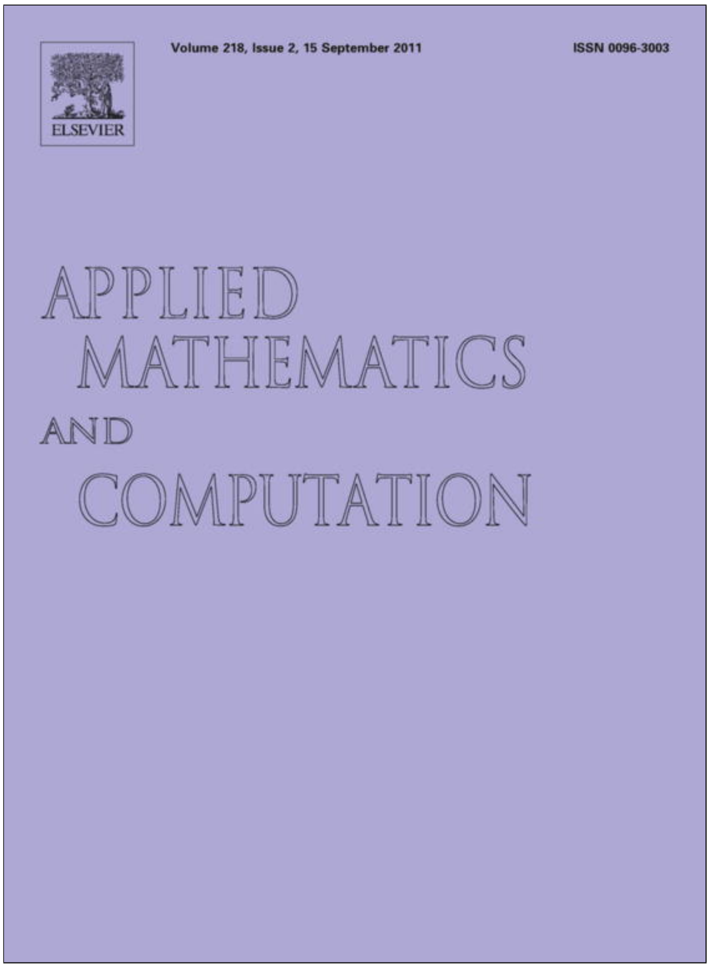
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Global stability of multigroup epidemic model with group mixing and nonlinear incidence rates

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ABSTRACT

In this paper, we introduce a basic reproduction number for a multigroup SEIR model with nonlinear incidence of infection and nonlinear removal functions between compartments. Then, we establish that global dynamics are completely determined by the basic reproduction number R_0. It shows that, the basic reproduction number R_0 is a global threshold parameter in the sense that if it is less than or equal to one, the disease free equilibrium is globally stable and the disease dies out; whereas if it is larger than one, there is a unique endemic equilibrium which is globally stable and thus the disease persists in the population. Finally, two numerical examples are also included to illustrate the effectiveness of the proposed result.

1. Introduction

Multigroup models have been proposed in the literature to describe the transmission dynamics of infectious diseases in heterogeneous host populations, such as measles, mumps, gonorrhea, HIV/AIDS, West-Nile virus and vector borne diseases such as Malaria. Heterogeneity in host population can be the result of many factors. Groups can be geographical such as communities, cities, and countries, or epidemiological, to incorporate differential infectivity or co-infection of multiple strains of the disease agent. Much research has been done on multigroup models, for example, see [1–6] and references therein. It is well known that global dynamics of multigroup models with higher dimensions, especially the global stability of the endemic equilibrium, is a very challenging problem. The question of uniqueness and global stability of the endemic equilibrium, when the basic reproduction number R_0 is greater than 1, has largely been open. Recently, the paper [7] proposed a graph-theoretic approach to the method of global Lyapunov functions and used it to establish the global stability of a unique endemic equilibrium of a multi-group SIR model with varying subpopulation sizes. Their result completely solved the open problem of the uniqueness and global stability of endemic equilibrium for this class of multi-group models. By using the results or ideas of [7], the papers [8–13] investigated uniqueness and global stability of the endemic equilibrium for several class of multigroup models, with the basic reproduction number R_0 is greater than 1, and some open problems were resolved.

In this paper, we consider a multigroup SEIR model with nonlinear incidence of infection and nonlinear removal functions between compartments. It covers many models in the literature, for example, the ones in [3,4,8,10,12,14,15]. The population is divided into n distinct groups ($n \geqslant 1$). For $1 \leqslant k \leqslant n$, the kth group is further partitioned into four compartments: the susceptible, exposed, infectious, and recovered, whose numbers of individuals at time t are denoted by $S_k(t)$, $E_k(t)$, $I_k(t)$ and $R_k(t)$, respectively. The new multigroup epidemic model with group mixing and nonlinear incidence rates as follows:

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$$\begin{cases} S_k' = \varphi_k(S_k) - \sum_{j=1}^{n} \beta_{kj} f_{kj}(S_k, I_j), \\ E_k' = \sum_{j=1}^{n} \beta_{kj} f_{kj}(S_k, I_j) - \mu_k g_k(E_k), \\ I_k' = \gamma_k g_k(E_k) - \alpha_k \psi_k(I_k), \\ R_k' = p_k \psi_k(I_k) - q_k(R_k), \end{cases} \tag{1}$$

where $\varphi_k(S_k)$ denotes the net growth of the susceptible class in the kth group, the nonlinear term $\beta_{kj}f(S_k,I_j)$ represents the cross infection from group j to group k; The matrix $B = (\beta_{ij})_{n\times n}$ is the irreducible contact matrix, where $\beta_{ij} \geqslant 0$; $\gamma_k g_k(E_k)$ accounts for the progression of individuals in group k from the exposed class into the infectious class; $\mu_k g_k(E_k), \alpha_k\psi_k(I_k)$ and $q_k(R_k)$ denote the removal of the exposed, infectious and recovered classes in the kth group, respectively, which include the mortality of individuals in the above-mentioned classes; $p_k\psi_k(I_k)$ denotes the production of the recovered individuals from infectious ones in the kth group. All constants μ_k, γ_k, α_k and p_k are assumed to be positive.

In Section 2, we first obtain that the basic reproduction number R_0 is a global threshold parameter in the sense that if it is less than or equal to one, then the disease free equilibrium is globally asymptotically stable and the disease dies out; whereas if it is larger than one, there is a unique endemic equilibrium which is globally asymptotically stable and thus the disease persists in the population. And in Section 3, some numerical simulations are showed to illustrate the effectiveness of the proposed result.

2. Main results

Since the variables R_k do not appear in the first three equations of (1), we can work on the reduced system as follows:

$$\begin{cases} S_k' = \varphi_k(S_k) - \sum_{j=1}^{n} \beta_{kj} f_{kj}(S_k, I_j), \\ E_k' = \sum_{j=1}^{n} \beta_{kj} f_{kj}(S_k, I_j) - \mu_k g_k(E_k), \\ I_k' = \gamma_k g_k(E_k) - \alpha_k \psi_k(I_k). \end{cases} \tag{2}$$

For the functions g_k, ψ_k and φ_k in (2), we assume that

- (G_1) g_k, ψ_k are local Lipschitz on $[0,\infty)$ with $g_k(0) = \psi_k(0) = 0$, g_k, ψ_k are continuous, positive, on $(0,\infty)$, the function $\frac{u}{\psi_k(u)}$ is non-increasing on $(0,\infty)$, and $\lim_{u\to 0^+} \frac{u}{\psi_k(u)} = \delta_k$ for positive constant $\delta_k > 0$;
- (G_2) φ_k are local Lipschitz on $[0,\infty)$ with $\varphi_k(0) > 0$, and the equation $\varphi_k(u) = 0$ admits a unique positive solution $u = S_k^0$ and $\varphi_k(S)(S - S_k^0) < 0$ for $S \neq S_k^0$;
- (G_3) there exist constants $D_k > 0$ and $M_k > 0$ such that

$$g_k(u) \geqslant \frac{1}{\mu_k} \max_{\xi\in[0,S_k^0]} \{\varphi_k(\xi)\}, \quad \text{for} \quad u \geqslant D_k,$$

$$\psi_k(u) \geqslant \frac{\gamma_k}{\alpha_k\mu_k} \max_{\xi\in[0,S_k^0]} \{\varphi_k(\xi)\}, \quad \text{for} \quad u \geqslant M_k.$$

The basic assumptions on functions $f_{kj}(S_k,I_j)$ are as follows:

- (H_1) $0 < \lim_{I_j\to 0^+} \frac{f_{kj}(S_k,I_j)}{I_j} = C_{kj}(S_k) \leqslant +\infty$, for $0 < S_k \leqslant S_k^0$;
- (H_2) $f_{kj}(S_k,I_j) \leqslant C_{kj}(S_k)I_j$ for all $I_j > 0$;
- (H_3) $C_{kj}(S_k) \leqslant C_{kj}(S_k^0)$, for $0 < S_k < S_k^0$, $k,j = 1,2,\ldots,n$.

In addition, we assume that

- (G_4) $\int_{0^+}^{1} \frac{1}{h(x)} dx = +\infty$, where $h \in \{C_{kk}, g_k, \psi_k\}$, $k = 1,2,\ldots,n$.

Typical examples of g_k and ψ_k are $g_k(E) = b_kE$ and $\psi_k(I) = c_kI$, and typical φ_k is $\varphi_k(S) = d_k - e_kS$. Examples of $f_{kj}(S_k,I_j)$ satisfying (H_1)–(H_3) include common incidence functions such as

$$f_{kj}(S_k, I_j) = S_kI_j \ [7,14\text{–}16], \quad f_{kj}(S_k, I_j) = f_j(I_j)S_k \ [17], \quad f_{kj}(S_k, I_j) = S_k^qI_j \ [18],$$

$$f_{kj}(S_k, I_j) = \frac{\eta S_kI_j}{1+\theta S_k} \ [2], \quad f_{kj}(S_k, I_j) = \frac{pS_kI_j}{1+\alpha I_j^2} \ [6], \quad f_{kj}(S_k, I_j) = \frac{\beta S_kI_j}{\varphi(I_j)} \ [19].$$

For each k, adding the equations in (2), one obtains that

$$(S_k+E_k)'=\varphi_k(S_k)-\mu_k g_k(E_k)\leqslant \max_{u\in[0,S_k^0]}\{\varphi_k(u)\}-\mu_k g_k(E_k)\leqslant 0,\quad \text{if } E_k\geqslant D_k$$

and

$$\left(S_k+E_k+\frac{\mu_k}{\gamma_k}I_k\right)'=\varphi_k(S_k)-\frac{\alpha_k\mu_k}{\gamma_k}\psi_k(I_k)\leqslant \max_{u\in[0,S_k^0]}\{\varphi_k(u)\}-\frac{\alpha_k\mu_k}{\gamma_k}\psi_k(I_k)\leqslant 0,\quad \text{if } I_k\geqslant M_k.$$

This indicates that the region

$$\Gamma=\left\{(S_1,E_1,I_1,S_2,E_2,I_2\ldots,S_n,E_n,I_n)\in\mathbb{R}_+^{3n}:S_k\leqslant S_k^0,S_k+E_k\leqslant S_k^0+D_k,S_k+E_k+\frac{\mu_k}{\gamma_k}I_k\leqslant S_k^0+D_k+\frac{\mu_k}{\gamma_k}M_k,k=1,2,\ldots,n\right\} \tag{3}$$

is positively invariant with respect to (2). Let $\mathring{\Gamma}$ denote the interior of Γ.

It is clear that $P_0=\left(S_1^0,0,0,S_2^0,0,0,\ldots,S_n^0,0,0\right)$ is a disease-free equilibrium of system (2). An equilibrium $P^*=(S_1^*,E_1^*,I_1^*,S_2^*,E_2^*,I_2^*,\ldots,S_n^*,E_n^*,I_n^*)$ in the interior $\mathring{\Gamma}$ of Γ is called an endemic equilibrium, where $S_k^*,\ E_k^*,\ I_k^*>0$ satisfy the following equilibrium equations

$$\varphi_k(S_k^*)=\sum_{j=1}^n\beta_{kj}f_{kj}(S_k^*,I_j^*)=\mu_k g_k(E_k^*), \tag{4}$$

$$\gamma_k g_k(E_k^*)=\alpha_k\psi_k(I_k^*). \tag{5}$$

Set $R_0=\rho(M_0)$ denote the spectral radius of the following matrix

$$M_0=M(S_1^0,S_2^0,\ldots,S_n^0)\equiv\left(\frac{\beta_{ij}\delta_j\gamma_i C_{ij}(S_i^0)}{\alpha_i\mu_i}\right)_{n\times n}.$$

If $C_{ij}(S_i^0)=+\infty$ for some i and j, we set $R_0=+\infty$. The parameter R_0 is referred to as the basic reproduction number.

We have the following basic result:

Theorem 2.1. *Assume that the functions g_k, ψ_k, φ_k and f_{ij} satisfy (G_1)–(G_4) and (H_1)–(H_3), and the matrix $B=(\beta_{ij})$ is irreducible.*

(1) If $R_0\leqslant 1$, then P_0 is the unique equilibrium of system (2) and it is globally asymptotically stable in Γ.
(2) If $R_0>1$, then P_0 is unstable and system (2) is uniformly persistent in $\mathring{\Gamma}$.

Proof. Let $Q(S,I)=\left(\frac{\gamma_i}{\alpha_i\mu_i}\frac{\beta_{ij}C_{ij}(S_i)I_j}{\psi_j(I_j)}\right)_{n\times n}$, where $S=(S_1,S_2,\ldots,S_n)$ and $I=(I_1,I_2,\ldots,I_n)$. Since $B=(\beta_{ij})_{n\times n}$ is irreducible, hence M_0 is also irreducible. One knows that there exist $\omega_k>0$, $k=1,2,\ldots,n$, such that

$$(\omega_1,\omega_2,\ldots,\omega_n)\rho(M_0)=(\omega_1,\omega_2,\ldots,\omega_n)M_0.$$

Let $V=\sum_{k=1}^n\frac{\omega_k(\gamma_k E_k+\mu_k I_k)}{\alpha_k\mu_k}$, we have

$$\begin{aligned}V'&=\sum_{k=1}^n\omega_k\left[\frac{\gamma_k\sum_{j=1}^n\beta_{kj}f_{kj}(S_k,I_j)}{\alpha_k\mu_k}-\psi_k(I_k)\right]\leqslant\sum_{k=1}^n\omega_k\left[\frac{\gamma_k}{\alpha_k\mu_k}\sum_{j=1}^n\beta_{kj}C_{kj}(S_k)I_j-\psi_k(I_k)\right]=(\omega_1,\omega_2,\ldots,\omega_n)[Q(S,I)\psi(I)-\psi(I)]\\&\leqslant(\omega_1,\omega_2,\ldots,\omega_n)[M_0\psi(I)-\psi(I)]=[\rho(M_0)-1](\omega_1,\omega_2,\ldots,\omega_n)\psi(I)\leqslant 0,\quad \text{if } \rho(M_0)\leqslant 1.\end{aligned}$$

Here $\psi(I)=(\psi_1(I_1),\psi_2(I_2),\ldots,\psi_n(I_n))$. If $\rho(M_0)<1$, then $V'=0$ if and only if $I=0$. If $\rho(M_0)=1$, then $V'=0$ implies

$$\sum_{k=1}^n\omega_k\left[\frac{\gamma_k}{\alpha_k\mu_k}\sum_{j=1}^n\frac{\beta_{kj}C_{kj}(S_k)I_j}{\psi_j(I_j)}\psi_j(I_j)\right]=\sum_{k=1}^n\omega_k\psi_k(I_k). \tag{6}$$

If $S\neq S^0\equiv(S_1^0,S_2^0,\ldots,S_n^0)$, then

$$\begin{aligned}\sum_{k=1}^n\omega_k\left[\frac{\gamma_k}{\alpha_k\mu_k}\sum_{j=1}^n\frac{\beta_{kj}C_{kj}(S_k)I_j}{\psi_j(I_j)}\psi_j(I_j)\right]&<\sum_{k=1}^n\omega_k\left[\frac{\gamma_k}{\alpha_k\mu_k}\sum_{j=1}^n\frac{\beta_{kj}C_{kj}(S_k^0)I_j}{\psi_j(I_j)}\psi_j(I_j)\right]\leqslant(\omega_1,\omega_2,\ldots,\omega_n)M_0\psi(I)\\&=(\omega_1,\omega_2,\ldots,\omega_n)\rho(M_0)\psi(I)=(\omega_1,\omega_2,\ldots,\omega_n)\psi(I),\end{aligned}$$

which implies that (6) has only the trivial solution $I=0$. Therefore, $V'=0$ if and only if $I=0$ or $S=S^0$ provided $\rho(M_0)\leqslant 1$. It can be verified that the only compact invariant subset of the set where $V'=0$ is the singleton $\{P_0\}$. By LaSalle's Invariance Principle, P_0 is globally asymptotically stable in Γ if $\rho(M_0)\leqslant 1$.

If $R_0 = \rho(M_0) > 1$ and $I \neq 0$, then

$$(\omega_1, \omega_2, \ldots, \omega_n)M_0 - (\omega_1, \omega_2, \ldots, \omega_n) = [\rho(M_0) - 1](\omega_1, \omega_2, \ldots, \omega_n) > 0$$

and thus, by continuity,

$$\sum_{k=1}^{n} \omega_k \left[\frac{\gamma_k \sum_{j=1}^{n} \beta_{kj} f_{kj}(S_k, I_j)}{\alpha_k \mu_k} - \psi_k(I_k)\right] > 0,$$

in a neighborhood of P_0 in $\mathring{\Gamma}$. This implies that P_0 is unstable. Using a uniform persistence result from [20] and a similar argument as in the Proof of Proposition 3.3 of [4], we can show that the instability of P_0 implies the uniform persistence of system (2) when $R_0 > 1$. This completes the Proof of Theorem 2.1. □

Next we show that the endemic equilibrium P^* of system (2) is unique and globally asymptotically stable when $R_0 > 1$. Note that the system (2) is uniformly persistent if $R_0 > 1$ from the Theorem 2.1, together with the uniform boundedness of solution of (2) in $\mathring{\Gamma}$, then system (2) admits at least one endemic equilibrium

$$P^* = (S_1^*, I_1^*, E_1^*, S_2^*, I_2^*, E_2^*, \ldots, S_n^*, I_n^*, E_n^*), \quad S_i^*, I_i^* \quad \text{and} \quad E_i^* > 0 \quad \text{for} \quad 1 \leqslant i \leqslant n.$$

To obtain our main global stability result, we further make the following assumptions:

(G_5) $(g_k(E_k) - g_k(E_k^*))(E_k - E_k^*) > 0$ for $E_k \neq E_k^*$, $E_k \geqslant 0$;
(G_6) $(\psi_k(I_k) - \psi_k(I_k^*))(I_k - I_k^*) > 0$ for $I_k \neq I_k^*$, $I_k \geqslant 0$;
(G_7) $(\varphi_k(S_k) - \varphi_k(S_k^*))(S_k - S_k^*) \leqslant 0$ for $S_k \geqslant 0$.

We notice that if g_k (or ψ_k) are increasing, then (G_5) (or (G_6)) holds; and if φ_k is decreasing, then (G_7) holds. For examples given earlier, all these conditions are easily satisfied. However the monotonicity of functions g_k, ψ_k and φ_k are not necessary for (G_5)–(G_7) to hold. In Section 3, we give an example of non-monotone φ_k but (G_7) is satisfied. Our main global stability result is:

Theorem 2.2. *Assume that the functions g_k, ψ_k, φ_k and f_{ij} satisfy (G_1)–(G_7) and (H_1)–(H_3), and the matrix $B = (\beta_{ij})$ is irreducible. If $R_0 > 1$ and $f_{kj}(S_k, I_j)$ also satisfy the following conditions*

(H_4) For $S_k \neq S_k^*$,

$$[\varphi_k(S_k) - \varphi_k(S_k^*)] \cdot [f_{kk}(S_k, I_k^*) - f_{kk}(S_k^*, I_k^*)] < 0;$$

(H_5) For S_k, $I_j > 0$,

$$(f_{kk}(S_k^*, I_k^*) f_{kj}(S_k, I_j) - f_{kj}(S_k^*, I_j^*) f_{kk}(S_k, I_k^*)) \cdot \left(\frac{f_{kk}(S_k^*, I_k^*) f_{kj}(S_k, I_j)}{\psi_j(I_j)} - \frac{f_{kj}(S_k^*, I_j^*) f_{kk}(S_k, I_k^*)}{\psi_j(I_j^*)}\right) \leqslant 0,$$

then there exists a unique endemic equilibrium P^* for system (2), and P^* is globally asymptotically stable in $\mathring{\Gamma}$.

Proof. We prove that P^* is globally asymptotically stable in $\mathring{\Gamma}$, which implies that the endemic equilibrium is unique. Let

$$V_k = \int_{S_k^*}^{S_k} \frac{f_{kk}(\xi, I_k^*) - f_{kk}(S_k^*, I_k^*)}{f_{kk}(\xi, I_k^*)} d\xi + \int_{E_k^*}^{E_k} \frac{g_k(\tau) - g_k(E_k^*)}{g_k(\tau)} d\tau + \frac{\mu_k}{\gamma_k} \int_{I_k^*}^{I_k} \frac{\psi_k(\tau) - \psi_k(E_k^*)}{\psi_k(\tau)} d\tau.$$

Using equilibrium Eqs. (4) and (5), one obtains that

$$V_k' = \left(1 - \frac{f_{kk}(S_k^*, I_k^*)}{f_{kk}(S_k, I_k^*)}\right)\left[\varphi_k(S_k) - \sum_{j=1}^{n} \beta_{kj} f_{kj}(S_k, I_j)\right] + \frac{g_k(E_k) - g_k(E_k^*)}{g_k(E_k)}\left[\sum_{j=1}^{n} \beta_{kj} f_{kj}(S_k, I_j) - \mu_k g_k(E_k)\right]$$
$$+ \frac{\mu_k}{\gamma_k} \frac{\psi_k(I_k) - \psi_k(I_k^*)}{\psi_k(I_k)} [\gamma_k g_k(E_k) - \alpha_k \psi_k(I_k)] = \varphi_k(S_k)\left(1 - \frac{f_{kk}(S_k^*, I_k^*)}{f_{kk}(S_k, I_k^*)}\right) + \sum_{j=1}^{n} \beta_{kj} f_{kj}(S_k, I_j) \frac{f_{kk}(S_k^*, I_k^*)}{f_{kk}(S_k, I_k^*)} + \mu_k g_k(E_k^*)$$
$$- \sum_{j=1}^{n} \beta_{kj} f_{kj}(S_k, I_j) \frac{g_k(E_k^*)}{g_k(E_k)} - \frac{\alpha_k \mu_k}{\gamma_k} \psi_k(I_k) - \mu_k g_k(E_k) \frac{\psi_k(I_k^*)}{\psi_k(I_k)} + \frac{\alpha_k \mu_k}{\gamma_k} \psi_k(I_k^*)$$
$$= \sum_{j=1}^{n} \beta_{kj} f_{kj}(S_k^*, I_j^*) \left[2 + \frac{f_{kk}(S_k^*, I_k^*) f_{kj}(S_k, I_j)}{f_{kk}(S_k, I_k^*) f_{kj}(S_k^*, I_j^*)} - \frac{g_k(E_k^*) f_{kj}(S_k, I_j)}{g_k(E_k) f_{kj}(S_k^*, I_j^*)}\right] + \varphi_k(S_k)\left(1 - \frac{f_{kk}(S_k^*, I_k^*)}{f_{kk}(S_k, I_k^*)}\right) - \frac{\alpha_k \mu_k}{\gamma_k} \psi_k(I_k) - \mu_k g_k(E_k) \frac{\psi_k(I_k^*)}{\psi_k(I_k)}$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} \beta_{kj} f_{kj}(S_k^*, I_j^*) \left[2 + \frac{f_{kk}(S_k^*, I_k^*) f_{kj}(S_k, I_j)}{f_{kk}(S_k, I_k^*) f_{kj}(S_k^*, I_j^*)} - \frac{g_k(E_k^*) f_{kj}(S_k, I_j)}{g_k(E_k) f_{kj}(S_k^*, I_j^*)} \right] + [\varphi_k(S_k) - \varphi_k(S_k^*)] \left(1 - \frac{f_{kk}(S_k^*, I_k^*)}{f_{kk}(S_k, I_k^*)} \right) \\
&\quad + \varphi_k(S_k^*) \left(1 - \frac{f_{kk}(S_k^*, I_k^*)}{f_{kk}(S_k, I_k^*)} \right) - \frac{\alpha_k \mu_k}{\gamma_k} \psi_k(I_k) - \mu_k g_k(E_k) \frac{\psi_k(I_k^*)}{\psi_k(I_k)} \\
&= \sum_{j=1}^{n} \beta_{kj} f_{kj}(S_k^*, I_j^*) \left[3 - \frac{f_{kk}(S_k^*, I_k^*)}{f_{kk}(S_k, I_k^*)} - \frac{g_k(E_k^*) f_{kj}(S_k, I_j)}{g_k(E_k) f_{kj}(S_k^*, I_j^*)} + \frac{f_{kk}(S_k^*, I_k^*) f_{kj}(S_k, I_j)}{f_{kk}(S_k, I_k^*) f_{kj}(S_k^*, I_j^*)} - \frac{\psi_k(I_k)}{\psi_k(I_k^*)} - \frac{\psi_k(I_k^*)}{\psi_k(I_k)} \frac{g_k(E_k)}{g_k(E_k^*)} \right] + [\varphi_k(S_k) \\
&\quad - \varphi_k(S_k^*)] \left(1 - \frac{f_{kk}(S_k^*, I_k^*)}{f_{kk}(S_k, I_k^*)} \right).
\end{aligned}
$$

Let $a_{kj} = \beta_{kj} f_{kj}(S_k^*, I_j^*)$, and

$$
F_{kj}(S_k, E_k, I_k, I_j) = 3 - \frac{f_{kk}(S_k^*, I_k^*)}{f_{kk}(S_k, I_k^*)} - \frac{g_k(E_k^*) f_{kj}(S_k, I_j)}{g_k(E_k) f_{kj}(S_k^*, I_j^*)} + \frac{f_{kk}(S_k^*, I_k^*) f_{kj}(S_k, I_j)}{f_{kk}(S_k, I_k^*) f_{kj}(S_k^*, I_j^*)} - \frac{\psi_k(I_k)}{\psi_k(I_k^*)} - \frac{\psi_k(I_k^*)}{\psi_k(I_k)} \frac{g_k(E_k)}{g_k(E_k^*)}.
$$

Then, by (H_4),

$$
V_k' \leqslant \sum_{j=1}^{n} a_{kj} F_{kj}(S_k, I_k, I_j). \tag{7}
$$

Let $\Phi(a) = 1 - a + \ln a$, then $\Phi(a) \leqslant 0$ for $a > 0$ and equality hold only at $a = 1$. Furthermore, under (H_5),

$$
\begin{aligned}
F_{kj}(S_k, E_k, I_k, I_j) &= G_k(I_k) - G_j(I_j) + \Phi\left(\frac{f_{kk}(S_k^*, I_k^*)}{f_{kk}(S_k, I_k^*)} \right) + \Phi\left(\frac{\psi_k(I_k^*)}{\psi_k(I_k)} \frac{g_k(E_k)}{g_k(E_k^*)} \right) + \Phi\left(\frac{\psi_j(I_j)}{\psi_j(I_j^*)} \frac{f_{kk}(S_k, I_k^*) f_{kj}(S_k^*, I_j^*)}{f_{kk}(S_k^*, I_k^*) f_{kj}(S_k, I_j)} \right) \\
&\quad + \Phi\left(\frac{g_k(E_k^*)}{g_k(E_k)} \frac{f_{kj}(S_k, I_j)}{f_{kj}(S_k^*, I_j^*)} \right) + \left[\frac{f_{kk}(S_k^*, I_k^*) f_{kj}(S_k, I_j)}{f_{kk}(S_k, I_k^*) f_{kj}(S_k^*, I_j^*)} - 1 \right] \cdot \left[1 - \frac{\psi_j(I_j)}{\psi_j(I_j^*)} \frac{f_{kk}(S_k, I_k^*) f_{kj}(S_k^*, I_j^*)}{f_{kk}(S_k^*, I_j^*) f_{kj}(S_k, I_j)} \right] \\
&\leqslant G_k(I_k) - G_j(I_j),
\end{aligned} \tag{8}
$$

where $G_k(I_k) = -\frac{\psi_k(I_k)}{\psi_k(I_k^*)} + \ln \frac{\psi_k(I_k)}{\psi_k(I_k^*)}$.

Obviously, the equalities in (7) and (8) hold if and only if

$$
\frac{f_{kk}(S_k^*, I_k^*)}{f_{kk}(S_k, I_k^*)} = 1, \quad [\varphi_k(S_k) - \varphi_k(S_k^*)] \left(1 - \frac{f_{kk}(S_k^*, I_k^*)}{f_{kk}(S_k, I_k^*)} \right) = 0
$$

and

$$
\left(\frac{f_{kk}(S_k^*, I_k^*) f_{kj}(S_k, I_j)}{f_{kk}(S_k, I_k^*) f_{kj}(S_k^*, I_j^*)} - 1 \right) \left(1 - \frac{\psi_j(I_j)}{\psi_j(I_j^*)} \frac{f_{kk}(S_k, I_k^*) f_{kj}(S_k^*, I_j^*)}{f_{kk}(S_k^*, I_k^*) f_{kj}(S_k, I_j)} \right) = 0,
$$

i.e., $S_k = S_k^*, I_k = I_k^*, k = 1, 2, \ldots, n$. We can show that V_k, F_{kj}, G_k and a_{kj} satisfy the assumptions of Theorem 3.1 and Corollary 3.3 in [10]. Therefore, the function $V = \sum_{k=1}^{n} c_k V_k$ as defined in the Theorem 3.1 of [10] is a Lyapunov function for system (2), namely, $V' \leqslant 0$ for all $(S_1, I_1, E_1, S_2, I_2, E_2, \ldots, S_n, I_n, E_n) \in \mathring{\Gamma}$. One can only show that the largest invariant subset where $V' = 0$ is the singleton $\{P^*\}$ using the same argument as in [8,10]. By LaSalle's Invariance Principle, P^* is globally asymptotically stable in $\mathring{\Gamma}$. This completes the Proof of Theorem 2.2. □

Remark 1. We present a complete proof for global asymptotic stability of unique equilibrium P^* of system (2). The paper [12] gives part of the proof for that problem when $f_{kj}(S_k, I_j) = C_k(S_k) g_j(I_j)$ in system (2).

3. Numerical example

Consider the system (2) when $k = 2$, one has a two-group model as follows:

$$
\begin{cases}
S_1' = \varphi_1(S_1) - \left[\beta_{11} \frac{S_1 I_1}{1 + I_1^2} + \beta_{12} \frac{S_1 I_2}{1 + I_2^2} \right], \\
E_1' = \left[\beta_{11} \frac{S_1 I_1}{1 + I_1^2} + \beta_{12} \frac{S_1 I_2}{1 + I_2^2} \right] - \mu_1 g_1(E_1), \\
I_1' = \gamma_1 g_1(E_1) - \alpha_1 \psi_1(I_1), \\
S_2' = \varphi_2(S_2) - \left[\beta_{21} \frac{S_2 I_1}{1 + I_1^2} + \beta_{22} \frac{S_2 I_2}{1 + I_2^2} \right], \\
E_2' = \left[\beta_{21} \frac{S_2 I_1}{1 + I_1^2} + \beta_{22} \frac{S_2 I_2}{1 + I_2^2} \right] - \mu_2 g_2(E_2), \\
I_2' = \gamma_2 g_2(E_2) - \alpha_2 \psi_2(I_2),
\end{cases} \tag{9}
$$

where

$$f_{kj}(S_k, I_j) = \frac{S_k I_j}{1 + I_j^2}, \quad k, j = 1, 2,$$
$$\varphi_1(u) = 2 - u, \quad \varphi_2(u) = 3 - u, \quad g_1(u) = g_2(u) = \psi_1(u) = \psi_2(u) = u \tag{10}$$

and

$$\gamma_1 = \frac{1}{2}, \quad \alpha_1 = 2, \quad \mu_1 = \frac{1}{4}, \quad \gamma_2 = 1, \quad \alpha_2 = \frac{1}{3}, \quad \mu_2 = 3. \tag{11}$$

If β_{ij} are chosen as

$$\beta_{11} = \frac{5}{24}, \quad \beta_{12} = \frac{1}{24}, \quad \beta_{21} = \frac{1}{36}, \quad \beta_{22} = \frac{5}{36}, \tag{12}$$

then we have $R_0 = 0.5 < 1$, hence $P_0 = (2,0,0,3,0,0)$ is the unique equilibrium of system (9) and it is globally stable in Γ from Theorem 2.1 (see Fig. 1 left panel). On the other hand, if β_{ij} are chosen as

$$\beta_{11} = \frac{1}{2}, \quad \beta_{12} = 1, \quad \beta_{21} = \frac{2}{3}, \quad \beta_{22} = \frac{1}{3}, \tag{13}$$

then we have $R_0 = 3 > 1$, hence $P^* = (1.145, 3.421, 0.855, 2.005, 0.332, 0.995)$ is a unique endemic equilibrium for system (9) and it is globally asymptotically stable in $\mathring{\Gamma}$ from Theorem 2.2 (see Fig. 1 right panel).

As a second example, we still consider (9) but with

$$f_{1j}(S_1, I_j) = S_1^2 I_j, \quad f_{2j}(S_2, I_j) = S_2^3 I_j \quad j = 1, 2,$$
$$\varphi_1(u) = \begin{cases} \frac{500}{27} + \left(u + \frac{1}{3}\right)\left(u - \frac{7}{3}\right)^2, & 0 \leqslant u \leqslant \frac{7}{3}, \\ (4 - u)(1 + u)^2, & u > \frac{7}{3}, \end{cases}$$
$$\varphi_2(u) = \begin{cases} \frac{256}{27} + \left(u + \frac{1}{6}\right)\left(u - \frac{2}{3}\right)^2, & 0 \leqslant u \leqslant \frac{2}{3}, \\ (2 - u)(2 + u)^2, & u > \frac{2}{3}, \end{cases} \tag{14}$$
$$g_1(u) = g_2(u) = u^2, \quad \psi_1(u) = \psi_2(u) = u$$

and

$$\gamma_1 = \frac{1}{8}, \quad \alpha_1 = 1, \quad \mu_1 = 2, \quad \gamma_2 = \frac{1}{4}, \quad \alpha_2 = 4, \quad \mu_2 = \frac{1}{2}. \tag{15}$$

Notice here the growth rates of susceptible class φ_k are not monotone decreasing, but one can check that the condition (G_7) is satisfied.

If β_{ij} are chosen as

$$\beta_{11} = \frac{3}{4}, \quad \beta_{12} = 2, \quad \beta_{21} = 0, \quad \beta_{22} = \frac{2}{3}, \tag{16}$$

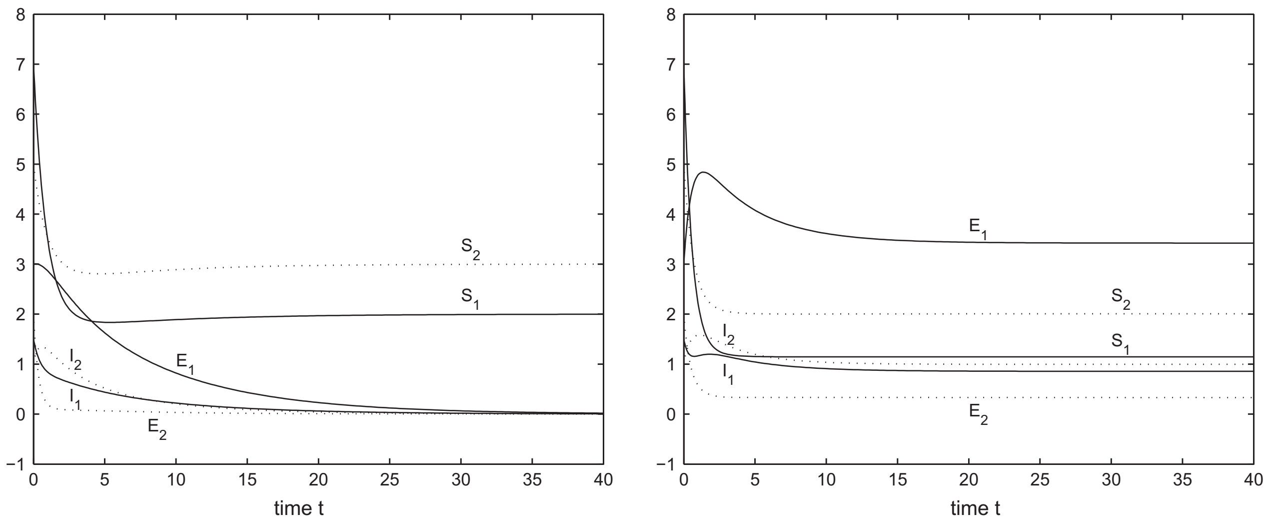

Fig. 1. Numerical simulation of (9) with functions and parameters in (10) and (11). (Left) β_{ij} as in (12), and P_0 is globally asymptotically stable; (Right) β_{ij} as in (13), and P^* is globally asymptotically stable. Initial condition in both graphs: $S_1(0) = 7$, $E_1(0) = 3$, $I_1(0) = 1.5$, $S_2(0) = 5$, $E_2(0) = 2$, $I_2(0) = 1$.

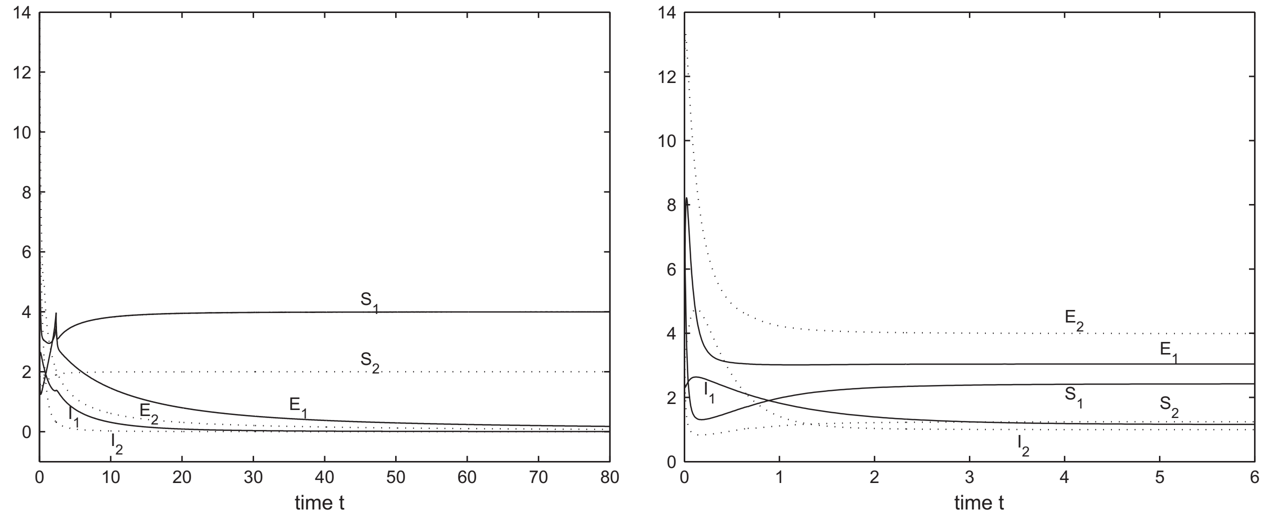

Fig. 2. Numerical simulation of (9) with functions and parameters in (14) and (15). (Left) β_{ij} as in (16), and P_0 is globally asymptotically stable; (Right) β_{ij} as in (17), and P^* is globally asymptotically stable. Initial condition in both graphs: $S_1(0) = 8$, $E_1(0) = 4$, $I_1(0) = 2.3$, $S_2(0) = 10$, $E_2(0) = 5.5$, $I_2(0) = 3$.

then we have $R_0 = 0.75 < 1$, hence $P_0 = (4, 0, 0, 2, 0, 0)$ is the unique equilibrium of system (9) and it is globally stable in Γ from Theorem 2.1 (see Fig. 2 left panel). On the other hand, if β_{ij} are chosen as

$$\beta_{11} = 1, \quad \beta_{12} = 2, \quad \beta_{21} = 1, \quad \beta_{22} = 3, \tag{17}$$

then we have $R_0 = 3.732 > 1$, hence $P^* = (2.422, 3.040, 1.159, 1.243, 3.990, 0.995)$ is a unique endemic equilibrium for system (9) and it is globally asymptotically stable in $\mathring{\Gamma}$ from Theorem 2.2 (see Fig. 2 right panel).

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