$. We define

$$v_2(u) = \big(1 - F'(k)\big)(k - u), \quad \text{and} \quad v_3(u) = \int_k^u \left(\frac{d}{p(s)} - 1\right) ds. \tag{2.2}$$

Lemma 2.1. *Suppose that F also satisfies*

(**H5**): *For any* $u \geqslant 0,\ F'(u)$ *is monotonically decreasing.*

Then the unstable manifold U of $(k, 0)$ *satisfies*

$$v_3(u) \leqslant v_1(u) \leqslant v_2(u), \quad a \leqslant u \leqslant k. \tag{2.3}$$

Proof. From Eq. (1.2), we have

$$\frac{dv}{du} = \frac{v}{ug(u) - vp(u)} \cdot (-d + p(u)).$$

Since the unstable manifold satisfies $0 < u_1(t) < k$ for all $t \in \mathbf{R}$, then along U, we have

$$\frac{dv}{du} \leqslant \frac{v}{-vp(u)} \cdot (-d + p(u)) = \frac{d}{p(u)} - 1.$$

Integrating along the portion of U from $u = k$ to some $u \in (a, k)$, we obtain

$$v \geqslant \int_k^u \left(\frac{d}{p(s)} - 1\right) ds = v_3(u),$$

if $(u, v) \in U$ and $a \leqslant u \leqslant k$.

For the upper bound, we notice that the tangent line of the unstable manifold is

$$v = \left(1 - \frac{kg'(k) + d}{p(k)}\right)(k - u) = \left(1 - F'(k) - \frac{d}{p(k)}\right)(k - u),$$

which is below $v = v_2(u)$. Hence we only need to show that the vector field $(f(u, v), g(u, v))$ points towards the region below the line $v = v_2(u)$ when $(u, v) = (u, v_2(u))$ and $a < u < k$. That is equivalent to

$$\left|\frac{dv}{du}\right| \leqslant 1 - F'(k), \quad (u, v) = (u, v_2(u)).$$

Let $l = 1 - F'(k)$, then for $(u, v) = (u, v_2(u)),\ a \leqslant u < k$,

$$\left|\frac{dv}{du}\right| = \frac{l(k - u)(p(u) - d)}{|F(u) - l(k - u)|p(u)} \leqslant \frac{l(k - u)}{|F(u) - l(k - u)|}.$$

From the mean-value theorem, (H1) and (H5), we have

$$F(u) = F(u) - F(k) = F'(\xi)(u - k) \geqslant F'(k)(u - k) = (1 - l)(u - k),$$

for some $\xi \in (u, k)$. Hence

$$\left|\frac{dv}{du}\right| \leqslant \frac{l(k - u)}{|(1 - l)(u - k) + l(u - k)|} = l,$$

which proves the upper bound $v_1(u) \leqslant v_2(u)$. □

From Lemma 2.1, the unstable manifold reaches its maximum when $u = a$, and the maximum value v_* can be estimated as

$$\int_k^a \left(\frac{d}{p(s)} - 1\right) ds \leqslant v_* \leqslant (1 - F'(k))(k - a). \tag{2.4}$$

From the phase portrait of the system, the limit cycle is below the unstable manifold U, then we also have the following estimate of the outer boundary of the limit cycle.

Lemma 2.2. *Let* $v_2(u)$ *be defined as in (2.2). Define*

$$v_4(u) = \begin{cases} v_2(u), & a \leqslant u \leqslant k, \\ v_2(a), & 0 \leqslant u \leqslant a. \end{cases} \tag{2.5}$$

Then the orbit of the limit cycle $\Sigma = \{(u(t), v(t)) : 0 \leqslant t \leqslant T\}$ *satisfies*

$$\Sigma \subset \{(u, v) : 0 < u < k, 0 < v < v_4(u)\} \equiv R_1.$$

By constructing a more precise region $R_2 \subset R_1$ containing Σ, we prove that for a sub-region R_3 containing $(a, F(a)),\ \Sigma \cap R_3 = \emptyset$. From (H3), there exists a unique $b = b(a) \in (a_*, k)$ such that $F(b) = F(a)$. Define

$$R_3 = \{(u,v) \in \mathbf{R}^2_+ : W(u,v) \leqslant W(b, F(b))\}, \tag{2.6}$$

where $W(u,v)$ is defined by the well-known Lyapunov function for (1.2) (when $(a, F(a))$ is locally asymptotically stable)

$$W(u,v) = \int_a^u \frac{p(\xi)-d}{p(\xi)}\,d\xi + \int_{F(a)}^v \frac{\eta - F(a)}{\eta}\,d\eta.$$

Lemma 2.3. *Let R_3 be defined as in (2.6). Then R_3 is a bounded convex subset of $\mathbf{R}^2_+$ containing $(a, F(a))$, and $\Sigma \cap R_3 = \emptyset$. In particular $\Sigma \subset R_2 \equiv R_1 \setminus R_3$.*

Proof. From the definition in $W(u,v)$, $W(u,v) = W_1(u) + W_2(v)$, where

$$W_1(u) = \int_a^u \frac{p(\xi)-d}{p(\xi)}\,d\xi, \quad \text{and} \quad W_2(v) = \int_{F(a)}^v \frac{\eta - F(a)}{\eta}\,d\eta.$$

Since $W_1'(u) = (p(u)-d)/p(u)$, then $W_1(u)$ is strictly decreasing in $(0,a)$ and is strictly increasing in (a,∞); similarly since $W_2'(v) = 1 - (F(a)/v)$, then $W_2(v)$ is strictly decreasing in $(0, F(a))$ and is strictly increasing in $(F(a),\infty)$. Hence W achieves the global minimum at the unique critical point $(a, F(a))$, and every level curve of $W(u,v)$ is a bounded closed curve. The level curves have convex boundary since W_1 and W_2 are both convex one-variable functions. For R_3 defined in (2.6), $(b, F(a))$ is the right-most point of R_3. Thus for any solution orbit $(u(t), v(t))$ passing through $(u,v) \in R_3 \setminus \{(b, F(a))\}$,

$$\begin{aligned} W'(u(t),v(t)) &= \tfrac{p(u)-d}{p(u)}(ug(u) - vp(u)) + \tfrac{v-F(a)}{v}(p(u)-d)v \\ &= (p(u)-p(a))(F(u)-F(a)) > 0. \end{aligned}$$

In particular, for $(u,v) \in \partial R_3 \setminus \{(b, F(a))\}$, the vector field $(f(u,v), g(u,v))$ points outwards. Hence from the properties of periodic orbit, $\Sigma \cap R_3 = \emptyset$. □

From Lemmas 2.2 and 2.3, we obtain an invariant region R_2 where the limit cycle is located in. Next we give some estimates for the extremal points on the orbit of limit cycle as $d \to 0^+$. Since $a = p^{-1}(d) \to 0$ when $d \to 0^+$, where p^{-1} is the inverse function of p, hence d and a are two equivalent parameters which tend to zero. Define

$$\begin{aligned} u_{a,-} &= \min\{u(t) : (u(t),v(t)) \in \Sigma\}, \quad u_{a,+} = \max\{u(t) : (u(t),v(t)) \in \Sigma\}, \\ v_{a,-} &= \min\{v(t) : (u(t),v(t)) \in \Sigma\}, \quad v_{a,+} = \max\{v(t) : (u(t),v(t)) \in \Sigma\}. \end{aligned} \tag{2.7}$$

Notice that both the upper and lower portions of the limit cycle are monotone functions, thus we define

$$\Sigma = \{(u, v_+(a,u)) : u_{a,-} \leqslant u \leqslant u_{a,+}\} \bigcup \{(u, v_-(a,u)) : u_{a,-} \leqslant u \leqslant u_{a,+}\}, \tag{2.8}$$

such that $v_-(a,u) < F(u) < v_+(a,u)$ for $u_{a,-} < u < u_{a,+}$. That is, $\{(u, v_+(a,u))\}$ is the upper portion of the limit cycle Σ, and $\{(u, v_-(a,u))\}$ is the lower portion. From the equations, it is easy to see that $u_{a,-}$ and $u_{a,+}$ are achieved when Σ intersects with the isocline $v = F(u)$, and $v_{a,-}$, $v_{a,+}$ are achieved when Σ intersects with the line $u = a$. Our estimates are mainly based on the inner boundary of the region R_2, *i.e.* the level curve $\Sigma_1 = \{(u,v) : W(u,v) = W(b, F(b))\}$. Hence we also define

$$\begin{aligned} u_{1,a} &= \min\{u : (u,v) \in \Sigma_1\}, \quad u_{2,a} = \max\{u : (u,v) \in \Sigma_1\}, \\ v_{1,a} &= \min\{v : (u,v) \in \Sigma_1\}, \quad v_{2,a} = \max\{v : (u,v) \in \Sigma_1\}, \end{aligned} \tag{2.9}$$

and

$$\Sigma_1 = \{(u, v_5(u)) : u_{1,a} \leqslant u \leqslant u_{2,a}\} \bigcup \{(u, v_6(u)) : u_{1,a} \leqslant u \leqslant u_{2,a}\}, \tag{2.10}$$

such that $v_6(u) < F(u) < v_5(u)$ for $u_{1,a} < u < u_{2,a}$. Notice that

$$\nabla W = \left(\frac{p(u)-d}{p(u)}, \frac{v - F(a)}{v}\right),$$

hence $v_{1,a}$ and $v_{2,a}$ are the two intersection points of $W(u,v) = W(b, F(b))$ with the line $u = a$. Also $u_{2,a} = b$, and $u_{1,a}$ satisfies $W(u_{1,a}, F(a)) = W(b, F(b))$ with $u_{1,a} < a$. We define

$$h_1(u,a) = \int_a^u \frac{p(\xi)-d}{p(\xi)}\,d\xi, \quad \text{and} \quad h_2(v, F(a)) = \int_{F(a)}^v \frac{\eta - F(a)}{\eta}\,d\eta. \tag{2.11}$$

From the monotonicity of $p(u)$, $h_1(\cdot, a)$ achieves its global minimum 0 at $u = a$. Similarly,

$$\frac{\partial h_2(v, F(a))}{\partial v} = 1 - \frac{F(a)}{v}, \tag{2.12}$$

hence $h_2(\cdot, F(a))$ achieves its global minimum 0 at $v = F(a)$. Since $h_2(v, F(a)) = v - F(a) - F(a)\ln(v/F(a))$, $\lim_{v\to 0^+} h_2(v, F(a)) = \lim_{v\to\infty} h_2(v, F(a)) = \infty$. Thus $h_2(v, F(a)) = C$ has exactly two roots for any $C > 0$.

Lemma 2.4. *Let $v_{1,a}$ and $v_{2,a}$ be defined as in (2.9), and let v_1 and v_2 be the two roots of $h_2(v, F(0)) = b(0)$ such that $v_1 < F(0) < v_2$, where $b(0)$ is the unique point in $(0, k)$ such that $F(b(0)) = F(0)$. Then*

$$\lim_{a\to 0^+} v_{1,a} = v_1, \quad \text{and} \quad \lim_{a\to 0^+} v_{2,a} = v_2. \tag{2.13}$$

Proof. From Lemma 2.3, $v_{a,-} < v_{1,a}$ and $v_{2,a} < v_{a,+}$. By definition $v = v_{i,a}$ $(i = 1, 2)$ satisfy $W(a, v) = W(b, F(b))$. From the form of $W(u, v)$ and (2.11), $v = v_{i,a}$ $(i = 1, 2)$ satisfy

$$h_2(v, F(a)) = h_1(b(a), a). \tag{2.14}$$

We prove that

$$\lim_{a\to 0^+} h_1(b(a), a) = b(0). \tag{2.15}$$

Since $p'(0)$ exists, for any $\varepsilon > 0$, there exists $\delta > 0$, such that for $0 < \xi < \delta$, we have

$$(p'(0) - \varepsilon)\xi \leqslant p(\xi) \leqslant (p'(0) + \varepsilon)\xi,$$

Then

$$h_1(b(a), a) = \int_a^{b(a)} \left(1 - \frac{p(a)}{p(\xi)}\right) d\xi = b(a) - a - p(a)\int_a^{\delta} \frac{d\xi}{p(\xi)} - p(a)\int_{\delta}^{b(a)} \frac{d\xi}{p(\xi)}.$$

Since

$$0 < \int_{\delta}^{b(a)} \frac{d\xi}{p(\xi)} < \frac{b(a) - \delta}{p(\delta)},$$

then $p(a)\int_{\delta}^{b(a)} \frac{d\xi}{p(\xi)} \to 0$ as $a \to 0^+$ from (H2). On the other hand,

$$\int_a^{\delta} \frac{d\xi}{p(\xi)} \leqslant \int_a^{\delta} \frac{d\xi}{(p'(0) - \varepsilon)\xi} = \frac{1}{p'(0) - \varepsilon}(\ln\delta - \ln a).$$

From L'Hospital's rule, we have $\lim_{a\to 0} p(a) \cdot \ln a = 0$. Then all estimates together imply that $\lim_{a\to 0^+} h_1(b(a), a) = b(0)$.

Since $v = v_{i,a}$ is continuously differentiable in a, and differentiating (2.14) with respect to a at $v = v_{i,a}$, and from (2.12), we obtain

$$\left(1 - \frac{F(a)}{v_{i,a}}\right)\frac{\partial v_{i,a}}{\partial a} - \ln\left(\frac{v_{i,a}}{F(a)}\right)F'(a) = \frac{p(b) - d}{p(b)}\frac{\partial b}{\partial a}. \tag{2.16}$$

When $a \to 0$, $d = p(a) \to 0$, and the right hand side of (2.16) tends to a negative limit $\frac{\partial b}{\partial a}(0)$. On the left hand side of (2.16), $F'(a) > 0$ when $0 < a < a^*$, and since $v_{1,a} < F(a), -\ln(v_{1,a}/F(a))F'(a) > 0$, thus $\partial v_{1,a}/\partial a > 0$. In particular, $\lim_{a\to 0^+} v_{1,a}$ exists. On the other hand, for a near 0, $v_{2,a}$ satisfies

$$[1 - F'(k)]k > v_2(a) > v_{2,a} > F(a) > F(0). \tag{2.17}$$

Hence $\{v_{2,a} : 0 < a < \delta\}$ for some small $\delta > 0$ is uniformly bounded, and there exists a decreasing sequence $a_n \to 0$ such that v_{2,a_n} converges to a limit v_∞, and v_∞ must satisfy $h_2(v_\infty, F(0)) = b(0)$ from (2.14) and (2.15). From the bound of $v_{2,a}$ in (2.17), v_∞ must be the larger root v_2 of $h_2(v, F(0)) = b(0)$. Since each subsequence of $\{v_{2,a}\}$ converges to v_2, then $\lim_{a\to 0^+} v_{2,a} = v_2$. The same argument yields that $\lim_{a\to 0^+} v_{1,a} = v_1$, which is the smaller root of $h_2(v, F(0)) = b(0)$. □

To obtain the global asymptotical behavior of the limit cycle Σ, we divide the orbit Σ into four segments by four reference points (see Fig. 1):

$$\begin{aligned} O_1 &= (a, v_{a,+}), \quad O_2 = (a, v_{a,-}), \\ O_3 &= (a_*, v_-(a, a_*)), \quad O_4 = (a_*, v_+(a, a_*)). \end{aligned} \tag{2.18}$$

Let $T = T(a)$ be the period of Σ. Then $T = T_1 + T_2 + T_3 + T_4$, where T_i is the time taken from O_i to O_{i+1} (with $O_5 = O_1$). We also assume that $u(0) = a$ and $v(0) = v_{a,+}$, *i.e.* the orbit starts from the highest point of $v(t)$.

Our main result in this section is

Theorem 2.5. *Assume that the condition (H1)–(H5) are satisfied. Let $\Sigma = \{(u(t), v(t)) : t \in \mathbf{R}\}$ be the orbit of the unique periodic solution of (1.2) when $0 < a < a_*$, the extremal points of Σ are defined as in (2.7), and O_i, T_i $(i = 1, 2, 3, 4)$ and the period T are defined as above. When $a > 0$ is sufficiently small (or equivalently $d > 0$ is small), then there exists constants $C_2, C_3 > 0$ independent of a, such that $C_3/p(a) \geqslant T \geqslant C_2/p(a)$. Moreover, for $a > 0$ sufficiently small, there exists $C_4 > 0$, such that*

$$\frac{C_3}{p(a)} \geqslant T_1 \geqslant \frac{C_4}{p(a)}, \quad T_2 = O(|\ln a|), \quad T_3 = O(1), \quad T_4 = O(|\ln a|). \tag{2.19}$$

as $a \to 0^+$.

Proof. We prove the theorem in several steps.

Step 1. We show that

$$T_1 \geqslant \frac{1}{d - p(u_{a,-})} \ln\left(\frac{v_{a,+}}{v_{a,-}}\right). \tag{2.20}$$

Define $p(u_{a,-}) = d(1-\delta_2)$ for some $0 < \delta_2 < 1$. Then for $0 < t < T_1,\ u_{a,-} \leqslant u(t) \leqslant a$, and from the equation of $v(t)$,

$$v' = v(-d + p(u)) \geqslant v(-d + p(u_{a,-})) = -d\delta_2 v.$$

Hence $v(t) \geqslant v(0)\exp(-d\delta_2 t)$, which leads to

$$T_1 \geqslant \frac{1}{\delta_2 d} \ln\left(\frac{v_{a,+}}{v_{a,-}}\right) = \frac{1}{d - p(u_{a,-})} \ln\left(\frac{v_{a,+}}{v_{a,-}}\right). \tag{2.21}$$

Step 2. We show there exist constants $\delta_3,\ \delta_4 > 0$ such that when $0 < a < \delta_4$,

$$0 < T_2 \leqslant \frac{1}{\delta_3}\int_a^{a_*} \frac{du}{p(u)}. \tag{2.22}$$

For $T_1 \leqslant t \leqslant T_1 + T_2$, we have $a \leqslant u(t) \leqslant a^*$. From the equation of $u(t)$,

$$u' = p(u)(v_0(u) - v) \geqslant p(u)(v_0(u) - v_6(u)), \tag{2.23}$$

which follows from Lemma 2.3 that the limit cycle is below the level curve $(u, v_6(u))$ in this portion. Since $v_0(u)$ is concave while $v_6(u)$ is convex, then the minimum of $v_7(u) = v_0(u) - v_6(u)$ on the interval $[a, a_*]$ must achieve at either $u = a$ or $u = a_*$. From the proof of Lemma 2.4, $v_6(a) \to v_1$, the smaller root of $h_2(v, F(0)) = b(0)$, and $v_0(a) = F(a) \to F(0)$ as $a \to 0^+$. Thus $v_7(a) \to F(0) - v_1 > 0$ as $a \to 0^+$. Similarly as $a \to 0^+,\ v_0(a_*) \to F(a_*)$, and $v_6(a_*) \to \tilde{v}_1$, which is the smaller root of $h_2(v, F(0)) = b(0) - a_*$, as we take the limit of $a \to 0^+$ in

$$W(a_*, v_6(a_*)) = h_1(a_*, a) + h_2(v_6(a_*), F(a)) = h_1(b(a), a).$$

Similar to the proof of $\lim_{a\to 0^+} h_1(b(a), a) = b(0)$, we have $\lim_{a\to 0^+} h_1(a_*, a) = a_*$, hence $h_2(v_6(a_*), F(a)) \to b(0) - a_*$, *i.e.* $v_6(a_*) \to \tilde{v}_1$.
Thus there exists $\delta_3,\ \delta_4 > 0$ such that when $0 < a < \delta_4$,

$$v_0(u) - v_6(u) \geqslant \min\{v_0(a) - v_6(a), v_0(a_*) - v_6(a_*)\} \geqslant \delta_3 > 0. \tag{2.24}$$

Now from (2.23) and (2.24), for $T_1 \leqslant t \leqslant T_1 + T_2$ we have

$$\frac{du}{p(u)} \geqslant \delta_3 dt, \quad \int_a^{a_*} \frac{du}{p(u)} \geqslant \delta_3 T_2, \tag{2.25}$$

which implies (2.22).
Step 3. We show that

$$0 < T_3 \leqslant \frac{1}{p(a_*) - d} \ln\left(\frac{v_+(a, a_*)}{v_-(a, a_*)}\right). \tag{2.26}$$

For this portion, $u(t) > a_*$. From the equation of v, we have

$$v' = v(-d + p(u)) \geqslant v(-d + p(a_*)).$$

Hence $v(t) \geqslant v(T_1 + T_2)\exp([p(a_*) - d]t)$, and in particular

$$v_+(a, a_*) \geqslant v_-(a, a_*)\exp([p(a_*) - d]T_3),$$

which implies (2.26).
Step 4. We show there exist constants $\delta_5, \delta_6 > 0$ such that when $0 < a < \delta_6$,

$$0 < T_4 \leqslant \frac{1}{\delta_5}\int_a^{a_*} \frac{du}{p(u)}. \tag{2.27}$$

This is similar to Step 2. Now we have

$$u' = p(u)(v_0(u) - v) \leqslant p(u)(v_0(u) - v_5(u)) \leqslant p(u)(v_0(a_*) - v_5(a_*)). \tag{2.28}$$

Here the first inequality is from Lemma 2.3, and the second inequality is from the fact that $v_0(u)$ is increasing while $v_5(u)$ is decreasing in $[a, a_*)$, and $v_0(u) < v_5(u)$. Similar to the proof of Step 2, we obtain that when $0 < a < \delta_6$,

$$|u'| \geqslant \delta_5 p(u).$$

The remaining part is same as Step 2.

Step 5. We show that for any $0 < \delta_7 < 1$, when $a > 0$ is sufficiently small, there exists a constant $C_1 > 0$ such that

$$T_1 \leqslant \frac{1}{\delta_7 p(a)} \ln\left(\frac{v_{a,+}}{v_{a,-}}\right) + C_1. \tag{2.29}$$

We reconsider the portion of Σ in $(0, T_1)$ again. Notice that when $a > 0$ is sufficiently small, $u_{a,-} < a$ and $\lim_{a\to 0} u_{a,-} = 0$. Hence for any $0 < \delta_7 < 1$, $a(1-\delta_7) < a$, thus the orbit does reach $u = a(1-\delta_7)$. We write $T_1 = T_{11} + T_{12} + T_{13}$, so that $u(T_{11}) = a(1-\delta_7)$, and $u(T_{12}) = a(1-\delta_7)$. That is, T_{11} and $T_{11} + T_{12}$ are the times that Σ reaches $u = a(1-\delta_7)$. We also define $v_{11} = v(T_{11})$ and $v_{12} = v(T_{11} + T_{12})$.
For $t \in (T_{11}, T_{11} + T_{12})$, when $a > 0$ is sufficiently small, similar to Step 1,

$$v' = v(-d + p(u)) \leqslant v(-d + p[a(1-\delta_7)]) = v\big(-d + p[p^{-1}(d)(1-\delta_7)]\big) = -d\delta_7 v.$$

Hence $v_{12} \leqslant v_{11}\exp(-d\delta_7 T_{12})$, and

$$T_{12} \leqslant \frac{1}{d\delta_7} \ln\left(\frac{v_{11}}{v_{12}}\right) \leqslant \frac{1}{p(a)\delta_7} \ln\left(\frac{v_{a,+}}{v_{a,-}}\right). \tag{2.30}$$

Next we estimate T_{11}. Similar to Step 4, for $a > 0$ small, $|u'| \geqslant \delta_8 p(u)$ for some $\delta_8 > 0$, if $0 < a < \delta_9$. Here the estimate of $v_0(u) - v_5(u)$ can be obtained using the same proof of Lemma 2.4. Indeed we can replace (2.14) by

$$h_1((1-\delta)a, a) + h_2(v_5((1-\delta)a), F(a)) = h_1(b(a), a), \tag{2.31}$$

for $0 < \delta < \delta_7$. Then the same arguments yield $|u'| \geqslant \delta_8 p(u)$, and an integration gives

$$\int_{(1-\delta_7)a}^{a} \frac{du}{p(u)} \geqslant \delta_8 T_{11}.$$

Hence T_{11} is bounded by a constant independent of a. Similarly we can prove T_{13} is bounded.
Step 6. We show that there exist constants $v_3, v_4 > 0$ such that $v_{a,+} < v_3$ and $v_4 < v_{a,-}$ for all small $a > 0$.
From Lemma 2.1 and (2.4), we obtain the estimate of upper bound of $v_{a,+}$ by letting $v_3 = 1 - F'(k)$. For the estimate of v_4, we notice that any solution orbit satisfies

$$\frac{du}{dv} = \frac{p(u)}{p(u) - d} \cdot \frac{v_0(u) - v}{v}. \tag{2.32}$$

Recall that $O_1 = (a, v_{a,+})$ and $O_2 = (a, v_{a,-})$ are the highest and lowest points on the orbit of the limit cycle Σ. Let the leftmost point on Σ be $O_5 = (u_{a,-}, v_*)$. Then from (2.32), we obtain that

$$\int_{v_{a,-}}^{v_*} \frac{v_0(u_2(v)) - v}{v}\,dv = \int_{a}^{u_{a,-}} \frac{p(u) - d}{p(u)}\,du = \int_{v_{a,+}}^{v_*} \frac{v_0(u_1(v)) - v}{v}\,dv, \tag{2.33}$$

where $(u_1(v), v)$, $v_* \leqslant v \leqslant v_{a,+}$, represents the orbit O_1O_5, and $(u_2(v), v)$, $v_{a,-} \leqslant v \leqslant v_*$, represents the orbit O_5O_2. For the last integral in (2.33)

$$\int_{v_{a,+}}^{v_*} \frac{v_0(u) - v}{v}\,dv = \int_{v_*}^{v_{a,+}} \frac{v - v_0(u)}{v}\,dv \leqslant \int_{v_*}^{v_{a,+}} \frac{v - v_*}{v}\,dv = v_{a,+} - v_* - v_*\ln v_{a,+} + v_*\ln v_*. \tag{2.34}$$

Since $v_2 - \delta_0 < v_{a,+} < v_3$ for small a, then the right hand side of (2.34) is bounded. On the other hand, for the first integral in (2.33),

$$\int_{v_{a,-}}^{v_*} \frac{v_0(u) - v}{v}\,dv \geqslant \int_{v_{a,-}}^{v_*} \frac{v_* - v}{v}\,dv = v_{a,-} - v_* - v_*\ln v_{a,-} + v_*\ln v_*. \tag{2.35}$$

Thus $-\ln v_{a,-}$ is bounded from above from (2.33)–(2.35), and consequently $v_{a,-}$ is bounded from below by some $v_4 > 0$ for all small $a > 0$.
Step 7. The completion of the proof. □

From Lemma 2.4 and Step 6, when $a > 0$ is small, $v_4 < v_{a,-} < v_1 + \delta_0$ and $v_2 - \delta_0 < v_{a,+} < v_3$, where v_1 and v_2 are the two roots of $h_2(v, F(0)) = b(0)$ such that $v_1 < F(0) < v_2$, since $u_{a,-} < a$ and $\lim_{a\to 0} a^{-1}u_{a,-}$=0. Thus from Step 1 and Step 5, for any $0 < \delta_9 < 1$, as long as $a > 0$ is sufficiently small,

$$\frac{1}{\delta_7 p(a)} \ln\left(\frac{v_3}{v_4}\right) + C_1 \geqslant T_1 \geqslant \frac{1-\delta_9}{p(a)} \ln\left(\frac{v_2 - \delta_0}{v_1 + \delta_0}\right). \tag{2.36}$$

Hence we obtain the estimate for T_1 in the theorem, since all constants except a are independent of a. The estimate for T_3 can also be obtained from Step 3 and Step 6 since $v_+(a, a_*) < v_{a,+} < v_3$ and $v_-(a, a_*) > v_{a,-} > v_4$. The estimates of T_i for $i = 2, 4$ are clear from Steps 2 and 4, and $T = \sum T_i = O(a^{-1})$. This completes the proof. □

Remark.

1. Our construction of an invariant region in Lemma 2.3 does not require the smallness of a—it holds as long as $0 < a < a_*$. This gives a direct proof of the existence of periodic orbit.
2. When defining O_3 and O_4, the choice of $u = a_*$ can be replaced by any fixed $u = \beta \in (0, a_*]$, and the results of Theorem 2.5 still hold with this change.

3. Examples

To visualize the relaxation oscillator profile of the limit cycle, we consider some specific examples of the model (1.2). The existence and uniqueness of the limit cycle have been proved in [8,13,22]. The Lotka–Volterra predator–prey systems with Holling type II functional response and Logistic growth in prey has been discussed completely in [11].

Here we consider a Lotka–Volterra predator–prey system with Holling type II functional response and weak Allee effect growth in prey: (here $e, m, d > 0, 0 \leqslant c < k$)

$$\begin{cases} \frac{du}{dt} = u(k-u)(u+c) - \frac{muv}{e+u}, \\ \frac{dv}{dt} = -dv + \frac{muv}{e+u}, \\ u(0) \geq 0, \quad v(0) \geq 0. \end{cases} \tag{3.1}$$

Then contraposing the model (1.2), we have $g(u) = (k-u)(u+c)$, $p(u) = mu/(e+u)$ and

$$F(u) = \frac{ug(u)}{p(u)} = \frac{(k-u)(u+c)(e+u)}{m}.$$

Then $F(0) = \frac{kec}{m} > 0$. If in addition,

(A1) $k(e+c) > ec$, holds, then we can get a unique $a_* = \frac{(k-e-c)+\sqrt{k^2+e^2+c^2+ke+kc-ec}}{3} \in (0, k)$, which satisfies $F'(a_*) = 0$, and

$$F'(u)(u-a_*) = \frac{-3u^2 + 2(k-e-c)u + ke + kc - ec}{m}(u-a_*) < 0,$$

for $u > 0,\ u \neq a_*$. Then (H1)–(H3) are satisfied. Furthermore,

$$F''(u) = \frac{-6u + 2(k-e-c)}{m}, \quad uF'''(u) + 2F''(u) = \frac{-18u + 4(k-e-c)}{m}.$$

Therefore, (H4)′ and (H5) are satisfied if
(A2) $k < e + c$.

Thus when (A1) and (A2) are satisfied, the system (3.1) has a unique limit cycle which is globally asymptotically orbital stable when $0 < a < a_*$, and the asymptotic behavior of the limit cycle of (3.1) can be obtained from Theorem 2.5. In particular we have that the period T of Eq. (3.1) satisfies $C_5 a^{-1} \geqslant T \geqslant C_4 a^{-1}$, where $C_4,\ C_5 > 0$ are the constants independent of a.

To visualize the asymptotic behavior of the limit cycle of (3.1), we show the phase portraits with fixed $k = 2,\ e = 1,\ m = 1,\ c = 1.5$ and varying d in Figs. 2 and 3.

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