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Existence and uniqueness of steady state solutions of a nonlocal diffusive logistic equation

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Abstract. In this paper, we consider a dynamical model of population biology which is of the classical Fisher type, but the competition interaction between individuals is nonlocal. The existence, uniqueness, and stability of the steady state solution of the nonlocal problem on a bounded interval with homogeneous Dirichlet boundary conditions are studied.

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1. Introduction

Fisher–KPP equation is a classical reaction-diffusion equation describing the dispersal and evolution of organisms in space, which was introduced by Fisher [13] and Kolmogoroff et al. [26] in 1937. It assumes passive diffusion and a logistic growth, thus takes the form:

$$\frac{\partial u(x,t)}{\partial t} = D\Delta u(x,t) + au(x,t) - bu^2(x,t), \tag{1.1}$$

where $u(x,t)$ is the concentration of the organism at location x and $t > 0, D$ is the coefficient of the diffusion process, a is the rate of population growth, and b reflects the crowding effect, which is associated with intraspecific competition for resources. It describes the temporal and spatial evolution of the density of individuals such as bacteria or rodents. The processes addressed are birth and death, nonlinear competition controlled by the environment leading to saturation of the population, and diffusion leading to spatial homogenization. These three respective processes are associated with parameters a, b, and D. The Fisher–KPP equation (1.1) describes the interaction between several important processes such as reproduction, competition for resources and diffusion, thus it is close to represent some universal natural principles and it has many applications in biology, physics, and chemistry [25,27,29].

The classical Fisher–KPP equation (1.1) only considers local effects in the competition term since typically the competition interaction occurs locally in space, that is, the individuals are assumed to compete for resource in their immediate neighborhood. However, in the reality, individuals sometimes compete for resource not only in their immediate neighborhood but also in a more board domain. For that reason, many people begin to generalize the equation by incorporating nonlocal effects in the competition term. Here, we consider a generalized Fisher–KPP equation on a spatial domain in the following form, which was introduced in [17,18,24]:

$$\frac{\partial u(x,t)}{\partial t} = D\Delta u(x,t) + au(x,t) - bu(x,t)\int\limits_{\Omega} f(x,y)u(y,t)\,\mathrm{d}y, \tag{1.2}$$

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where $f(x, y)$ is a nonnegative distribution function which we call the influence function, and Ω is the domain for the nonlocal interaction. According to the expression of this equation, we find that it features competitive interactions linking the density $u(x, t)$ at point x with the density $u(y, t)$ at another point y through an influence function $f(x, y)$.

Here, we restrict our attention only to the one-dimensional problem where $\Omega = (-L, L)$ for $L > 0$, and the exterior of the domain is hostile thus a zero boundary condition is imposed:

$$\begin{cases} u_t = Du_{xx} + au - bu \displaystyle\int_{-L}^{L} f(x, y)u(y, t)\mathrm{d}y, & x \in (-L, L),\ t > 0, \\ u(-L, t) = u(L, t) = 0, & t > 0, \\ u(x, 0) = g_1(x), & x \in (-L, L). \end{cases} \tag{1.3}$$

By using the dimensionless new variables

$$s = \frac{Dt}{L^2},\ \ z = \frac{x}{L},\ \ v = \frac{bLu}{a}, \tag{1.4}$$

we obtain the dimensionless equation (while still using the variables t, x, u):

$$\begin{cases} u_t = u_{xx} + \lambda u - \lambda u \displaystyle\int_{-1}^{1} f(x, y)u(y, t)\mathrm{d}y, & x \in (-1, 1),\ t > 0, \\ u(-1, t) = u(1, t) = 0, & t > 0, \\ u(x, 0) = g(x), & x \in (-1, 1), \end{cases} \tag{1.5}$$

where

$$\lambda = \frac{aL^2}{D},\ \ g(x) = g_1(xL) \cdot \frac{bL}{a}.$$

In the following, we shall only consider (1.5) where $\lambda > 0$, and we assume that the influence function $f : \Omega \times \Omega \to \mathbb{R}$ satisfies [here $\Omega = (-1, 1)$]:

(f1) $f \in L^2(\Omega \times \Omega)$, and $f(x, y) \geq 0$ for almost all $(x, y) \in \Omega \times \Omega$.

We notice that in applications, f often only depends on the distance $|x - y|$, hence an additional stronger condition on f is

(f2) $f(x, y) = f_1(|x - y|)$, where $f_1 : [0, 2] \to (0, \infty)$ is nondecreasing and piecewise continuous, and $\int_0^2 f_1(y)\mathrm{d}y > 0$.

Our main result is that under the mild condition (f1), the set of steady state solutions of (1.5) possesses a similar global bifurcation structure as that of classical Fisher–KPP equation in one-dimensional case. Moreover, we show that the positive steady state solution must be stable and unique, again similar to the classical case. On the other hand, through an explicit example, we show that steady state solutions of (1.5) are not uniformly bounded as the classical case, and even for the positive solution case, the *a priori* bound may depend on the kernel function. It is not clear to us whether the mild condition (f1) is adequate to guarantee the existence of a positive steady state solution. But under the stronger condition (f2), the existence can be proved (see details in Sect. 3). Moreover, under (f2), we can show that the unique positive solution is symmetric and strictly decreasing on $(0, 1)$.

Nonlocal integral–differential equations as models of biological dispersal and interaction have been a focus of recent research, see excellent reviews in [3,16,22]. Nonlocal models arising from phase transitions in material sciences were considered in [4–7], and the existence of stationary and traveling wave patterns were proved under various assumptions. Spectral properties of the nonlocal integral–differential operators were studied in [9,10,12,14,15], as well as some bifurcation problems of nonlocal equations, see also [1,19]. In [2,21,34], the traveling wave solutions of nonlocal integral–differential equations were constructed.

In Sect. 2, we prove bifurcation results for the steady state solutions, and we also give explicit steady state solutions for some special kernels. And, we prove the stability thus uniqueness of the positive steady state solution in Sect. 3, and we also prove the existence and profile of the positive steady state solution under the stronger condition (f2).

2. Bifurcation of steady state solutions

To illustrate the global picture of the set of steady state solutions of (1.5), we first consider a simple case that $f(x,y)\equiv 1$, then the equation takes the form:

$$\begin{cases} u_t = u_{xx} + \lambda u - \lambda u \displaystyle\int_{-1}^{1} u(y)\mathrm{d}y, & x\in(-1,1),\ t>0,\\ u(-1,t)=u(1,t)=0, & t>0,\\ u(x,0)=g_1(x), & x\in(-1,1), \end{cases} \tag{2.1}$$

The steady state solution equation of (2.1) is

$$\begin{cases} u''(x) + \lambda u(x)\left(1-\displaystyle\int_{-1}^{1} u(y)\mathrm{d}y\right)=0, & x\in(-1,1),\\ u(-1)=u(1)=0. \end{cases} \tag{2.2}$$

Indeed, we notice that the problem $u''+\lambda(1-k)u=0$ with $u(-1)=u(1)=0$ has a positive solution only when $\lambda(1-k)=\lambda_1=\dfrac{\pi^2}{4}$ from the linear eigenvalue problem, and the positive solution $u(x)$ must be $k_1\cos\dfrac{\pi x}{2}$. Thus from calculation, we find that for fixed $\lambda>0$, the unique positive solution of (2.2) is

$$u(x)=k_1\cos\frac{\pi x}{2},\quad \text{where}\quad k_1=\frac{(4\lambda-\pi^2)\pi}{16\lambda}. \tag{2.3}$$

Apparently, $u(x)>0$ only when $\lambda>\dfrac{\pi^2}{4}$, and moreover, $k_1\to\dfrac{\pi}{4}$ when $\lambda\to\infty$. So $u(x)<\dfrac{\pi}{4}$ for any positive solution of (2.2). In fact, for each odd number n, we can obtain all solutions of (2.2) when $\lambda(1-k)=\lambda_n=\dfrac{n^2\pi^2}{4}$, which are

$$u_n(x)=k_n\cos\frac{n\pi x}{2},\quad \text{where}\quad k_n=\frac{(4\lambda-n^2\pi^2)n\pi}{16\lambda},\quad n=1,3,5,\ldots, \tag{2.4}$$

and the corresponding transcritical bifurcation point where $\{(\lambda,u_n(x))\}$ crosses the line of trivial solutions is $\lambda(1-k)=\lambda_n=\dfrac{n^2\pi^2}{4}$. On the other hand, if n is even, since $\int_{-1}^{1}\cos(n\pi x/2)\mathrm{d}x=0$, then we have a line of solutions $(\lambda=\lambda_n=\dfrac{n^2\pi^2}{4}, u=k\cos\dfrac{n\pi x}{2})$, for any $k\in\mathbb{R}$. Hence, the steady state solutions of (2.1) are all explicitly solved. In the next section, we study the stability of the steady state solutions (Fig. 1).

In fact, the explicit form of solutions to (2.2) can be extended to all $f(x,y)\equiv f_2(y)=f_2(|y|)$. Let the Fourier series of f_2 on $(-1,1)$ be

$$f_2(y)=\sum_{n=1}^{\infty}c_n\cos\frac{n\pi y}{2},\quad \text{where}\quad c_n=\int_{-1}^{1}\cos\frac{n\pi y}{2}f_2(y)\mathrm{d}y.$$

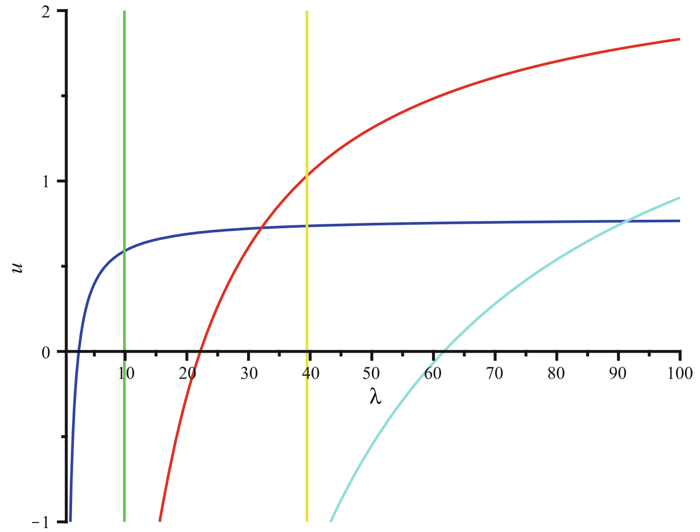

Fig. 1. Bifurcation diagram of solutions to (2.2): the *horizontal axis* is λ, the *vertical axis* is k, and the five curves are $(\lambda, u_i(x))$ bifurcating from λ_i with $i = 1, 2, 3, 4, 5$

Then for each $n \in \mathbb{N}$, the equation

$$\begin{cases} u''(x) + \lambda u(x)\left(1 - \displaystyle\int_{-1}^{1} f_2(y)u(y)\mathrm{d}y\right) = 0, \quad x \in (-1,1), \\ u(-1) = u(1) = 0, \end{cases} \tag{2.5}$$

has a curve of solutions $(\lambda, u_n(x))$ where

$$u_n(x) = k_n \cos\frac{n\pi x}{2}, \quad \text{and} \quad k_n = \frac{4\lambda - n^2\pi^2}{4\lambda \displaystyle\int_{-1}^{1} \cos\frac{n\pi y}{2} f_2(y)\mathrm{d}y}, \tag{2.6}$$

if $\int_{-1}^{1} \cos\frac{n\pi y}{2} f_2(y)\mathrm{d}y \neq 0$; and if $\int_{-1}^{1} \cos\frac{n\pi y}{2} f_2(y)\mathrm{d}y = 0$, then (2.5) has a vertical line of solutions $(\lambda_n, k\cos\frac{n\pi x}{2})$ for $k \in \mathbb{R}$. From this rather special (2.5), we can see that the set of steady state solutions of the nonlocal equation is curves in the space $(\lambda, u) \in \mathbb{R}^+ \times C^2[-1,1]$, and these curves (continua) in general are not bounded.

Motivated by the explicit solvable example above, we consider the general steady state equation of (1.5):

$$\begin{cases} u''(x) + \lambda u(x)\left(1 - \displaystyle\int_{-1}^{1} f(x,y)u(y)\mathrm{d}y\right) = 0, \quad x \in (-1,1), \\ u(-1) = u(1) = 0. \end{cases} \tag{2.7}$$

Obviously, $u = 0$ is a trivial steady state solution of (1.5) for any $\lambda > 0$. First, we show the nonexistence of the positive solutions of (2.7) for λ small

Theorem 2.1. *Suppose that f satisfies (f1). Then the problem (2.7) has no positive solutions when $\lambda \leq \pi^2/4$.*

Proof. We prove the theorem by contradiction. We assume that there exists a positive solution u_1 of (2.7) for $\lambda \le \pi^2/4$. Then, we have

$$\begin{cases} u_1''(x) + \lambda u_1(x)\left(1 - \int_{-1}^{1} f(x,y)u_1(y)\mathrm{d}y\right) = 0, & x \in (-1,1), \\ u_1(-1) = u_1(1) = 0. \end{cases} \tag{2.8}$$

Let (μ_1, ϕ_1) be the principle eigenpair of

$$\begin{cases} \phi'' + \mu\phi = 0, & x \in (-1,1), \\ \phi(-1) = \phi(1) = 0, \end{cases} \tag{2.9}$$

such that $\phi_1 > 0$, then

$$\begin{cases} \phi_1'' + \mu_1\phi_1 = 0, & x \in (-1,1), \\ \phi_1(-1) = \phi_1(1) = 0. \end{cases} \tag{2.10}$$

We can easily see that $\mu_1 = \pi^2/4$. Multiplying the equation in (2.8) by ϕ_1 and (2.10) by u_1, and subtracting and integrating from -1 to 1, then we obtain

$$\int_{-1}^{1} (\mu_1 - \lambda)\phi_1(x)u_1(x)\mathrm{d}x = \int_{-1}^{1} \lambda u_1(x)\phi_1(x) \int_{-1}^{1} f(x,y)u_1(y)\mathrm{d}y\mathrm{d}x. \tag{2.11}$$

Since $\phi_1 > 0, u_1 > 0$, $\lambda < \mu_1 = \pi^2/4$, then we find that the left hand side of (2.11) is < 0, and the right hand side is > 0, which is a contradiction. So (2.7) has no positive solution for $\lambda \le \pi^2/4$. □

For the existence of positive solutions and other solutions of (2.7), we use the local and global bifurcation theory. We recall the following abstract bifurcation theorem (see [8,28,31–33]):

Theorem 2.2. *Let X, Y be Banach spaces, and let $F : \mathbb{R} \times X \to Y$ be continuously differentiable. Suppose that $F(\lambda, u_0) = 0$ for $\lambda \in \mathbb{R}$, the partial derivative $F_{\lambda u}$ exists and is continuous. At $(\lambda_0, u_0), F$ satisfies*

(F1) *$dim N(F_u(\lambda_0, u_0)) = codim R(F_u(\lambda_0, u_0)) = 1$, and $N(F_u(\lambda_0, u_0)) = span\{w_0\}$.*

(F3) *$F_{\lambda u}(\lambda_0, u_0)[w_0] \notin R(F_u(\lambda_0, u_0))$, where $w_0 \in N(F_u(\lambda_0, u_0))$.*

Then

1. *The solutions of $F(\lambda, u) = 0$ near (λ_0, u_0) consists precisely of the curves $u = u_0$ and $(\lambda(s), u(s)), s \in I = (-\delta, \delta)$, where $(\lambda(s), u(s))$ are continuously differentiable functions such that $\lambda(0) = \lambda_0, u(0) = u_0, u'(0) = w_0$. Moreover, if F is C^2 in u, then $\lambda(s)$ is differentiable, and*
$$\lambda'(0) = -\frac{\langle l, F_{uu}(\lambda_0, u_0)[w_0, w_0]\rangle}{2\langle l, F_{\lambda u}(\lambda_0, u_0)[w_0]\rangle}, \tag{2.12}$$
where $l \in Y^*$ such that $R(F_u(\lambda_0, u_0)) = \{v \in Y : \langle l, v\rangle = 0\}$.
2. *If in addition, $F_u(\lambda, u)$ is a Fredholm operator for all $(\lambda, u) \in \mathbb{R} \times X$, then the curve $\{(\lambda(s), u(s)) : s \in I\}$ is contained in $\mathcal{C}$, which is a connect component of $\bar{S}$ where $S = \{(\lambda, u) \in \mathbb{R} \times X : F(\lambda, u) = 0, u \ne u_0\}$; and either $\mathcal{C}$ is not compact or $\mathcal{C}$ contains a point (λ_*, u_0) with $\lambda_* \ne \lambda_0$.*

In Theorem 2.2, if $\lambda'(0) \ne 0$, then a transcritical bifurcation occurs; while $\lambda'(0) = 0$ and $\lambda''(0) \ne 0$, a pitchfork bifurcation occurs.

To put (2.7) into the framework of Theorem 2.3, we define $X = \{u \in C^2[-1,1] : u(\pm 1) = 0\}, Y = C[-1,1]$, and $F : \mathbb{R} \times X \to Y$ by

$$F(\lambda, u) = u'' + \lambda u \left(1 - \int_{-1}^{1} f(x,y)u(y)\mathrm{d}y\right). \tag{2.13}$$

Our main existence result is as follows:

Theorem 2.3. *Suppose that f satisfies (f1). Let S be the set of solutions to (2.7) with $u \neq 0$. Then for each $n \in \mathbb{N}$, there exists a connected component $\mathcal{C}_n$ of $\overline{S}$ satisfying*

1. $(\lambda_n, 0) \equiv (n^2\pi^2/4, 0) \in \mathcal{C}_n$;
2. *Near $(\lambda, u) = (\lambda_n, 0)$, the solution set of (2.7) consists of two parts: a line of constant solutions $(\lambda, 0)$ and a curve of nonconstant solutions $\{(\lambda_n(s), u_n(s)) : |s| < \delta_n\}$, such that $u_n(s) = s\cos(n\pi x/2) + sz_n(s)$, where $\lambda : I_n \to \mathbb{R}, z_n : I_n \to Z_n$ are C^1 functions, $I_n = (-\delta_n, \delta_n), Z_n$ is a subspace of Y orthogonal to span$\{\cos(n\pi x/2)\}, \lambda_n(0) = \lambda_n$ and $z(0) = 0$;*
3. *For each $(\lambda, u) \in \mathcal{C}_n$ and $(\lambda, u) \neq (\lambda_n, 0), u(x)$ has exactly $n-1$ simple zeroes in $(-1, 1)$, all zeroes of $u(x)$ in $[-1, 1]$ are simple, and $\mathcal{C}_n$ is unbounded in $\mathbb{R}^+ \times X$.*

Proof. For any $\lambda \in \mathbb{R}, F(\lambda, 0) = 0$. It is easy to verify that F is C^2 in λ and u. For any $\phi \in X$, we have $F_u(\lambda, 0)[\phi] = \phi'' + \lambda\phi$. Thus, the singular points of F_u are given by the equation (2.9). Hence, potential bifurcation points are given by $\lambda_n = n^2\pi^2/4$.

It is well known that $dim N(F_u(\lambda_n, 0)) = codim R(F_u(\lambda_n, 0)) = 1$. In fact

$$N(F_u(\lambda_n, 0)) = span\left\{\cos\frac{n\pi x}{2}\right\}, \quad \text{and} \quad R(F_u(\lambda_n, 0)) = \left\{v \in Y : \int_{-1}^{1} \cos\frac{n\pi x}{2} v(x)\mathrm{d}x = 0\right\}.$$

Hence, $F_u(\lambda_n, 0)$ is a Fredholm operator with index zero and (F1) holds.

We also have

$$F_{\lambda u}(\lambda_n, 0)\left[\cos\frac{n\pi x}{2}\right] = \cos\frac{n\pi x}{2}, \tag{2.14}$$

and

$$\int_{-1}^{1} \cos\frac{n\pi x}{2}\cos\frac{n\pi x}{2}\mathrm{d}x \neq 0, \tag{2.15}$$

then we have $F_{\lambda u}(\lambda_n, 0)[\cos(n\pi x/2)] \notin R(F_u(\lambda_n, 0))$, thus (F3) holds. Hence, a local bifurcation of $F(\lambda, u) = 0$ occurs at $(\lambda, u) = (\lambda_n, 0)$ (part 1 of Theorem 2.2), and the solution set of (2.7) near $(\lambda_n, 0)$ consists of the trivial solutions $\{(\lambda, 0)\}$ and a smooth curve $\Sigma_n = \{(\lambda_n(s), u_n(s) : |s| < \delta_n\}$. Moreover, the direction and type of the bifurcation can be determined by

$$\lambda_n'(0) = -\frac{\lambda_n \int_{-1}^{1}\int_{-1}^{1} f(x, y)w_n^2(x)w_n(y)\mathrm{d}x\mathrm{d}y}{\int_{-1}^{1} w_n^2(x)\mathrm{d}x}, \tag{2.16}$$

where $w_n(x) = \cos(n\pi x/2)$.

Now, we turn to the global bifurcation. In general, we have

$$F_u(\lambda, u)[\phi] = \phi''(x) + \lambda\phi(x)\left(1 - \int_{-1}^{1} f(x, y)u(y)\mathrm{d}y\right) - \lambda u(x)\int_{-1}^{1} f(x, y)\phi(y)\mathrm{d}y. \tag{2.17}$$

Since $F_1[\phi] = \phi'' + \lambda\phi\left(1 - \int_{-1}^{1} u(y)f(x, y)\mathrm{d}y\right)$ is a Fredholm operator with index zero, and $F_u(\lambda, u)$ is a rank-one (thus compact) perturbation of F_1, then $F_u(\lambda, u)$ is also a Fredholm operator with index zero. Therefore, the global bifurcation result in part 2 of Theorem 2.2 also holds: each local curve Σ_n bifurcating from $(\lambda, 0)$ is contained in a connected component $\mathcal{C}_n$ of $\overline{S}$ where $S = \{(\lambda, u) \in \mathbb{R}\times X : F(\lambda, u) = 0, u \neq 0\}$.

To obtain a more precise structure of the solution set, we follow the classical approach in Rabinowitz [30]. For $n \in \mathbb{N}$, let S_n^+ denote the set of functions $u \in X$ such that $u(x)$ has exactly $n-1$ simple zeroes in $(-1, 1)$, all zeroes of $u(x)$ in $[-1, 1]$ are simple, and $u(x)$ is positive in a deleted neighborhood of $x = -1$. Set $S_n^- = -S_n^+$ and $S_n = S_n^+ \bigcup S_n^-$. We claim that for $n \in \mathbb{N}$, the connected component $\mathcal{C}_n$ of $\overline{S}$ is contained in $(\mathbb{R}^+ \times S_n) \bigcup \{(\lambda_n, 0)\}$, and $\mathcal{C}_n$ is unbounded in $(\mathbb{R}^+ \times S_n) \bigcup \{(\lambda_n, 0)\}$. From part 1 of Theorem 2.3, near $(\lambda_n, 0)$, the solutions on $\Sigma_n \subset \mathcal{C}_n$ are in form of $u_n(s) = s\cos(n\pi x/2) + o(s)$, hence $\Sigma_n \subset (\mathbb{R}^+ \times S_n) \bigcup \{(\lambda_n, 0)\}$. If $\mathcal{C}_n \subset (\mathbb{R}^+ \times S_n) \bigcup \{(\lambda_n, 0)\}$, then it must be unbounded since it cannot contain another $(\lambda_j, 0)$ with $j \neq n$. If $\mathcal{C}_n \not\subset (\mathbb{R}^+ \times S_n) \bigcup \{(\lambda_n, 0)\}$, then there exists $(\overline{\lambda}, \overline{u}) \in \mathcal{C}_n \bigcap (\mathbb{R}^+ \times \partial S_n)$, $(\overline{\lambda}, \overline{u}) \neq (\lambda_n, 0)$, and $(\overline{\lambda}, \overline{u}) = \lim_{m\to\infty}(\lambda^m, u^m)$ with $(\lambda^m, u^m) \in \mathcal{C}_n \bigcap (\mathbb{R}^+ \times S_n)$. Then $\overline{u}$ must have a double zero, which implies that $\overline{u} \equiv 0$ from the uniqueness of solution to the ordinary differential equation $\overline{u}'' + \overline{\lambda}(1 - k(x))\overline{u} = 0$ where $k(x) = \int_{-1}^{1} f(x, y)\overline{u}(y)\mathrm{d}y$ (see Lemma 12 in [30]). This implies that $(\overline{\lambda}, \overline{u}) \neq (\lambda_j, 0)$ for some $j \neq n$, but that is impossible since $(\lambda^m, u^m) \in \mathcal{C}_n \bigcap (\mathbb{R}^+ \times S_n)$. This reaches a contradiction, hence $\mathcal{C}_n \subset (\mathbb{R}^+ \times S_n) \bigcup \{(\lambda_n, 0)\}$ and $\mathcal{C}_n$ is unbounded. □

Remark 2.4.
1. Following the unilateral global bifurcation theorem in [11] (see also [33]), it can be shown that sufficiently close to the bifurcation point, the connected component $\mathcal{C}_n$ can be locally decomposed into two sub-continua $\mathcal{C}_n^{\pm}$; moreover, in the sufficiently small neighborhood of bifurcation point, the two sub-continua must coincide with the branch of solutions given in above theorem. Furthermore, the solutions on these sub-continua have the required sign. Again, following the argument in [11,33], the alternatives stated above for the component $\mathcal{C}_n$ can in fact be applied to each of the sub-continua $\mathcal{C}_n^{\pm}$.
2. We notice that the bifurcation at the first eigenvalue $\lambda = \lambda_1$ is always a transcritical bifurcation with $\lambda_1'(0) > 0$ from (2.16) since $w_1(x) > 0$ for $x \in (-1, 1)$ and $f \not\equiv 0$. But other bifurcation branches can even be vertical as we have shown for (2.5).

3. Stability, uniqueness, and existence of positive solution

The linear stability of an equilibrium solution u of (1.5) can be determined by the following linearized eigenvalue problem

$$\begin{cases} \phi''(x) + \lambda\phi(x) - \lambda\phi(x)\displaystyle\int_{-1}^{1} f(x, y)u(y)\mathrm{d}y \\ \quad -\lambda u(x)\displaystyle\int_{-1}^{1} f(x, y)\phi(y)\mathrm{d}y = \mu\phi(x), \qquad x \in (-1, 1), \\ \phi(-1) = \phi(1) = 0. \end{cases} \tag{3.1}$$

A solution u of (2.7) is stable if all eigenvalues of (3.1) are negative, otherwise it is unstable. In general, (3.1) is a nonlocal eigenvalue problem, but when $u = 0$, it becomes a standard eigenvalue problem, and we can easily deduce that $u = 0$ is linearly stable when $\lambda < \pi^2/4$, and $u = 0$ is unstable if $\lambda \geq \pi^2/4$.

We define for $\phi \in X$,

$$L[\phi] = \phi''(x) + \lambda\phi(x) - \lambda\phi(x)\int_{-1}^{1} f(x, y)u(y)\mathrm{d}y - \lambda u(x)\int_{-1}^{1} f(x, y)\phi(y)\mathrm{d}y. \tag{3.2}$$

We summarize some known results regarding the spectral properties of L:

1. With $D(L) = H_0^1(\Omega) \cap H^2(\Omega) \subset L^2(\Omega)$, L is a densely defined, closed, self-adjoint operator with compact resolvent, L is a Fredholm operator of index 0, the spectrum $\sigma(L)$ consists of isolated eigenvalues $\{\mu_i\}$ $(i = 1, 2, \cdots)$ each with finite multiplicity, so that $Re(\mu_1) > Re(\mu_2) \geq Re(\mu_3) \geq \cdots \to -\infty$, and L has only finitely many complex eigenvalues;

2. The adjoint operator L^* is defined by

$$L^*[\xi] = \xi''(x) + \lambda\xi(x) - \lambda\xi(x)\int_{-1}^{1} f(x,y)u(y)\mathrm{d}y - \lambda\int_{-1}^{1} f(x,y)u(y)\xi(y)\mathrm{d}y, \tag{3.3}$$

$\sigma(L^*) = \sigma(L)$, and an eigenvalue of L is also an eigenvalue of L^* with same multiplicity.
3. The principal eigenvalue μ_1 is real valued, μ_1 is a simple eigenvalue with a positive eigenfunction $\phi_1(x)$, and eigenfunctions corresponding to other eigenvalues are sign changing.

Parts 1 and 2 can be found in Freitas [14,15], and part 3 follows from a Krein–Rutman type result in Huang [23] (Propositions 2.3 and 2.4), or Davidson and Dodds [9] (Theorems 4.1 and 5.1). For the latter result, we use the space $Z = L^2(\Omega)$ and $D(L) = H_0^1(\Omega) \cap H^2(\Omega)$, then the standard positive cone Z_+ is generating and normal. One can easily check that L is a closed and resolvent positive operator (see definition in [23]), then Propositions 2.3 and 2.4 in [23] imply the results stated in part 3 above. The eigenfunctions in Z clearly also belong to X from the regularity theory of elliptic equations.

Our main stability result is for the positive steady state solutions:

Theorem 3.1. *Suppose that f satisfies (f1). Then any positive solution $u(x)$ of (2.7) for $\lambda > \pi^2/4$ is stable.*

Proof. Let $u(x)$ be a positive solution of (2.7), and let (μ_1, ϕ_1) and (μ_1, ξ_1) be the corresponding principal eigenpairs of L and L^*, respectively. We notice that u and ξ_1 satisfy the equations

$$u'' + \lambda u - \lambda u\int_{-1}^{1} f(x,y)u(y)\mathrm{d}y = 0, \quad u(\pm 1) = 0, \tag{3.4}$$

and

$$\xi_1'' + \lambda\xi_1 - \lambda\xi_1\int_{-1}^{1} f(x,y)u(y)\mathrm{d}y - \lambda\int_{-1}^{1} f(x,y)u(y)\xi_1(y)\mathrm{d}y = \mu_1\xi_1, \quad \xi_1(\pm 1) = 0. \tag{3.5}$$

Multiplying (3.5) by u and (3.4) by ξ_1, subtracting and integrating, we obtain

$$\mu_1\int_{-1}^{1} u(x)\xi_1(x)\mathrm{d}x = -\lambda\int_{-1}^{1}\int_{-1}^{1} f(x,y)u(x)u(y)\xi_1(y)\mathrm{d}x\mathrm{d}y. \tag{3.6}$$

Since $u(x) > 0$ and $\xi_1(x) > 0$ in $(-1, 1)$, then $\mu_1 < 0$ and the positive steady state solution u must be stable. □

The stability result implies the uniqueness of positive solution of (2.7).

Corollary 3.2. *Suppose that f satisfies (f1). Then for any $\lambda > \pi^2/4$, (2.7) has at most one positive solution.*

Proof. Suppose that $u_*(\lambda, x)$ is a positive solution of (2.7) for $\lambda > \pi^2/4$, then from Theorem 3.1, $u(\lambda, x)$ is stable thus nondegenerate. The implicit function theorem can be applied to (λ, u_*) to obtain a curve Σ of positive solutions in $\mathbb{R}_+ \times X$ containing (λ, u_*). We extend Σ to the maximum connected component of positive solutions in $\mathbb{R}_+ \times X$, then every solution on Σ is stable, and hence Σ is a curve parameterized by λ. But from Theorem 2.1, Σ cannot be extended to the left beyond $\lambda = \pi^2/4$. Therefore, Σ must connect to the trivial solution $u = 0$ at some $\lambda = \lambda_n$. Since all solutions on Σ are positive, then Σ can only connect to $u = 0$ at $\lambda = \lambda_1 = \pi^2/4$, and $\Sigma = S_1^+$ which is defined in the proof of Theorem 2.3. Since this argument works for any positive solution $u_*(\lambda, x)$, then any positive solution of (2.7) is on S_1^+. This also implies that S_1^+ is a curve parameterized by λ. □

Notice that we do not claim that for all $\lambda > \pi^2/4$, the equation (2.7) always has a positive solution. It is well known that if one can show that all positive solutions are *a priori* bounded, then S_1^+ can be extended to $\lambda = \infty$. But it is not clear whether such an *a priori* estimate holds for the positive solutions of (2.7) under condition (f1). We notice that from the explicit example in Sect. 2, the sign-changing solutions are in fact not uniformly bounded, unlike the corresponding reaction-diffusion equations. In the following, we prove that under a stronger condition (f2), more properties of positive solutions of (2.7) can be shown, and in particular, we can show the existence of a positive solution of (2.7) for all $\lambda > \pi^2/4$.

Theorem 3.3. *Suppose that f satisfies (f1) and (f2). Then*

1. *Any positive solution $u(\lambda, x)$ of (2.7) is an even function for $x \in (-1, 1)$, and $u(\lambda, x)$ is strictly decreasing in x for $x \in (0, 1]$.*
2. *For every $\lambda > \pi^2/4$, (2.7) has a unique positive solution $u(\lambda, x)$, and it satisfies*

$$f_1(0)\int_{-1}^{1} u(x)dx \le 1 - \frac{\pi^2}{4\lambda}. \tag{3.7}$$

Proof. For a positive solution u of (2.7), define $k(x) = \int_{-1}^{1} f_1(|x-y|)u(y)\mathrm{d}y$. If (f2) is satisfied, then it is easy to verify that whether $u(x)$ is a solution of (2.7), so is $u(-x)$. But from Corollary 3.2, the positive solution of (2.7) is unique, then $u(-x) = u(x)$ and hence any positive solution must be an even function.

Next, we prove $u'(x) < 0$ for $x \in (0, 1]$. From Hopf boundary lemma and the equation of $u, u'(1) < 0$, thus there exists $\delta > 0$ such that $u'(x) < 0$ in $(1-\delta, 1]$. We use the well-known moving plane method. For $\theta \in (0, 1)$, define $x^\theta = 2\theta - x$ and $u_\theta(x) = u(x^\theta)$. Let $w_\theta = u_\theta(x) - u(x)$. Then $w_\theta(x) > 0$ for $x \in (1-\theta, 1], w_\theta(\theta) = 0$ and $w'_\theta(\theta) < 0$ if $\theta > 0$ is small. Let

$$\theta_0 = \sup\{\theta > 0 : w_\mu(x) > 0,\ \ x \in (1-\mu, 1],\ \text{ and }\ w'_\mu(\mu) < 0\ \text{ for }\ \mu \in (0, \theta)\}.$$

If $\theta_0 < 1$, then at $\theta = \theta_0$, either $w_\theta(x) = 0$ for some $x \in (1-\theta, 1)$ or $w'_\theta(\theta) = 0$. But w_θ satisfies the equation:

$$w'' + \lambda[1 - k(x^\theta)]w + \lambda u\int_{-1}^{1}[f_1(|x-y|) - f_1(|x^\theta - y|)]u(y)\mathrm{d}y = 0.$$

From the condition (f2), one can show that

$$\int_{-1}^{1}[f_1(|x-y|) - f_1(|x^\theta - y|)]u(y)\mathrm{d}y > 0.$$

Therefore, from the maximum principle, neither $w_\theta(x) = 0$ for some $x \in (1-\theta, 1)$ nor $w'_\theta(\theta) = 0$ can occur. Hence, we must have $\theta_0 = 1$, and $u'(x) < 0$ for $x \in (0, 1]$.

We claim that $0 < k(x) < f_1(2)/f_1(0)$ for $x \in [-1, 1]$. From (f2), it is clear that $k(x) > 0$. From the last paragraph, we know that u achieves its maximum at $x = 0$ and from the maximal principle, $k(0) < 1$. From (f2), we obtain that

$$k(x) = \int_{-1}^{1} f_1(|x-y|)u(y)\mathrm{d}y \le f_1(2)\int_{-1}^{1} u(y)\mathrm{d}y$$
$$\le \frac{f_1(2)}{f_1(0)}\int_{-1}^{1} f_1(|y|)u(y)\mathrm{d}y = \frac{f_1(2)}{f_1(0)}k(0) < \frac{f_1(2)}{f_1(0)}.$$

Note that $k(x) \geq k(0)$ from (f2), and $k(0) < 1$ implies that

$$f_1(0) \int_{-1}^{1} u(x)\mathrm{d}x \leq k(0) < 1.$$

Multiplying (2.7) by u and integrating on $[-1, 1]$, we obtain that

$$\lambda \int_{-1}^{1} u^2(x)\mathrm{d}x = \int_{-1}^{1} [u'(x)]^2\mathrm{d}x + \lambda \int_{-1}^{1} k(x)u^2(x)\mathrm{d}x. \tag{3.8}$$

From $k(x) \geq k(0)$ and the Poincaré inequality, the equation (3.8) becomes

$$\lambda \int_{-1}^{1} u^2(x)\mathrm{d}x \geq \frac{\pi^2}{4} \int_{-1}^{1} u^2(x)\mathrm{d}x + \lambda f_1(0) \int_{-1}^{1} u(x)\mathrm{d}x \int_{-1}^{1} u^2(x)\mathrm{d}x,$$

and this is equivalent to (3.7).

At last we prove the existence of a positive solution $u(\lambda, x)$ for all $\lambda > \pi^2/4$ by showing that S_1^+ can be extended to $\lambda = \infty$. Suppose this is not true, then S_1^+ can be at most extended to some $\lambda^* > \pi^2/4$. We claim that if that is the case, then $||u(\lambda, \cdot)||_{L^2} \to \infty$ as $\lambda \to (\lambda^*)^+$. Indeed, if $||u(\lambda, \cdot)||_{L^2}$ is bounded, and we also know that $k(x)$ is bounded, then $||u(\lambda, \cdot)||_{H^1}$ is bounded from (3.8), and a weak solution would exist at $\lambda = \lambda^*$, so we can extend S_1^+ beyond λ^*, which is a contradiction.

We choose an increasing sequence $\lambda^n \to \lambda^*$ so that $||u(\lambda^n, \cdot)||_{L^2} \to \infty$, and we define $v_n(x) = u(\lambda^n, x)/||u(\lambda^n, \cdot)||_{L^2}$. Also, we define $k_n(x) = \int_{-1}^{1} f_1(|x-y|)u(\lambda^n, y)\mathrm{d}y$. So v_n satisfies

$$v_n'' + \lambda^n(1 - k_n(x))v_n = 0, \quad x \in (-1, 1), \quad v_n(\pm 1) = 0.$$

Since $||v_n||_{L^2} = 1, \{1 - k_n(x)\}$ is bounded, and $\lambda^n \to \lambda^*$, then $\{v_n\}$ is bounded in $H_0^1(-1, 1)$, and hence, $\{v_n\}$ has a subsequence converging strongly in $L^2(-1, 1)$ and weakly in $H_0^1(-1, 1)$ to a limit $v \in H_0^1(-1, 1)$, and $||v||_{L^2} = 1$. Moreover, from Sobolev embedding theorem, $v \in C^{\alpha}[-1, 1]$ for some $\alpha \in (0, 1), v(x)$ is nonnegative, $v(x)$ is even, and v is decreasing for $x \in (0, 1)$. Hence, there exist ϵ and $\delta > 0$ such that $v(x) > \epsilon$ when $|x| < \delta$. This implies that for n large, $k_n(0) \geq \dfrac{||u(\lambda^n, \cdot)||_{L^2}}{2} \int_{-\delta}^{\delta} f_1(|y|)v(y)\mathrm{d}y \to \infty$, which contradicts with the boundedness of k_n. Therefore, the assumption that $||u(\lambda, \cdot)||_{L^2} \to \infty$ as $\lambda \to (\lambda^*)^+$ does not hold, and S_1^+ can be extended to $\lambda = \infty$. □

Remark 3.4. The symmetry of the solution in Theorem 3.3 follows from the uniqueness of solution, and it does not rely on the condition (f2). But (f2) is needed for the monotonicity of the solution. Note that in the classical paper [20], a similar monotonicity condition on x is required for the function $f(x, u)$. But we conjecture that the monotonicity still holds without (f2).

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