EXISTENCE, UNIQUENESS AND STABILITY OF POSITIVE SOLUTIONS FOR A CLASS OF SEMILINEAR ELLIPTIC SYSTEMS

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Abstract. We consider the stability of positive solutions to semilinear elliptic systems under a new general sublinear condition and its variants. Using the stability result and bifurcation theory, we prove the existence and uniqueness of positive solution and obtain the precise global bifurcation diagram of the system being a single monotone solution curve.

1. Introduction

We consider the positive solutions of a semilinear elliptic system:

\[
\begin{aligned}
\Delta u + \lambda f(u, v) &= 0, \quad x \in \Omega, \\
\Delta v + \lambda g(u, v) &= 0, \quad x \in \Omega, \\
u(x) = v(x) &= 0, \quad x \in \partial \Omega,
\end{aligned}
\]

where \( \lambda > 0 \) is a positive parameter, \( \Omega \) is a bounded smooth domain in \( \mathbb{R}^n \) for \( n \geq 1 \), and \( f \) and \( g \) are smooth real-valued functions defined on \( \mathbb{R}^+ \times \mathbb{R}^+ = \)

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[0, ∞) × [0, ∞) satisfying \( f_v(u, v) \geq 0 \) and \( g_u(u, v) \geq 0 \) for \((u, v) \in \mathbb{R}^+ \times \mathbb{R}^+\), which implies that the system is cooperative.

The existence, uniqueness and stability of positive solutions of sublinear semilinear elliptic systems have been recently studied in [2], [3], [26]. In [2], the stability of a positive solution was established under the condition

\[
(1.2) \quad f(u, v) > f_v(u, v)v + g_v(u, v)u, \quad g(u, v) > g_u(u, v)u + f_u(u, v)v. 
\]

The sublinear condition (1.2) involves both \( f \) and \( g \) in the two inequalities, which is sometimes hard to achieve. In this article, we continue the effort in [2] to prove the stability of positive solution to (1.1) under some more reasonable sublinear conditions, and once again the stability implies the uniqueness of the positive solution. We also prove corresponding existence results using bifurcation and continuation theory.

For the scalar semilinear elliptic equation:

\[
(1.3) \quad \left\{ \begin{array}{ll}
\Delta u + \lambda f(u) = 0, & x \in \Omega, \\
u(x) = 0, & x \in \partial \Omega,
\end{array} \right.
\]

the exact multiplicity of positive solutions has been previously considered by many people, see for example, [14], [15], [18], [19]. In recent years, there have been some results on the existence and uniqueness of solution to the semilinear cyclic elliptic system:

\[
(1.4) \quad \left\{ \begin{array}{ll}
\Delta u + \lambda f(v) = 0, & x \in \Omega, \\
\Delta v + \lambda g(u) = 0, & x \in \Omega, \\
u(x) = v(x) = 0, & x \in \partial \Omega.
\end{array} \right.
\]

Dalmaso [7], [8] obtained the existence and uniqueness result for a more special sublinear system, and it was extended by Shi and Shivaji [26]. The uniqueness of positive solution for large \( \lambda \) was proved in Hai [9], [10], Hai and Shivaji [11]. If \( \Omega \) is a finite ball or the whole space, then the positive solutions of systems are radially symmetric and decreasing in radial direction by the result of Troy [28], see also [1], [17]. Hence the system can be converted into a system of ODEs. Several authors have taken that approach for the existence of the solutions, see Serrin and Zou [21], [22]. Much success has been achieved for Lane–Emden systems. Using the scaling invariant, the uniqueness of the radial positive solution for the Lane–Emden system has been shown in Dalmaso [7], [8], Korman and Shi [16]. Cui, Wang and Shi [5], [6] considered cyclic systems with three equations, and the uniqueness and existence of positive solutions were obtained.

The approach in this article includes several ingredients. We recall the maximum principle and prove the main stability result in Section 2. In Sections 3 and 4, we use the stability result and bifurcation theory to prove the existence and uniqueness of positive solution for two types of semilinear system. We also
obtain the precise global bifurcation diagrams of the system and the bifurcation diagram is a single monotone solution curve in all cases. We use $W^{2,p}(\Omega)$ and $W^{2,p}_{\text{loc}}(\Omega)$ for the standard Sobolev space, $C(\Omega)$ for the space of continuous functions defined on $\Omega$, and $C_0(\Omega) = \{u \in C(\Omega) : u(x) = 0, x \in \partial \Omega\}$. We use $N(L)$ and $R(L)$ to denote the null space and the range space of linear operator $L$.

2. Stability and linearized equations

Let $(u, v)$ be a positive solution of (1.1). The stability of $(u, v)$ is determined by the eigenvalue equation:

\[
\begin{aligned}
\Delta \xi + \lambda f_u(u, v)\xi + \lambda f_v(u, v)\eta &= -\mu \xi, \quad x \in \Omega, \\
\Delta \eta + \lambda g_u(u, v)\xi + \lambda g_v(u, v)\eta &= -\mu \eta, \quad x \in \Omega, \\
\xi(x) &= \eta(x) = 0, \quad x \in \partial \Omega,
\end{aligned}
\]

which can be written as

\[
L u = H u + \mu u,
\]

where

\[
u = \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \quad L u = \begin{pmatrix} -\Delta \xi \\ -\Delta \eta \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} f_u(u, v) & f_v(u, v) \\ g_u(u, v) & g_v(u, v) \end{pmatrix}.
\]

For linear elliptic systems of cooperative type, the maximum principle holds and here we recall some known results:

**Lemma 2.1.** Let $X = [W^{2,p}_{\text{loc}}(\Omega) \cap C_0(\Omega)]^2$, and let $Y = [L^p(\Omega)]^2$ for $p > n$. Suppose that $L$ and $H$ are given as in (2.3), the partial derivatives of $f$ and $g$ are continuous on $\mathbb{R}^+ \times \mathbb{R}^+$, and $f_v(u, v) \geq 0$, $g_u(u, v) \geq 0$ for $(u, v) \in \mathbb{R}^+ \times \mathbb{R}^+$. Then:

(a) $\mu_1 = \inf \{\mu \in \text{spt}(L-H) \}$ is a real eigenvalue of $L-H$, where $\text{spt}(L-H)$ is the spectrum of $L-H$.

(b) For $\mu = \mu_1$, there exists a unique eigenfunction $u_1 \in [W^{2,p}_{\text{loc}}(\Omega) \cap C_0(\Omega)]^2$ of $L-H$ (up a constant multiple), and $u_1 > 0$ in $\Omega$.

(c) For $\mu < \mu_1$, the equation $Lu = Hu + \mu u + f$ has a unique solution $u \in X$ for any $f \in Y$, and $u > 0$ as long as $f \geq (\neq) 0$.

(d) (Maximum principle) For $\mu \leq \mu_1$, suppose that there exists $u \in [W^{2,p}_{\text{loc}}(\Omega) \cap C(\Omega)]^2$, satisfies $Lu \geq Hu + \mu u$ in $\Omega$, $u \geq 0$ on $\partial \Omega$, then $u \geq 0$ in $\Omega$.

(e) If there exists $u \in [W^{2,p}(\Omega) \cap C(\Omega)]^2$, satisfies $Lu \geq Hu$ and $u \geq 0$ in $\Omega$, and either $u \not\equiv 0$ on $\partial \Omega$ or $Lu \not\equiv Hu$ in $\Omega$, then $\mu_1 > 0$.

For the result and proof of Lemma 2.1, see Sweers [26], Proposition 3.1 and Theorem 1.1. Moreover, from a standard compactness argument, the eigenvalues $\{\mu_i\}$ of $L-H$ are countably many, and $\text{Re}(\mu_i - \mu_1) \to \infty$ as $i \to \infty$. We notice
that $\mu_i$ is not necessarily real-valued for $i \geq 2$. We call a solution $(u, v)$ is **stable** if $\mu_1 > 0$, and otherwise it is **unstable** ($\mu_1 \leq 0$).

For our purpose in this section, we also need to consider the adjoint operator of $L - H$. Let the transpose matrix of the Jacobian be

$$H^T = \begin{pmatrix} f_u(u, v) & g_u(u, v) \\ f_v(u, v) & g_v(u, v) \end{pmatrix}.$$  

Then evidently the results in Lemma 2.1 also hold for the eigenvalue problem

$$Lu^* = H^Tu^* + \mu u^*,$$

which is

$$\begin{cases} 
\Delta \xi^* + \lambda f_u(u, v)\xi^* + \lambda g_u(u, v)\eta^* = 0, & x \in \Omega, \\
\Delta \eta^* + \lambda f_v(u, v)\xi^* + \lambda g_v(u, v)\eta^* = -\mu \xi^*, & x \in \Omega, \\
\xi^*(x) = \eta^*(x) = 0, & x \in \partial\Omega.
\end{cases}$$

It is easy to verify that $L - H^T$ is the adjoint operator of $L - H$, while both are considered as operators defined on subspaces of $[L^2(\Omega)]^2$. By using the well-known functional analytic techniques (see [12], [26]), one can show that:

**Lemma 2.2.** Let $X, Y, L, H$ and $f$, $g$ be same as in Lemma 2.1. Then the principal eigenvalue $\mu_1$ of $L - H$ is also a real eigenvalue of $L - H^T$, $\mu_1 = \inf\{\mu \in \text{spt}(L - H^T)\}$, and for $\mu = \mu_1$, there exists a unique eigenfunction $u_1^* \in [W^{2,p}_0(\Omega) \cap C_0(\Omega)]^2$ of $L - H^T$ (up a constant multiple), and $u_1^* > 0$ in $\Omega$.

Now we are ready to establish the main stability result:

**Theorem 2.3.** Suppose that $(u, v)$ is a positive solution of (1.1), and $f$ and $g$ are smooth real-valued functions defined on $\mathbb{R}^+ \times \mathbb{R}^+$ satisfying $f_u(u, v) \geq 0$ and $g_u(u, v) \geq 0$ for $(u, v) \in \mathbb{R}^+ \times \mathbb{R}^+$. Then $(u, v)$ is stable if $(f, g)$ satisfies one of the following conditions: for any $(u, v) \in \mathbb{R}^+ \times \mathbb{R}^+$,

- $(A_1)$ $f(u, v) > f_u(u, v)u + f_v(u, v)v$, $g(u, v) > g_u(u, v)u + g_v(u, v)v$; or
- $(A_2)$ $f(u, v) > f_u(u, v)u + g_u(u, v)v$, $g(u, v) > g_u(u, v)u + f_v(u, v)v$; or
- $(A_3)$ $f(u, v) > f_u(u, v)u + g_v(u, v)v$, $g(u, v) > g_u(u, v)u + f_v(u, v)v$; or
- $(A_4)$ $f(u, v) > g_u(u, v)u + g_v(u, v)v$, $g(u, v) > f_u(u, v)u + g_v(u, v)v$.

**Proof.** The result under $(A_3)$ has been proved in [3], hence here we prove the stability result when $(f, g)$ satisfies one of $(A_1)$, $(A_2)$ and $(A_4)$. Let $(u, v)$ be a positive solution of (1.1), and let $(\mu_1, \xi, \eta)$ and $(\mu_1, \xi^*, \eta^*)$ be the corresponding principal eigen-pair of (2.1) and (2.6) respectively, such that $\xi, \eta, \xi^*, \eta^* > 0$ in $\Omega$.

First we assume that $(f, g)$ satisfies $(A_1)$. Multiplying the equation of $u$ in (1.1) by $\xi^*$, the equation of $\xi^*$ in (2.6) by $u$, integrating over $\Omega$ and subtracting, we obtain that

$$\lambda \int_{\Omega} f \xi^* \, dx = \lambda \int_{\Omega} (f_u \xi^* + g_u \eta^*) u \, dx + \mu_1 \int_{\Omega} u \xi^* \, dx.$$
Similarly from the equation of $v$ and $\eta^*$, we find
\begin{equation}
\lambda \int_{\Omega} q\eta^* \, dx = \lambda \int_{\Omega} (f_v \xi^* + g_v \eta^*) v \, dx + \mu_1 \int_{\Omega} v\eta^* \, dx.
\end{equation}

Adding (2.7) and (2.8), we get
\begin{equation}
\mu_1 \int_{\Omega} (u \xi^* + \nu \eta^*) \, dx = \lambda \int_{\Omega} [f - f_u u - f_v \nu] \eta^* \, dx + \lambda \int_{\Omega} [g - g_u u - g_v \nu] \xi^* \, dx.
\end{equation}

Hence $\mu_1 > 0$ if (A1) is satisfied.

Secondly we assume that $(f, g)$ satisfies (A2). Similar to the proof above, multiplying the equation of $u$ in (1.1) by $\xi$, multiplying the equation of $\xi$ in (2.1) by $u$, integrating over $\Omega$ and subtracting, and also doing the same operations for the equations of $v$ and $\eta$, we can get
\begin{equation}
\mu_1 \int_{\Omega} (u \xi + v \eta) \, dx = \lambda \int_{\Omega} [f - f_u u - g_v \nu] \xi \, dx + \lambda \int_{\Omega} [g - g_u u - f_v \nu] \eta \, dx,
\end{equation}

which implies $\mu_1 > 0$ if (A2) is satisfied.

Finally we assume that $(f, g)$ satisfies (A4). We repeat the above calculation for the equations of $u$ and $\eta^*$, and the equations of $v$ and $\xi^*$, then we obtain
\begin{equation}
\mu_1 \int_{\Omega} (u \xi + v \eta) \, dx = \lambda \int_{\Omega} [f - f_u u - g_v \nu] \xi \, dx + \lambda \int_{\Omega} [g - g_u u - f_v \nu] \eta \, dx.
\end{equation}

Therefore $\mu_1 > 0$ if (A4) is satisfied.

On the other hand, the same proof also implies the following instability result under the opposite condition of (A_i) for $i = 1, 2, 3$ and 4:

**Theorem 2.4.** Suppose that $(u, v)$ is a positive solution of (1.1), and $f$ and $g$ are smooth real-valued functions defined on $\mathbb{R}^+ \times \mathbb{R}^+$ satisfying $f_v(u, v) \geq 0$ and $g_u(u, v) \geq 0$ for $(u, v) \in \mathbb{R}^+ \times \mathbb{R}^+$. Then $(u, v)$ is unstable if $(f, g)$ satisfies one of the following conditions:

- $(A_1')$ $f(u, v) < f_u(u, v)u + f_v(u, v)v$, $g(u, v) < g_u(u, v)u + g_v(u, v)v$; or
- $(A_2')$ $f(u, v) < f_u(u, v)u + g_u(u, v)v$, $g(u, v) < g_u(u, v)u + f_v(u, v)v$; or
- $(A_3')$ $f(u, v) < f_u(u, v)u + g_u(u, v)v$, $g(u, v) < g_u(u, v)u + f_u(u, v)v$; or
- $(A_4')$ $f(u, v) < g_u(u, v)u + g_v(u, v)v$, $g(u, v) < f_v(u, v)u + f_v(u, v)v$,

for any $(u, v) \in \mathbb{R}^+ \times \mathbb{R}^+$.

**Remark 2.5.** (a) Theorems 2.3 and 2.4 are generalizations of corresponding results for the positive solutions of scalar equation:

$$\Delta u + \lambda h(u) = 0, \quad x \in \Omega, \quad u(x) = 0, \quad x \in \partial \Omega.$$

In [19], the function $h(u)$ is called a sublinear function if $h(u) > uh'(u)$, and it is superlinear if $h(u) < uh'(u)$. It was proved in Proposition 3.14 of [19] that a positive solution $u$ is stable if $h$ is sublinear, and $u$ is unstable if $h$ is superlinear.
The conditions \((A_i)\) (or \((A'_i)\)) for \(1 \leq i \leq 4\) are the generalization of sublinearity (or superlinearity) to two-variable vector fields.

(b) The conditions \((A_i)\) for \(1 \leq i \leq 4\) can be written in a vector form
\[
F(u) > J_i(u)u,
\]
where \(u = (u, v)^T\), and
\[
J_1 = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}, \quad J_2 = \begin{pmatrix} f_u & g_u \\ g_v & f_v \end{pmatrix}, \quad J_3 = \begin{pmatrix} g_v & f_v \\ g_u & f_u \end{pmatrix}, \quad J_4 = \begin{pmatrix} g_v & g_u \\ f_v & f_u \end{pmatrix}.
\]
Notice that \(J_1\) is the original Jacobian matrix of the vector field \((f(u), g(u))\), and \(J_i\) (\(2 \leq i \leq 4\)) are reflections of \(J_1\) with respect to the two diagonal lines. The condition with original Jacobian is clearly more natural as the conditions for \(f\) and \(g\) are separate. Hence the sublinearity can be defined for a single two-variable function \(f(u, v)\) or \(g(u, v)\). Other conditions are defined for the whole vector field \((f, g)\).

(c) We can weaken the strict inequalities in \((A_i)\) or \((A'_i)\) to \(\geq\) or \(\leq\) respectively, but assume that the strict inequalities hold at least for \((u, v)\) in a set of positive measure. On the other hand, if we have \(f \equiv f_uu + f_vv\) and \(g \equiv g_uu + g_vv\), then \(f\) and \(g\) are necessarily linear functions of \(u\) and \(v\), and the corresponding positive solution \((u, v)\) is neutrally stable with \(\mu_1 = 0\). Indeed \((u, v)\) is the principal eigenfunction for linear \(f\) and \(g\).

(d) If a solution \((u, v)\) is stable, then it is necessarily a non-degenerate solution. That is, any eigenvalue \(\mu_i\) of (2.1) has positive real part. But when a solution is proved to be unstable, it can be a degenerate one with zero or pure imaginary eigenvalues.

3. Application: positive nonlinearities

In this section, we consider the uniqueness and existence of positive solutions for the following problem:
\[
\begin{align*}
\Delta u + \lambda(f_1(v) + f_2(u)) &= 0, \quad x \in \Omega, \\
\Delta v + \lambda(g_1(u) + g_2(v)) &= 0, \quad x \in \Omega, \\
u(x) = v(x) = 0, \quad x \in \partial\Omega.
\end{align*}
\]
Suppose that each of the functions \(f_1, f_2, g_1\) and \(g_2\) is a smooth real-valued function defined on \(\mathbb{R}^+\) and satisfies (denote \(f_1, f_2, g_1\) or \(g_2\) by \(h\)):

(B1) \(h(0) \geq 0\);
(B2) \(h'(x) \geq 0, \quad (h(x)/x)' \leq 0, \text{ for all } x \geq 0, \text{ and } (h(x)/x)' \not\equiv 0 \text{ for any open interval } (a, b) \subset \mathbb{R}^+\).

Here let \((\lambda_1, \varphi_1)\) be the principal eigen-pair of
\[
-\Delta \varphi = \lambda \varphi, \quad x \in \Omega, \quad \varphi(x) = 0, \quad x \in \partial\Omega,
\]
such that \(\varphi_1(x) > 0 \text{ in } \Omega \text{ and } \|\varphi_1\|_\infty = 1\). Then we have the following result about this sublinear problem:
Theorem 3.1. Assume that each of $f_1$, $f_2$, $g_1$ and $g_2$ satisfies (B1), (B2) and
\begin{equation}
(B3) \quad \lim_{x \to \infty} \frac{h(x)}{x} = 0.
\end{equation}

(a) If at least one of $f_i(0)$ and $g_i(0)$ ($i=1,2$) is positive, then (3.1) has a unique positive solution $(u, v)$ for all $\lambda > 0$;
(b) If $h(0) = 0$, and $h'(0) \geq 0$, then for some $\lambda_* > 0$, (3.1) has no positive solution when $\lambda \leq \lambda_*$, and (3.1) has a unique positive solution $(u(\lambda), v(\lambda))$ for $\lambda > \lambda_*$.\n
Moreover, \{$(\lambda, u(\lambda), v(\lambda)) : \lambda > \lambda_*$\} (in the first case, we assume $\lambda_* = 0$) is a smooth curve so that $u(\lambda), v(\lambda)$ are strictly increasing in $\lambda$, and $(u(\lambda), v(\lambda)) \to (0,0)$ as $\lambda \to \lambda_*^+$.\n
Proof. Our proof follows that of Theorem 6.1 in [26]. First we extend $f_i$, $g_i$ to be defined on $\mathbb{R}$ for $u, v < 0$ properly so they are continuous differentiable on $\mathbb{R}$. From the assumptions, $f(u,v) = f_1(v) + f_2(u)$, $g(u,v) = g_1(u) + g_2(v)$ satisfy (A$_1$). Hence from Theorem 2.3, any positive solution of (3.1) is stable.

We define
\begin{equation}
F(\lambda, u, v) = \left( \frac{\Delta u + \lambda [f_1(v) + f_2(u)]}{\lambda [g_1(u) + g_2(v)]} \right),
\end{equation}
where $\lambda \in \mathbb{R}$ and $u, v \in C^{2,\alpha}_0(\Omega)$. Here $f, g$ are at least $C^1$, then $F: \mathbb{R} \times X \to Y$ is continuously differentiable, where $X = [C^{2,\alpha}_0(\Omega)]^2$ and $Y = [C^\alpha(\Omega)]^2$. For weak solutions $(u,v)$, one can also consider $X = [W^{2,p}(\Omega) \cap W^{1,p}(\Omega)]^2$ and $Y = [L^p(\Omega)]^2$ where $p > 1$ is properly chosen.

Apparently $(\lambda, u(\lambda), v(\lambda)) = (0,0,0)$ is a solution of (3.1). We apply the implicit function theorem at $(\lambda, u, v) = (0,0,0)$. The Fréchet derivative of $F$ is given by
\begin{equation}
F_{(u,v)}(\lambda, u, v) \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \Delta \phi + \lambda [f_1'(v)\psi + f_2'(u)\phi] \\ \Delta \psi + \lambda [g_1'(u)\phi + g_2'(v)\psi] \end{pmatrix}.
\end{equation}
Then $F_{(u,v)}(0,0,0)(\phi, \psi)^T = (\Delta \phi, \Delta \psi)^T$ is an isomorphism from $X$ to $Y$, and the implicit function theorem implies that $F(\lambda, u, v) = 0$ has a unique solution $(\lambda, u(\lambda), v(\lambda))$ for $\lambda \in (0, \delta)$ for some small $\delta > 0$, and $(u'(0), v'(0))$ is the unique solution of
\begin{equation}
\begin{cases}
\Delta \phi + \lambda (f_1(0) + f_2(0)) = 0, & x \in \Omega, \\
\Delta \psi + \lambda (g_1(0) + g_2(0)) = 0, & x \in \Omega, \\
\phi(x) = \psi(x) = 0, & x \in \partial \Omega.
\end{cases}
\end{equation}
Then $(u'(0), v'(0)) = ((f_1(0) + f_2(0))e, (g_1(0) + g_2(0))e)$ where $e$ is the unique positive solution of
\begin{equation}
\Delta e + 1 = 0, \quad e(x) = 0, \quad x \in \partial \Omega.
\end{equation}
corresponding eigenvector \((k, \phi)\), there exists a positive principal eigenvalue \(\chi\) in the matrix operator of \(F\) and the corresponding eigenvector \((\phi, \psi)\). Thus, \(\phi, \psi\) are positive. Therefore by using the Perron–Frobenius theorem (see [23, Theorem 5.3.1]), there exists a positive principal eigenvalue \(\chi\) and the system (3.9) by \((\phi, \psi)\) is one-dimensional. Suppose that \((\phi, \psi)\) is a positive constant.

Next we assume that \(h(0) = 0\), and \(h'(0) > 0\) for each of \(f_1, f_2, g_1\) and \(g_2\). Then the linearized operator at \((\lambda, 0, 0)\) is

\[
F_{(u,v)}(\lambda, 0, 0) \left( \begin{array}{c} \Phi \\ \Psi \end{array} \right) = \left( \begin{array}{c} \Delta \Phi + \lambda [f'_2(0) \Phi + f'_1(0) \Psi] \\ \Delta \Psi + \lambda [g'_1(0) \Phi + g'_2(0) \Psi] \end{array} \right)
= \left( \begin{array}{c} \Delta \Phi \\ \Delta \Psi \end{array} \right) + \lambda \left( \begin{array}{cc} f'_2(0) & f'_1(0) \\ g'_1(0) & g'_2(0) \end{array} \right) \left( \begin{array}{c} \Phi \\ \Psi \end{array} \right)
= \left( \begin{array}{c} \Delta \Phi \\ \Delta \Psi \end{array} \right) + \lambda J \left( \begin{array}{c} \Phi \\ \Psi \end{array} \right),
\]

where \(J = \left( \begin{array}{cc} f'_2(0) & f'_1(0) \\ g'_1(0) & g'_2(0) \end{array} \right)\). Since \(h'(0) > 0\) for each \(h = f_i, g_i\), then all entries of matrix \(J\) are positive. Therefore by using the Perron–Frobenius theorem (see [23, Theorem 5.3.1]), there exists a positive principal eigenvalue \(\chi\) and the corresponding eigenvector \((1, k)^T\) of \(J\) for some \(k > 0\), such that \((\varphi_1, k \varphi_1)^T\) is a positive eigenvector of \(F_{(u,v)}(\lambda, 0, 0)\) where \(\lambda_* = \lambda_1 / \chi\). Similarly, the adjoint eigenvalue equation:

\[
\Delta \phi_1 + \lambda_* J^T \phi_1 = \lambda_* \phi_1,
\]

where \((\phi_1, \psi_1)^T = (\varphi_1, k \varphi_1)^T\). Inner-producting the system (3.8) by \((\phi, \psi)\), the system (3.9) by \((\phi, \psi)\), integrating over \(\Omega\) and subtracting, we obtain

\[
\int_{\Omega} (h_1 \phi_1 + h_2 \psi_1) dx = \int_{\Omega} (h_1 \varphi_1 + k_* h_2 \varphi_1) dx = 0.
\]

Hence \((h_1, h_2)^T \in R(F_{(u,v)}(\lambda_*, 0, 0))\) if and only if (3.14) holds, which implies that the codimension of \((R(F_{(u,v)}(\lambda_*, 0, 0))\) is one.
Next we verify that \( F_{\lambda(u,v)}(\lambda, 0, 0)(\varphi_1, k\varphi_1)^T \notin R(F_{(u,v)}(\lambda_*, 0, 0)) \). Indeed

\[
F_{\lambda(u,v)}(\lambda, 0, 0) \begin{pmatrix} \varphi_1 \\ k\varphi_1 \end{pmatrix} = J \begin{pmatrix} 1 \\ k \end{pmatrix} \varphi_1 = \chi J \begin{pmatrix} 1 \\ k \end{pmatrix} \varphi_1.
\]

But

\[
0 = \chi J \int_\Omega (1 + kk_*)\varphi_1^2 \, dx > 0,
\]
hence from (3.10), \( F_{\lambda(u,v)}(\lambda, 0, 0)[\varphi_1, k\varphi_1]^T \notin R(F_{(u,v)}(\lambda_*, 0, 0)) \).

Applying a bifurcation from simple eigenvalue theorem of Crandall–Rabinowitz [4], we conclude that \((\lambda_*, 0, 0)\) is a bifurcation point for (3.1), and the nontrivial solutions of \( F(\lambda, u, v) = (0, 0) \) near \((\lambda_*, 0, 0)\) are in form of

\[
\{(\lambda(s), u(s), v(s)) : s \in (-\delta, \delta) \}
\]

where \( u(s) = s\varphi_1 + o(s) \), \( v(s) = ks\varphi_1 + o(s) \).

From the stability of positive solutions, each positive solution is stable thus non-degenerate.

We claim that (3.1) has no positive solution when \( \lambda \leq \lambda_* \). We assume that \((u, v)\) is a positive solution of (3.1) and recall that \((\varphi_1, k_\ast \varphi_1)\) satisfies

\[
\Delta \begin{pmatrix} \varphi_1 \\ k_\ast \varphi_1 \end{pmatrix} + \lambda_* \begin{pmatrix} f_1'(0) \\ f_1'(0) \end{pmatrix} \begin{pmatrix} \varphi_1 \\ k_\ast \varphi_1 \end{pmatrix} = 0.
\]

Multiplying the system (3.1) by \((\varphi_1, k_\ast \varphi_1)\), the system (3.13) by \((u, v)\), integrating over \( \Omega \) and subtracting, and by using (B2) and \( h(0) = 0 \) for \( h = f, g \), we obtain

\[
\lambda_* \int_\Omega [(f_1'(0)u + f_2'(0)v)\varphi_1 + (g_1'(0)u + g_2'(0)v)k_\ast \varphi_1] \, dx
\]

\[
= \lambda \int_\Omega [f_1(u) + f_2(u))\varphi_1 + (g_1(u) + g_2(v))k_\ast \varphi_1] \, dx
\]

\[
< \lambda \int_\Omega [(f_1'(0)u + f_2'(0)v)\varphi_1 + (g_1'(0)u + g_2'(0)v)k_\ast \varphi_1] \, dx.
\]

Hence (3.1) has no positive solution when \( \lambda \leq \lambda_* \), and the bifurcating solution \((\lambda(s), u(s), v(s))\) must satisfy \( \lambda(s) > \lambda_* \) for \( s \in (0, \delta) \). Hence the curve \( \{(\lambda(s), u(s), v(s)) : s \in (0, \delta)\} \) can also be parameterized as \((\lambda, u, v)\) for \( \lambda \in (\lambda_*, \lambda_* + \delta) \). Since any positive solution is stable, then with implicit function theorem, we can extend this curve to a largest \( \lambda^* \leq \infty \).

Let \( \Gamma = \{(\lambda, u, v) : \lambda_* < \lambda < \lambda^* \} \). We show that \((u, v)\) is strictly increasing in \( \lambda \) for \( \lambda \in (\lambda_*, \lambda^*) \). In fact, \((\partial u / \partial \lambda, \partial v / \partial \lambda)\) satisfies the equation:

\[
F_{(u,v)}(\lambda, u, v) \begin{pmatrix} \partial u/\partial \lambda \\ \partial u/\partial \lambda \\ \partial v/\partial \lambda \end{pmatrix} = \begin{pmatrix} \Delta u/\partial \lambda \\ \partial v/\partial \lambda \end{pmatrix} + \lambda \begin{pmatrix} f_1'(u) \\ g_1'(u) \\ f_2'(v) \\ g_2'(v) \end{pmatrix} \begin{pmatrix} \partial u/\partial \lambda \\ \partial v/\partial \lambda \end{pmatrix}
\]

\[
= -\begin{pmatrix} f_1(u) + f_2(u) \\ g_1(u) + g_2(v) \end{pmatrix},
\]
hence \((\partial u_\lambda/\partial \lambda, \partial v_\lambda/\partial \lambda) > 0\) from the maximum principle (Lemma 2.1(c)) and the fact that \(\mu_1((\lambda, u_\lambda, v_\lambda)) > 0\) from stability of positive solutions. We claim that \(\lambda^* = \infty\). Suppose not, then \(\lambda^* < \infty\), and \(\|u_\lambda, v_\lambda\|_\infty < \infty\), then one can show that the curve \(\Gamma\) can be extended to \(\lambda = \lambda^*\) from some standard elliptic estimates, then from implicit function theorem, \(\Gamma\) can be extended beyond \(\lambda = \lambda^*\), which is a contradiction; if \(\lambda^* < \infty\), and \(\|u_\lambda, v_\lambda\|_\infty = \infty\), a contradiction can be derived with the solution curve can not blow-up at a finite \(\lambda^*\) (see similar arguments for scalar equation in [24]). Hence we must have \(\lambda^* = \infty\).

If there is another positive solution for some \(\lambda > \lambda_*\), then the arguments above show this solution also belongs to a solution curve defined for \(\lambda \in (\lambda_*, \infty)\), and the solutions on the curve are increasing in \(\lambda\), but the nonexistence of positive solutions for \(\lambda < \lambda_*\) and the local bifurcation at \(\lambda = \lambda_*\) excludes the possibility of another solution curve. Hence the positive solution is unique for all \(\lambda > \lambda_*\). □

We remark that nonlinearity \(h(x)\) satisfying (B1), (B2) and (B3) appears very often in applied problems such as ecological studies and chemical reactions. For example, the Michaelis–Menten type functions \(h(x) = ax/(1+bx)\) for \(a, b > 0\) and \(h(x) = 1 - e^{-ax}\) for \(a > 0\), see [25].

4. Application: logistic type system

In this section, we consider the following semilinear elliptic system:

\[
\begin{aligned}
\Delta u + \lambda (au - h_1(u) + f_1(v)) &= 0, \quad x \in \Omega, \\
\Delta v + \lambda (bv - h_2(v) + g_1(u)) &= 0, \quad x \in \Omega, \\
u(x) = v(x) &= 0, \quad x \in \partial \Omega,
\end{aligned}
\]

(4.1)

where \(a > 0\), \(b > 0\). Suppose that each of the functions \(f_1\), \(g_1\), \(h_1\) and \(h_2\) is a smooth real-valued function defined on \(\mathbb{R}^+\). Moreover, we assume that each of \(f_1(v)\) and \(g_1(u)\) still satisfies (B1) and (B2) as defined in Section 3; and for \(i = 1, 2\), \(h_i(x)\) satisfies:

(H1) \(h_i(0) = h_i'(0) = 0\);
(H2) \(h_i'(x) \geq 0, (h_i(x)/x)' \geq 0\), for all \(x \geq 0\);
(H3) There exists a function \(h_3 \in C^1(\mathbb{R}^+)\) such that for \(u, v \geq 0\) and \(u + v \geq 1\),

\[h_1(u) + h_2(v) \geq h_3(u + v) \quad \text{and} \quad \lim_{x \to \infty} \frac{h_3(x)}{x} = \infty.\]

Our main result in this section is as follows:

**Theorem 4.1.** Assume that each of \(f_1(v)\) and \(g_1(u)\) satisfies (B1) and (B2), \(f_1(0) = g_1(0) = 0\), and \(h_i(x)\) (for \(i = 1, 2\)) satisfies (H1)–(H3). Then there exists \(\lambda_* > 0\) such that, (4.1) has no positive solution when \(\lambda \leq \lambda_*\); (4.1) has a unique positive solution \((u(\lambda), v(\lambda))\) for \(\lambda > \lambda_*\), and \(\|u(\lambda) + v(\lambda)\|_{\infty} \leq K\), where the constant \(K\) depends only on \(a, b, g_1'(0), f_1'(0)\) and \(h_3\). Moreover,
Hence (4.1) has no positive solution when \( \lambda = \lambda_* \) owing to (H3), we get that, if \( \|u\|^{\infty} \) exists, \( \lambda_* = \chi/\chi_J \), \( \chi_J \) is the positive principal eigenvalue of the matrix

\[
J_1 = \left( \begin{array}{cc} a & f_1'(0) \\ g_1'(0) & b \end{array} \right).
\]

Note that from (4.2) \( \Delta \left( \varphi_1 \right) + \lambda_* \left( \begin{array}{c} a \\ g_1'(0) \end{array} \right) \left( \begin{array}{c} \varphi_1 \\ k_1' \end{array} \right) = 0 \), where \( (1, k_1')^T \) is the corresponding positive eigenvector of \( J_1^T \) with eigenvalue \( \chi_J \). Inner-producting the system (4.1) by \( (\varphi_1, k_1') \), the system (4.2) by \( (u, v) \), integrating over \( \Omega \) and subtracting, and by using (B2) and (H2), we obtain that

\[
\lambda_* \int_{\Omega} \left[ (au + f_1'(0)v)\varphi_1 + (g_1'(0)u + bv)k_1' \varphi_1 \right] dx = \lambda \int_{\Omega} \left[ (au - h_1(u) + f_1(v))\varphi_1 + (bv - h_2(v) + g_2(u))k_1' \varphi_1 \right] dx < \lambda \int_{\Omega} \left[ (au + f_1'(0)v)\varphi_1 + (bv + g_2'(0)v)k_1' \varphi_1 \right] dx.
\]

Hence (4.1) has no positive solution when \( \lambda \leq \lambda_* \).

Next we claim that there exists a positive constant \( K \) which depends only on \( a, g_1'(0), f_1'(0), b \) and \( h_1 \) such that any positive solution \( (u, v) \) of (4.1) satisfies \( \|u + v\|^{\infty} \leq K \). Actually, adding the equation of \( u \) and the equation of \( v \) in (4.1), owing to (H3), we get that, if \( x \in \Omega \) and \( u(x) + v(x) \geq 1 \), then

\[
-\Delta (u + v) = \lambda \left[ (au + g_1'(0)u + bv + f_1'(0)v - h_1(u) - h_2(v)) \right]
\leq \lambda \left[ (au + g_1'(0)u + bv + f_1'(0)v - h_1(u) - h_2(v)) \right]
\leq \lambda \left[ (M(u + v) - h_1(u) - h_2(v)) \right] \leq \lambda \left[ (M(u + v) - h_3(u + v)) \right],
\]

where \( M = \max \{(a + g_2'(0)), (f_1'(0) + d)\} \). Because of \( \lim_{x \to \infty} \frac{h_3(x)}{x} = \infty \), there exists \( K > 0 \) such that for \( x > K \), \( Mx - h_3(x) < 0 \). By the maximum principle, we obtain \( \|u + v\|^{\infty} \leq K \).

From the assumptions, \( au - h_1(u) + f_1(v), dv - h_2(v) + g_1(u) \) satisfy (A1). Hence from Theorem 2.3, any positive solution of (4.1) is stable. Thus, we can extend the solution branch \( \Sigma \) for \( \lambda > \lambda_* \). By using the global bifurcation theorem of Rabinowitz [20], we can conclude that either \( \Sigma \) is unbounded or \( \Sigma \) contains another bifurcation point \( (\lambda^*, 0, 0) \) with \( \lambda^* \neq \lambda_* \). But latter case cannot
happen as \( \lambda = \lambda_* \) is the only \( \lambda \) so that the corresponding linearized operator has a positive eigenvector. Hence \( \Sigma \) is unbounded. Since all positive solutions \((u,v)\) of (4.1) are uniformly bounded for \( \lambda > \lambda_* \), then \( \Sigma \) must be unbounded in \( \lambda \) direction. Similar to the proof of Theorem 3.1, we obtain that (4.1) has a unique positive solution for any \( \lambda > \lambda_* \). \( \square \)

**Example 4.2.** Consider

\[
\begin{aligned}
\Delta u + \lambda (au - u^p + cv) &= 0, \quad x \in \Omega, \\
\Delta v + \lambda (bv - v^q + du) &= 0, \quad x \in \Omega, \\
u(x) = v(x) &= 0, \quad x \in \partial \Omega,
\end{aligned}
\]

where \( a, b, c, d > 0 \), and \( p, q > 1 \). Then (4.5) is the classical cooperative logistic system when \( p = q = 2 \).

It is easy to see that the system (4.5) satisfies the conditions (B1), (B2), (H1) and (H2). We only need to verify the condition (H3). In fact, when \( p = q = 2 \), \( u^2 + v^2 \geq (u + v)^2/2 \), thus \( h_3(u + v) = (u + v)^2/2 \). When \( p \neq q \), without loss of generality, we assume that \( 1 < q \leq p \). Consider the function \( j(u) = u^p + (1 - u)^q \), then for \( u \in [0,1] \), \( j(u) \) achieves a global minimum value \( m_* > 0 \) at some \( u_* \in (0,1) \). Thus \( u^p + v^q \geq m_* \) for any \( u + v = 1, u, v \geq 0 \). This implies that for any \( u, v \geq 0 \) and \( u + v \geq 1 \), define \( V = u + v \), then \( u/V + v/V = 1 \), and \((u/V)^p + (v/V)^q \geq m_* \). It follows that

\[
\frac{u^p}{V^p} + \frac{v^q}{V^q} \geq \frac{u^p}{V^p} + \frac{v^q}{V^q} \geq m_*.
\]

Hence we can define \( h_3(u + v) = m_*(u + v)^q \) which satisfies (H3). Therefore, Theorem 4.1 implies the existence and uniqueness result for the positive solutions of (4.5), when \( \lambda > \lambda_1/\chi_J \) and \( \chi_J \) is the principal eigenvalue of the positive matrix

\[
J = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.
\]

**References**


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