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Existence of positive solutions to Schrödinger–Poisson type systems with critical exponent

Fuyi Li[*,‡], Yuhua Li[*,§] and Junping Shi[*,†,¶]

[*] *School of Mathematical Sciences*
Shanxi University
Taiyuan 030006, Shanxi, P. R. China

[†] *Department of Mathematics*
College of William and Mary
Williamsburg, VA 23187-8795, USA

[‡] *fyli@sxu.edu.cn*
[§] *yhli@sxu.edu.cn*
[¶] *jxshix@wm.edu*

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The existence of positive solutions to Schrödinger–Poisson type systems in $\mathbb{R}^3$ with critically growing nonlocal term is proved by using variational method which does not require usual compactness conditions. A key ingredient of the proof is a new Brézis–Lieb type convergence result.

Keywords: Schrödinger–Poisson type system; critical exponent; variational method; Brézis–Lieb type convergence lemma.

Mathematics Subject Classification 2010: 35J50, 35J47

1. Introduction

The Schrödinger–Poisson system is a standard model in quantum mechanics describing electrons moving on a positive charged background [14, 22]. In this paper, we consider positive solutions to the following nonlinear Schrödinger–Poisson type system

$$\begin{cases} -\Delta u + bu + q\phi|u|^3 u = f(u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = |u|^5 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $b \geq 0$ and $q \in \mathbb{R}$ are parameters. The investigation of (1.1) is motivated by recent studies of Schrödinger–Poisson system

$$\begin{cases} -\Delta u + bu + q\phi g(u) = f(u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = 2G(u) & \text{in } \mathbb{R}^3, \end{cases} \tag{1.2}$$

where $|g(t)| \le C(|t| + |t|^s)$ for some $s \in [1, 4)$ (see, for example, [5, 19]). In [19], the system (1.2) was studied by using monotonicity technique for $b > 0$ and $q \ge 0$. For the case of $q = \pm 1$, the system (1.2) on a bounded domain Ω was considered in [5]. Most of them focus on the case of $b > 0, q > 0$ and $g(t) = t$, that is

$$\begin{cases} -\Delta u + bu + q\phi u = f(u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3 \end{cases} \tag{1.3}$$

(see, for example, [1–3, 6, 7, 11, 13, 16, 18, 23, 24, 27, 28, 32]). For $b = q = 1$ and $f(t) = |t|^{s-1}t$, the existence of a positive radial solution of (1.3) for $s \in (2, 5)$ was proved in [12, 23], while the nonexistence was also shown for $s \le 2$. Multiple solutions to (1.3) were also possible for the case of $q > 0$ and $f(t) = |t|^{s-1}t$ (see [3, 23]). These results can be extended to the case of more general f. In [2, 6], the following conditions on f were assumed:

(f_1) $f \in C(\mathbb{R}, \mathbb{R})$ and $-\infty < \liminf_{t\to 0^+} \frac{f(t)}{t} \le \limsup_{t\to 0^+} \frac{f(t)}{t} = -m < 0$;
(f_2) $-\infty \le \limsup_{t\to\infty} \frac{f(t)}{t^5} \le 0$;
(f_3) there exists $\alpha > 0$ such that $F(\alpha) = \int_0^\alpha f(t)dt > 0$

and it was shown that system (1.3) with $b = 0$ and $q > 0$ has a positive radial solution for $q \in (0, q_0)$ for some $q_0 > 0$. Related results for more general f were also obtained in [1, 28, 32], and solutions of Schrödinger–Poisson systems with nonconstant potential (where b is a function of x) were considered in [17, 20, 21, 30, 31]. However, these aforementioned papers all considered the Schrödinger–Poisson type system with subcritical nonlocal term which assumes that $\lim_{t\to\infty} G(t)/t^5 = 0$.

To the best of our knowledge, the Schrödinger–Poisson system with critical exponent $G(t) = |t|^5$ was only studied by [4]. In [4], a Schrödinger–Poisson type system with a critically growing nonlinearity on a bounded domain was considered, and the equation studied there was in the following form:

$$\begin{cases} -\Delta u = \lambda u + q\phi |u|^3 u & \text{in } B_R, \\ -\Delta \phi = q|u|^5 & \text{in } B_R, \\ u = \phi = 0 & \text{on } \partial B_R, \end{cases} \tag{1.4}$$

where B_R is a ball in $\mathbb{R}^n$ with radius R. Note that $s = 5$ is the critical exponent for the Sobolev embedding, and the critically growing nonlinearity in (1.4) appears in a nonlocal way as the second equation can be solved by a Green's function.

In this paper, $H^1(\mathbb{R}^3)$ and $\mathcal{D}^{1,2}(\mathbb{R}^3)$ are the usual Sobolev spaces, and the norms and inner products are defined by (here $b > 0$ is a fixed parameter)

$$\|u\|_{H^1(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} [|\nabla u|^2 + bu^2]\right)^{1/2}, \quad (u, v) = \int_{\mathbb{R}^3} [\nabla u \cdot \nabla v + buv], \quad u, v \in H^1(\mathbb{R}^3),$$

$$\|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^{1/2}, \quad (u, v) = \int_{\mathbb{R}^3} \nabla u \cdot \nabla v, \quad u, v \in \mathcal{D}^{1,2}(\mathbb{R}^3).$$

For convenience, we define $H_b = H_r^1(\mathbb{R}^3)$ for the case $b > 0$, and $H_b = \mathcal{D}_r^{1,2}(\mathbb{R}^3)$ when $b = 0$, where $H_r^1(\mathbb{R}^3)$ and $\mathcal{D}_r^{1,2}(\mathbb{R}^3)$ denote the space of radial functions of $H^1(\mathbb{R}^3)$ and $\mathcal{D}^{1,2}(\mathbb{R}^3)$ respectively. Indeed we consider the system (1.1) in the subspace $H_r^1(\mathbb{R}^3)$ of $H^1(\mathbb{R}^3)$ when $b > 0$, or in the subspace $\mathcal{D}_r^{1,2}(\mathbb{R}^3)$ of $\mathcal{D}^{1,2}(\mathbb{R}^3)$ when $b = 0$. Throughout the paper, we use $\|\cdot\|$ as the norm of H_b, and $(\cdot\,,\cdot)$ as the inner product in H_b. In this paper we consider the system (1.1) which is the critical exponent case for $\mathbb{R}^3$, and we assume the following conditions for the nonlinearity f which are weaker than the ones in previous work:

(H_1) $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $\lim_{t\to 0^+} \frac{f(t)}{bt+t^5} = 0$, here $\mathbb{R}_+ = [0, +\infty)$;
(H_2) $\lim_{t\to\infty} \frac{f(t)}{t^5} = 0$;
(H_3) there is a function $z \in H_b$ such that $\int_{\mathbb{R}^3} F(z) > b\int_{\mathbb{R}^3} z^2$, where $F(z) = \int_0^z f(t)dt$;
(H_4) there exist $r \in (4,6), A > 0, B > 0$ such that $F(t) \geq At^r - Bt^2$ for $t \geq 0$.

Sometimes the system (1.1) is labeled in two cases: $b > 0$ (positive mass), and $b = 0$ (zero mass). If $b = 0$, then the condition (H_1) is equivalent to $\lim_{t\to 0^+} \frac{f(t)}{t^5} = 0$; and if $b > 0$, then the condition (H_1) is equivalent to $\lim_{t\to 0^+} \frac{f(t)}{t} = 0$. In our result, we consider both the positive and zero mass cases. We also remark that the condition (H_3) is weaker than $\lim_{t\to\infty} \frac{F(t)}{t} = \infty$.

Our main results in this paper are as follows.

Theorem 1.1. *Suppose that $b \geq 0$ is fixed. Then there exists $q_0 > 0$ such that for any $q \in [0, q_0)$, the system* (1.1) *possesses at least one positive radially symmetric solution (u, ϕ) in $H_b \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ if the conditions* (H_1), (H_2) *and* (H_3) *hold. For $q = -1$, the system* (1.1) *possesses at least one positive radially symmetric solution (u, ϕ) in $H_b \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ if the conditions* (H_1), (H_2) *and* (H_4) *hold.*

Theorem 1.1 appears to be the first existence result for the system (1.1) which is the critical exponent case. For the subcritical case,

$$\begin{cases} -\Delta u + bu + q\phi|u|^{s-1}u = f(u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = |u|^{s+1} & \text{in } \mathbb{R}^3, \end{cases} \tag{1.5}$$

where $b > 0$ and $s \in [1, 4)$, we also have the following similar results.

Theorem 1.2. *Suppose that $b > 0$ is fixed and $s \in [1, 4)$. There exists $q_0 > 0$ such that for any $q \in [0, q_0)$, the system* (1.5) *possesses at least one positive radially symmetric solution (u, ϕ) in $H_b \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ if the conditions* (H_1), (H_2) *and* (H_3) *hold. For any $q \in (-\infty, 0)$, the system* (1.5) *possesses at least one positive radially symmetric solution (u, ϕ) in $H_b \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ if the conditions* (H_1) *and* (H_2) *hold.*

We remark that the results for the critical and subcritical cases appear to be similar, but the subcritical case has been studied in many papers for the case of $q > 0$ though not the case of $q < 0$. The critical case has not been studied in either $q > 0$ or $q < 0$ case. From the technical point of view, there are several difficulties to prove our results for the critical exponent case. Firstly, our result does not assume the classical Ambrosetti–Rabinowitz hypothesis

(f_4) there exists $\mu > 2$ such that $0 < \mu F(t) \leq f(t)t$ for all $t > 0$

and this makes it difficult to prove the boundedness of Palais–Smale (PS) sequences. Secondly, the critically growing nonlinearity in the equation sets an obstacle when showing the convergence. Lastly, for the zero mass case, the space $\mathcal{D}^{1,2}(\mathbb{R}^3)$ can only be embedded into $L^6(\mathbb{R}^3)$, which makes it difficult to deal with the integral $\int_{\mathbb{R}^3} \phi|u|^5$ in the energy function. Our result can be regarded as a generalization of the classical result in [8] for the semilinear scalar equation

$$-\Delta u = f(u) \quad \text{in } \mathbb{R}^3. \tag{1.6}$$

Note that when $b = 0$ and $q = 0$, (1.1) is reduced to (1.6). Then according to Theorem 1.2, we have the following result.

Corollary 1.3. *If the conditions* (H_1) (*with* $b = 0$), (H_2) *and* (H_3) *hold, then Eq.* (1.6) *possesses at least one positive solution.*

Since Corollary 1.3 is a classical result proved in [8] for the zero mass case, then Theorem 1.2 is an extension of the classical result to the nonlocal case.

From the Gagliardo–Nirenberg–Sobolev inequality, $\mathcal{D}^{1,2}(\mathbb{R}^3)$ can be continuously embedded into $L^6(\mathbb{R}^3)$, hence we can equivalently define $\mathcal{D}^{1,2}(\mathbb{R}^3) = \{u \in L^6(\mathbb{R}^3) : |\nabla u| \in L^2(\mathbb{R}^3)\}$. In the following we frequently use the inequality:

$$S|u|^2_{L^6(\mathbb{R}^3)} \leq \|u\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^3)} \tag{1.7}$$

and if $u \in H^1(\mathbb{R}^3)$,

$$S|u|^2_{L^6(\mathbb{R}^3)} \leq \|u\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^3)} \leq \|u\|^2_{H^1(\mathbb{R}^3)}. \tag{1.8}$$

Note that we can choose the best embedding constant S here (see [26, 29]). We denote by $|\cdot|_p$ the usual $L^p(\mathbb{R}^3)$ norm for $p \geq 1$. In this paper, we only consider positive solutions to (1.1), then we may assume that $f(t) = 0$ for $t < 0$. In Sec. 2, we recall some basic setup and some useful lemmas. In Sec. 3, we consider the case of $q > 0$ and prove the first half of Theorem 1.1. We consider the case of $q < 0$ and prove the second half of Theorem 1.1 in Sec. 4. At the end of Sec. 4 we show that how proofs for the critical case can be adapted to the subcritical case.

2. Preliminaries

In this section, we first provide the basic variational setup for (1.1), and we follow it with several technical ingredients in preparation of the proof of our main results. In Sec. 2.2, we prove a Brézis–Lieb type convergence lemma which is the key in the convergence proof. Some other related convergence results and a monotonicity technique method are recalled in Secs. 2.3 and 2.4 respectively. In Sec. 2.5, a new Pohozaev identity which contains terms depending on the norms of functions in H_b is proved. Note that the Brézis–Lieb type convergence lemma and the

Pohozaev identity with norm terms are new techniques which may be useful in other applications.

2.1. *Setup*

For a fixed $u \in H_b$, the second equation of (1.1) is a Poisson equation which is uniquely solvable for ϕ. Then the system (1.1) can be reduced to the first equation with ϕ represented by the solution of the Poisson equation. This is the basic strategy of solving (1.1). To be more precise about the solution ϕ of the Poisson equation, we have the following lemma for the critical case which is well known for the subcritical case of (1.5) with $1 \leq s < 4$ (see [2, 12, 23]).

Lemma 2.1. *For every $u \in L^6(\mathbb{R}^3)$, there exists a unique $\phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ which is the solution of*

$$-\Delta\phi = |u|^5 \quad \text{in } \mathbb{R}^3. \tag{2.1}$$

Moreover,

(i) $\|\phi_u\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^3)} = \int_{\mathbb{R}^3} \phi_u|u|^5$;
(ii) $\phi_u(x) > 0$ *for every* $x \in \mathbb{R}^3$;
(iii) *for any* $\theta > 0, \phi_{u_\theta} = \theta^2(\phi_u)_\theta$, *where* $u_\theta(\cdot) = u(\cdot/\theta)$;
(iv) *for any* $t > 0$, $\phi_{tu} = t^5\phi_u$;
(v) *for any* $u \in L^6(\mathbb{R}^3)$,

$$\|\phi_u\|_{\mathcal{D}^{1,2}(\mathbb{R}^3)} \leq S^{-1/2}|u|_6^5$$

and

$$\int_{\mathbb{R}^3} \phi_u|u|^5 \leq S^{-1}|u|_6^{10},$$

where S is defined in (1.7);
(vi) *if u is radially symmetric, so is* ϕ_u;
(vii) *if $u_n \rightharpoonup u$ in $L^6(\mathbb{R}^3)$ and $u_n \to u$ a.e. in $\mathbb{R}^3$ as $n \to \infty$, then $\phi_{u_n} \rightharpoonup \phi_u$ in* $\mathcal{D}^{1,2}(\mathbb{R}^3)$;
(viii) *if $u_n \to u$ in $L^6(\mathbb{R}^3)$ as $n \to \infty$, then $\phi_{u_n} \to \phi_u$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ and* $\int_{\mathbb{R}^3} \phi_{u_n} |u_n|^5 \to \int_{\mathbb{R}^3} \phi_u|u|^5$.

Proof. The existence and uniqueness of ϕ_u follow from the Lax–Milgram theorem. The conclusions (i), (ii), (iv) and (vi) are clear from simple calculation.

(iii) It follows from (2.1) that $-\Delta\phi(x/\theta) = |u(x/\theta)|^5 = |u_\theta(x)|^5$ in $\mathbb{R}^3$. So $-\theta^2\Delta(\phi(x/\theta)) = |u_\theta(x)|^5$. This implies that $\phi_{u_\theta}(\cdot) = \theta^2\phi_u(\cdot/\theta)$.

(v) For any $u \in L^6(\mathbb{R}^3)$,

$$\|\phi_u\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^3)} = \int_{\mathbb{R}^3} \phi_u|u|^5 \leq |\phi_u|_6|u|_6^5 \leq S^{-1/2}\|\phi_u\|_{\mathcal{D}^{1,2}(\mathbb{R}^3)}|u|_6^5$$

and

$$\int_{\mathbb{R}^3} \phi_u |u|^5 \leq |\phi_u|_6 |u|_6^5 \leq S^{-1/2} \|\phi_u\|_{\mathcal{D}^{1,2}(\mathbb{R}^3)} |u|_6^5 \leq S^{-1} |u|_6^{10}.$$

(vii) For any $v \in \mathcal{D}^{1,2}(\mathbb{R}^3)$, $v \in L^6(\mathbb{R}^3)$ from the Sobolev embedding. Since $u_n \rightharpoonup u$ in $L^6(\mathbb{R}^3)$ and $u_n \to u$ a.e. in $\mathbb{R}^3$ as $n \to \infty$, then $|u_n|^5 \rightharpoonup |u|^5$ in $L^{6/5}(\mathbb{R}^3)$. So

$$(\phi_{u_n}, v) = \int_{\mathbb{R}^3} |u_n|^5 v \to \int_{\mathbb{R}^3} |u|^5 v = (\phi_u, v).$$

Therefore, $\phi_{u_n} \rightharpoonup \phi_u$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$.

(viii) For any $v \in \mathcal{D}^{1,2}(\mathbb{R}^3)$, since

$$\begin{aligned}
|(\phi_{u_n} - \phi_u, v)| &= \left| \int_{\mathbb{R}^3} (|u_n|^5 - |u|^5) v \right| \\
&\leq 5 \int_{\mathbb{R}^3} |u + \theta(u_n - u)|^4 |u_n - u||v| \\
&\leq 40(|u_n|_6^4 + |u|_6^4)|u_n - u|_6 |v|_6 \\
&\leq 40 S^{-1/2}(|u_n|_6^4 + |u|_6^4)|u_n - u|_6 \|v\|_{\mathcal{D}^{1,2}(\mathbb{R}^3)},
\end{aligned}$$

where $\theta \in [0,1]$, $\phi_{u_n} \to \phi_u$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ and then $\phi_{u_n} \to \phi_u$ in $L^6(\mathbb{R}^3)$. Hence $\int_{\mathbb{R}^3} \times \phi_{u_n} |u_n|^5 \to \int_{\mathbb{R}^3} \phi_u |u|^5$. □

The results in Lemma 2.1 imply that the system (1.1) is reduced to a nonlocal semilinear elliptic equation

$$-\Delta u + bu + q\phi_u |u|^3 u = f(u), \quad \text{in } \mathbb{R}^3, \tag{2.2}$$

where ϕ_u is defined in Lemma 2.1.

We look for positive solutions of (2.2) in the space H_b. Define a functional J_q in the space H_b by

$$J_q(u) = \frac{1}{2}\|u\|^2 + \frac{1}{10} q \int_{\mathbb{R}^3} \phi_u |u|^5 - \int_{\mathbb{R}^3} F(u), \quad u \in H_b. \tag{2.3}$$

Then from (H_1), (H_2) and (H_3) we have that J_q is well defined on H_b and is of C^1 class for any $q \in \mathbb{R}$, and

$$\langle J_q'(u), v \rangle = (u, v) + q \int_{\mathbb{R}^3} \phi_u |u|^3 uv - \int_{\mathbb{R}^3} f(u) v, \quad u, v \in H_b.$$

It is standard to verify that a critical point u of the functional J_q corresponds to a weak solution (u, ϕ_u) of (1.1). Hence in the following, we consider critical points of J_q using variational method.

2.2. *A Brézis–Lieb type convergence lemma*

To prove the convergence of nonlocal term in the critically growing Schrödinger–Poisson system (1.1), we need the following key lemma which is inspired by the Brézis–Lieb convergence lemma (see [10]).

Lemma 2.2. *Let $r \geq 1$ and let Ω be an open subset of $\mathbb{R}^N$. Suppose that $u_n \rightharpoonup u$ in $L^r(\Omega)$, and $u_n \to u$ a.e. in Ω as $n \to \infty$, then for $p \in [1, r]$, as $n \to \infty$,*

$$|u_n|^p - |u_n - u|^p - |u|^p \to 0 \quad \text{in } L^{r/p}(\Omega), \tag{2.4}$$

$$|u_n|^{p-1}u_n - |u_n - u|^{p-1}(u_n - u) - |u|^{p-1}u \to 0 \quad \text{in } L^{r/p}(\Omega). \tag{2.5}$$

Proof. If $p = r$, the conclusions are proved in [10]. If $p = 1 < r$, since

$$||u_n| - |u_n - u| - |u|| \leq 2|u|$$

and $u_n \to u$ a.e. in Ω as $n \to \infty$, by Lebesgue's Dominated Convergence Theorem, we have $|u_n| - |u_n - u| - |u| \to 0$ in $L^r(\Omega)$.

In the following, we assume that $p \in (1, r)$. Let $v_n = u_n - u$. Then $v_n \rightharpoonup 0$ in $L^r(\Omega)$ and $u_n = v_n + u$. From the intermediate value theorem, there exists $\theta \in L^\infty(\Omega)$ and $0 \leq \theta \leq 1$ such that

$$||v_n + u|^p - |v_n|^p| = p|v_n + \theta u|^{p-1}|u| \leq 2^{p-1}p(|v_n|^{p-1}|u| + |u|^p).$$

For any $\varepsilon > 0$, by using Young's inequality we obtain that, there exists $C_\varepsilon > 0$ such that

$$||v_n + u|^p - |v_n|^p - |u|^p| \leq \varepsilon|v_n|^p + C_\varepsilon|u|^p.$$

We consider the sequence defined by

$$h_n = \max\{||v_n + u|^p - |v_n|^p - |u|^p| - \varepsilon|v_n|^p, 0\},$$

which satisfies

$$h_n \to 0 \quad \text{a.e. in } \Omega, \quad 0 \leq h_n \leq C_\varepsilon|u|^p \in L^{r/p}(\Omega).$$

Then by Lebesgue's Dominated Convergence Theorem, we have

$$\int_\Omega h_n^{r/p} \to 0.$$

From the definition of h_n, it follows that

$$||v_n + u|^p - |v_n|^p - |u|^p| \leq h_n + \varepsilon|v_n|^p.$$

Thus, we obtain the following inequality

$$\limsup_{n\to\infty} \int_\Omega ||v_n + u|^p - |v_n|^p - |u|^p|^{r/p} \leq C\varepsilon,$$

where C is a positive constant so that $\int_\Omega |v_n|^r \leq C$. This implies that

$$||v_n + u|^p - |v_n|^p - |u|^p| \to 0 \quad \text{in } L^{r/p}(\Omega),$$

which is equivalent to (2.4).

For (2.5), the case of $p = 1$ is trivial as now

$$|u_n|^{p-1}u_n - |u_n - u|^{p-1}(u_n - u) - |u|^{p-1}u = 0.$$

If $p \in (1, r]$, then from the intermediate value theorem, there exists $\theta \in L^\infty(\Omega)$ and $0 \le \theta \le 1$ such that

$$\||v_n + u|^{p-1}(v_n + u) - |v_n|^{p-1}v_n| = p|v_n + \theta u|^{p-1}|u| \le 2^{p-1}p(|v_n|^{p-1}|u| + |u|^p).$$

Then we can follow a similar proof as in the last paragraph to prove (2.5). □

By using the basic convergence result in Lemma 2.2, we have the following convergence estimates which are also related to ϕ_u.

Lemma 2.3. *If $u_n \rightharpoonup u$ in $L^6(\mathbb{R}^3)$ and $u_n \to u$ a.e. in $\mathbb{R}^3$, then as $n \to \infty$,*

$$|u_n|^5 - |u_n - u|^5 - |u|^5 \to 0 \quad \text{in } L^{6/5}(\mathbb{R}^3), \tag{2.6}$$

$$|u_n|^3u_n - |u_n - u|^3(u_n - u) - |u|^3u \to 0 \quad \text{in } L^{3/2}(\mathbb{R}^3), \tag{2.7}$$

$$\phi_{u_n} - \phi_{u_n-u} - \phi_u \to 0 \quad \text{in } \mathcal{D}^{1,2}(\mathbb{R}^3), \tag{2.8}$$

$$\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^5 - \int_{\mathbb{R}^3} \phi_{u_n-u}|u_n - u|^5 - \int_{\mathbb{R}^3} \phi_u|u|^5 \to 0. \tag{2.9}$$

Proof. The convergences in (2.6) and (2.7) follow from Lemma 2.2. Let $v_n = u_n - u$. Since for every $w \in \mathcal{D}^{1,2}(\mathbb{R}^3)$,

$$\begin{aligned} |\langle \phi_{v_n+u} - \phi_{v_n} - \phi_u, w\rangle| &= \left| \int_{\mathbb{R}^3} w(|v_n + u|^5 - |v_n|^5 - |u|^5) \right| \\ &\le |w|_6 ||v_n + u|^5 - |v_n|^5 - |u|^5|_{6/5}, \end{aligned}$$

then

$$\phi_{v_n+u} - \phi_{v_n} - \phi_u \to 0, \quad \text{in } \mathcal{D}^{1,2}(\mathbb{R}^3),$$

which implies (2.8). Since $u_n \rightharpoonup u$ in $L^6(\mathbb{R}^3)$ and $u_n \to u$ a.e. $x \in \mathbb{R}^3$, then $u_n - u \rightharpoonup 0$ in $L^6(\mathbb{R}^3)$ and $u_n - u \to 0$ a.e. $x \in \mathbb{R}^3$. By Lemma 2.1(vii), $\phi_{u_n-u} \rightharpoonup 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$. Therefore, as $n \to \infty$,

$$\begin{aligned} &\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^5 - \int_{\mathbb{R}^3} \phi_{u_n-u}|u_n - u|^5 - \int_{\mathbb{R}^3} \phi_u|u|^5 \\ &\quad = \int_{\mathbb{R}^3} [\phi_{u_n} - \phi_{u_n-u} - \phi_u]|u_n|^5 + \int_{\mathbb{R}^3} \phi_{u_n-u}[|u_n|^5 - |u_n - u|^5 - |u|^5] \\ &\qquad + \int_{\mathbb{R}^3} \phi_{u_n-u}|u|^5 + \int_{\mathbb{R}^3} \phi_u|u_n|^5 - \int_{\mathbb{R}^3} \phi_u|u|^5 \\ &\quad \to 0. \end{aligned}$$

□

2.3. *Convergence of nonlinear terms*

For the convergence of nonlinear terms, we first recall the following lemma from [8].

Lemma 2.4. *Let $N \geq 3$. Every radial function u in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is almost everywhere equal to a function U, continuous for $x \neq 0$, such that*

$$|U(x)| \leq C_N |x|^{(2-N)/2} \|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}, \quad |x| \geq 1,$$

where C_N only depends on N.

The convergence of $\int_{\mathbb{R}^3} f(u_n)u_n \to \int_{\mathbb{R}^3} f(u)u$ and $\int_{\mathbb{R}^3} F(u_n) \to \int_{\mathbb{R}^3} F(u)$ needs the following compactness result due to [25], see also [8].

Theorem 2.5. *Let $P, Q : \mathbb{R} \to \mathbb{R}$ be two continuous functions satisfying*

$$\frac{P(s)}{Q(s)} \to 0, \quad |s| \to \infty.$$

Let $\{u_n\}$ be a sequence of measurable functions on $\mathbb{R}^N$ such that

$$\sup_n \int_{\mathbb{R}^N} |Q(u_n)| < \infty$$

and

$$P(u_n) \to v \quad \text{a.e. on } \mathbb{R}^N.$$

Then for any bounded Borel set B one has

$$\int_B |P(u_n) - v| \to 0.$$

If one further assumes that

$$\frac{P(s)}{Q(s)} \to 0, \quad s \to 0$$

and $u_n(x) \to 0$ as $|x| \to \infty$, uniformly with respect to n, then $\{P(u_n)\}$ converges to v in $L^1(\mathbb{R}^N)$.

Lemma 2.6. *Suppose that* (H_1) *and* (H_2) *hold. If $u_n \rightharpoonup u$ in H_b, then*

$$\int_{\mathbb{R}^3} F(u_n) \to \int_{\mathbb{R}^3} F(u) \quad \text{and} \quad \int_{\mathbb{R}^3} f(u_n)u_n \to \int_{\mathbb{R}^3} f(u)u.$$

Proof. Let $P(s) = F(s)$ and $Q(s) = \frac{1}{2}bs^2 + \frac{1}{6}s^6$. By the conditions (H_1), (H_2), Lemma 2.4 and Theorem 2.5, the conclusion of Lemma 2.6 holds. □

2.4. *Monotonicity technique*

Here we recall a monotonicity method due to Struwe [26] and Jeanjean [15] which will be used in the proof. The version here is from [15].

Theorem 2.7. *Let $(X, \|\cdot\|)$ be a Banach space and $I \subset \mathbb{R}_+$ be an interval. Consider the family of C^1 functionals on X*

$$J_\lambda = A - \lambda B, \quad \lambda \in I,$$

with B nonnegative and either $A(u) \to \infty$ or $B(u) \to \infty$ as $\|u\| \to \infty$ and such that $J_\lambda(0) = 0$.

For any $\lambda \in I$ we set

$$\Gamma_\lambda = \{\gamma \in C([0,1], X) : \gamma(0) = 0, J_\lambda(\gamma(1)) < 0\}.$$

If for every $\lambda \in I$ the set Γ_λ is nonempty and

$$c_\lambda = \inf_{\gamma \in \Gamma_\lambda} \max_{t \in [0,1]} J_\lambda(\gamma(t)) > 0,$$

then for almost every $\lambda \in I$ there is a sequence $\{u_n\} \subset X$ such that

(i) $\{u_n\}$ *is bounded*;
(ii) $J_\lambda(u_n) \to c_\lambda$;
(iii) $J'_\lambda(u_n) \to 0$ *in the dual X^{-1} of X.*

2.5. *A generalized Pohozaev identity*

Pohozaev type integral identities have been used in previous work as an important tool (see [8, 18]). Here we prove a generalized Pohozaev identity which involves functions of norms of solutions.

Lemma 2.8. *If $u \in H_b$ is a weak solution of*

$$\begin{cases} -a_1(u)\Delta u + a_2(u)u + a_3(u)\phi|u|^{p-1}u = \lambda f(u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = |u|^{p+1} & \text{in } \mathbb{R}^3, \end{cases} \tag{2.10}$$

where $p \in [1,4]$, $a_1, a_2, a_3 \in C(H_b, \mathbb{R})$ and $a_1(u) \neq 0$ for all $u \neq 0$, then the following Pohozaev type identity holds:

$$\frac{1}{2}a_1(u)\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{3}{2}a_2(u)\int_{\mathbb{R}^3} |u|^2 + \frac{5}{2(p+1)}a_3(u)\int_{\mathbb{R}^3} \phi|u|^{p+1} = 3\lambda\int_{\mathbb{R}^3} F(u). \tag{2.11}$$

Proof. By the assumption, $a_1(u), a_2(u), a_3(u)$ are constants for a given $u \in H_b$. Since $u \in H_b$ is a weak solution of (2.10), by the elliptic standard regularity results, we have $u \in W^{2,6/5}_{\text{loc}}(\mathbb{R}^3)$. The first equation of (2.10) can be rewritten to $-a_1(u)\Delta u = (\lambda f(u)/u - a_2(u) - a_3(u)\phi|u|^{p-1})u$. By the conditions (H_1) and (H_2), $|f(u)| \leq C(|u|^5 + |u|)$ for some positive constant C so $f(u)/u \in L^{3/2}_{\text{loc}}(\mathbb{R}^3)$. Since $\phi, u \in L^6(\mathbb{R}^3)$, we have that $\phi|u|^{p-1} \in L^{3/2}_{\text{loc}}(\mathbb{R}^3)$ as long as $p \in [1,4]$. Now the Brézis–Kato theorem [9] implies that $u \in L^r_{\text{loc}}(\mathbb{R}^3)$ for any $r \in [1,\infty)$, and consequently $u \in W^{2,r}_{\text{loc}}(\mathbb{R}^3)$ for any $r \in [1,\infty)$ from the L^p estimates. Again the elliptic regularity theory implies that $u, \phi \in C^2(\mathbb{R}^3)$. It follows that, for $R > 0$ and $B_R = \{x \in \mathbb{R}^3 : |x| < R\}$, we have

$$\int_{B_R} -\Delta u(x \cdot \nabla u) = -\frac{1}{2}\int_{B_R} |\nabla u|^2 - \frac{1}{R}\int_{\partial B_R} |x \cdot \nabla u|^2 + \frac{R}{2}\int_{\partial B_R} |\nabla u|^2,$$

$$\int_{B_R} u(x \cdot \nabla u) = -\frac{3}{2}\int_{B_R} |u|^2 + \frac{R}{2}\int_{\partial B_R} |u|^2,$$

$$\int_{B_R} \phi|u|^{p-1}u(x \cdot \nabla u) = -\frac{1}{p+1}\int_{B_R} |u|^{p+1} x \cdot \nabla\phi - \frac{3}{p+1}\int_{B_R} \phi|u|^{p+1} + \frac{R}{p+1}\int_{\partial B_R} \phi|u|^{p+1},$$

$$\int_{B_R} f(u)(x \cdot \nabla u) = -3\int_{B_R} F(u) + R\int_{\partial B_R} F(u),$$

$$\int_{B_R} -\Delta\phi(x \cdot \nabla\phi) = -\frac{1}{2}\int_{B_R} |\nabla\phi|^2 - \frac{1}{R}\int_{\partial B_R} |x \cdot \nabla\phi|^2 + \frac{R}{2}\int_{\partial B_R} |\nabla\phi|^2.$$

Multiplying the first equation of (2.10) by $x \cdot \nabla u$, multiplying the second equation by $x \cdot \nabla\phi$, integrating on B_R, and using the estimates above we get that

$$\begin{aligned} &a_1(u)\left(-\frac{1}{2}\int_{B_R} |\nabla u|^2 - \frac{1}{R}\int_{\partial B_R} |x \cdot \nabla u|^2 + \frac{R}{2}\int_{\partial B_R} |\nabla u|^2\right) \\ &\quad + a_2(u)\left(-\frac{3}{2}\int_{B_R} |u|^2 + \frac{R}{2}\int_{\partial B_R} |u|^2\right) \\ &\quad + a_3(u)\left(-\frac{1}{p+1}\int_{B_R} |u|^{p+1} x \cdot \nabla\phi - \frac{3}{p+1}\int_{B_R} \phi|u|^{p+1} + \frac{R}{p+1}\int_{\partial B_R} \phi|u|^{p+1}\right) \\ &= -3\lambda\int_{B_R} F(u) + \lambda R\int_{\partial B_R} F(u) \end{aligned}$$

and

$$\int_{B_R} |u|^{p+1}(x \cdot \nabla\phi) = -\frac{1}{2}\int_{B_R} |\nabla\phi|^2 - \frac{1}{R}\int_{\partial B_R} |x \cdot \nabla\phi|^2 + \frac{R}{2}\int_{\partial B_R} |\nabla\phi|^2.$$

It follows that

$$\begin{aligned} &a_1(u)\left(-\frac{1}{2}\int_{B_R} |\nabla u|^2 - \frac{1}{R}\int_{\partial B_R} |x \cdot \nabla u|^2 + \frac{R}{2}\int_{\partial B_R} |\nabla u|^2\right) \\ &\quad + a_2(u)\left(-\frac{3}{2}\int_{B_R} |u|^2 + \frac{R}{2}\int_{\partial B_R} |u|^2\right) \end{aligned}$$

$$+ a_3(u)\left(\frac{1}{2(p+1)}\int_{B_R}|\nabla\phi|^2 + \frac{1}{(p+1)R}\int_{\partial B_R}|x\cdot\nabla\phi|^2 - \frac{R}{2(p+1)}\int_{\partial B_R}|\nabla\phi|^2\right.$$
$$\left. - \frac{3}{p+1}\int_{B_R}\phi|u|^{p+1} + \frac{R}{p+1}\int_{\partial B_R}\phi|u|^{p+1}\right)$$
$$= -3\lambda\int_{B_R}F(u) + \lambda R\int_{\partial B_R}F(u).$$

Letting $R\to\infty$, we obtain that

$$a_1(u)\left(-\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2\right) - \frac{3}{2}a_2(u)\int_{\mathbb{R}^3}|u|^2$$
$$+ a_3(u)\left(\frac{1}{2(p+1)}\int_{\mathbb{R}^3}|\nabla\phi|^2 - \frac{3}{p+1}\int_{\mathbb{R}^3}\phi|u|^{p+1}\right) = -3\lambda\int_{\mathbb{R}^3}F(u).$$

By using that $\int_{\mathbb{R}^3}|\nabla\phi|^2 = \int_{\mathbb{R}^3}\phi|u|^{p+1}$, we obtain (2.11). □

3. The Case of Positive q

In this section, we consider the case of $q > 0$ for (1.1), and we assume that the conditions (H_1), (H_2) and (H_3) are satisfied. Using the setup in Sec. 2.1, we seek for critical points of the function J_q defined in (2.3). To overcome the difficulty of finding bounded PS sequences for the associated functional J_q, following [16, 18], we use a cut-off function $\psi \in C^\infty(\mathbb{R}_+, [0,1])$ satisfying

$$\begin{cases} \psi(t) = 1, & t\in[0,1],\\ \psi(t) = 0, & t\in[2,\infty),\\ |\psi'|_\infty \le 2 \end{cases}$$

and study the following modified functional $J_q^T : H_b \to \mathbb{R}$ defined by

$$J_q^T(u) = \frac{1}{2}\|u\|^2 + \frac{1}{10}qh_T(u)\int_{\mathbb{R}^3}\phi_u|u|^5 - \int_{\mathbb{R}^3}F(u), \quad u\in H_b,$$

where for every $T > 0$,

$$h_T(u) = \psi\left(\frac{\|u\|^2}{T^2}\right).$$

With this penalization, for $T > 0$ sufficiently large and q sufficiently small, we are able to find a critical point u of J_q^T such that $\|u\| \le T$ and so u is also a critical point of J_q.

We will apply the monotonicity method described in Theorem 2.7. In our case, $X = H_b$,

$$A(u) = \frac{1}{2}\|u\|^2 + \frac{1}{10}qh_T(u)\int_{\mathbb{R}^3}\phi_u|u|^5, \quad B(u) = \int_{\mathbb{R}^3}F(u).$$

So the perturbed functional which we shall study is

$$J^T_{q,\lambda}(u) = \frac{1}{2}\|u\|^2 + \frac{1}{10}qh_T(u)\int_{\mathbb{R}^3}\phi_u|u|^5 - \lambda\int_{\mathbb{R}^3}F(u)$$

and

$$\begin{aligned}\langle (J^T_{q,\lambda})'(u), v\rangle &= (u,v) + qh_T(u)\int_{\mathbb{R}^3}\phi_u|u|^3uv \\ &\quad + \frac{q}{5T^2}\psi'\left(\frac{\|u\|^2}{T^2}\right)(u,v)\int_{\mathbb{R}^3}\phi_u|u|^5 - \lambda\int_{\mathbb{R}^3}f(u)v. \qquad (3.1)\end{aligned}$$

Lemmas 3.1 and 3.2 below imply that $J^T_{q,\lambda}$ satisfies the conditions of Theorem 2.7. First we show that the path set Γ_λ as defined in Theorem 2.7 is not empty.

Lemma 3.1. *For every $\lambda \in I \equiv [1/2, 1]$, $q \geq 0$ and $T > 0$, define the path set*

$$\Gamma_\lambda = \{\gamma \in C([0,1], H_b) : \gamma(0) = 0, J^T_{q,\lambda}(\gamma(1)) < 0\}.$$

Let $z \in H_b$ be the function defined in (H$_3$)*. Choose $\theta > 0$ satisfying*

$$\theta^2\left(\int_{\mathbb{R}^3}F(z) - b\int_{\mathbb{R}^3}z^2\right) = 2\left(|\nabla z|_2^2 + \frac{1}{5}\int_{\mathbb{R}^2}\phi_z|z|^5\right).$$

If $q \in [0, 1/\theta^4]$, then $\Gamma_\lambda \neq \emptyset$.

Proof. Let $\lambda \in I$. Set $w(\cdot) = z(\cdot/\theta)$. Define $\gamma : [0,1] \to H_b$ in the following way

$$\gamma(t) = \begin{cases} 0, & t = 0, \\ w(\cdot/t), & t \in (0,1]. \end{cases}$$

It is easy to see that γ is a continuous path connecting 0 and w. Moreover, we have that

$$\begin{aligned} J^T_{q,\lambda}(\gamma(1)) &= \frac{1}{2}|\nabla w|_2^2 + \frac{1}{2}b|w|_2^2 + \frac{1}{10}qh_T(w)\int_{\mathbb{R}^3}\phi_w|w|^5 - \lambda\int_{\mathbb{R}^3}F(w) \\ &\leq \frac{1}{2}\theta|\nabla z|_2^2 + \frac{1}{2}\theta^3 b|z|_2^2 + \frac{1}{10}q\theta^5\int_{\mathbb{R}^3}\phi_z|z|^5 - \frac{1}{2}\theta^3\int_{\mathbb{R}^3}F(z) \\ &\leq \frac{1}{2}\theta|\nabla z|_2^2 + \frac{1}{2}\theta^3 b|z|_2^2 + \frac{1}{10}\theta\int_{\mathbb{R}^3}\phi_z|z|^5 - \frac{1}{2}\theta^3\int_{\mathbb{R}^3}F(z) \\ &= -\frac{1}{4}\theta^3\left(\int_{\mathbb{R}^3}F(z) - b\int_{\mathbb{R}^3}z^2\right) < 0. \end{aligned}$$

□

Secondly we show that the critical level c_λ is uniformly bounded from below by a positive lower bound for all $\lambda \in I$.

Lemma 3.2. *Suppose that $\lambda \in I$, $q \in [0, 1/\theta^4]$ and $T > 0$, and define*

$$c_\lambda = \inf_{\gamma\in\Gamma_\lambda}\max_{t\in[0,1]} J^T_{q,\lambda}(\gamma(t)).$$

Then there exists a constant $c > 0$ such that $c_\lambda \geq c$ for all $\lambda \in I$.

Proof. From the conditions (H_1) and (H_2), there exists $C_0 > 0$ such that

$$|f(t)t| \leq \frac{1}{4}b|t|^2 + C_0|t|^6 \quad \text{and} \quad |F(t)| \leq \frac{1}{4}b|t|^2 + C_0|t|^6, \quad t \in \mathbb{R}. \tag{3.2}$$

Then for any $u \in H_b$ and $\lambda \in I$, we have

$$\begin{aligned} J^T_{q,\lambda}(u) &\geq \frac{1}{2}\|u\|^2 + \frac{1}{10}qh_T(u)\int_{\mathbb{R}^3}\phi_u|u|^5 - \frac{1}{4}b\int_{\mathbb{R}^3}|u|^2 - C_0\int_{\mathbb{R}^3}|u|^6 \\ &\geq \frac{1}{2}\|u\|^2 - \frac{1}{4}b\int_{\mathbb{R}^3}|u|^2 - C_0\int_{\mathbb{R}^3}|u|^6. \end{aligned}$$

By the Sobolev's embedding theorem, we conclude that there exists $\rho > 0$ such that $J^T_{q,\lambda}(u) > 0$ for any $\lambda \in I$ and $u \in H_b$ with $0 < \|u\| \leq \rho$. In particular, for $\|u\| = \rho$, we have $J^T_{q,\lambda}(u) \geq c > 0$. For any given $\lambda \in I$ and $\gamma \in \Gamma_\lambda$, by the definition of Γ_λ, we have $\|\gamma(1)\| > \rho$. According to the continuity of $\gamma(t)$, we then deduce that there exists $t_\gamma \in (0,1)$ such that $\|\gamma(t_\gamma)\| = \rho$. Therefore, for any $\lambda \in I$,

$$c_\lambda \geq \inf_{\gamma\in\Gamma_\lambda} J^T_{q,\lambda}(\gamma(t_\gamma)) \geq c > 0. \qquad \square$$

Now we can show that the modified functional $J^T_{q,\lambda}$ satisfies the PS condition.

Lemma 3.3. *Assume that q and T satisfy*

$$\frac{128}{5}qS^{-6}T^8 \leq 1, \tag{3.3}$$

where S is defined in (1.7)*. Then for any $\lambda \in I$, each bounded PS sequence of the functional $J^T_{q,\lambda}$ admits a convergent subsequence.*

Proof. Let $\lambda \in I$ and let $\{u_n\}$ be a bounded PS sequence of $J^T_{q,\lambda}$, that is, $\{u_n\}$ and $\{J^T_{q,\lambda}(u_n)\}$ are bounded, and $(J^T_{q,\lambda})'(u_n) \to 0$ in H'_b, where H'_b is the dual space of H_b. Since $\{u_n\}$ is bounded, we may assume that there exists $u \in H_b$ such that

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } H_b, \\ u_n &\to u \quad \text{a.e. on } \mathbb{R}^3. \end{aligned}$$

It follows from Lemma 2.6 that

$$\int_{\mathbb{R}^3} f(u_n)(u_n - u) \to 0.$$

Since $u_n \rightharpoonup u$ in H_b, in view of the Sobolev's embedding theorem, we may assume that

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } L^6(\mathbb{R}^3), \\ \phi_{u_n} &\rightharpoonup \phi_u \quad \text{in } \mathcal{D}^{1,2}_r(\mathbb{R}^3). \end{aligned}$$

Let $v_n = u_n - u$. Since $|v_n| \rightharpoonup 0$ in $L^6(\mathbb{R}^3)$ and $|u|^3 u v_n \to 0$ in $L^{6/5}(\mathbb{R}^3)$, then $\int_{\mathbb{R}^3} \phi_{u_n} |u|^3 u v_n \to 0$. By using Lemma 2.3, we obtain that

$$\begin{aligned}\int_{\mathbb{R}^3} \phi_{u_n} |u_n|^3 u_n(u_n - u) &= \int_{\mathbb{R}^3} \phi_{u_n} |v_n|^5 + \int_{\mathbb{R}^3} \phi_{u_n} |u|^3 u v_n + o(1) \\ &= \int_{\mathbb{R}^3} \phi_{u_n} |v_n|^5 + o(1) \\ &= \int_{\mathbb{R}^3} \phi_{v_n} |v_n|^5 + \int_{\mathbb{R}^3} \phi_u |v_n|^5 + o(1) \\ &= \int_{\mathbb{R}^3} \phi_{v_n} |v_n|^5 + o(1).\end{aligned}$$

Thus,

$$\begin{aligned}&\langle (J_{q,\lambda}^T)'(u_n), u_n - u\rangle \\ &\quad = (u_n, u_n - u) + q h_T(u_n) \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^3 u_n (u_n - u) \\ &\qquad + \frac{q}{5T^2} \psi'\left(\frac{\|u_n\|^2}{T^2}\right)(u_n, u_n - u) \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^5 - \lambda \int_{\mathbb{R}^3} f(u_n)(u_n - u) \\ &\quad = \left(1 + \frac{q}{5T^2}\psi'\left(\frac{\|u_n\|^2}{T^2}\right)\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^5\right)(u_n, u_n - u) \\ &\qquad + q h_T(u_n) \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^3 u_n (u_n - u) + o(1) \\ &\quad = \left(1 + \frac{q}{5T^2}\psi'\left(\frac{\|u_n\|^2}{T^2}\right)\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^5\right)\|v_n\|^2 + q h_T(u_n) \int_{\mathbb{R}^3} \phi_{v_n} |v_n|^5 + o(1),\end{aligned}$$

and then

$$\left(1 + \frac{q}{5T^2}\psi'\left(\frac{\|u_n\|^2}{T^2}\right)\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^5\right)\|v_n\|^2 + q h_T(u_n) \int_{\mathbb{R}^3} \phi_{v_n} |v_n|^5 \to 0.$$

From Lemma 2.1 and (3.3), we have

$$\left| \frac{q}{5T^2}\psi'\left(\frac{\|u_n\|^2}{T^2}\right)\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^5 \right| \le \frac{64}{5} q S^{-6} T^8 \le \frac{1}{2},$$

it follows that $\|v_n\| \to 0$. This implies that $u_n \to u$ in H_b. The proof is completed. □

From the results in Lemmas 3.1–3.3 and Theorem 2.7, we can now show the existence of a critical point for the modified functional $J_{q,\lambda}^T$.

Lemma 3.4. *Suppose that* q *and* T *satisfy* (3.3). *Then for almost every* $\lambda \in I$, *there exists* $u^\lambda \in H_b$ *such that* $(J_{q,\lambda}^T)'(u^\lambda) = 0$ *and* $J_{q,\lambda}^T(u^\lambda) = c_\lambda$.

Proof. By using the results in Lemmas 3.1–3.3 and Theorem 2.7, for almost every $\lambda \in I$, there exists a bounded sequence $\{u_n^\lambda\} \subset H_b$ such that $J_{q,\lambda}^T(u_n^\lambda) \to c_\lambda$ and

$(J^T_{q,\lambda})'(u^\lambda_n) \to 0$. By Lemma 3.3, we can assume that there exists $u^\lambda \in H_b$ such that $u^\lambda_n \to u^\lambda$ in H_b, then the assertion follows. □

According to Lemma 3.4, there exist sequences $\{\lambda_n\} \subset I$ with $\lambda_n \to 1^-$ and $\{u_n\} \subset H_b$ such that

$$J^T_{q,\lambda_n}(u_n) = c_{\lambda_n}, \quad (J^T_{q,\lambda_n})'(u_n) = 0.$$

The following lemma shows that this critical sequence $\{(\lambda_n, u_n)\}$ satisfies $\|u_n\| \le T$ for all n, which is the key for this paper.

Lemma 3.5. *Let u_n be a critical point of J^T_{q,λ_n} at the energy level c_{λ_n}, where $\{\lambda_n\} \subset I$. Then there exists $T_0 > 0$ sufficiently large so that for any $T \ge T_0$, there exists $q_T = q(T)$ satisfying*

$$q(T) \le \min\left\{\frac{5}{128}S^6 T^{-8}, \theta^{-4}\right\}, \tag{3.4}$$

such that for any $q \in [0, q_T)$, subject to a subsequence, $\|u_n\| \le T$ for all $n \in \mathbb{N}$.

Proof. We define

$$a_1(u) = \left(1 + \frac{q}{5T^2}\psi'\left(\frac{\|u\|^2}{T^2}\right)\int_{\mathbb{R}^3}\phi_u|u|^5\right),$$

$$a_2(u) = b\left(1 + \frac{q}{5T^2}\psi'\left(\frac{\|u\|^2}{T^2}\right)\int_{\mathbb{R}^3}\phi_u|u|^5\right)$$

and

$$a_3(u) = qh_T(u) = q\psi\left(\frac{\|u\|^2}{T^2}\right).$$

By (viii) of Lemma 2.1, $a_1, a_2, a_3 \in C(H_b, \mathbb{R})$ hence the assumptions in Lemma 2.8 are satisfied.

Since (λ_n, u_n) satisfies that $(J^T_{q,\lambda_n})'(u_n) = 0$, that is (2.10) with $(\lambda, u, \phi) = (\lambda_n, u_n, \phi_{u_n})$ with a_1, a_2, a_3 defined above, then it follows from (2.11) in Lemma 2.8 that the following Pohozaev type identity holds:

$$\begin{aligned}
&\left(1 + \frac{q}{5T^2}\psi'\left(\frac{\|u_n\|^2}{T^2}\right)\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^5\right)\left(\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u_n|^2 + \frac{3}{2}b\int_{\mathbb{R}^3}|u_n|^2\right)\\
&\quad + \frac{1}{2}qh_T(u_n)\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^5\\
&= 3\lambda_n\int_{\mathbb{R}^3}F(u_n).
\end{aligned} \tag{3.5}$$

By using $J^T_{q,\lambda_n}(u_n) = c_{\lambda_n}$, we have that

$$\frac{1}{2}\|u_n\|^2 + \frac{1}{10}qh_T(u_n)\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^5 - \lambda_n\int_{\mathbb{R}^3}F(u_n) = c_{\lambda_n}. \tag{3.6}$$

Combining (3.5) and (3.6), we obtain that

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u_n|^2 &\leq \left(1-\frac{q}{10T^2}\psi'\left(\frac{\|u_n\|^2}{T^2}\right)\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^5\right)\int_{\mathbb{R}^3}|\nabla u_n|^2\\ &= 3c_{\lambda_n}+\frac{1}{5}qh_T(u_n)\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^5\\ &\quad+\frac{3bq}{10T^2}\psi'\left(\frac{\|u_n\|^2}{T^2}\right)\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^5\int_{\mathbb{R}^3}|u_n|^2. \end{aligned}\tag{3.7}$$

We now estimate the right-hand side of (3.7). By the min–max definition of the mountain pass level, Lemma 3.1 and (H_3), we have that

$$\begin{aligned}c_{\lambda_n} &\leq \max_{t\in[0,1]} J^T_{q,\lambda_n}(\gamma(t))\\ &\leq \max_{t\in[0,1]}\left\{\frac{1}{2}t\theta|\nabla z|_2^2+\frac{1}{2}bt^3\theta^3|z|_2^2-t^3\theta^3\lambda_n\int_{\mathbb{R}^3}F(z)\right\}\\ &\quad+\max_{t\in[0,1]}\frac{1}{10}q\psi\left(\frac{t\theta|\nabla z|_2^2+bt^3\theta^3|z|_2^2}{T^2}\right)t^5\theta^5\int_{\mathbb{R}^3}\phi_z|z|^5\\ &\leq \frac{1}{2}\theta|\nabla z|_2^2+\frac{1}{10}\theta\int_{\mathbb{R}^3}\phi_z|z|^5\equiv C_1, \end{aligned}\tag{3.8}$$

where γ is the path defined in Lemma 3.1. We also have that

$$\frac{1}{5}qh_T(u_n)\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^5\leq\frac{32}{5}qS^{-6}T^{10},\tag{3.9}$$

$$\left|\frac{3bq}{10T^2}\psi'\left(\frac{\|u_n\|^2}{T^2}\right)\right|\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^5\int_{\mathbb{R}^3}|u_n|^2\leq\frac{96}{5}bqS^{-6}T^8|u_n|_2^2\leq\frac{192}{5}qS^{-6}T^{10}.\tag{3.10}$$

Thus it follows from (3.7), (3.8), (3.9) and (3.10) that

$$\int_{\mathbb{R}^3}|\nabla u_n|^2\leq 6C_1+\frac{64}{5}qS^{-6}T^{10}+\frac{384}{5}qS^{-6}T^{10}=6C_1+\frac{448}{5}qS^{-6}T^{10}.\tag{3.11}$$

On the other hand, since $\langle (J^T_{q,\lambda_n})'(u_n),u_n\rangle=0$, we have that

$$\begin{aligned}\int_{\mathbb{R}^3}|\nabla u_n|^2+b\int_{\mathbb{R}^3}|u_n|^2+qh_T(u_n)\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^5\\ +\frac{q}{5T^2}\psi'\left(\frac{\|u_n\|^2}{T^2}\right)\|u_n\|^2\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^5\\ =\lambda_n\int_{\mathbb{R}^3}f(u_n)u_n.\end{aligned}$$

It follows from (3.4) and (3.2) that

$$\begin{aligned}\frac{1}{2}b\int_{\mathbb{R}^3}|u_n|^2 &\le \left(1+\frac{q}{5T^2}\psi'\left(\frac{\|u_n\|^2}{T^2}\right)\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^5\right)b\int_{\mathbb{R}^3}|u_n|^2\\ &= \lambda_n\int_{\mathbb{R}^3}f(u_n)u_n-\left(1+\frac{q}{5T^2}\psi'\left(\frac{\|u_n\|^2}{T^2}\right)\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^5\right)\int_{\mathbb{R}^3}|\nabla u_n|^2\\ &\quad -qh_T(u_n)\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^5\\ &\le \frac{1}{4}b\int_{\mathbb{R}^3}|u_n|^2+C_0\int_{\mathbb{R}^3}|u_n|^6,\end{aligned}$$

where C_0 is defined in (3.2). Therefore,

$$\begin{aligned}\frac{1}{4}b\int_{\mathbb{R}^3}|u_n|^2 \le C_0|u_n|_6^6 &\le C_0S^{-3}\left(\int_{\mathbb{R}^3}|\nabla u_n|^2\right)^3\\ &\le C_0S^{-3}\left(6C_1+\frac{448}{5}qS^{-6}T^{10}\right)^3. \qquad (3.12)\end{aligned}$$

Thus from (3.11) and (3.12), we obtain that

$$\begin{aligned}\|u_n\|^2 &\le 6C_1+\frac{448}{5}qS^{-6}T^{10}+4C_0S^{-3}\left(6C_1+\frac{448}{5}qS^{-6}T^{10}\right)^3\\ &= C_2+C_3qT^{10}+C_4q^2T^{20}+C_5q^3T^{30},\end{aligned}$$

where

$$C_2=6C_1+864C_0S^{-3}C_1^3,\quad C_3=\frac{448}{5}S^{-6}+432C_0C_1^2S^{-9}\frac{448}{5},$$

$$C_4=12C_0C_1S^{-15}\left(\frac{448}{5}\right)^2,\quad C_5=4C_0S^{-21}\left(\frac{448}{5}\right)^3.$$

Choose $T_0^2 \ge C_2+1$, $T\ge T_0$ and $q_T=q(T)$ such that $C_3qT^{10}+C_4q^2T^{20}+C_5q^3T^{30}<1$. Then the conclusion holds. □

Now we can complete the proof of Theorem 1.1 for the case of $q>0$.

Proof of Theorem 1.1 for $q>0$**.** Let T_0 and $q_T=q(T)$ be chosen as in Lemma 3.5, and let u_n be a critical point for $J_{q,\lambda_n}^{T_0}$ at the level c_{λ_n}. Then from Lemma 3.5 we may assume that

$$\|u_n\|\le T_0.$$

Therefore

$$J_{q,\lambda_n}^{T_0}(u_n)=\frac{1}{2}\|u_n\|^2+\frac{1}{10}q\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^5-\lambda_n\int_{\mathbb{R}^3}F(u_n).$$

Since $\lambda_n \to 1$ as $n \to \infty$, we can show that $\{u_n\}$ is a PS sequence of J_q. Indeed, the boundedness of $\{u_n\}$ implies that $\{J_q(u_n)\}$ is bounded. Also for $v \in H_b$, we have

$$\langle J_q'(u_n), v\rangle = \langle (J_{q,\lambda_n}^{T_0})'(u_n), v\rangle + (\lambda_n - 1)\int_{\mathbb{R}^3} f(u_n)v.$$

It follows that $J_q'(u_n) \to 0$ and hence $\{u_n\}$ is a bounded PS sequence of J_q. By Lemma 3.3, $\{u_n\}$ has a convergent subsequence. Without loss of generality, we may assume that $u_n \to u$. Consequently, $J_q'(u) = 0$. According to Lemma 3.2, we have that $J_q(u) = \lim_{n\to\infty} J_q(u_n) = \lim_{n\to\infty} J_{q,\lambda_n}^{T_0}(u_n) \geq c > 0$, and u is a positive solution by the condition (H_1). The proof is completed. □

4. The Case of Negative *q*

In this section, we assume $q < 0$, and we assume that the conditions (H_1), (H_2) and (H_4) are satisfied. Indeed let $p = -q$, we may consider the following equivalent system

$$\begin{cases} -\Delta u + bu - p\phi|u|^3u = f(u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = |u|^5 & \text{in } \mathbb{R}^3. \end{cases} \tag{4.1}$$

Here we define

$$A(u) = \frac{1}{2}\|u\|^2, \quad B(u) = \frac{1}{10}p\int_{\mathbb{R}^3}\phi_u|u|^5 + \int_{\mathbb{R}^3} F(u)$$

and

$$J_{p,\lambda}(u) = \frac{1}{2}\|u\|^2 - \frac{1}{10}p\lambda\int_{\mathbb{R}^3}\phi_u|u|^5 - \lambda\int_{\mathbb{R}^3} F(u).$$

Then we have

$$\langle (J_{p,\lambda})'(u), v\rangle = (u,v) - p\lambda\int_{\mathbb{R}^3}\phi_u|u|^3uv - \lambda\int_{\mathbb{R}^3} f(u)v. \tag{4.2}$$

The following Lemmas 4.1 and 4.2 imply that $J_{p,\lambda}$ satisfies the conditions of Theorem 2.7. First we show that the path set $\Gamma_{p,\lambda}$ is not empty.

Lemma 4.1. *For every $\lambda \in I \equiv [1/2, 1]$ and $p > 0$, define the path set*

$$\Gamma_{p,\lambda} = \{\gamma \in C([0,1], H_b) : \gamma(0) = 0, J_{p,\lambda}(\gamma(1)) < 0\}.$$

Then $\Gamma_{p,\lambda} \neq \emptyset$.

Proof. Let $\lambda \in I$. For any $u \in H_b \setminus \{0\}$ and $t > 0$,

$$\begin{aligned} J_{p,\lambda}(tu) &= \frac{1}{2}t^2\|u\|^2 - \frac{1}{10}t^{10}p\lambda\int_{\mathbb{R}^3}\phi_u|u|^5 - \lambda\int_{\mathbb{R}^3} F(tu) \\ &\leq \frac{1}{2}t^2\|u\|^2 - \frac{1}{20}t^{10}p\lambda\int_{\mathbb{R}^3}\phi_u|u|^5. \end{aligned}$$

Hence $J_{p,\lambda}(tu) \to -\infty$ as $t \to \infty$. □

Secondly we show that the critical level $c_{p,\lambda}$ is uniformly bounded from below by a positive lower bound for all $\lambda \in I$.

Lemma 4.2. *For every $\lambda \in I$ and $p > 0$, define*

$$c_{p,\lambda} = \inf_{\gamma\in\Gamma_{p,\lambda}} \max_{t\in[0,1]} J_{p,\lambda}(\gamma(t)).$$

Then there exists a constant $c_p > 0$ such that $c_{p,\lambda} \geq c_p$ for all $\lambda \in I$.

Proof. For any $u \in H_b$ and $\lambda \in I$, the estimates (3.2) hold from the conditions (H_1) and (H_2). Then we have

$$\begin{aligned} J_{p,\lambda}(u) &\geq \frac{1}{2}\|u\|^2 - \frac{1}{10}p\int_{\mathbb{R}^3}\phi_u|u|^5 - \frac{1}{4}b\int_{\mathbb{R}^3}|u|^2 - C_0\int_{\mathbb{R}^3}|u|^6 \\ &\geq \frac{1}{2}\|u\|^2 - \frac{1}{10}pS^{-1}|u|_6^{10} - \frac{1}{4}b|u|_2^2 - C_0|u|_6^6 \\ &\geq \frac{1}{2}\|u\|^2 - \frac{1}{10}pS^{-6}\|u\|^{10} - \frac{1}{4}\|u\|^2 - C_0S^{-3}\|u\|^6. \end{aligned}$$

Hence we can reach the desired conclusion in the same way as in the proof of Lemma 3.2. □

Next we give an estimate of the upper bound of the critical level $c_{p,\lambda}$ by using a test function.

Lemma 4.3. *For any $p > 0$ and $\lambda \in [1/2, 1]$, let $c_{p,\lambda}$ be defined as in Lemma 4.2. Then we have*

$$c_{p,\lambda} < \frac{1}{15}S^{3/2}\frac{(5+p\lambda)^{3/2}}{(6p\lambda)^{1/2}}. \tag{4.3}$$

Proof. It is well known that the best Sobolev embedding constant S is attained by the functions

$$\xi_\varepsilon(x) = \frac{\varepsilon^{1/4}}{(\varepsilon + |x|^2)^{1/2}}$$

for $\varepsilon > 0$. We define $u_\varepsilon(x) = \xi_\varepsilon(x)\psi(x)$, where $\psi \in C_0^\infty(B_{2r}(0))$ such that $0 \leq \psi(x) \leq 1$, and $\psi(x) = 1$ on $B_r(0)$ for some $r > 0$. By simple calculations, we can derive that as $\varepsilon \to 0^+$,

$$\int_{\mathbb{R}^3}|\nabla u_\varepsilon|^2 = \int_{\mathbb{R}^3}\frac{|x|^2}{(1+|x|^2)^3} + O(\varepsilon^{1/2}) := K_1 + O(\varepsilon^{1/2}), \tag{4.4}$$

$$\int_{\mathbb{R}^3}|u_\varepsilon|^6 = \int_{\mathbb{R}^3}\frac{1}{(1+|x|^2)^3} + O(\varepsilon^{3/2}) := K_2 + O(\varepsilon^{3/2}) \tag{4.5}$$

and for every $\varepsilon > 0$,

$$\int_{\mathbb{R}^3} |u_\varepsilon|^t = \begin{cases} \bar{K}_t \varepsilon^{\frac{6-t}{4}}, & t \in (3,6), \\ \bar{K}_3 \varepsilon^{\frac{3}{4}} |\ln \varepsilon|, & t = 3, \\ \bar{K}_t \varepsilon^{\frac{t}{4}}, & t \in [2,3), \end{cases} \tag{4.6}$$

where $K_1, K_2, \bar{K}_t$ $(2 \le t < 6)$ are positive constants, and $S = K_1 K_2^{-1/3}$. By (4.4) and (4.5), we have

$$\frac{\int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2}{\left(\int_{\mathbb{R}^3} |u_\varepsilon|^6\right)^{1/3}} = S + O(\varepsilon^{1/2}). \tag{4.7}$$

Multiplying the second equation of (4.1) by $|u|$ and integrating, we obtain that

$$|u|_6^6 = \int_{\mathbb{R}^3} \nabla\phi_u \cdot \nabla |u| \le \frac{1}{2}|\nabla \phi_u|_2^2 + \frac{1}{2}|\nabla |u||_2^2 = \frac{1}{2}\int_{\mathbb{R}^3} \phi_u |u|^5 + \frac{1}{2}|\nabla u|_2^2. \tag{4.8}$$

Define

$$I_{p,\lambda}(u) = \left(\frac{1}{2} + \frac{1}{10}p\lambda\right)|\nabla u|_2^2 + \frac{1}{2}b|u|_2^2 - \frac{1}{5}p\lambda|u|_6^6 - \lambda A|u|_r^r + \lambda B|u|_2^2$$

and

$$H_{p,\lambda}(u) = \left(\frac{1}{2} + \frac{1}{10}p\lambda\right)|\nabla u|_2^2 + \frac{1}{2}b|u|_2^2 - \frac{1}{5}p\lambda|u|_6^6 + \lambda B|u|_2^2,$$

where A and B are defined in the condition (H_4). From (4.8), Lemma 4.2 and the condition (H_4), it can be derived that

$$c_{p,\lambda} \le \sup_{t\in[0,\infty)} J_{p,\lambda}(tu_\varepsilon) \le \sup_{t\in[0,\infty)} I_{p,\lambda}(tu_\varepsilon).$$

Since

$$\begin{aligned} \frac{d}{dt}[I_{p,\lambda}(tu_\varepsilon)] &= \left(1 + \frac{1}{5}p\lambda\right)t|\nabla u_\varepsilon|_2^2 + bt|u_\varepsilon|_2^2 - \frac{6}{5}p\lambda t^5|u_\varepsilon|_6^6 \\ &\quad - r\lambda A t^{r-1}|u_\varepsilon|_r^r + 2\lambda B t|u_\varepsilon|_2^2, \end{aligned} \tag{4.9}$$

then there exists a unique $t_\varepsilon > 0$ such that

$$\frac{d}{dt}[I_{p,\lambda}(tu_\varepsilon)] = 0.$$

Thus

$$\sup_{t\in[0,\infty)} I_{p,\lambda}I(tu_\varepsilon) = I_{p,\lambda}(t_\varepsilon u_\varepsilon) = H_{p,\lambda}(t_\varepsilon u_\varepsilon) - \lambda t_\varepsilon^r A|u_\varepsilon|_r^r \le H_{p,\lambda}(t'_\varepsilon u_\varepsilon) - \lambda t_\varepsilon^r A|u_\varepsilon|_r^r,$$

where t'_ε is the unique positive solution of the equation

$$\frac{d}{dt}[H_{p,\lambda}(tu_\varepsilon)] = 0.$$

From

$$\frac{d}{dt}[H_{p,\lambda}(tu_\varepsilon)] = \left(1+\frac{1}{5}p\lambda\right)t|\nabla u_\varepsilon|_2^2 + bt|u_\varepsilon|_2^2 - \frac{6}{5}p\lambda t^5|u_\varepsilon|_6^6 + 2\lambda Bt|u_\varepsilon|_2^2, \quad (4.10)$$

it is easy to see that $t_\varepsilon < t'_\varepsilon$.

From (4.4)–(4.7) and (4.10), we obtain that, for small $\varepsilon > 0$,

$$\begin{aligned} t'_\varepsilon &= \sqrt[4]{\frac{(5+p\lambda)|\nabla u_\varepsilon|_2^2 + 5b|u_\varepsilon|_2^2 + 10\lambda B|u_\varepsilon|_2^2}{6p\lambda|u_\varepsilon|_6^6}} \\ &= \sqrt[4]{\frac{(5+p\lambda)K_1 + O(\varepsilon^{1/2})}{6p\lambda K_2 + O(\varepsilon^{3/2})}} = \sqrt[4]{\frac{5+p\lambda}{6p\lambda}\frac{K_1}{K_2} + O(\varepsilon^{1/2})}. \end{aligned}$$

And from (4.4)–(4.7) and (4.9), we calculate that

$$\begin{aligned} \left(1+\frac{1}{5}p\lambda\right)K_1 + O(\varepsilon^{1/2}) &= \left(1+\frac{1}{5}p\lambda\right)|\nabla u_\varepsilon|_2^2 + b|u_\varepsilon|_2^2 + 2\lambda B|u_\varepsilon|_2^2 \\ &= \frac{6}{5}p\lambda t_\varepsilon^4|u_\varepsilon|_6^6 + r\lambda At_\varepsilon^{r-2}|u_\varepsilon|_r^r \\ &\le \frac{6}{5}p\lambda t_\varepsilon^{r-2}(t'_\varepsilon)^{6-r}|u_\varepsilon|_6^6 + r\lambda At_\varepsilon^{r-2}|u_\varepsilon|_r^r \\ &\le \left(\frac{6}{5}p\lambda(t'_\varepsilon)^{6-r}|u_\varepsilon|_6^6 + r\lambda A|u_\varepsilon|_r^r\right)t_\varepsilon^{r-2} \\ &= \left(\frac{6}{5}p\lambda(t'_\varepsilon)^{6-r}K_2 + O(\varepsilon^{(6-r)/4})\right)t_\varepsilon^{r-2} \end{aligned}$$

and hence we have that $t'_\varepsilon > t_\varepsilon \ge K_{p,\lambda} > 0$, where $K_{p,\lambda}$ is a positive constant only depending on p, λ and K_1, K_2. Summarizing these estimates, we get

$$\begin{aligned} \sup_{t\in[0,\infty)} I_{p,\lambda}(tu_\varepsilon) &= H_{p,\lambda}(t_\varepsilon u_\varepsilon) - \lambda At_\varepsilon^r|u_\varepsilon|_r^r \le H_{p,\lambda}(t'_\varepsilon u_\varepsilon) - \lambda t_\varepsilon^r A|u_\varepsilon|_r^r \\ &\le \left(\frac{1}{2}+\frac{1}{10}p\lambda\right)(t'_\varepsilon)^2|\nabla u_\varepsilon|_2^2 + \frac{1}{2}b(t'_\varepsilon)^2|u_\varepsilon|_2^2 + B\lambda(t'_\varepsilon)^2|u_\varepsilon|_2^2 \\ &\quad - \frac{1}{5}p\lambda(t'_\varepsilon)^6|u_\varepsilon|_6^6 - \lambda At_\varepsilon^r|u_\varepsilon|_r^r \\ &\le \frac{1}{15}S^{3/2}\frac{(5+p\lambda)^{3/2}}{(6p\lambda)^{1/2}} + O(\varepsilon^{1/2}) - \lambda AK_{p,\lambda}^r\bar{K}_r\varepsilon^{(6-r)/4} \\ &< \frac{1}{15}S^{3/2}\frac{(5+p\lambda)^{3/2}}{(6p\lambda)^{1/2}}. \end{aligned}$$

The last inequality holds if we choose $\varepsilon > 0$ small enough as $r \in (4,6)$ here. Thus we obtain the estimate of $c_{p,\lambda}$ in (4.3). □

Now we can show that the modified functional $J_{p,\lambda}$ satisfies the PS condition.

Lemma 4.4. *For any $\lambda \in I$ and p satisfying $p\lambda = 1$, each bounded $(PS)_{c_{p,\lambda}}$ sequence of the functional $J_{p,\lambda}$ admits a convergent subsequence.*

Proof. Let $\lambda \in I$ and $\{u_n\}$ be a bounded $(\mathrm{PS})_{c_{p,\lambda}}$ sequence of $J_{p,\lambda}$, that is, $\{u_n\}$ is bounded and $J_{p,\lambda}(u_n) \to c_{p,\lambda}$, $(J_{p,\lambda})'(u_n) \to 0$ in H_b', where H_b' is the dual space of H_b. Since $\{u_n\}$ is bounded, we may assume that there exists $u \in H_b$ such that

$$\begin{aligned} u_n &\rightharpoonup u, \quad \text{in } H_b, \\ u_n &\to u, \quad \text{a.e. on } \mathbb{R}^3. \end{aligned}$$

From Lemma 2.6, we have

$$\int_{\mathbb{R}^3} f(u_n)u_n \to \int_{\mathbb{R}^3} f(u)u, \quad \int_{\mathbb{R}^3} F(u_n) \to \int_{\mathbb{R}^3} F(u). \tag{4.11}$$

Since $u_n \rightharpoonup u$ in H_b, in view of Sobolev's embedding theorems, we have that

$$J_{p,\lambda}'(u) = 0.$$

By using Lemma 2.8 and (4.11), we obtain that

$$\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{3}{2}b\int_{\mathbb{R}^3} |u|^2 - \frac{1}{2}p\lambda\int_{\mathbb{R}^3} \phi_u|u|^5 = 3\lambda\int_{\mathbb{R}^3} F(u),$$

hence,

$$\begin{aligned} J_{p,\lambda}(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{10}p\lambda\int_{\mathbb{R}^3} \phi_u|u|^5 - \lambda\int_{\mathbb{R}^3} F(u) \\ &\geq \frac{1}{3}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{1}{15}p\lambda\int_{\mathbb{R}^3} \phi_u|u|^5 \geq 0. \end{aligned}$$

Now let $v_n = u_n - u$, by using Lemma 2.3 and (4.11), we have

$$\begin{aligned} 0 &\leftarrow \langle (J_{p,\lambda})'(u_n), u_n\rangle \\ &= \langle (J_{p,\lambda})'(u_n), u_n\rangle - \langle (J_{p,\lambda})'(u), u\rangle \\ &= \|u_n\|^2 - \|u\|^2 - p\lambda\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^5 + p\lambda\int_{\mathbb{R}^3} \phi_u|u|^5 - \lambda\int_{\mathbb{R}^3} f(u_n)u_n + \lambda\int_{\mathbb{R}^3} f(u)u \\ &= \|v_n\|^2 - p\lambda\int_{\mathbb{R}^3} \phi_{v_n}|v_n|^5 + o(1) \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} c_{p,\lambda} - J_{p,\lambda}(u) &= J_{p,\lambda}(u_n) - J_{p,\lambda}(u) + o(1) \\ &= \frac{1}{2}\|u_n\|^2 - \frac{1}{2}\|u\|^2 - \frac{1}{10}p\lambda\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^5 + \frac{1}{10}p\lambda\int_{\mathbb{R}^3} \phi_u|u|^5 \\ &\quad - \lambda\int_{\mathbb{R}^3} F(u_n) + \lambda\int_{\mathbb{R}^3} F(u) + o(1) \\ &= \frac{1}{2}\|v_n\|^2 - \frac{1}{10}p\lambda\int_{\mathbb{R}^3} \phi_{v_n}|v_n|^5 + o(1). \end{aligned} \tag{4.13}$$

Suppose that $v_n \not\to 0$ in H_b, we may assume that $\|v_n\|^2 \to l > 0$. So (4.12) and the estimate

$$p\lambda \int_{\mathbb{R}^3} \phi_{v_n} |v_n|^5 \le p\lambda S^{-1} |v_n|_6^{10} \le p\lambda S^{-6} \|v_n\|^{10},$$

implies that

$$l \le p\lambda S^{-6} l^5 \quad \text{or} \quad l \ge \frac{S^{3/2}}{(p\lambda)^{1/4}}.$$

Hence, we have that

$$c_{p,\lambda} \ge c_{p,\lambda} - J_{p,\lambda}(u) = \frac{1}{2} l - \frac{1}{10} l + o(1) = \frac{2}{5} l + o(1) \ge \frac{2}{5} \frac{S^{3/2}}{(p\lambda)^{1/4}} + o(1). \tag{4.14}$$

Since $p\lambda = 1$, by Lemma 4.3, we have

$$c_{p,\lambda} < \frac{1}{15} S^{3/2} \frac{(5 + p\lambda)^{3/2}}{(6p\lambda)^{1/2}} = \frac{2}{5} \frac{S^{3/2}}{(p\lambda)^{1/4}}, \tag{4.15}$$

which is a contradiction with (4.14). Therefore $v_n \to 0$ in H_b, or equivalently, $u_n \to u$ in H_b as $n \to \infty$. □

From the results in Lemmas 4.1–4.4 and Theorem 2.7, we can now show the existence of a critical point for the functional $J_{p,\lambda}$.

Lemma 4.5. *For almost every $\lambda \in I$ and $p = 1/\lambda$, there exists $u_p^\lambda \in H_b$ such that $(J_{p,\lambda})'(u_p^\lambda) = 0$ and $J_{p,\lambda}(u_p^\lambda) = c_{p,\lambda}$.*

Proof. By Theorem 2.7, for almost every $\lambda \in I$, there exists a bounded sequence $\{u_n^\lambda\} \subset H_b$ such that $J_{p,\lambda}(u_n^\lambda) \to c_{p,\lambda}$ and $(J_{p,\lambda})'(u_n^\lambda) \to 0$. From Lemma 4.4, for $p = 1/\lambda$, we can assume that there exists $u^\lambda \in H_b$ such that $u_n^\lambda \to u^\lambda$ in H_b, then the assertion follows. □

From Lemma 4.5, there exists a sequence $\{(\lambda_n, p_n, u_n)\} \subset I \times \mathbb{R} \times H_b$ such that as $n \to \infty$, $\lambda_n \to 1^-$, $p_n \lambda_n = 1$ and

$$J_{p_n,\lambda_n}(u_n) = c_{p_n,\lambda_n}, \quad (J_{p_n,\lambda_n})'(u_n) = 0.$$

We now show that the sequence $\{u_n\}$ is uniformly.

Lemma 4.6. *Let u_n be a critical point of J_{p_n,λ_n} at the level c_{p_n,λ_n} as defined above, where $\lambda_n \in I$ and $p_n \lambda_n = 1$. Then there exists $M > 0$ such that $\|u_n\| \le M$ for all $n \in \mathbb{N}$.*

Proof. Firstly, since $(J_{p_n,\lambda_n})'(u_n) = 0$, it follows from (2.11) in Lemma 2.8 that u_n satisfies the following Pohozaev type identity:

$$\frac{1}{2} \int_{\mathbb{R}^3} |\nabla u_n|^2 + \frac{3}{2} b \int_{\mathbb{R}^3} |u_n|^2 - \frac{1}{2} p_n \lambda_n \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^5 = 3\lambda_n \int_{\mathbb{R}^3} F(u_n). \tag{4.16}$$

By using $J_{p_n,\lambda_n}(u_n) = c_{p_n,\lambda_n}$, we have that

$$\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_n|^2 + \frac{1}{2}b\int_{\mathbb{R}^3} |u_n|^2 - \frac{1}{10}p_n\lambda_n\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^5 - \lambda_n\int_{\mathbb{R}^3} F(u_n) = c_{p_n,\lambda_n}. \tag{4.17}$$

Thus we obtain from (4.16) and (4.17) that

$$\int_{\mathbb{R}^3} |\nabla u_n|^2 + \frac{1}{5}p_n\lambda_n\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^5 = 3c_{p_n,\lambda_n}. \tag{4.18}$$

We now estimate the right-hand side of (4.18). Since

$$J_{p_n,\lambda_n}(u) = \frac{1}{2}\|u\|^2 - \frac{1}{10}\int_{\mathbb{R}^3} \phi_u|u|^5 - \lambda_n\int_{\mathbb{R}^3} F(u) \le J_{2,1/2}(u), \quad u \in H_b.$$

Hence for any $\gamma \in \Gamma_{2,1/2}$, we have that $\gamma \in \Gamma_{p_n,\lambda_n}$, and then

$$c_{p_n,\lambda_n} \le \max_{t\in[0,1]} J_{2,1/2}(\gamma(t)), \quad \gamma \in \Gamma_{2,1/2}.$$

So

$$c_{p_n,\lambda_n} \le c_{2,1/2} \equiv \frac{1}{3}C_6.$$

Then (4.18) implies that

$$|\nabla u_n|_2^2 \le C_6, \quad p_n\lambda_n\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^5 \le 5C_6. \tag{4.19}$$

On the other hand, from the equation $(J_{p_n,\lambda_n})'(u_n) = 0$, we have the relation

$$\int_{\mathbb{R}^3} |\nabla u_n|^2 + b\int_{\mathbb{R}^3} |u_n|^2 - p_n\lambda_n\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^5 - \lambda_n\int_{\mathbb{R}^3} f(u_n)u_n = 0. \tag{4.20}$$

It follows from (4.20) and (3.2) that

$$\begin{aligned} b\int_{\mathbb{R}^3} |u_n|^2 &= \lambda_n\int_{\mathbb{R}^3} f(u_n)u_n - \int_{\mathbb{R}^3} |\nabla u_n|^2 + p_n\lambda_n\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^5 \\ &\le \frac{1}{2}b\int_{\mathbb{R}^3} |u_n|^2 + C_0\int_{\mathbb{R}^3} |u_n|^6 + 5C_6 \\ &\le \frac{1}{2}b\int_{\mathbb{R}^3} |u_n|^2 + C_0S^{-3}C_6^3 + 5C_6, \end{aligned} \tag{4.21}$$

which implies

$$\frac{1}{2}b\int_{\mathbb{R}^3} |u_n|^2 \le C_0S^{-3}C_6^3 + 5C_6. \tag{4.22}$$

Combining (4.19) and (4.22), we obtain that

$$\|u_n\|^2 \le 11C_6 + 2C_0S^{-3}C_6^3 \equiv M^2.$$ □

Now we are at the position to prove the second part of Theorem 1.1 where q is negative.

Proof of Theorem 1.1 for $q < 0$. Let $\lambda_n \in [1/2, 1]$, $p_n\lambda_n = 1$ and $\lambda_n \to 1$ as $n \to \infty$. Let u_n be a critical point of J_{p_n,λ_n} at the critical level c_{p_n,λ_n}. Since $\lambda_n \to 1$, then we can show that $\{u_n\}$ is a PS sequence of $J_{1,1}$. Indeed, by Lemma 4.6, we may assume that $\{u_n\}$ is bounded, which also implies that $\{J_{1,1}(u_n)\}$ is bounded. Also for any $v \in H_b$,

$$\langle J'_{1,1}(u_n), v\rangle = \langle (J'_{p_n,\lambda_n}(u_n), v\rangle + (\lambda_n - 1)\int_{\mathbb{R}^3} f(u_n)v.$$

It follows that $J'_{1,1}(u_n) \to 0$, and thus $\{u_n\}$ is a bounded $(\text{PS})_{c_{1,1}}$ sequence of $J_{1,1}$. Therefore Lemma 4.4 holds and $\{u_n\}$ has a convergent subsequence. Without loss of generality, we assume that $u_n \to u$ and consequently $J'_{1,1}(u) = 0$. From the proof of Lemma 4.2, we have that $J_{p_n,\lambda_n}(u_n) \geq c_{2,1/2}$, and hence

$$J_{1,1}(u) = \lim_{n\to\infty} J_{1,1}(u_n) = \lim_{n\to\infty} J_{p_n,\lambda_n}(u_n) \geq c_{2,1/2} > 0.$$

Therefore u is a positive solution by the condition (H_1), and the proof is completed. □

To conclude the paper we briefly remark on the proof of Theorem 1.2. For the subcritical equation (1.5) with $s \in [1, 4)$, there exists $q_0 > 0$ such that (1.5) has a positive solution for any $q \in [0, q_0)$. This has been proved in [19]. For the case of $q < 0$, by using the same method in this section for the critical case, we can prove that (1.5) has a positive solution for any $q < 0$ as the results in Lemmas 4.1, 4.2, 4.4, 4.5, 4.6 hold for any $q < 0$. We need only to notice that H_b can be compactly embedded into $L_r^{s+1}(\mathbb{R}^3)$. So the conclusion of Lemma 4.4 hold for all $q < 0$ and $\lambda \in I$.

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