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MULTIPLE POSITIVE SOLUTIONS FOR P-LAPLACIAN EQUATION WITH WEAK ALLEE EFFECT GROWTH RATE

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Abstract. A p-Laplacian equation with weak Allee effect growth rate and Dirichlet boundary condition is considered. The existence, multiplicity and bifurcation of positive solutions are proved with comparison and variational techniques. The existence of multiple positive solutions implies that the related ecological system may exhibit bistable dynamics.

1. Introduction

Consider a boundary-value problem

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda u^{p-1} g(x,u) = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{P_λ}$$

where $p > 1$ and λ is a nonnegative parameter, and $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain for $N \geq 1$.

The existence and multiplicity (or uniqueness) of positive solutions to (P_λ) have been considered by many people (see, e.g., [1, 5, 6, 7, 9, 10, 14, 15, 17, 20, 21, 22] and the references therein). In many previous studies, the nonlinear function $g(x,u)$ was assumed to be nonincreasing in u, and under this condition, it can be shown that (P_λ) has at most one positive solution (see, e.g., [1, 7, 9, 10]). On the other hand, the uniqueness of a positive solution no longer holds if $g(x,u)$ is not nonincreasing in u. In [21, 22], it was shown that (P_λ) has at least two positive solutions if $g(x,u) = g(u) =$

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$u^{q-p}(1-u^r)$, where $2 < p < q$, $r > 0$. In this case, $g(u)$ increases for small u and decreases for large u, $g(u) > 0$ for $u \in (0,1)$. In [18], the existence of a maximum positive solution was shown when $u^{p-1}g(x,u) = au^{p-1} - u^{\gamma-1} - c \cdot h(x)$ with $\gamma > p > 1$. In this case $g(x,u)$ is also not monotone.

The monotonicity of the function g also arises from the studies of ecological population models. In this context, $g(x,u)$ is the growth rate per capita of the population at $x \in \Omega$. When g is nonincreasing in u, then it is logistic-type growth; and when g is increasing for small u and decreases for larger u, then it is of an Allee effect growth. Here we follow the definitions and conditions on g in [20], which were motivated by ecological applications. Throughout this paper we assume that $g(x,u)$ satisfies the following conditions:

- $(g1)$ $g \in C(\overline{\Omega} \times [0,\infty), \mathbb{R})$;
- $(g2)$ For any $x \in \overline{\Omega}$, there exists $u_1(x) \geq 0$ such that $g(x,\cdot)$ is increasing in $[0, u_1(x)]$, $g(x,\cdot)$ is decreasing in $[u_1(x),\infty)$, and there exists $N_1 > 0$ such that $g(x,u_1(x)) \leq N_1$ for all $x \in \overline{\Omega}$;
- $(g3)$ For any $x \in \overline{\Omega}$, there exists $u_2(x) > u_1(x)$ such that $g(x,u_2(x)) = 0$, and there exists $M > 0$ such that $u_2(x) \leq M$ for all $x \in \overline{\Omega}$.

In addition $g(x,u)$ can take one of the following three forms (following [20]):

- $(g4a)$ **Logistic.** $g(x,0) > 0$, $u_1(x) = 0$, and $g(x,\cdot)$ is decreasing in $[0,\infty)$;
- $(g4b)$ **Weak Allee effect.** $g(x,0) \geq 0$, $g(x,0) \not\equiv 0$, $u_1(x) > \delta_1$ for some $\delta_1 > 0$, $g(x,\cdot)$ is increasing in $[0,u_1(x)]$, and $g(x,\cdot)$ is decreasing in $[u_1(x),\infty)$;
- $(g4c)$ **Strong Allee effect.** $g(x,0) < 0$, $u_1(x) > 0$, $g(x,u_1(x)) > 0$, $g(x,\cdot)$ is increasing in $[0,u_1(x)]$, and $g(x,\cdot)$ is decreasing in $[u_1(x),\infty)$.

In this paper, we consider the positive solutions of (P_λ) with g satisfying $(g1)$–$(g3)$ and $(g4b)$, that is, the weak Allee effect case. For the case of $p = 2$, Shi and Shivaji [20] considered the existence, multiplicity, and bifurcation of positive solutions of (P_λ) with Allee effect. The more general p-Laplacian case is considerably more difficult, as bifurcation theory based on linearization cannot be easily applied here due to the degeneracy of the p-Laplacian operator when $u = 0$. Here we combine the techniques from comparison methods, variational methods, and properties of p-Laplacian equations to prove the existence, multiplicity, and bifurcation of positive solutions of (P_λ) with Allee effect for $p > 1$.

In Section 2 we recall some basic setups and preliminary results, and in Section 3 we state and prove the main results. In the paper, we denote by $C^1(\overline{\Omega})$ the space of all continuously differentiable functions $u : \Omega \to \mathbb{R}$ such that it and all its partial derivatives possess a continuous extension up to the boundary. The closed linear subspace of $C^1(\overline{\Omega})$ consisting of all functions $u \in C^1(\overline{\Omega})$ with $u|_{\partial\Omega} \equiv 0$ is denoted by $C^1_0(\overline{\Omega})$. Given $0 < \gamma \leq 1$, we denote by $C^{1,\gamma}(\overline{\Omega})$ the Hölder space of all functions $u \in C^1(\overline{\Omega})$ such that it and all its partial derivatives are γ-Hölder continuous on $\overline{\Omega}$. The closed linear subspace of $C^{1,\gamma}(\overline{\Omega})$ consisting of all functions $u \in C^{1,\gamma}(\overline{\Omega})$ satisfying $u|_{\partial\Omega} \equiv 0$ is denoted by $C^{1,\gamma}_0(\overline{\Omega})$. Given $1 \leq r \leq \infty$, we denote by $L^r(\Omega)$ the Lebesgue space of all Lebesgue-measurable functions $u : \Omega \to \mathbb{R}$. We denote by $W^{1,r}(\Omega)$ the Sobolev space of all functions $u \in L^r(\Omega)$ for which all weak partial derivatives also belong to $L^r(\Omega)$. The norms $\|\cdot\|_{C^1}$ in $C^1(\overline{\Omega})$, $\|\cdot\|_{C^{1,\gamma}}$ in $C^{1,\gamma}(\overline{\Omega})$, $\|\cdot\|_r$ in $L^r(\Omega)$, and $\|\cdot\|_{W^{1,r}}$ in $W^{1,r}(\Omega)$ are defined in natural ways, respectively. Finally, for $1 \leq r < \infty$, the closure in $W^{1,r}(\Omega)$ of the set of all C^1 functions $u : \Omega \to \mathbb{R}$ with compact support is denoted by $W^{1,r}_0(\Omega)$, whose norm is $\|u\|_{W^{1,r}_0} = \|\nabla u\|_r$, and we denote by $W^{-1,r'}(\Omega)$ the dual space of $W^{1,r}_0(\Omega)$. Here $1/r + 1/r' = 1$.

2. Preliminaries

Consider a boundary-value problem

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) + f(x,u) = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{Q}$$

where $p > 1$, and suppose $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function which satisfies the growth condition

$$|f(x,s)| \leq C|s|^{q-1} + b(x) \quad \text{for } (x,s) \in \Omega \times \mathbb{R}, \tag{2.1}$$

where $C \geq 0$ is constant, $q > 1$, $b \in L^{q'}(\Omega)$, and $1/q + 1/q' = 1$.

Now we recall some well-known facts regarding the p-Laplacian operator (see, e.g., [8]). Define $-\Delta_p : W^{1,p}_0(\Omega) \to W^{-1,p'}(\Omega)$ by

$$\langle -\Delta_p u, v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v \, dx \quad \text{for all } u, v \in W^{1,p}_0(\Omega),$$

and define $N_f : W^{1,p}_0(\Omega) \to W^{-1,p'}(\Omega)$ by

$$\langle N_f u, v\rangle = \int_\Omega f(x,u) v \, dx \quad \text{for all } u, v \in W^{1,p}_0(\Omega).$$

Then the operator $-\Delta_p$ is a one-to-one correspondence between $W^{1,p}_0(\Omega)$ and $W^{-1,p'}(\Omega)$, with inverse $(-\Delta_p)^{-1}$ monotone, bounded, and continuous, and the operator $N_f : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is completely continuous if f satisfies (2.1) with $q \in (1, p^*)$. Here

$$p^* = \begin{cases} \dfrac{Np}{N-p}, & p < N, \\ \infty, & p \geq N. \end{cases}$$

A function $u \in W_0^{1,p}(\Omega)$ is said to be a solution of problem (Q) if

$$-\Delta_p u = N_f u \tag{2.2}$$

in the sense of $W^{-1,p'}(\Omega)$; i.e.,

$$\langle -\Delta_p u, v\rangle = \langle N_f u, v\rangle \quad \text{for all } v \in W_0^{1,p}(\Omega)$$

or

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v \, dx = \int_\Omega f(x,u) v \, dx \quad \text{for all } v \in W_0^{1,p}(\Omega).$$

Consequently, problem (2.2) can be equivalently written as

$$u = (-\Delta_p)^{-1} N_f u.$$

Note that the operator $(-\Delta_p)^{-1}N_f$ is completely continuous on $W_0^{1,p}(\Omega)$ if $q \in (1, p^*)$.

Define $\Phi : W_0^{1,p}(\Omega) \to \mathbb{R}$ by

$$\Phi(u) = \int_\Omega F(x, u(x)) \, dx \quad \text{for } u \in W_0^{1,p}(\Omega),$$

where $F : \Omega \times \mathbb{R} \to \mathbb{R}$ is defined by

$$F(x,s) = \int_0^s f(x,\tau)\, d\tau \quad \text{for } (x,s) \in \Omega \times \mathbb{R}.$$

Then Φ is continuously Fréchet differentiable on $W_0^{1,p}(\Omega)$ if $q \in (1, p^*)$, and

$$\Phi'(u)\phi = \int_\Omega f(x,u)\phi \, dx \quad \text{for } u, \phi \in W_0^{1,p}(\Omega).$$

Consequently, the functional $I : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{p}\int_\Omega |\nabla u|^p \, dx - \Phi(u) \quad \text{for } u \in W_0^{1,p}(\Omega)$$

is continuously Fréchet differentiable on $W_0^{1,p}(\Omega)$ if $q \in (1, p^*)$, and

$$I'(u)\phi = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla \phi \, dx - \int_\Omega f(x,u)\phi \, dx \quad \text{for all } \phi \in W_0^{1,p}(\Omega).$$

Thus critical points of I are solutions of (Q).

We define $u \in W^{1,p}(\Omega)$ to be a sub-solution to problem (Q) if $u \le 0$ on $\partial\Omega$ and

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\phi\,dx - \int_\Omega f(x,u)\phi\,dx \le 0 \quad \text{for all } \phi \in W_0^{1,p}(\Omega),\ \phi \ge 0.$$

Similarly, $u \in W^{1,p}(\Omega)$ is a super-solution to problem (Q) if in the above the reverse inequalities hold.

We recall the following existence result based on the super-/sub-solution method ([11, Theorem 4.11]):

Theorem 2.1. *Assume that $f(x,s)$ satisfies (2.1) with $q = p$, and assume that $\rho \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ is a sub-solution and $\psi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ is a super-solution to (Q) such that $\rho \le \psi$. Then (Q) has a minimal solution u_* and a maximal solution u^* in the order interval $[\rho,\psi]$ such that any solution u of (Q) in $[\rho,\psi]$ satisfies $u_* \le u \le u^*$.*

Now consider the nonlinear eigenvalue problem

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda m(x)|u|^{p-2}u = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{2.3}$$

where $m \in L^\infty(\Omega)$ with $m(x) \ge 0$ in Ω, and there exists $\Omega_0 \subseteq \Omega$ such that $|\Omega_0| > 0$ and $m(x) > 0$ for $x \in \Omega_0$. Then it is well known that the principal eigenvalue $\lambda_1(m)$ for problem (2.3) exists, and $\lambda_1(m) \in (0,\infty)$ is a simple isolated eigenvalue for problem (2.3) with associated eigenfunction $\phi_1(m) > 0$ in Ω (see, e.g., [4]). If $m(x) \equiv 1$, then we denote the principal eigenvalue and the associated positive eigenfunction by λ_1 and ϕ_1, respectively. Recall $g(x,u)$ is the function in the equation (P_λ), and in the following we denote $\lambda_1(g(x,0))$ and $\phi_1(g(x,0))$ by λ_1^* and ϕ_1^*, respectively.

3. Main result

From now on, we assume that $g(x,s)$ satisfies the conditions $(g1)$–$(g3)$ and $(g4b)$. Let u be a nonnegative solution of (P_λ). Then, by the same argument as in the proof of [22, Proposition 2.1], we have $0 \le u(x) \le M$ for any $x \in \Omega$, and thus $u \in C_0^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0,1)$ by Lieberman's regularity result [16, Theorem 1]. It follows from the maximum principle due to Vázquez [23, Theorem 5] that $u > 0$ in Ω and $\partial u/\partial\nu < 0$ on $\partial\Omega$ if $u \not\equiv 0$ in Ω.

First we prove a nonexistence result.

Lemma 3.1. *Assume that* $(g1)$–$(g3)$ *are satisfied, and* u_λ *is a positive solution of* (P_λ); *then* $\lambda > \lambda_1/N_1$. *Here,* N_1 *is the constant in the condition* $(g2)$.

Proof. Assume on the contrary that there exists a positive solution u_λ of (P_λ) such that $\lambda \le \lambda_1/N_1$. Let $u = u_\lambda$, $v = \phi_1$, $A = B = 1$, $a(x) = \lambda g(x, u_\lambda(x))$, and $b(x) = \lambda_1$ in [2, Theorem 1]. Then $a(x) \le b(x)$ in Ω, and

$$\int_\Omega L(u_\lambda, \phi_1)\, dx \le 0$$

since $\phi_1 > 0$ in Ω. Here

$$L(u_\lambda, \phi_1) = |\nabla u_\lambda|^p - p\left(\frac{u_\lambda}{\phi_1}\right)^{p-1} |\nabla \phi_1|^{p-2} \nabla \phi_1 \nabla u_\lambda + (p-1)\left(\frac{u_\lambda}{\phi_1}\right)^p |\nabla \phi_1|^p. \tag{3.1}$$

On the other hand, $L(u_\lambda, \phi_1) \ge 0$ by Picone's identity (see, e.g., [3, Theorem 1.1]). Thus $L(u_\lambda, \phi_1) = 0$, almost everywhere in Ω, which implies $u_\lambda = k\phi_1$ for some constant k, and one can easily proceed to a contradiction. □

To study the bifurcation problem of (P_λ), we define a modified equation in which the nonlinearity is truncated for large u. For any $\delta > 0$, let us consider the following problem:

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda g(x,0)|u|^{p-2}u + h_\delta(x, u; \lambda) = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{3.2}$$

where $p > 1$, $\lambda \in \mathbb{R}$,

$$h_\delta(x, u; \lambda) = \begin{cases} 0, & \lambda \in (-\infty, 0], \\ \lambda(\bar{g}(x,u) - g(x,0))|u|^{p-2}u, & \lambda \in (0, \lambda_1^* + \delta], \\ (\lambda_1^* + \delta)(\bar{g}(x,u) - g(x,0))|u|^{p-2}u, & \lambda \in (\lambda_1^* + \delta, \infty), \end{cases}$$

and

$$\bar{g}(x,u) = \begin{cases} g(x,0), & (x,u) \in \overline{\Omega} \times (-\infty, 0], \\ g(x,u), & (x,u) \in \overline{\Omega} \times (0, M], \\ g(x,M), & (x,u) \in \overline{\Omega} \times (M, \infty). \end{cases}$$

Fix a small $\delta > 0$; $h_\delta : \Omega \times \mathbb{R} \times \mathbb{R}$ is a Carathéodory function. Furthermore, there exists a constant $C = C(\delta) > 0$ such that

$$|h_\delta(x, u; \lambda)| \le C|u|^{p-1}$$

for almost every $x \in \Omega$ and all $(u, \lambda) \in \mathbb{R} \times \mathbb{R}$, and

$$h_\delta(x, u; \lambda)/|u|^{p-1} \to 0 \text{ as } u \to 0$$

uniformly for almost every $x \in \Omega$ and uniformly in λ from bounded intervals in $\mathbb{R}$. From [13, Proposition 3.5 and Lemma 3.6], for $\delta > 0$ the pair $(\lambda_1^*, 0)$ is a bifurcation point for problem (3.2) in the sense that there exists a sequence $\{(\mu_n, v_n)\}_{n=1}^{\infty} \subset \mathbb{R} \times W_0^{1,p}(\Omega)$ such that $(\mu_n, v_n) \to (\lambda_1^*, 0)$ and v_n is a positive solution of (3.2) with $\lambda = \mu_n$. One can easily see that for $0 \le \lambda \le \lambda_1^* + \delta$, all positive solutions u of (3.2) or (P_λ) satisfy $0 \le u \le M$, and u is a positive solution of (P_λ) if and only if u is a positive solution of (3.2). Thus we get the following proposition.

Proposition 3.2. *Assume that* $(g1)$–$(g3)$ *and* $(g4b)$ *are satisfied. Then the pair* $(\lambda_1^*, 0)$ *is a bifurcation point for problem* (P_λ) *in the sense that there exists a sequence* $\{(\mu_n, v_n)\}_{n=1}^{\infty} \subset \mathbb{R} \times W_0^{1,p}(\Omega)$ *such that* $(\mu_n, v_n) \to (\lambda_1^*, 0)$ *and* v_n *is a positive solution of* (P_{μ_n}).

Put $\mathcal{A} := \{\lambda : (P_\lambda) \text{ has a positive solution}\}$. Then by Proposition 3.2, $\mathcal{A}$ is non-empty, and by Lemma 3.1, $\lambda_* := \inf \mathcal{A} \in [\lambda_1/N_1, \infty)$. To show that $\lambda_* < \lambda_1^*$, we prove the following result on the "bifurcation direction" of solutions of (P_λ) at $\lambda = \lambda_1^*$.

Lemma 3.3. *Assume that* $(g1)$–$(g3)$ *and* $(g4b)$ *are satisfied, and* u_λ *is a positive solution of* (P_λ) *with* $\|u_\lambda\|_\infty < \delta_1$*; then* $\lambda < \lambda_1^*$. *Here,* δ_1 *is the constant in the condition* $(g4b)$.

Proof. Assume on the contrary that there exists a positive solution u_λ of (P_λ) such that $\lambda \ge \lambda_1^*$ and $\|u_\lambda\|_\infty < \delta_1$. Note that $u_\lambda > 0$ in Ω by the maximum principle. Let $u = \phi_1^*$, $v = u_\lambda$, $A = B = 1$, $a(x) = \lambda_1^* g(x, 0)$, and $b(x) = \lambda g(x, u_\lambda(x))$ in [2, Theorem 1]. Then $a(x) \le b(x)$ in Ω, and

$$\int_\Omega L(\phi_1^*, u_\lambda)\, dx \le 0,$$

where L is defined in (3.1), since $u_\lambda > 0$ in Ω. By the same argument as in the proof of Lemma 3.1, the proof is complete. □

Let $\{(\mu_n, v_n)\}_{n=1}^{\infty}$ be a sequence such that v_n is a positive solution of (P_{μ_n}) (or (3.2) with $\lambda = \mu_n$) and $\mu_n \in (0, \lambda_1^* + \delta)$. Here, $\delta \in (0, \lambda_2^* - \lambda_1^*)$ is a constant and λ_2^* is the second eigenvalue of (2.3) with $m(x) \equiv g(x, 0)$. Then, by [13, Lemma B.2], the following three statements are equivalent:

(*i*) $\|v_n\|_{W_0^{1,p}} \to 0$ as $n \to \infty$;
(*ii*) $\|v_n\|_\infty \to 0$ as $n \to \infty$;
(*iii*) $\|v_n\|_{C^{1,\beta}} \to 0$ as $n \to \infty$, for some $\beta \in (0, 1)$;

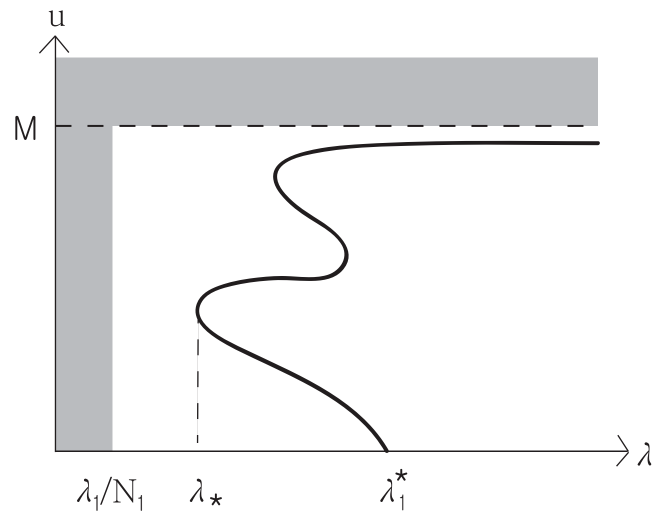

FIGURE 1. The bifurcation diagram of the positive solutions of (P_λ).

and in all three cases we have $\mu_n \to \lambda_1^*$ as $n \to \infty$. From Proposition 3.2 and Lemma 3.3, one can conclude that the bifurcation at $(\lambda_1^*, 0)$ is subcritical, and thus $\lambda_* \in [\lambda_1/N_1, \lambda_1^*)$ (see Figure 1).

Theorem 3.4. *Assume that $(g1)$–$(g3)$ and $(g4b)$ are satisfied. Then $\lambda_* \in (\lambda_1/N_1, \lambda_1^*)$ and there exists $\lambda^* \in [\lambda_*, \lambda_1^*)$ such that (P_λ) has at least one positive solution for $\lambda \geq \lambda^*$.*

Proof. Let $\mathcal{B} := \{\lambda : (P_\mu)$ has a positive solution for all $\mu \in [\lambda, \infty)\}$. Then $\mathcal{B}$ is non-empty, and $\lambda^* := \inf \mathcal{B}$ is well defined. Indeed, for sufficiently small $\delta > 0$, there exists $\bar{\lambda} \in (\lambda_1^* - \delta, \lambda_1^*)$ such that $(P_{\bar{\lambda}})$ has a positive solution $\bar{u}$ satisfying $\|\bar{u}\|_\infty < \delta_1$, and $g(x, \bar{u}(x)) \geq 0$ for all $x \in \Omega$. Then the solution $\bar{u}$ is a sub-solution of (P_λ) for all $\lambda \geq \bar{\lambda}$. Clearly, M is a super-solution of (P_λ) for all $\lambda > 0$, and (P_λ) has at least one positive solution for all $\lambda \geq \bar{\lambda}$ in view of Theorem 2.1. Consequently, $\lambda^* \in [\lambda_*, \lambda_1^*)$ and (P_λ) has at least one positive solution for all $\lambda > \lambda^*$.

Now we prove that (P_{λ^*}) has a positive solution. By the definition of λ^*, there exists a sequence $\{(\mu_n, v_n)\}_{n=1}^\infty$ such that $\mu_n \to \lambda^*$ as $n \to \infty$ and v_n is a positive solution of (P_{μ_n}). Clearly the sequence $\{(\mu_n, v_n)\}_{n=1}^\infty$ is bounded in $\mathbb{R} \times W_0^{1,p}(\Omega)$. Note that the operator $(-\Delta_p)^{-1}T : [0,\infty) \times W_0^{1,p}(\Omega) \to$

$W_0^{1,p}(\Omega)$ is completely continuous. Here $T : [0,\infty) \times W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is defined by $T(\lambda, u) = \lambda N_{\bar{f}}(u)$, $\bar{f}(x,s) = \bar{g}(x,s)|s|^{p-2}s$, and $\bar{g}(x,u)$ is the function defined in the equation (3.2). By a standard argument, (P_{λ^*}) has a nonnegative solution v^*. Then v^* is a positive solution of (P_{λ^*}). Indeed, if $v^* \equiv 0$, $v_n \to 0$ in $W_0^{1,p}(\Omega)$ as $n \to \infty$. By [13, Lemma B.2], $\mu_n \to \lambda_1^*$ as $n \to \infty$; this contradicts the fact that $\lambda^* < \lambda_1^*$. Similarly, we can prove that $\lambda_* > \lambda_1/N_1$, and thus the proof is complete. □

The result in Theorem 3.4 shows that, in contrast with the logistic case, the bifurcation point $\lambda = \lambda_1^*$ is not the threshold value of the existence/non-existence of a positive solution of (P_λ). This is a basic difference between the equation with weak Allee effect growth rate and the one with logistic growth.

In the following we shall show that when $\lambda \in (\lambda_*, \lambda_1^*)$, equation (P_λ) indeed has two positive solutions, if an additional condition is satisfied. We employ the variational method in this part. From now on we assume that g satisfies

$(g3')$ There exists $M > u_1(x)$ for any $x \in \overline{\Omega}$, such that $g(x, M) = 0$

instead of the condition $(g3)$. Note that this is equivalent to letting $u_2(x) \equiv M$ in $(g3)$. Then all nonnegative solutions u of (P_λ) satisfy that $0 \le u(x) \le M$ for all $x \in \Omega$, and thus $g(x, u(x)) \ge 0$ for all $x \in \Omega$. In this case, $\lambda_* = \lambda^*$, and (P_λ) has a maximal positive solution $u_m(\lambda)$ for all $\lambda \ge \lambda_*$ in view of Theorem 2.1. Moreover, $u_m(\lambda)$ is nondecreasing with respect to λ; *i.e.*, $u_m(\lambda^1)(x) \le u_m(\lambda^2)(x)$ for all $x \in \Omega$ if $\lambda_* \le \lambda^1 < \lambda^2$.

Define a modified functional $\hat{\Phi}_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ by

$$\hat{\Phi}_\lambda(u) := \frac{1}{p}\int_\Omega |\nabla u|^p \, dx - \lambda \int_\Omega \hat{F}(x,u)\, dx, \quad u \in W_0^{1,p}(\Omega),$$

where $\hat{F}(x,u) := \int_0^u \hat{f}(x,s)ds$, $\hat{f}(x,s) = \hat{g}(x,s)|s|^{p-2}s$ and

$$\hat{g}(x,s) := \begin{cases} 0, & (x,s) \in \overline{\Omega} \times (-\infty, 0], \\ g(x,s), & (x,u) \in \overline{\Omega} \times (0, M], \\ 0, & (x,s) \in \overline{\Omega} \times (M, \infty). \end{cases}$$

Note that if u is any critical point of $\hat{\Phi}_\lambda$, then $0 \le u(x) \le M$ for all $x \in \Omega$, and thus u is a nonnegative solution of (P_λ).

Since $g(x,s)$ satisfies the conditions $(g1)$–$(g3)$ and $(g4b)$, for any $\delta > 0$, there exists $C_\delta > 0$ such that

$$|\hat{g}(x,s)| \le g(x,0) + \delta + C_\delta |s|^{q^*-p}, \quad (x,s) \in \Omega \times \mathbb{R},$$

and

$$|\hat{f}(x,s)| \le (g(x,0)+\delta)|s|^{p-1} + C_\delta |s|^{q^*-1}, \quad (x,s) \in \Omega \times \mathbb{R}, \tag{3.3}$$

where $q^* \in (p, p^*)$. Thus the functional $\hat{\Phi}_\lambda$ is C^1 in $W_0^{1,p}(\Omega)$. Moreover, the functional $\hat{\Phi}_\lambda$ satisfies the Palais–Smale condition. Indeed, let $\{v_n\}_{n=1}^\infty$ be any sequence in $W_0^{1,p}(\Omega)$ such that $\{\hat{\Phi}_\lambda(v_n)\}$ is bounded and $\hat{\Phi}'_\lambda(v_n) \to 0$ as $n \to \infty$. Then it follows from the boundedness of $\hat{F}$ that $\{v_n\}_{n=1}^\infty$ is bounded in $W_0^{1,p}(\Omega)$. By [8, Lemma 2 on page 363], the sequence $\{v_n\}_{n=1}^\infty$ has a convergent subsequence, and thus the functional $\hat{\Phi}_\lambda$ satisfies the Palais–Smale condition.

Lemma 3.5. *Assume that $(g1)$–$(g3)$ and $(g4b)$ are satisfied, and let $\lambda \in (0, \lambda_1^*)$. Then the trivial solution 0 is a local minimizer of $\hat{\Phi}_\lambda$ in $W_0^{1,p}(\Omega)$.*

Proof. By (3.3), for any $\delta > 0$, there exists $C_\delta > 0$ such that

$$|\hat{F}(x,s)| \le \frac{1}{p}(g(x,0)+\delta)|s|^p + \frac{C_\delta}{q^*}|s|^{q^*}, \quad (x,s) \in \Omega \times \mathbb{R}.$$

By the Sobolev inequality, for $u \in W_0^{1,p}(\Omega)$,

$$\begin{aligned} \hat{\Phi}_\lambda(u) &\ge \frac{1}{p}\left[\|u\|^p_{W_0^{1,p}} - \lambda \int_\Omega (g(x,0)+\delta)|u|^p\right] - \frac{C_\delta \lambda}{q^*}\int_\Omega |u|^{q^*}\,dx \\ &\ge \frac{1}{p}\left[1 - \frac{\lambda}{\lambda_1^*} - \delta C_1 - C_\delta C_1 \|u\|^{q^*-p}_{W_0^{1,p}}\right]\|u\|^p_{W_0^{1,p}}, \end{aligned}$$

for some constant $C_1 > 0$. Thus for each $\lambda \in (0, \lambda_1^*)$, there exist positive constants δ and ρ such that $\hat{\Phi}_\lambda(u) > 0 = \hat{\Phi}_\lambda(0)$ if $0 < \|u\|_{W_0^{1,p}} \le \rho$. □

Fix $\lambda \in (\lambda_*, \lambda_1^*)$ and let μ_i $(i = 1, 2)$ be the constants satisfying $\lambda_* < \mu_1 < \lambda < \mu_2$. If we assume g satisfies the condition

$(g5)$ $kg(x,u) \le g(x,ku)$ for $0 < k < 1$ and $0 \le u \le M$,

then $\varphi_1 := \epsilon_1 u_m(\mu_1)$ is a sub-solution and $\varphi_2 := \epsilon_2 u_m(\mu_2)$ is a super-solution of (P_λ), respectively. Here ϵ_1 and ϵ_2 are the constants satisfying

$$\frac{\mu_1}{\lambda} < \epsilon_1 < 1 < \epsilon_2 < \frac{\mu_2}{\lambda},$$

and we note that the condition $(g5)$ implies that

$(g5')$ $kg(x,u) \ge g(x,ku)$ for $k > 1$ and $0 \le u \le M$.

Let

$$\mathcal{C} := \{u \in C_0^1(\overline{\Omega}) : \varphi_1(x) \le u(x) \le \varphi_2(x),\ x \in \Omega\}. \tag{3.4}$$

Then $u_m(\lambda)$ is an interior point of $\mathcal{C}$ with respect to the C^1-topology by the maximum principle due to Vázquez [23, Theorem 5].

Lemma 3.6. *Let $\lambda \in (\lambda_*, \lambda_1^*)$ be fixed, and assume that $(g1)$, $(g2)$, $(g3')$, $(g4b)$, and $(g5)$ are satisfied. If (P_λ) has no solution in $\mathcal{C}$ except for $u_m(\lambda)$, then $u_m(\lambda)$ is a local minimizer of $\hat{\Phi}_\lambda$ in $W_0^{1,p}(\Omega)$.*

Proof. Assume that (P_λ) has no solution in $\mathcal{C}$ except for $u_m(\lambda)$. If we show that $u_m(\lambda)$ is a local minimizer of $\hat{\Phi}_\lambda$ in $C_0^1(\overline{\Omega})$, then it is a local minimizer of $\hat{\Phi}_\lambda$ in $W_0^{1,p}(\Omega)$ in view of [12, Theorem 1.2] (or [5, Lemma 2.2]), since $0 \le \hat{g}(x,s)|s|^{p-2}s \le N_1 M^{p-1}$ for all $(x,s) \in \Omega \times \mathbb{R}$. To prove that $u_m(\lambda)$ is a local minimizer of $\hat{\Phi}_\lambda$ in $C_0^1(\overline{\Omega})$, let us define the functional $\tilde{\Phi}_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ by

$$\tilde{\Phi}_\lambda(u) := \frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \lambda \int_\Omega \tilde{F}(x,u)\,dx, \quad u \in W_0^{1,p}(\Omega),$$

where $\tilde{F}(x,u) := \int_0^u \tilde{f}(x,s)ds$, and

$$\tilde{f}(x,s) := \begin{cases} \hat{g}(x,\varphi_1(x))\varphi_1^{p-1}, & \text{if } x \in \overline{\Omega},\ s < \varphi_1(x), \\ \hat{g}(x,s)s^{p-1}, & \text{if } x \in \overline{\Omega},\ \varphi_1(x) \le s \le \varphi_2(x), \\ \hat{g}(x,\varphi_2(x))\varphi_2^{p-1}, & \text{if } x \in \overline{\Omega},\ s > \varphi_2(x). \end{cases}$$

Since $\tilde{\Phi}_\lambda$ is weakly lower-semicontinuous and coercive on $W_0^{1,p}(\Omega)$, $\tilde{\Phi}_\lambda$ has a global minimizer $u_0 \in W_0^{1,p}(\Omega)$. Then u_0 is a solution of

$$\begin{cases} \operatorname{div}(|\nabla u_0|^{p-2}\nabla u_0) + \lambda \tilde{f}(x,u_0) = 0, & x \in \Omega, \\ u_0 = 0, & x \in \partial\Omega, \end{cases} \tag{3.5}$$

and $u_0 \in C_0^1(\overline{\Omega})$ by Lieberman's regularity result [16, Theorem 1]. Moreover, u_0 satisfies that $\varphi_1(x) \le u_0(x) \le \varphi_2(x)$ for all $x \in \Omega$. We only prove the left-hand inequality that $\varphi_1(x) \le u_0(x)$ for all $x \in \Omega$, since the right-hand inequality that $u_0(x) \le \varphi_2(x)$ for all $x \in \Omega$ can be proved in a similar manner. If it is not true, $\Omega_1 := \{x \in \Omega : u_0(x) < \varphi_1(x)\}$ is a nonempty open set in $\mathbb{R}^N$ as $u_0, \varphi_1 \in C_0^1(\overline{\Omega})$. Since φ_1 is a sub-solution of (P_λ) and u_0 is a solution of (3.5), we have

$$\int_{\Omega_1} |\nabla\varphi_1|^{p-2}\nabla\varphi_1\nabla\psi\,dx \le \lambda \int_{\Omega_1} \hat{g}(x,\varphi_1)\varphi_1^{p-1}\psi\,dx$$

and

$$\begin{aligned}\int_{\Omega_1} |\nabla u_0|^{p-2}\nabla u_0 \nabla\psi\, dx &= \lambda \int_{\Omega_1} \tilde{f}(x,u_0)\psi\, dx,\\ &= \lambda\int_{\Omega_1} \hat{g}(x,\varphi_1)\varphi_1^{p-1}\psi\, dx,\end{aligned}$$

where $\psi := \max\{\varphi_1 - u_0, 0\} \in W_0^{1,p}(\Omega)$. Then

$$\int_{\Omega_1} (|\nabla\varphi_1|^{p-2}\nabla\varphi_1 - |\nabla u_0|^{p-2}\nabla u_0)(\nabla\varphi_1 - \nabla u_0)\, dx \le 0. \tag{3.6}$$

It is well known that the following inequality holds:

$$(|x|^{p-2}x - |y|^{p-2}y)(x-y) \ge 0 \quad \text{for all } x, y \in \mathbb{R}^N,$$

where the equality holds if and only if $x = y$. It follows from (3.6) that $\nabla u_0(x) = \nabla\varphi_1(x)$ for all $x \in \Omega_1$, which contradicts the facts that Ω_1 is a nonempty open set in $\mathbb{R}^N$ and $u_0 \equiv \varphi_1$ on $\partial\Omega_1$. Thus we have $\varphi_1(x) \le u_0(x) \le \varphi_2(x)$ for all $x \in \Omega$, which implies that u_0 is a solution of (P_λ), and it must be the same as $u_m(\lambda)$ by the assumption that (P_λ) has no solution in $\mathcal{C}$ except for $u_m(\lambda)$. Consequently, $u_m(\lambda)$ is a global minimizer of $\tilde{\Phi}$ in $W_0^{1,p}(\Omega)$.

Since $u_m(\lambda)$ is an interior point of $\mathcal{C}$, for sufficiently small $\epsilon > 0$, any $u \in C_0^1(\overline{\Omega})$ with $\|u - u_m(\lambda)\|_{C^1} < \epsilon$ satisfies $u \in \mathcal{C}$. On the other hand, for any $u \in \mathcal{C}$,

$$\begin{aligned}\hat{\Phi}_\lambda(u) - \tilde{\Phi}_\lambda(u) &= \lambda\int_\Omega [\tilde{F}(x,u) - \hat{F}(x,u)]\, dx\\ &= \lambda\int_\Omega\int_0^{u(x)} [\tilde{f}(x,s) - \hat{g}(x,s)s^{p-1}]\, ds\, dx\\ &= \lambda\int_\Omega\int_0^{\varphi_1(x)} [\hat{g}(x,\varphi_1(x))\varphi_1^{p-1} - \hat{g}(x,s)s^{p-1}]\, ds\, dx,\end{aligned}$$

which is a constant independent of u. Thus, since $u_m(\lambda)$ is a global minimizer of $\tilde{\Phi}_\lambda$ in $W_0^{1,p}(\Omega)$, $u_m(\lambda)$ is a local minimizer of $\hat{\Phi}_\lambda$ in $C_0^1(\overline{\Omega})$, which completes the proof in view of [12, Theorem1.2] (or [5, Lemma 2.2]). □

Now we can prove our main result of multiple positive solutions of (P_λ):

Theorem 3.7. *Assume that* $(g1)$, $(g2)$, $(g3')$, $(g4b)$, *and* $(g5)$ *are satisfied. Then there exists* $\lambda_* \in (\lambda_1/N_1, \lambda_1^*)$ *such that* (P_λ) *has no positive solutions for* $\lambda < \lambda_*$, (P_λ) *has at least two positive solutions of* (P_λ) *for* $\lambda_* < \lambda < \lambda_1^*$, *and* (P_λ) *has at least one positive solution for* $\lambda \ge \lambda_1^*$ *and* $\lambda = \lambda_*$.

Proof. By Theorem 3.4, it suffices to show that (P_λ) has at least two positive solutions for $\lambda_* < \lambda < \lambda_1^*$. Let $\lambda \in (\lambda_*, \lambda_1^*)$ be fixed and let $\mathcal{C}$ be the set defined by (3.4). We will show that (P_λ) has another positive solution distinct from $u_m(\lambda)$. If it is not true, we may assume that there exists no solution in $\mathcal{C}$ except for $u_m(\lambda)$. From Lemmas 3.5 and 3.6, we have two local minimizers 0 and $u_m(\lambda)$ of $\hat{\Phi}_\lambda$ in $W_0^{1,p}(\Omega)$. Then there exists a third critical point of $\hat{\Phi}_\lambda$ in $W_0^{1,p}(\Omega)$ in view of the extended mountain-pass theorem by Pucci and Serrin [19, Theorem 4]. Thus we get the second positive solution of (P_λ) distinct from $u_m(\lambda)$, which completes the proof. □

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