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The existence, bifurcation and stability of positive stationary solutions of a diffusive Leslie–Gower predator–prey model with Holling-type II functional responses

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ABSTRACT

In this paper, we revisit a diffusive Leslie–Gower predator–prey model with Holling-type II functional responses and Dirichlet boundary condition. It is shown that multiple positive steady state solutions exist under certain conditions on the parameters, while for another parameter region, the positive steady state solution is unique and locally asymptotically stable. Results are proved by using bifurcation theory, fixed point index theory, energy estimates and asymptotic behavior analysis.

1. Introduction

In this paper, we revisit the following steady state equation of the diffusive Leslie–Gower predator–prey model with Holling-type II functional responses and Dirichlet boundary condition which was considered in [43]:

$$\begin{cases} -\Delta u = u\left(a_1 - bu - \dfrac{c_1 v}{u+k_1}\right), & x \in \Omega, \\ -\Delta v = v\left(a_2 - \dfrac{c_2 v}{u+k_2}\right), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain with a smooth boundary $\partial\Omega$, the parameters a_i, b, c_i, k_i $(i = 1, 2)$ are positive numbers, and u and v are the population densities of the prey and predator species.

The system (1.1) is based on a classical predator–prey model of Leslie and Gower [24]:

$$\begin{cases} \dfrac{du}{dt} = u\,(a_1 - bu - c_1 v)\,, \\ \dfrac{dv}{dt} = v\left(a_2 - \dfrac{c_2 v}{u}\right), \end{cases} \tag{1.2}$$

which was regarded as a prototypical predator–prey system in the ecological studies. But the interaction terms in (1.2) are unbounded, which is not reasonable in the real world. By using Holling type II functional response [19] in both prey and

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predator interaction terms, a Leslie–Gower predator–prey system with saturated functional responses is obtained in the following form (see [1]):

$$\begin{cases} \dfrac{du}{dt} = u\left(a_1 - bu - \dfrac{c_1 v}{u+k_1}\right), \\ \dfrac{dv}{dt} = v\left(a_2 - \dfrac{c_2 v}{u+k_2}\right). \end{cases} \tag{1.3}$$

The model (1.3) is proposed based on the biological fact that if the predator v is more capable of switching from its favorite food (the prey u) to other food options, then it has better ability to survive when the prey population is low; here a_1 and a_2 are the growth rates per capita of prey u and predator v, respectively; b measures the strength of intraspecific competition among individuals of species u, and it is related to the carrying capacity of the prey; c_1 is the maximum value of the per capita reduction rate of u due to v, and c_2 is the maximum growth per capita of v due to predation of u; k_1 and k_2 measure the extent to which environment provides protection to prey u and predator v, respectively.

On the other hand, the spatial component of ecological interactions has been identified as an important factor in how ecological communities are shaped, and understanding the role of space is challenging both theoretically and empirically [30]. Empirical evidence suggests that the spatial scale and structure of environment can influence population interactions [6]. The reaction–diffusion model

$$\begin{cases} u_t - d_1\Delta u = u\left(a_1 - bu - \dfrac{c_1 v}{u+k_1}\right), & x \in \Omega,\ t > 0, \\ v_t - d_2\Delta v = v\left(a_2 - \dfrac{c_2 v}{u+k_2}\right), & x \in \Omega,\ t > 0, \end{cases} \tag{1.4}$$

which corresponds to the ODE dynamics of (1.3) was proposed in [3,4]. Here d_1 and d_2 are the diffusion coefficients of prey u and predator v respectively. Hopf and Turing bifurcations of (1.4) with no-flux boundary condition were analyzed in [3,4] (see also [38,42]) and numerical simulations showed rich pattern formation dynamics and self-organization of various patterns.

It is known that elliptic systems with homogeneous Dirichlet boundary condition are usually more difficult to analyze, as shown in [5,7,10,13–16,22,23,25,28,29] for several other diffusive ecological interaction models, since the only possible constant steady state is $(0, 0)$ and the ODE dynamics is not embedded in the PDE models. There are more references for the corresponding Neumann boundary value problems, which we do not list here but refer to [6,17,31].

The positive steady state solutions of problem (1.4) with $d_1 = d_2 = 1$ and homogeneous Dirichlet boundary condition, *i.e.* system (1.1), were first considered in [43] by one of the authors. The existence of one positive solution to (1.1) under some conditions on the parameters was showed in [43], while the nonexistence of positive solutions to (1.1) in some other parameter regions was also proved. The multiplicity or uniqueness of positive solutions to (1.1) is not known except some very limited cases. In this paper we will prove some further results on the existence, multiplicity, uniqueness and bifurcation structure of positive solutions to (1.1). In particular, we prove that when fixing other parameters a_2, b, c_2, k_2, but choosing small c_1, k_1 with $c_1/k_1 \geq K > 0$, then for a_1 in a certain interval, there exist at least two positive solutions to (1.1) (see Section 3 for more details). The existence of multiple steady state solutions indicate that the system (1.4) can have possible bistable dynamics. In Section 5, a numerical simulation of (1.4) shows that for the same system parameters, two solution trajectories of (1.4) with slightly different initial conditions converge to two different steady state solutions. The choice of parameters in the numerical simulation is guided by theoretical work in Section 3. On the other hand, in Section 4, the uniqueness of a positive solution to (1.1) is proved under some conditions, which shows the richness of the dynamics of (1.4) for different parameters.

We remark that for reaction–diffusion Lotka–Volterra or more generally Gause type predator–prey systems with Dirichlet boundary conditions, the positive steady state solution is often unique if the domain Ω is one-dimensional or spherical (see [7,10,23]), while in some other situations multiplicity of positive steady solutions have been shown [15,16]. We also comment that another variation of (1.3) was proposed by Tanner [41], and the diffusive Holling–Tanner predator–prey system has also been considered extensively [8,9,27,32,33].

The organization of the remaining part of the paper is as follows. We introduce some basic notations and recall some previous results in Section 2. In Section 3, we study the multiplicity of positive solutions of problem (1.1), and the uniqueness of the positive solution under certain condition is studied in Section 4. The conclusion with some numerical simulations is in Section 5. In the paper, we use $\|\cdot\|_X$ as the norm of Banach space X, $\langle \cdot, \cdot \rangle$ as the duality pair of a Banach space X and its dual space X^*. For a linear operator L, we use $\mathcal{N}(L)$ as the null space of L and $\mathcal{R}(L)$ as the range space of L, and we use $L[w]$ to denote the image of w under the linear mapping L. For a multilinear operator L, we use $L[w_1, w_2, \ldots, w_k]$ to denote the image of $(w_1, w_2, \ldots, w_k)$ under L, and when $w_1 = w_2 = \cdots = w_k$, we use $L[w_1]^k$ instead of $L[w_1, \ldots, w_1]$. For a nonlinear operator F, we use F_u as the partial derivative of F with respect to u.

2. Preliminaries

In order to state the main results of this paper and [43], we introduce some notations and basic facts which are well-known (see for example [2,6,31]). For any $q \in C(\overline{\Omega})$, the linear eigenvalue problem

$$\begin{cases} -\Delta u + q(x)u = \rho u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega \end{cases} \tag{2.1}$$

has an infinite sequence of eigenvalues, $\rho_1 < \rho_2 \le \rho_3 \le \cdots$, which are bounded below. It is also known that the principal eigenvalue

$$\rho = \rho_1 = \rho_1(-\Delta + q(x)) \tag{2.2}$$

is simple, and all solutions of (2.2) with $\rho = \rho_1(-\Delta + q(x))$ are multiples of a particular eigenfunction, which does not change sign in Ω and its normal derivative does not vanish on the boundary $\partial\Omega$. Furthermore ρ_1 is strictly increasing in the sense that for $q_1(x), q_2(x) \in C(\bar{\Omega})$, $q_1(x) \le q_2(x)$ and $q_1(x) \not\equiv q_2(x)$ implies that $\rho_1(-\Delta + q_1(x)) < \rho_1(-\Delta + q_2(x))$. In particular we denote $\lambda_1 = \rho_1(-\Delta)$ and its corresponding normalized positive eigenfunction $\omega(x)$ satisfies $\max_{x\in\Omega} \omega(x) = 1$.

Next consider the logistic type problem

$$\begin{cases} -\Delta\phi + q(x)\phi = a\phi - f(x)\phi^2, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega, \end{cases} \tag{2.3}$$

where $a > 0$, $q(x), f(x)$ are continuous functions on $\bar{\Omega}$ and $f(x) > 0$. It is well known that if $a \le \rho_1(-\Delta + q(x))$, $\phi = 0$ is the unique nonnegative solution of (2.3), while (2.3) has a unique positive solution if $a > \rho_1(-\Delta + q(x))$. In the paper we denote the unique positive solution of

$$\begin{cases} -\Delta u = u(a - \sigma u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega \end{cases} \tag{2.4}$$

by $\theta(a, \sigma)$ when $a > \lambda_1$, where a, σ are positive constants. In particular we denote $\theta(a_1, b)$ by $\hat{u}(x)$ when $a_1 > \lambda_1$ and $\theta(a_2, c_2/k_2)$ by $\hat{v}(x)$ when $a_2 > \lambda_1$.

With the notations introduced above, one of the main results derived in [43] can be summarized as follows.

Theorem 2.1 (*[43, Theorem 1.1]*). *Assume that $a_2 > \lambda_1$, and $b, c_1, c_2, k_1, k_2 > 0$. If (1.1) has a positive solution, then $a_1 > \lambda_1$; moreover if $a_1 > \rho_1\left(-\Delta + \frac{c_1}{k_1}\hat{v}\right)$, then (1.1) has a positive solution.*

The result in Theorem 2.1 shows that there is a gap between the sufficient condition and necessary condition for the existence of positive solutions of problem (1.1). What happens when a_1 is in this gap? In this paper, we provide an answer to this question and also give further discussion on the positive solutions to (1.1). In the following we always assume that $a_1 > \lambda_1$ and $a_2 > \lambda_1$, since apparently (1.1) has no positive solution if $a_1 \le \lambda_1$ or $a_2 \le \lambda_1$.

For later applications, we recall the following *a priori* estimates of nonnegative solutions of (1.1). The proof can be found in Lemma 2.1 in [43], and we omit it here.

Lemma 2.2. *Let (u, v) be a nonnegative solution to problem (1.1). Then*

$$0 \le u(x) < \frac{a_1}{b}, \qquad 0 \le v(x) < \frac{a_2}{c_2}\left(\frac{a_1}{b} + k_2\right), \quad x \in \Omega.$$

If in addition $a_1 > \lambda_1 + \frac{a_1 c_1}{k_1 \kappa}$ and (u, v) is a positive solution of (1.1), then

$$\theta\left(a_1 - \frac{a_1 c_1}{k_1 \kappa}, b\right)(x) \le u(x) \le \hat{u}(x), \qquad \hat{v}(x) \le v(x) \le \theta(a_2, \kappa)(x), \quad x \in \Omega, \tag{2.5}$$

where $\theta(a, \sigma)$ is the unique positive solution of (2.4), and $\kappa = \frac{bc_2}{a_1 + bk_2}$.

Finally we review the fixed point index theory (see [12,26,36]) which will be used in Section 3. Let E be a Banach space and $\mathcal{W} \subset E$ is a closed convex set. Then $\mathcal{W}$ is called a total wedge if $\gamma\mathcal{W} \subset \mathcal{W}$ for all $\gamma \ge 0$ and $\overline{\mathcal{W} - \mathcal{W}} = E$. For $y \in \mathcal{W}$, define $\mathcal{W}_y = \{x \in E : y + \gamma x \in \mathcal{W}$ for some $\gamma > 0\}$ and $S_y = \{x \in \overline{\mathcal{W}_y} : -x \in \overline{\mathcal{W}_y}\}$. Then $\overline{\mathcal{W}_y}$ is a wedge containing $\mathcal{W}, y, -y$, while S_y is a closed subset of E containing y. Let T be a compact linear operator on E which satisfies $T(\overline{\mathcal{W}_y}) \subset \overline{\mathcal{W}_y}$. We say that T has property α on $\overline{\mathcal{W}_y}$ if there is a $t \in (0, 1)$ and a $\omega \in \overline{\mathcal{W}_y} \setminus S_y$ such that $(I - tT)\omega \in S_y$. Let $G : \mathcal{W} \to \mathcal{W}$ be a compact operator with a fixed point $y \in \mathcal{W}$ which is Fréchet differentiable at y. Let $\mathcal{L} = G'(y)$ be the Fréchet derivative of G at y. Then $\mathcal{L}$ maps $\overline{\mathcal{W}_y}$ into itself. We denote by $\text{index}_{\mathcal{W}}(G, \mathcal{O})$ the degree of $I - G$ in $\mathcal{O}$ relative to $\mathcal{W}$ and $\text{index}_{\mathcal{W}}(G, y)$ the fixed point index of G at y relative to $\mathcal{W}$, where $\mathcal{O} \subset \mathcal{W}$ is a bounded open set with respect to the relative topology of $\mathcal{W}$. The following result is standard in the theory of fixed point index.

Lemma 2.3 ([12,26,36]). *Assume that $I - \mathcal{L}$ has no non-trivial kernel in $\overline{W}_y$. Then, we have the following.*

1. *If $\mathcal{L}$ has property α on $\overline{W}_y$, then* $\text{index}_W(F, y) = 0$.
2. *If $\mathcal{L}$ does not have property α on $\overline{W}_y$, then* $\text{index}_W(F, y) = (-1)^\sigma$*, where σ is the sum of multiplicities of all eigenvalues of $\mathcal{L}$ which is greater than* 1.

Now we recall some related results which have been proved in [43]. Let $E = C_0(\bar{\Omega}) \times C_0(\bar{\Omega})$ and $\mathcal{W} = \{(u, v) \in E : u, v \geq 0\}$ be the positive cone in E such that $E = \overline{\mathcal{W} - \mathcal{W}}$, where $C_0(\bar{\Omega}) := \{u \in C(\bar{\Omega}) : u(x) = 0,\ x \in \partial\Omega\}$. Define the set $\mathcal{O}$ and the operator G by

$$\mathcal{O} = \left\{(u, v) \in W : 0 \leq u(x) < \frac{a_1}{b},\ 0 \leq v(x) < \frac{a_2}{c_2}\left(\frac{a_1}{b} + k_2\right)\right\},$$
$$G(u, v) = (-\Delta + M)^{-1} \begin{pmatrix} u\left(a_1 - bu - \dfrac{c_1 v}{u + k_1}\right) + Mu \\ v\left(a_2 - \dfrac{c_2 v}{u + k_2}\right) + Mv \end{pmatrix}, \tag{2.6}$$

where M is a large positive constant to ensure $F : \overline{\mathcal{O}} \to W$. The set $\mathcal{O}$ contains all nonnegative solutions of (1.1) by Lemma 2.2, and G is compact by the standard regularity theory of elliptic equations [18]. Furthermore, a positive solution of (1.1) is equivalent to a positive fixed point of G. Then the degree $\text{index}_W(G, \mathcal{O})$ and the fixed point indices of G at trivial solution $(0, 0)$, semi-trivial solutions $(\hat{u}, 0)$ and $(0, \hat{v})$ are as follows.

Lemma 2.4 ([43]). *Assume $a_1 > \lambda_1$ and $a_2 > \lambda_1$. Then*

1. $\text{index}_W(G, \mathcal{O}) = 1$;
2. $\text{index}_W(G, (0, 0)) = 0$;
3. $\text{index}_W(G, (\hat{u}, 0)) = 0$;
4. $\text{index}_W(G, (0, \hat{v})) = 0$ *if* $a_1 > \tilde{a}_1$ *and* $\text{index}_W(G, (0, \hat{v})) = 1$ *if* $a_1 < \tilde{a}_1$.

3. Multiplicity and bifurcation of positive solutions

In this section, we use the bifurcation method to show that (1.1) may have multiple positive solutions. To simplify the notation we define

$$\tilde{a}_1 \equiv \rho_1\left(-\Delta + \frac{c_1}{k_1}\hat{v}(x)\right).$$

Our main result in this section is that, there exists a positive constant $a_1^* \in (\lambda_1, \tilde{a}_1)$ such that problem (1.1) has at least two positive solutions for $a_1 \in (\tilde{a}_1 - \varepsilon, \tilde{a}_1)$ for some small $\varepsilon > 0$, and has at least one positive solution for $a_1 \in [a_1^*, \tilde{a}_1]$.

From the definitions in Section 2, (1.1) has a semi-trivial non-negative solution $(u, v) = (0, \hat{v}(x))$ for any $a_1 > 0$ as long as $a_2 > \lambda_1$. Here we use a_1 as a bifurcation parameter, and consider the bifurcation of positive solutions from the branch of semi-trivial solutions: $\{(a_1, 0, \hat{v}) : a_1 > \lambda_1\}$. By linearizing (1.1) at $(a_1, 0, \hat{v})$, we obtain the following eigenvalue problem:

$$\begin{cases} \Delta\xi + a_1\xi - \dfrac{c_1\hat{v}}{k_1}\xi = \mu\xi, & x \in \Omega, \\ \Delta\eta + \dfrac{c_2\hat{v}^2}{k_2}\xi + a_2\eta - \dfrac{2c_2\hat{v}}{k_2}\eta = \mu\eta, & x \in \Omega, \\ \xi = \eta = 0, & x \in \partial\Omega. \end{cases} \tag{3.1}$$

A necessary condition for bifurcation is that the principal eigenvalue μ_1 of (3.1) is zero, which occurs if $a_1 = \tilde{a}_1$.

Let Φ be the positive eigenfunction corresponding to $\tilde{a}_1$, *i.e.*, $(\tilde{a}_1, \Phi)$ satisfies

$$\begin{cases} -\Delta\Phi + \dfrac{c_1}{k_1}\hat{v}(x)\Phi = \tilde{a}_1\Phi, & x \in \Omega, \\ \Phi = 0, & x \in \partial\Omega. \end{cases} \tag{3.2}$$

We assume that Φ is normalized so that $\int_\Omega \Phi^2 dx = 1$. Since

$$\rho_1\left(-\Delta + 2\frac{c_2}{k_2}\hat{v} - a_2\right) > \rho_1\left(-\Delta + \frac{c_2}{k_2}\hat{v} - a_2\right) = 0,$$

then $-\Delta + 2\frac{c_2}{k_2}\hat{v} - a_2$ is invertible, and $\left(-\Delta + 2\frac{c_2}{k_2}\hat{v} - a_2\right)^{-1}$ maps positive functions to positive functions because of the maximum principle. Define

$$\Psi = \frac{c_2}{k_2^2}\left(-\Delta + 2\frac{c_2}{k_2}\hat{v} - a_2\right)^{-1}(\hat{v}^2\Phi) \tag{3.3}$$

then both Φ and Ψ are positive in Ω.

With the functions defined above, we have the following result regarding the bifurcation of positive solutions of (1.1) from $(a, 0, \hat{v}(x))$ at $a_1 = \tilde{a}_1$.

Theorem 3.1. *Assume $a_1 > \lambda_1$ and $a_2 > \lambda_1$. Then $a_1 = \tilde{a}_1$ is a bifurcation value of (1.1) where positive solutions bifurcate from the line of semi-trivial solutions $\Gamma_0 = \{(a_1, 0, \hat{v}) : a_1 > 0\}$; near $(\tilde{a}_1, 0, \hat{v})$, all the positive solutions of (1.1) lie on a smooth curve $\Gamma_1 = \{(a_1(s), u(s), v(s)) : s \in (0, \delta)\}$ for some $\delta > 0$ such that*

$$\begin{cases} a_1(s) = \tilde{a}_1 + s\tilde{a}_2 + s\tilde{a}_3(s), \\ u(s) = s\Phi + su_1(s, x), \\ v(s) = \hat{v}(x) + s\Psi + sv_1(s, x), \end{cases} \tag{3.4}$$

where $s \mapsto (\tilde{a}_3(s), u_1(s, x), v_1(s, x))$ is a smooth function from $(0, \delta)$ to $\mathbb{R} \times X \times X$ for $X = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ with $p > N$, such that $\tilde{a}_3(0) = 0$, $u_1(0, x) = v_1(0, x) = 0$, and

$$\tilde{a}_2 = \frac{1}{k_1}\int_\Omega \left[\left(bk_1 - \frac{c_1}{k_1}\hat{v}\right)\Phi^3 + c_1\Phi^2\Psi\right]dx. \tag{3.5}$$

Moreover $a_1 = \tilde{a}_1$ is the unique bifurcation value for which positive solutions bifurcate from Γ_0.

Proof. We apply a bifurcation result of Crandall and Rabinowitz [11]. Let $X = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and let $Y = L^p(\Omega)$, where $p > N$. Define a nonlinear mapping $F : \mathbb{R} \times X \times X \to Y \times Y$ by

$$F(a_1, u, v) = \begin{pmatrix} \Delta u + u\left(a_1 - bu - \dfrac{c_1 v}{u + k_1}\right) \\ \Delta v + v\left(a_2 - \dfrac{c_2 v}{u + k_2}\right) \end{pmatrix}. \tag{3.6}$$

We consider the bifurcation at $(a_1, u, v) = (\tilde{a}_1, 0, \hat{v})$. From straightforward calculations, we obtain that

$$\begin{aligned} &F_{(u,v)}(a_1, u, v)[\xi, \eta] = \begin{pmatrix} \Delta\xi + a_1\xi - 2bu\xi - \dfrac{c_1k_1v}{(u+k_1)^2}\xi - \dfrac{c_1u}{u+k_1}\eta \\ \Delta\eta + \dfrac{c_2v^2}{(u+k_2)^2}\xi + a_2\eta - \dfrac{2c_2v}{u+k_2}\eta \end{pmatrix}, \\ &F_{a_1}(a_1, u, v) = \begin{pmatrix} u \\ 0 \end{pmatrix}, \qquad F_{a_1(u,v)}(a_1, u, v)[\xi, \eta] = \begin{pmatrix} \xi \\ 0 \end{pmatrix}, \\ &F_{(u,v)(u,v)}(a_1, u, v)[\xi, \eta]^2 = \begin{pmatrix} -2b\xi^2 + 2\dfrac{c_1k_1v}{(u+k_1)^3}\xi^2 - 2\dfrac{c_1k_1}{(u+k_1)^2}\xi\eta \\ -2\dfrac{c_2v^2}{(u+k_2)^3}\xi^2 + 4\dfrac{c_2v}{(u+k_2)^2}\xi\eta - 2\dfrac{c_2}{u+k_2}\eta^2 \end{pmatrix}. \end{aligned} \tag{3.7}$$

At $(a_1, u, v) = (\tilde{a}_1, 0, \hat{v})$, it is easy to verify that the kernel $\mathcal{N}(F_{u,v}(\tilde{a}_1, 0, \hat{v})) = \text{span}\{(\Phi, \Psi)\}$, the range space

$$\mathcal{R}(F_{u,v}(\tilde{a}_1, 0, \hat{v})) = \left\{(f, g) \in Y \times Y : \int_\Omega f(x)\Phi(x)dx = 0\right\},$$

and

$$F_{a_1(u,v)}(\tilde{a}_1, 0, \hat{v})[\Phi, \Psi] = (\Phi, 0) \notin \mathcal{R}(F_{u,v}(\tilde{a}_1, 0, \hat{v})) \quad \text{since } \int_\Omega \Phi^2(x)dx \neq 0.$$

Thus we can apply [11, Theorem 1.7] to conclude that the set of positive solutions to (1.1) near $(\tilde{a}_1, 0, \hat{v})$ is a smooth curve

$$\Gamma_1 = \{(a_1(s), u(s), v(s)) : s \in (0, \delta)\}, \tag{3.8}$$

such that $a_1(0) = \tilde{a}_1$, $u(s) = \Phi s + o(s)$, $v(s) = \hat{v} + \Psi s + o(s)$. Moreover, $a_1'(0)$ can be calculated by (see for example [39]):

$$\begin{aligned} a_1'(0) &= \tilde{a}_2 = -\frac{\langle l, F_{(u,v)(u,v)}(\tilde{a}_1, 0, \hat{v})[\Phi, \Psi]^2\rangle}{2\langle l, F_{a_1(u,v)}(\tilde{a}_1, 0, \hat{v})[\Phi, \Psi]\rangle} \\ &= \frac{2b\int_\Omega \Phi^3 dx - 2c_1 k_1^{-2}\int_\Omega \hat{v}\Phi^3 dx + 2c_1 k_1^{-1}\int_\Omega \Phi^2\Psi dx}{2\int_\Omega \Phi^2 dx} \\ &= \frac{1}{k_1}\int_\Omega \left[\left(bk_1 - \frac{c_1}{k_1}\hat{v}\right)\Phi^3 + c_1\Phi^2\Psi\right] dx, \end{aligned} \tag{3.9}$$

where l is a linear functional on $Y \times Y$ defined as $\langle l, [f, g]\rangle = \int_\Omega f(x)\Phi(x)dx$.

Finally we prove that $a_1 = \tilde{a}_1$ is the unique bifurcation point where positive solutions of problem (1.1) bifurcate from $(0, \hat{v})$. Suppose that there is a sequence $\{(a_{1n}, u_n, v_n)\}_{n=1}^\infty$ of positive solutions of (1.1) with

$$\lim_{n\to\infty}(a_{1n}, u_n, v_n) = (\tilde{a}, 0, \hat{v}) \in \mathbb{R} \times X. \tag{3.10}$$

Then, we find from the first equation of (1.1) with $a_1 = a_{1n}$ that, for every $n \geq 1$,

$$-\Delta\left(\frac{u_n}{\|u_n\|_{L^p(\Omega)}}\right) = a_{1n}\frac{u_n}{\|u_n\|_{L^p(\Omega)}} - b\frac{u_n^2}{\|u_n\|_{L^p(\Omega)}} - \frac{c_1 v_n}{(u_n + k_1)}\cdot\frac{u_n}{\|u_n\|_{L^p(\Omega)}}, \tag{3.11}$$

or, equivalently,

$$\begin{aligned} \frac{u_n}{\|u_n\|_{L^p(\Omega)}} &= (a_{1n} - \tilde{a})(-\Delta)^{-1}\left(\frac{u_n}{\|u_n\|_{L^p(\Omega)}}\right) \\ &\quad + (-\Delta)^{-1}\left(\tilde{a}\frac{u_n}{\|u_n\|_{L^p(\Omega)}} - b\frac{u_n^2}{\|u_n\|_{L^p(\Omega)}} - \frac{c_1 v_n}{(u_n + k_1)}\cdot\frac{u_n}{\|u_n\|_{L^p(\Omega)}}\right). \end{aligned} \tag{3.12}$$

By the compactness of $(-\Delta)^{-1}$, it is easy to see that, along some subsequence, re-labeled by n, we have that

$$\lim_{n\to\infty}\frac{u_n}{\|u_n\|_{L^p(\Omega)}} = \phi > 0 \tag{3.13}$$

for some $\phi \in W^{2,p}(\Omega)$ with $\|\phi\|_{L^p(\Omega)} = 1$. Thus, passing to the limit as $n \to \infty$ in the previous identities, we find that

$$\phi = (-\Delta)^{-1}\left(\tilde{a}\phi - \frac{c_1}{k_1}\hat{v}\phi\right), \tag{3.14}$$

or, equivalently,

$$\begin{cases} -\Delta\phi + \dfrac{c_1}{k_1}\hat{v}\phi = \tilde{a}\phi, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega. \end{cases} \tag{3.15}$$

Therefore, $\tilde{a} = \tilde{a}_1$. The proof is completed. □

Next we discuss the stability of the positive solutions bifurcating from the semi-trivial solutions.

Theorem 3.2. *Assume the conditions of Theorem 3.1 are satisfied, and let $\tilde{a}_2$ be defined as in (3.5). If $\tilde{a}_2 \neq 0$, then for $s \in (0, \delta)$, the positive solution $(a_1(s), u(s), v(s))$ bifurcating from $(\tilde{a}_1, 0, \hat{v})$ is non-degenerate. Moreover, $(u(s), v(s))$ is unstable if $\tilde{a}_2 < 0$, and stable if $\tilde{a}_2 > 0$.*

Proof. Denote $a_1 = a_1(s)$ and $(u, v) = (u(s), v(s))$. Then the corresponding linearized problem at (u, v) can be written as

$$\mathfrak{L}(s)\begin{pmatrix}\xi \\ \eta\end{pmatrix} = \mu(s)\begin{pmatrix}\xi \\ \eta\end{pmatrix}, \qquad \mathfrak{L}(s) = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22}\end{pmatrix}, \tag{3.16}$$

where

$$M_{11} = -\Delta - a_1 + 2bu + \frac{c_1 k_1 v}{(u + k_1)^2}, \qquad M_{12} = \frac{c_1 u}{u + k_1},$$
$$M_{21} = -\frac{c_2 v^2}{(u + k_2)^2}, \qquad M_{22} = -\Delta - a_2 + \frac{2c_2 v}{u + k_2}.$$

Then letting $s \to 0^+$, we get

$$\mathfrak{L}(s) \to \mathfrak{L}_0 = \begin{pmatrix} -\Delta - \tilde{a}_1 + \frac{c_1}{k_1}\hat{v} & 0 \\ -\frac{c_2}{k_2^2}\hat{v}^2 & -\Delta - a_2 + \frac{2c_2}{k_2}\hat{v} \end{pmatrix}. \tag{3.17}$$

Since $\tilde{a}_1 = \rho_1\left(-\Delta + \frac{c_1}{k_1}\hat{v}\right)$, 0 is the first eigenvalue of the operator $-\Delta - \tilde{a}_1 + \frac{c_1}{k_1}\hat{v}$. On the other hand, since $\rho_1\left(-\Delta - a_2 + 2\frac{c_2}{k_2}\hat{v}\right) > \rho_1\left(-\Delta - a_2 + \frac{c_2}{k_2}\hat{v}\right) = 0$, 0 is the first eigenvalue of $\mathfrak{L}_0$ with the corresponding eigenfunction (Φ, Ψ). Moreover, all the other eigenvalues of $\mathfrak{L}_0$ are positive and apart from 0. By the perturbation theory of linear operators [21], we know that for the small $s > 0$, $\mathfrak{L}(s)$ has a unique eigenvalue $\mu(s)$ satisfying $\mu(s) \to 0$ as $s \to 0^+$ and all other eigenvalues of $\mathfrak{L}(s)$ have positive real parts and apart from 0. In the following, we denote $\mathfrak{L}(s) = \mathfrak{L}$ and $\mu(s) = \mu$.

Now we determine the sign of $\mathrm{Re}(\mu)$ for $s > 0$ small enough. Let (ξ, η) be the corresponding eigenfunction to μ such that $(\xi, \eta) \to (\Phi, \Psi)$ as $s \to 0^+$, then (ξ, η) satisfies

$$\begin{cases} -\Delta\xi - \left(a_1 - 2bu - \dfrac{c_1k_1v}{(u+k_1)^2}\right)\xi + \dfrac{c_1u}{u+k_1}\eta = \mu\xi, & x \in \Omega, \\ -\Delta\eta - \left(a_2 - \dfrac{2c_2v}{u+k_2}\right)\eta - \dfrac{c_2v^2}{(u+k_2)^2}\xi = \mu\eta, & x \in \Omega, \\ \xi = \eta = 0, & x \in \partial\Omega. \end{cases} \tag{3.18}$$

Multiplying the first equation of (3.18) by u and integrating the result over Ω, we obtain

$$-\int_\Omega u\Delta\xi dx - \int_\Omega u\xi\left(a_1 - 2bu - \frac{c_1k_1v}{(u+k_1)^2}\right)dx + \int_\Omega \frac{c_1u^2}{u+k_1}\eta dx = \mu\int_\Omega u\xi dx. \tag{3.19}$$

By multiplying ξ to the first equation of (1.1) with $(u, v) = (u(s), v(s))$ and integrating the result over Ω, we have

$$-\int_\Omega u\Delta\xi dx = -\int_\Omega \xi\Delta u dx = \int_\Omega \xi u\left(a_1 - bu - \frac{c_1v}{u+k_1}\right)dx. \tag{3.20}$$

Combining (3.20) with (3.19) yields

$$\mu\int_\Omega u\xi dx = \int_\Omega \xi u^2\left(b - \frac{c_1v}{(u+k_1)^2}\right)dx + \int_\Omega \frac{c_1u^2}{u+k_1}\eta dx. \tag{3.21}$$

Recall that $(u, v) = \left(\Phi s + O(s^2), \hat{v} + \Psi s + O(s^2)\right)$ and $(\xi, \eta) \to (\Phi, \Psi)$ as $s \to 0^+$. Taking the real part in (3.21), then dividing the results by s^2 and letting $s \to 0^+$, we have

$$\begin{aligned} \lim_{s\to 0^+}\frac{\mathrm{Re}(\mu)}{s} &= \frac{\int_\Omega \Phi^3\left(b - \frac{c_1}{k_1^2}\hat{v}\right)dx + \int_\Omega \frac{c_1}{k_1}\Phi^2\Psi dx}{\int_\Omega \Phi^2 dx} \\ &= \frac{1}{k_1}\int_\Omega\left[\left(bk_1 - \frac{c_1}{k_1}\hat{v}\right)\Phi^3 + c_1\Phi^2\Psi\right]dx \\ &= \tilde{a}_2 \neq 0, \end{aligned} \tag{3.22}$$

which implies $\mathrm{Re}(\mu) \neq 0$ for $s > 0$ small. Since all the other eigenvalues of $\mathfrak{L}$ have positive real parts and apart from 0, then the stability assertions follow from (3.22). □

By using the bifurcation results proved above and the fixed point index used in [43], we have the following multiplicity result on the positive solutions of (1.1).

Theorem 3.3. *Assume the conditions of Theorem 3.1 are satisfied, and let $\tilde{a}_2$ be defined as in (3.5). If $\tilde{a}_2 < 0$, there exists a positive constant $a_1^* \in (\lambda_1, \tilde{a}_1)$ and $\varepsilon \in (0, \tilde{a}_1 - a_1^*]$ such that problem (1.1) has at least two positive solutions for $a_1 \in (\tilde{a}_1 - \varepsilon, \tilde{a}_1)$, and has at least one positive solution for $a_1 \in [a_1^*, \tilde{a}_1]$.*

Proof. From Theorem 3.1, (1.1) has a curve $\Gamma = \{(a_1(s), u(s), v(s)) : s \in (0, \delta)\}$ of positive solutions near $(\tilde{a}_1, 0, \hat{v})$. Since $\tilde{a}_2 < 0$, we get $a_1(s) < \tilde{a}_1$ for $s > 0$ small. Assume that (1.1) has a unique positive solution $(\tilde{u}, \tilde{v})$ when $a_1 < \tilde{a}_1$ but near $\tilde{a}_1$. By Theorem 3.1, we know that $(\tilde{u}, \tilde{v})$ must be the positive solution bifurcating from $(\tilde{a}_1, 0, \hat{v})$. That is $(\tilde{u}, \tilde{v}) = (u(s), v(s))$, which is non-degenerate by Theorem 3.2. Therefore $(I - G_{(u,v)})(\tilde{u}, \tilde{v}) : \overline{W}_{(\tilde{u},\tilde{v})} \to \overline{W}_{(\tilde{u},\tilde{v})}$ is invertible.

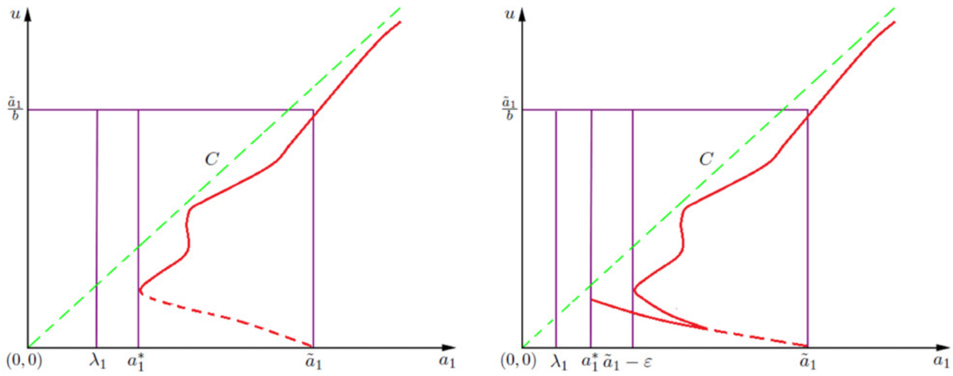

Fig. 1. Possible bifurcation diagrams of positive solutions when $\tilde{a}_2 < 0$. Left: $\tilde{a}_1 = a_1^* + \varepsilon$; right: $\tilde{a}_1 > a_1^* + \varepsilon$. Here the line C is $u = a_1/b$.

Since $\mathcal{W}_{(\tilde{u},\tilde{v})} \setminus S_{(\tilde{u},\tilde{v})} = \emptyset$, $G_{(u,v)}(\tilde{u}, \tilde{v})$ does not have property α on $\mathcal{W}_{(\tilde{u},\tilde{v})}$. Consequently, $\text{index}_{\mathcal{W}}(G, (\tilde{u}, \tilde{v})) = 1$ or -1. Notice that $\lambda_1 < a_1 < \tilde{a}_1$ and $a_2 > \lambda_1$, applying Lemma 2.4, we have

$$\begin{aligned} 1 &= \text{index}_{\mathcal{W}}(G, \mathcal{O}) \\ &= \text{index}_{\mathcal{W}}(G, (0, 0)) + \text{index}_{\mathcal{W}}(G, (\hat{u}, 0)) + \text{index}_{\mathcal{W}}(G, (0, \hat{v})) + \text{index}_{\mathcal{W}}(G, (\tilde{u}, \tilde{v})) \\ &= 0 + 0 + 1 \pm 1, \end{aligned}$$

which is a contradiction. So when $a_1 < \tilde{a}_1$ and near to $\tilde{a}_1$, there exist at least two positive solutions of (1.1).

From a global bifurcation result of Rabinowitz [34] (see also [40]), the curve Γ of the bifurcating positive solutions is contained in a connected component S_0 of the set of positive solutions of (1.1). Moreover either the closure of S_0 contains another trivial solution on $\{(a_1, 0, \hat{v}) : a_1 > 0\}$, or S_0 is unbounded. By Theorem 3.1, $a_1 = \tilde{a}_1$ is the unique bifurcation value to positive solutions of problem (1.1) from the line of trivial solutions $\{(a_1, 0, \hat{v}) : a_1 > 0\}$, so the first alternative is not possible and S_0 must be unbounded. Furthermore, $0 < u < \tilde{a}_1/b$ for $\lambda_1 \le a_1 \le \tilde{a}_1$ by Lemma 2.2. Finally, there is no positive solution when $a_1 \le \lambda_1$ by Theorem 2.1. Thus the projection of S_0 contains an interval $[a_1^*, \infty)$ for some a_1^* satisfying $\lambda_1 < a_1^* < \tilde{a}_1$. In particular, (1.1) has at least one positive solution for $a_1 \in [a_1^*, \tilde{a}_1]$. □

Two possible bifurcation diagrams are shown in Fig. 1. We also remark that $\tilde{a}_2 < 0$ can be achieved by fixing $c_2, k_2 > 0, a_2 > \lambda_1$, letting $c_1 = k_1 = \varepsilon > 0$. Then as $\varepsilon \to 0$, $\hat{v}$, Φ and Ψ are all independent of ε, while $\lim_{\varepsilon\to 0^+} k_1\tilde{a}_2 = -\int_\Omega \hat{v}\Phi^3 dx < 0$. So $\tilde{a}_2$ is negative if $\varepsilon > 0$ is small enough.

4. Uniqueness of a positive solution

In this section, we study the uniqueness of the positive solution to problem (1.1) under some conditions on the parameters. Our first result is as follows.

Theorem 4.1. *Suppose that $a_1 > \lambda_1$, $a_2 > \lambda_1$, and $b, c_2, k_2 > 0$ are fixed parameters, then there exists a constant $\delta > 0$ such that when $0 < c_1/k_1 \le \delta$, the problem (1.1) has a unique positive solution which is locally asymptotically stable.*

For the proof of Theorem 4.1, we first prove the following lemma about the asymptotic behavior of positive solutions of (1.1) when c_1/k_1 is sufficiently small.

Lemma 4.2. *Assume that $a_1 > \lambda_1$ and $a_2 > \lambda_1$.*

1. *Suppose that (u_i, v_i) is a positive solution of (1.1) with $c_1 = c_{1i}$ and $k_1 = k_{1i}$, and $c_{1i}/k_{1i} \to 0$ as $i \to \infty$, then (u_i, v_i) converges to $(\hat{u}, v^*)$ uniformly as $i \to \infty$, where v^* is the unique positive solution of*

$$\begin{cases} -\Delta v = v\left(a_2 - \dfrac{c_2 v}{\hat{u} + k_2}\right), & x \in \Omega, \\ v = 0, & x \in \partial\Omega. \end{cases} \tag{4.1}$$

2. *There exists a positive constant δ small enough such that any positive solution of (1.1) is non-degenerate and linearly stable if $c_1/k_1 \le \delta$.*

Proof. 1. It is clear that $(\hat{u}, v^*)$ is the unique positive solution of the following problem

$$\begin{cases} -\Delta u = u(a_1 - bu), & x \in \Omega, \\ -\Delta v = v\left(a_2 - \dfrac{c_2 v}{u + k_2}\right), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega. \end{cases} \tag{4.2}$$

Assume that $c_{1i}/k_{1i} \to 0$ as $i \to \infty$, and (u_i, v_i) is a positive solution of (1.1) with $c_1 = c_{1i}$ and $k_1 = k_{1i}$. By Lemma 2.2, $\|u_i\|_{L^\infty(\Omega)} \le a_1/b$ and $\|v_i\|_{L^\infty(\Omega)} \le a_2k_2/c_2$ and the upper bounds are both independent of i (thus independent of c_{1i} and k_{1i}), then $\|(u_i, v_i)\|_{[C^{2+\alpha}(\Omega)]^2}$ with $0 < \alpha < 1$ are uniformly bounded by regularity theory of elliptic equations [18]. Then there exists a subsequence of $\{(u_i, v_i)\}_{i=1}^\infty$, relabeled by itself, and two non-negative functions $u, v \in C^{2+\beta}(\bar{\Omega})$ with $0 < \beta < \alpha$ such that $(u_i, v_i) \to (u, v)$ in $[C^{2+\beta}(\bar{\Omega})]^2$ as $i \to \infty$. Then (u, v) must be a solution of (4.2). From the strong maximum principle, we know that each of u and v is either >0 in Ω or $\equiv 0$ in Ω. So if we can show that $u, v > 0$ in Ω, then the proof is completed as the positive solution of (4.2) is unique hence it must be the limit of the sequence $\{(u_i, v_i)\}$.

To the contrary, we assume that $u \equiv 0$, $\|u_i\|_\infty \to 0$ as $i \to \infty$. Let $\bar{u}_i = u_i/\|u_i\|_{L^\infty(\Omega)}$, then $\bar{u}_i$ satisfies

$$\begin{cases} -\Delta \bar{u}_i = \bar{u}_i \left(a_1 - bu_i - \dfrac{c_{1i}v_i}{u_i + k_{1i}}\right), & x \in \Omega, \\ \bar{u}_i = 0, & x \in \partial\Omega. \end{cases} \tag{4.3}$$

Similar to the arguments above, $\|u_i\|_{C^{2+\alpha}(\Omega)}$ are uniformly bounded, thus there exists a subsequence of $\{\bar{u}_i\}_{i=1}^\infty$, relabeled by itself, and a non-negative function $\bar{u} \in C^{2+\beta}(\bar{\Omega})$ with $0 < \beta < \alpha$ such that $\bar{u}_i \to \bar{u}$ in $C^{2+\beta}(\bar{\Omega})$ as $i \to \infty$. Obviously, $\|\bar{u}\|_{L^\infty(\Omega)} = 1$ and $\bar{u}$ satisfies

$$\begin{cases} -\Delta \bar{u} = a_1\bar{u}, & x \in \Omega, \\ \bar{u} = 0, & x \in \partial\Omega. \end{cases} \tag{4.4}$$

Therefore $a_1 = \lambda_1$ must hold, which contradicts with the assumption that $a_1 > \lambda_1$.

On the other hand, if we assume that $v \equiv 0$, the same arguments as above show that there exists a non-negative function $\bar{v} \in C^{2+\beta}(\bar{\Omega})$ such that $\bar{v}_i := v_i/\|v_i\|_{L^\infty(\Omega)} \to \bar{v}$ in $C^{2+\beta}(\bar{\Omega})$ as $i \to \infty$. Furthermore, $\|\bar{v}\|_{L^\infty(\Omega)} = 1$ and $\bar{v}$ satisfies

$$\begin{cases} -\Delta \bar{v} = a_2\bar{v}, & x \in \Omega, \\ \bar{v} = 0, & x \in \partial\Omega. \end{cases} \tag{4.5}$$

Therefore, $a_2 = \lambda_1$, which again contradicts with the assumption that $a_2 > \lambda_1$. Hence the limit $(u, v) > (0, 0)$ must be the unique positive solution of (4.2).

2. To the contrary, we assume that there exist sequences $\{c_{1i}\}$ and $\{k_{1i}\}$ such that $c_{1i}/k_{1i} \to 0$, μ_i with $\text{Re}(\mu_i) \le 0$ and (ξ_i, η_i) with $\|\xi_i\|^2_{L^2(\Omega)} + \|\eta_i\|^2_{L^2(\Omega)} = 1$ satisfying

$$\begin{cases} -\Delta \xi_i - \left(a_1 - 2bu_i - \dfrac{c_{1i}k_{1i}v_i}{(u_i + k_{1i})^2}\right)\xi_i + \dfrac{c_{1i}u_i}{u_i + k_{1i}}\eta_i = \mu_i\xi_i, & x \in \Omega, \\ -\Delta \eta_i - \dfrac{c_2v_i^2}{(u_i + k_2)^2}\xi_i - \left(a_2 - \dfrac{2c_2v_i}{u_i + k_2}\right)\eta_i = \mu_i\eta_i, & x \in \Omega, \\ \xi_i = \eta_i = 0, & x \in \partial\Omega, \end{cases} \tag{4.6}$$

where (u_i, v_i) is a positive solution of (1.1) with $c_1 = c_{1i}$ and $k = k_{1i}$. Multiplying (4.6) $_1$ by $\bar{\xi}_i$ and (4.6) $_2$ by $\bar{\eta}_i$, integrating the results over Ω, and then adding the results, we obtain

$$\begin{aligned} \mu_i = {} & \int_\Omega \left(|\nabla\xi_i|^2 + |\nabla\eta_i|^2\right)dx + \int_\Omega \left(\frac{c_{1i}k_{1i}v_i}{(u_i + k_{1i})^2} + 2bu_i - a_1\right)|\xi_i|^2dx \\ & + \int_\Omega \left(\frac{c_{1i}u_i}{u_i + k_{1i}}\eta_i\bar{\xi}_i - \frac{c_2v_i^2}{(u_i + k_2)^2}\xi_i\bar{\eta}_i\right)dx + \int_\Omega \left(\frac{2c_2v_i}{u_i + k_2} - a_2\right)|\eta_i|^2dx. \end{aligned} \tag{4.7}$$

Since $\text{Re}(\mu_i) \le 0$, $\|\xi_i\|^2_{L^2(\Omega)} + \|\eta_i\|^2_{L^2(\Omega)} = 1$ and $\{(u_i, v_i)\}$ is uniformly bounded, then we obtain that both $\text{Re}(\mu_i)$ and $\text{Im}(\mu_i)$ are uniformly bounded. So $\{\mu_i\}_{i=1}^\infty$ are uniformly bounded and there exists a subsequence of $\{\mu_i\}_{i=1}^\infty$, denoted by itself, such that $\lim_{i\to\infty} \mu_i = \mu$ with $\text{Re}(\mu) \le 0$. Using the boundedness of $\{\mu_i\}_{i=1}^\infty$ and L^p-theory of elliptic equations [18], we get that for any $p > n$, $\{\xi_i\}_{i=1}^\infty$ and $\{\eta_i\}_{i=1}^\infty$ are uniformly bounded in $W^{2,p}(\Omega)$. Since $W^{2,p}(\Omega)$ is embedded in $C^1(\bar{\Omega})$ compactly, there exist subsequences of $\{\xi_i\}_{i=1}^\infty$ and $\{\eta_i\}_{i=1}^\infty$, denoted by themselves, such that $\lim_{i\to\infty} \xi_i = \xi$ and $\lim_{i\to\infty} \eta_i = \eta$ in $C^1(\bar{\Omega})$ and $\|\xi\|^2_{L^2(\Omega)} + \|\eta\|^2_{L^2(\Omega)} = 1$. Letting $i \to \infty$ in (4.6), we obtain that (μ, ξ, η) satisfies the following equation in the sense of distribution

$$\begin{cases} -\Delta \xi - \left(a_1 - 2b\hat{u}\right)\xi = \mu\xi, & x \in \Omega, \\ -\Delta \eta - \left(a_2 - \dfrac{2c_2}{\hat{u} + k_2}v^*\right)\eta - \dfrac{c_2(v^*)^2}{(\hat{u} + k_2)^2}\xi = \mu\eta, & x \in \Omega, \\ \xi = \eta = 0, & x \in \partial\Omega. \end{cases} \tag{4.8}$$

Since $\xi, \eta \in C^1(\bar{\Omega})$, by the regularity theory of elliptic equations [18], we get $(\xi, \eta) \in C^{2+\alpha}(\bar{\Omega}) \times C^{2+\alpha}(\bar{\Omega})$ for some $\alpha \in (0, 1)$ and (μ, ξ, η) satisfies (4.8) in the classical sense. Furthermore μ is a real number with $\mu \le 0$.

If $\xi \not\equiv 0$, we can see that μ is an eigenvalue of the following problem

$$\begin{cases} -\Delta\phi - \left(a_1 - 2b\hat{u}\right)\phi = \mu\phi, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega. \end{cases} \tag{4.9}$$

So we have $0 \ge \mu \ge \rho_1\left(-\Delta + 2b\hat{u} - a_1\right) > \rho_1\left(-\Delta + b\hat{u} - a_1\right) = 0$, which is a contradiction. On the other hand, if $\xi \equiv 0, \eta \not\equiv 0$, then μ is an eigenvalue of the following problem

$$\begin{cases} -\Delta\phi - \left(a_2 - \dfrac{2c_2}{\hat{u} + k_2}v^*\right)\phi = \mu\phi, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega. \end{cases} \tag{4.10}$$

So, $0 \ge \mu \ge \rho_1\left(-\Delta + \frac{2c_2}{\hat{u}+k_2}v^* - a_2\right) > \rho_1\left(-\Delta + \frac{c_2}{\hat{u}+k_2}v^* - a_2\right) = 0$ since v^* is the unique solution of problem (4.1), again a contradiction. This shows that the stated result holds. □

Now we are ready to prove Theorem 4.1 by using Lemmas 2.4 and 4.2.

Proof of Theorem 4.1. The existence of a positive solution easily follows from Theorem 2.1 as since $a_1 > \lambda_1$ is fixed and $\rho_1\left(-\Delta + \frac{c_1}{k_1}\hat{v}\right) \to \lambda_1$ as $c_1/k_1 \to 0$. Hence we only need to show the uniqueness and local stability. Recall that G is the operator defined in (2.6) and $\mathcal{O}$ is the region that positive solutions lie in. By the compactness, G has at most finitely many positive fixed points in the region $\mathcal{O}$. We denote all the positive fixed points of G in $\mathcal{O}$ by (u_i, v_i) for $i = 1, 2, \ldots, \ell$. From part 2 of Lemma 4.2, we have $\text{index}_W(F, (u_i, v_i)) = 1$ for each $i \in \{1, 2, \ldots, \ell\}$. According to the additive property of Leray–Schauder degree, we get

$$\begin{aligned} 1 &= \text{index}_W(F, \mathcal{O}) \\ &= \text{index}_W(F, (0,0)) + \text{index}_W(F, (\hat{u}, 0)) + \text{index}_W(F, (0, \hat{v})) + \sum_{i=1}^{l} \text{index}_W(F, (u_i, v_i)) \\ &= 0 + 0 + 0 + \ell = \ell. \end{aligned}$$

Hence $\ell \equiv 1$ which asserts the uniqueness. The local stability has been proved in Lemma 4.2. □

The next theorem uses some specific inequalities on parameters to ensure the existence of a unique positive solution of problem (1.1).

Theorem 4.3. *Assume $a_1 > \lambda_1 + \frac{a_1 c_1}{k_1 \kappa}$, where $\kappa = \frac{bc_2}{a_1 + bk_2}$. Let*

$$\Lambda = \max\left\{\sup_{x\in\Omega}\frac{\hat{u}(x)}{\hat{v}(x)},\ \sup_{x\in\Omega}\frac{\theta(a_2, \kappa)(x)}{\theta\left(a_1 - \frac{a_2 c_1}{k_1\kappa}, b\right)(x)}\right\}. \tag{4.11}$$

Then problem (1.1) has a unique positive solution provided that

$$a_2 c_1 < bk_1^2\kappa, \tag{4.12}$$

and

$$\left[\frac{c_1^2}{k_1^2} + \frac{c_2^2}{k_2^4} - \frac{2c_1c_2}{k_2}\max\left\{\frac{1}{k_1}, \frac{1}{k_2}\right\} + \frac{4c_1c_2}{k_1}\min\left\{\frac{1}{k_1}, \frac{1}{k_2}\right\}\right]\Lambda \le \frac{4bc_2}{k_2}. \tag{4.13}$$

Proof. The existence of a positive solution follows from the first result of Theorem 2.1 since $a_1 > \lambda_1 + \frac{a_1c_1}{k_1\kappa} \ge \rho_1\left(-\Delta + \frac{c_1}{k_1}\hat{v}\right)$ and $a_2 > \lambda_1$. We only have to prove the uniqueness. Assume that (1.1) has two different positive solutions (u_1, v_1) and (u_2, v_2). Let $A = u_1 - u_2$ and $B = v_1 - v_2$, then $A \not\equiv 0$ or $B \not\equiv 0$ and (A, B) satisfies

$$\begin{cases} -\Delta A - \left(a_1 - bu_1 - \dfrac{c_1 v_1}{u_1 + k_1}\right)A + bu_2 A - \dfrac{c_1 u_2 v_2}{(u_1 + k_1)(u_2 + k_1)}A + \dfrac{c_1 u_2}{u_1 + k_1}B = 0, & x \in \Omega, \\ -\Delta B - \left(a_2 - \dfrac{c_2 v_2}{u + k_2}\right)B + \dfrac{c_2 v_1}{u_2 + k_2}B - \dfrac{c_2 v_1^2}{(u_1 + k_2)(u_2 + k_2)}A = 0, & x \in \Omega, \\ A = B = 0, & x \in \partial\Omega. \end{cases} \tag{4.14}$$

Since (u_1, v_1) is a positive solution of (1.1), we get $\rho_1\left(-\Delta - \left(a_1 - bu_1 - \frac{c_1 v_1}{u_1+k_1}\right)\right) = 0$. From the variational characterization of the principal eigenvalue, we have

$$\int_\Omega \left[-\Delta\xi - \left(a_1 - bu_1 - \frac{c_1 v_1}{u_1+k_1}\right)\xi\right]\xi dx \geq 0, \quad \text{for any } \xi \in H^2(\Omega)\cap H_0^1(\Omega). \tag{4.15}$$

In particular, (4.15) holds for $\xi = A$. Then combine the first equation of (4.14) with (4.15), we obtain

$$\int_\Omega \left[\left(b - \frac{c_1 v_2}{(u_1+k_1)(u_2+k_2)}\right)u_2 A^2 + \frac{c_1 u_2 AB}{u+k_1}\right] dx \leq 0. \tag{4.16}$$

Similarly, we get

$$\int_\Omega \left[\frac{c_2 v_1 B^2}{u_2+k_2} - \frac{c_2 v_1^2 AB}{(u_1+k_2)(u_2+k_2)}\right] dx \leq 0. \tag{4.17}$$

Let

$$I := \int_\Omega \left[\left(b - \frac{c_1 v_2}{(u_1+k_1)(u_2+k_1)}\right)u_2 A^2 + \left(\frac{c_1 u_2}{u_1+k_1} - \frac{c_2 v_1^2}{(u_1+k_2)(u_2+k_2)}\right)AB + \frac{c_2 v_1}{u_2+k_2}B^2\right] dx.$$

From (4.16) and (4.17), we obtain that

$$I \leq 0. \tag{4.18}$$

In the following, we will prove $I > 0$ to get a contradiction, if stated conditions are satisfied. From Lemma 2.2, we have

$$\theta\left(a_1 - \frac{a_1 c_1}{k_1\kappa}, b\right)(x) \leq u_i(x) \leq \hat{u}(x), \qquad \hat{v}(x) \leq v_i(x) \leq \theta(a_2,\kappa)(x), \quad x \in \Omega. \tag{4.19}$$

Since $\frac{c_1 v_2}{(u_1+k_1)(u_2+k_1)} \leq \frac{a_2 c_1}{k_1^2\kappa} < b$, we get that

$$b - \frac{c_1 v_2}{(u_1+k_1)(u_2+k_1)} > 0. \tag{4.20}$$

Let

$$\triangle := \left(\frac{c_1 u_2}{u_1+k_1} - \frac{c_2 v_1^2}{(u_1+k_2)(u_2+k_2)}\right)^2 - 4\left(b - \frac{c_1 v_2}{(u_1+k_1)(u_2+k_1)}\right)\frac{c_2 u_2 v_1}{u_2+k_1}.$$

By using the estimates in (4.19), we obtain that

$$\begin{aligned}\triangle &= u_2 v_1\left[\frac{c_1^2 u_2}{v_1(u_1+k_1)^2} + \frac{c_2^2 v_1}{u_2(u_1+k_2)^2(u_2+k_2)^2} - \frac{2c_1c_2v_1}{(u_1+k_1)(u_1+k_2)(u_2+k_2)}\right.\\ &\qquad \left. - \frac{4bc_2}{u_2+k_2} + \frac{4c_1c_2v_2}{(u_1+k_1)(u_2+k_1)(u_2+k_2)}\right]\\ &< u_2 v_1\left\{\left[\frac{c_1^2}{k_1^2} + \frac{c_2^2}{k_2^4} - \frac{2c_1c_2}{k_2}\max\left\{\frac{1}{k_1},\frac{1}{k_2}\right\} + \frac{4c_1c_2}{k_1}\min\left\{\frac{1}{k_1},\frac{1}{k_2}\right\}\right]\Lambda - \frac{4bc_2}{k_2}\right\}\\ &\leq 0.\end{aligned} \tag{4.21}$$

Combining (4.20), (4.21) with $A \not\equiv 0$ or $B \not\equiv 0$, we obtain that

$$I > 0, \tag{4.22}$$

which contradicts with (4.18). The proof is completed. □

For the quantities in Theorem 4.3, we remark that since $a_1 > \lambda_1 + \frac{a_1 c_1}{k_1\kappa}$ and $a_2 > \lambda_1$, $\theta\left(a_1 - \frac{a_1 c_1}{k_1\kappa}, b\right)(x)$ and $\theta(a_2,\kappa)(x)$ exist and both of them are positive in Ω. Thus the condition (4.11) is well-defined. Moreover the conditions (4.12) and (4.13) are satisfied if c_1/k_1 is sufficiently small, which is consistent with Theorem 4.1, but the result in Theorem 4.3 gives a more explicit estimate for the small constant δ in Theorem 4.1.

5. Conclusions

In this paper we prove some further existence, multiplicity results for the steady state equation (1.1) of the diffusive Leslie–Gower predator–prey model with Holling-type II functional responses and Dirichlet boundary condition (1.4). To

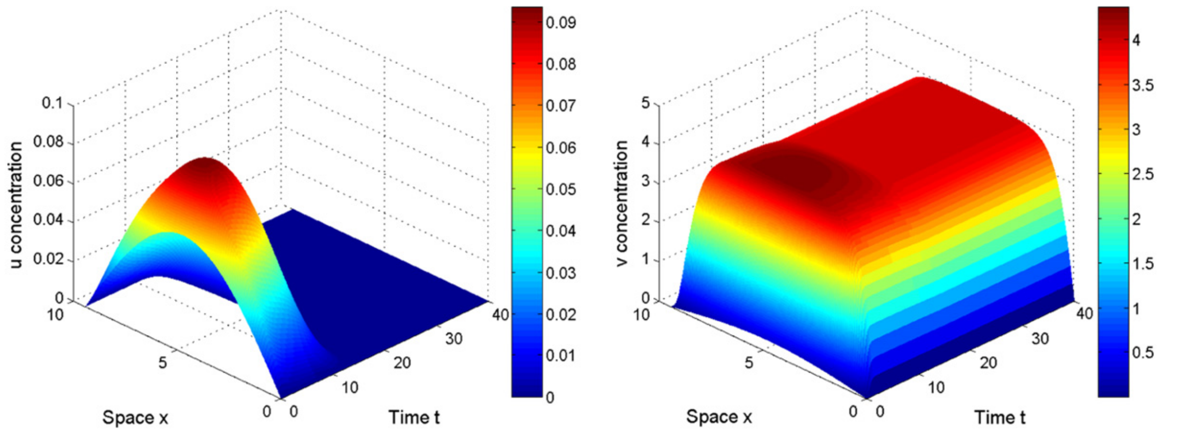

Fig. 2. Numerical simulation of the system (1.4) with $N = 1$, $\Omega = (0, 3\pi)$, parameters $d_1 = d_2 = 1$, $c_1 = k_1 = 0.01$, $b = c_2 = k_2 = 1$, $a_1 = 0.7$, $a_2 = 4$ and initial values $u_0(x) = 0.06\sin(x/3)$, $v_0(x) = 0.3\sin(x/3)$. As $t \to \infty$, the solution converges to the semi-trivial steady state solution $(0, \hat{v}(x))$.

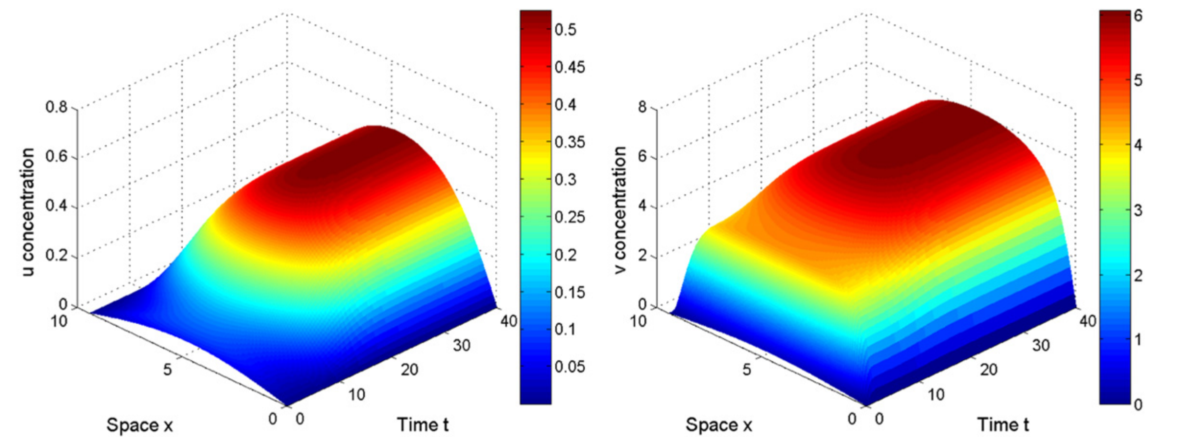

Fig. 3. Numerical simulation of the system (1.4) with $N = 1$, $\Omega = (0, 3\pi)$, parameters $d_1 = d_2 = 1$, $c_1 = k_1 = 0.01$, $b = c_2 = k_2 = 1$, $a_1 = 0.7$, $a_2 = 4$ and initial values $u_0(x) = 0.07\sin(x/3)$, $v_0(x) = 0.3\sin(x/3)$. As $t \to \infty$, the solution converges to a positive steady state solution.

illustrate the multiplicity results shown in Section 3, we show an example of numerical simulation in Figs. 2 and 3. In both figures, we consider the one-dimensional spatial domain $\Omega = (0, 3\pi)$, so the principal eigenvalue $\lambda_1 = 1/9 \approx 0.11$. For the parameters we choose $d_1 = d_2 = 1$, $c_1 = k_1 = 0.01$, $b = c_2 = k_2 = 1$, $a_1 = 0.7$ and $a_2 = 4$, so $a_1 > \lambda_1$ and $a_2 > \lambda_1$ are satisfied, and following the remark at the end of Section 3, we choose c_1, k_1 small but c_1/k_1 much larger. One can indeed show that $\tilde{a}_2 < 0$ in this case. In Fig. 2, we use the initial condition $u_0(x) = 0.07\sin(x/3)$, $v_0(x) = 0.3\sin(x/3)$, and in Fig. 3, we use $u_0(x) = 0.07\sin(x/3)$ and the same v_0. Then one can see that a small difference in the initial prey populations triggers a drastic difference of asymptotic fates, and it also shows that there exist two stable asymptotic states in this case. The existence of at least two stable steady states has profound impact on the ecological conservation, as a sudden collapse of the ecosystem is more likely to occur in such systems [20,35,37].

The bistability of the dynamics of (1.4) only holds when the parameters are carefully chosen. In Section 3, we have shown that if the parameters $a_1 > \lambda_1$, b, c_2, $k_2 > 0$ are fixed, $c_1, k_1 > 0$ are small and $c_1/k_1 \geq K > 0$, then the bistability occurs for $a_2 \in (a_2^*, a_2^{**})$ for some a_2^*, a_2^{**} satisfying $0 < a_2^* < a_2^{**}$. For many other choices of parameters, the system (1.1) may have zero or one positive solution; see the related results in [43] and the uniqueness result in Section 4. In particular, the uniqueness result in Section 4 requires $0 < c_1/k_1 \leq \delta$ for some $\delta > 0$. Thus the bistability and uniqueness results indicate that the quantity c_1/k_1 may play an important role in the number of positive solutions of (1.1).

We also comment that for the case $k_2 = 0$ (the Holling–Tanner model), time-periodic orbits of (1.4) with Neumann boundary conditions have been found [26]. The periodic orbits are usually found through Hopf bifurcations, but Hopf bifurcations from spatially nonhomogeneous steady state solutions are usually difficult to obtain because of the inhomogeneity of the steady state solutions. Thus the existence of time-periodic orbits of (1.4) with Dirichlet boundary condition remains an open question.

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