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Existence and Uniqueness of Positive Solutions for a Class of Semilinear Elliptic Systems

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Abstract The authors prove the uniqueness and existence of positive solutions for the semilinear elliptic system which involves nonlinearities with sublinear growth conditions.

Keywords Semilinear elliptic systems, positive solution, existence, uniqueness

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1 Introduction

The purpose of this paper is to study the positive solutions of the semilinear elliptic system with homogeneous Dirichlet boundary condition:

$$\begin{cases} \Delta u + a(x)f(v) = 0, & x \in \Omega, \\ \Delta v + b(x)g(w) = 0, & x \in \Omega, \\ \Delta w + c(x)h(u) = 0, & x \in \Omega, \\ u = v = w = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ denotes a bounded domain of class $C^{2,\alpha}, \alpha \in (0,1)$, $a(x)$, $b(x)$, $c(x)$ are continuous positive functions on $\overline{\Omega}$, and f, g, h are continuous and nondecreasing.

Our main result is that when the nonlinearities f, g, h have a sublinear growth rate, then there exists a positive solution of (1.1); and moreover we prove that with some additional conditions and for the ball domain, the positive solution is unique. Notice that with the conditions here, a positive solution of (1.1) on a ball is necessarily radially symmetric from the result of Troy [1].

Similar results have been proved for systems with two equations, and mostly for autonomous systems (nonlinearities independent of spatial variable x). Dalmasso [2] proved the existence of positive solution with two equations, see also Dalmasso [3], Hai and Shivaji [4]. The uniqueness problem for systems with two equations was considered in An [5], Dalmasso [2, 6], Hai [7, 8], Hai and Shivaji [9], Korman and Shi [10], Shi and Shivaji [11]. Cui et al. [12] prove a uniqueness result for a system with three equations. These results can be extended in several ways. For systems with n equations with arbitrary n, see for example, the results of Ali et al. [13], Maniwa [14], O'Reagan and Wang [15]. For extension to quasilinear systems, see, for example, Hai [16, 17], Hai and Shivaji [18], Wang [19]. Also many authors have considered elliptic systems from population biology, see, for example, [20–23].

We comment that compared to the case of systems with two equations, there are some extra difficulties in the study of systems with three or more equations. For example, some systems with two equations could have a variational structure (see Serrin and Zou [24]), but not for most systems with three or more equations. This difficulty also arises in our previous result of uniqueness [12].

We prove our existence result in Section 2, and the uniqueness result in Section 3. Throughout the paper, we assume that each of the functions f, g and h satisfies the following hypotheses:

(H1) Each of the functions f, g and h satisfies that (denote f, g or h by ψ): $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing, $\psi \in C^\beta(\mathbb{R}^+)$ for $\beta \in (0,1)$ if $n \geq 2$ and $\psi \in C^0(\mathbb{R}^+)$, and $\psi(0) \geq 0$, where $\mathbb{R}^+ = [0,\infty)$.

2 Existence

Our main existence result is as follows:

Theorem 2.1 *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ be a bounded domain with $C^{2,\alpha}$ boundary, $\alpha \in (0,1)$, and let $a(x)$, $b(x)$, $c(x)$ be continuous positive functions on $\overline{\Omega}$. Suppose that f, g, h satisfy* (H1) *and the following*

(H2) *Each of f, g and h (denoted by ψ) satisfies $\lim_{t\to 0^+} \frac{\psi(t)}{t} = \infty$ and $\lim_{t\to\infty} \frac{\psi(t)}{t} = 0$. Then* (1.1) *has at least one positive solution $(u,v,w) \in \left(C^{2,\alpha}(\overline{\Omega})\right)^3$ if $n \geq 2$ and $(u,v,w) \in \left(C^2(\overline{\Omega})\right)^3$ if $n = 1$.*

To prove the theorem, we recall some preliminaries which can be found in Chapter 6 of [25]. For $f \in C(\overline{\Omega})$, we define

$$u(x) = \int_\Omega G(x,y) f(y) dy, \quad x \in \overline{\Omega},$$

where $G(x,y)$ denotes the Green's function of the operator $-\Delta$ on Ω with zero boundary conditions.

Lemma 2.2 *Let $f \in C(\overline{\Omega})$ if $n=1$ and $f \in C^{\beta}(\overline{\Omega})$ if $n \geq 2$ with $\beta \in (0,1)$. Then*

(1) *u is the unique solution of*

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

and there exists a constant $C^1 > 0$ such that

$$\|u\|_{C^2(\overline{\Omega})} \leq C^1 \|f\|_{C(\overline{\Omega})}, \quad n = 1,$$
$$\|u\|_{C^{2,\beta}(\overline{\Omega})} \leq C^1 \|f\|_{C^{\beta}(\overline{\Omega})}, \quad n \geq 2.$$

(2) *If $f \geq 0$ and $f \not\equiv 0$, there exists $C' > 0$ such that*

$$u(x) \geq C' d(x, \partial\Omega) \quad \text{for } x \in \overline{\Omega}.$$

Let $\psi \in C^{2,\alpha}(\overline{\Omega})$ be the unique solution of

$$-\Delta\psi = 1, \quad \text{in } \Omega, \quad \psi = 0, \quad \text{on } \partial\Omega.$$

Then by Lemma 2.2, there exists $C_0 > 0$ such that $\|\psi\|_\infty \leq C_0$. Let $a_0 = \inf_{x\in\overline{\Omega}} a(x)$, $b_0 = \inf_{x\in\overline{\Omega}} b(x)$, $c_0 = \inf_{x\in\overline{\Omega}} c(x)$ and $a_1 = \sup_{x\in\overline{\Omega}} a(x)$, $b_1 = \sup_{x\in\overline{\Omega}} b(x)$, $c_1 = \sup_{x\in\overline{\Omega}} c(x)$.

Proof of Theorem 2.1 First note that $(u,v,w) \in \left(C^{2,\alpha}(\overline{\Omega})\right)^3$ is a solution to (1.1) if and only if $(u,v,w) \in \left(C^2(\overline{\Omega})\right)^3$ and satisfies

$$\begin{cases} u(x) = \displaystyle\int_\Omega G(x,y)a(y)f(v(y))dy, \\ v(x) = \displaystyle\int_\Omega G(x,y)b(y)g(w(y))dy, \\ w(x) = \displaystyle\int_\Omega G(x,y)c(y)h(u(y))dy. \end{cases}$$

Let $\phi \in C_0^\infty(\Omega)$ be a function such that $\phi \not\equiv 0$ and $0 \leq \phi \leq 1$. We define

$$m = \min_{x \in \text{supp}(\phi)} d(x, \partial\Omega).$$

By Lemma 2.2, there exists $C' > 0$ such that

$$\int_\Omega G(x,y)\phi(y)dy \geq C' d(x,\partial\Omega), \quad x \in \overline{\Omega}. \tag{2.1}$$

By (H2), we can find $l_i > 0$ $(i = 1,2,3)$ such that

$$a_0 C' f(l_1 m) \geq l_1, \quad b_0 C' g(l_2 m) \geq l_2, \quad c_0 C' h(l_3 m) \geq l_3.$$

The above inequalities also hold if $l \leq l_i$. Similarly, (H2) also implies that there exists $l_i' > l_i \max_{x\in\overline{\Omega}} d(x,\partial\Omega)$ $(i=1,2,3)$ such that

$$a_1 C_0 f(l_1') \leq l_1', \quad b_1 C_0 g(l_2') \leq l_2', \quad c_1 C_0 h(l_3') \leq l_3'.$$

And the above inequalities also hold if $l \geq l_i'$. Now let $l_0 = \min(l_1, l_2, l_3)$, $l_0' = \max(l_1', l_2', l_3')$. Then

$$a_0 C' f(l_0 m) \geq l_0, \quad a_1 C_0 f(l_0') \leq l_0'. \tag{2.2}$$

We define a set

$$Z = \{(u, v, w) \in (C(\overline{\Omega}))^3 : l_0 d(x, \partial\Omega) \leq u(x) \leq l_0', \\ l_0 d(x, \partial\Omega) \leq v(x) \leq l_0', l_0 d(x, \partial\Omega) \leq w(x) \leq l_0'\}.$$

Then Z is a bound closed convex subset of the Banach space $X = (C(\overline{\Omega}))^3$ equipped with the sup norm. Let $T : Z \to Z$ be defined by

$$\begin{aligned} T(u, v, w) &= (T_1(v), T_2(w), T_3(u)) \\ &= \left(\int_\Omega G(x, y) a(y) f(v(y)) dy, \int_\Omega G(x, y) b(y) g(w(y)) dy, \int_\Omega G(x, y) c(y) h(u(y)) dy \right), \end{aligned}$$

for $x \in \overline{\Omega}$. Let $(u, v, w) \in Z$; by (2.1) and (2.2) we have

$$\begin{aligned} T_1(v(x)) &= \int_\Omega G(x, y) a(y) f(v(y)) dy \geq \int_\Omega G(x, y) a(y) f(v(y)) \phi(y) dy \\ &\geq a_0 f(l_0 m) \int_\Omega G(x, y) \phi(y) dy \geq a_0 f(l_0 m) C' d(x, \partial\Omega) \\ &\geq l_0 d(x, \partial\Omega). \end{aligned}$$

On the other hand, we also have

$$\begin{aligned} T_1(v(x)) &= \int_\Omega G(x, y) a(y) f(v(y)) dy \leq a_1 f(l_0') \int_\Omega G(x, y) dy \\ &\leq a_1 f(l_0') C_0 \leq l_0'. \end{aligned}$$

Similarly, we can obtain

$$l_0 d(x, \partial\Omega) \leq T_2(w(x)) \leq l_0', \quad l_0 d(x, \partial\Omega) \leq T_3(u(x)) \leq l_0'.$$

Hence T is a mapping from Z to Z, and $T(Z)$ is a precompact set by Arzela–Ascoli Theorem. Therefore, by using Schauder fixed-point theorem, there exists $(u, v, w) \in Z$ such that $T(u, v, w) = (u, v, w)$. This proves the existence of a positive solution of (1.1). □

3 Uniqueness

In this section we consider the uniqueness of positive solutions for the following system:

$$\begin{cases} \Delta u + a(|x|) f(v) = 0, & x \in B, \\ \Delta v + b(|x|) g(w) = 0, & x \in B, \\ \Delta w + c(|x|) h(u) = 0, & x \in B, \\ u = v = w = 0, & x \in \partial B, \end{cases} \tag{3.1}$$

where B is the unit ball in $\mathbb{R}^n$ with $n \geq 2$. Again $f, g, h : \mathbb{R}^+ \to \mathbb{R}^+$ and $a, b, c : \mathbb{R}^+ \to (0, +\infty)$ are continuous functions. According to a result of Troy [1], positive solutions of system (3.1) are radially symmetric and decreasing in radial direction. Hence positive solutions of (3.1) satisfy

$$\begin{cases}(r^{n-1}u')' = -a(r)r^{n-1}f(v(r)), & r\in(0,1),\\ (r^{n-1}v')' = -b(r)r^{n-1}g(w(r)), & r\in(0,1),\\ (r^{n-1}w')' = -c(r)r^{n-1}h(u(r)), & r\in(0,1),\\ u'(0)=v'(0)=w'(0)=u(1)=v(1)=w(1)=0. \end{cases} \tag{3.2}$$

We make the following assumptions:

(H3) Each of f, g and h (denoted by ψ) satisfies $\psi:\mathbb{R}^+\to\mathbb{R}^+$ is continuous, nondecreasing, C^1 on $(0,\infty)$ and

$$\lim_{x\to 0+}\sup x\psi'(x)<\infty.$$

(H4) There exist nonnegative numbers p,q,r,A,B,C with $A,B,C>0$, $pqr<1$ such that

$$\lim_{x\to 0+}\inf\frac{f(x)}{x^p}>0,\quad \lim_{x\to 0+}\inf\frac{g(x)}{x^q}>0,\quad \lim_{x\to 0+}\inf\frac{h(x)}{x^r}>0,$$

and

$$\lim_{x\to\infty}\frac{f(x)}{x^p}=A,\quad \lim_{x\to\infty}\frac{g(x)}{x^q}=B,\quad \lim_{x\to\infty}\frac{h(x)}{x^r}=C \tag{3.3}$$

and for any $p_1>p, q_1>q, r_1>r$, $f(x)/x^{p_1}, g(x)/x^{q_1}$ and $h(x)/x^{r_1}$ are non-increasing for x large.

(H5) $a,b,c:\mathbb{R}^+\to(0,\infty)$ are continuous functions. Let $a_0=\min_{x\in[0,1]}a(x)>0$, $b_0=\min_{x\in[0,1]}b(x)>0$, $c_0=\min_{x\in[0,1]}c(x)>0$ and $a_1=\max_{x\in[0,1]}b(x)>0$, $b_1=\max_{x\in[0,1]}b(x)>0$, $c_1=\max_{x\in[0,1]}c(x)>0$. There exist positive numbers L_1,L_2,L_3 independent of $a(\cdot)$, $b(\cdot)$, $c(\cdot)$ such that

$$\frac{a_1}{a_0}\le L_1,\quad \frac{b_1}{b_0}\le L_2,\quad \frac{c_1}{c_0}\le L_3.$$

Our uniqueness result is:

Theorem 3.1 *Assume* (H3), (H4) *and* (H5) *hold. Then there exists a positive number* σ *such that* (3.2) *has a unique positive solution if* $\min(a_0b_0^pc_0^{pq}, b_0c_0^qa_0^{qr}, c_0a_0^rb_0^{rp})\ge\sigma$.

Before giving the proof of Theorem 3.1, we introduce two lemmas.

Lemma 3.2 ([7, Lemma 3]) *Let* $H(x)$ *be continuous on* $\mathbb{R}^+$ *and* C^1 *on* $(0,\infty)$ *such that*

$$\lim_{x\to 0+}\sup xH'(x)<\infty.$$

Let M,ε,μ *be positive numbers with* $\varepsilon<1$. *Then there exists a positive number* M' *such that*

$$|H(\nu x)-\nu^\mu H(x)|\le M'(1-\nu)$$

for $\varepsilon\le\nu\le 1$ *and* $0\le x\le M$.

The following lemma establishes the upper and lower estimates for positive solutions of system (3.2).

Lemma 3.3 *Let (u,v,w) be a positive solution of system (3.2). Then there exist positive constants M_i, $1\le i\le 6$, and σ independent of u,v,w such that*

$$M_1(a_0b_0^pc_0^{pq})^{(\frac{1}{1-pqr})}(1-r)\le u(r)\le M_2(a_0b_0^pc_0^{pq})^{(\frac{1}{1-pqr})}(1-r),\quad 0<r<1,$$
$$M_3(b_0c_0^qa_0^{qr})^{(\frac{1}{1-pqr})}(1-r)\le v(r)\le M_4(b_0c_0^qa_0^{qr})^{(\frac{1}{1-pqr})}(1-r),\quad 0<r<1,$$
$$M_5(c_0a_0^rb_0^{rp})^{(\frac{1}{1-pqr})}(1-r)\le w(r)\le M_6(c_0a_0^rb_0^{rp})^{(\frac{1}{1-pqr})}(1-r),\quad 0<r<1,$$

if $\min(a_0b_0^pc_0^{pq},b_0c_0^qa_0^{qr},c_0a_0^rb_0^{rp})\ge\sigma$.

Proof Let (u,v,w) be a positive solution of system (3.2). By integrating equations, we obtain

$$u(r)=\int_r^1\frac{1}{s^{n-1}}\left(\int_0^s\tau^{n-1}a(\tau)f(v(\tau))d\tau\right)ds,$$
$$v(r)=\int_r^1\frac{1}{s^{n-1}}\left(\int_0^s\tau^{n-1}b(\tau)g(w(\tau))d\tau\right)ds,$$
$$w(r)=\int_r^1\frac{1}{s^{n-1}}\left(\int_0^s\tau^{n-1}c(\tau)h(u(\tau))d\tau\right)ds.$$

From now on, we shall denote $C_i, i=1,2,\dots,$ positive constants independent of u,v,w,a_0,b_0,c_0.

Since w is decreasing and g is nondecreasing,

$$\begin{aligned}v\left(\frac12\right)&\ge\int_{\frac12}^1\frac{1}{s^{n-1}}\left(\int_0^{\frac12}\tau^{n-1}b(\tau)g(w(\tau))d\tau\right)ds\\&\ge\frac12 b_0g\left(w\left(\frac12\right)\right)\int_0^{\frac12}\tau^{n-1}d\tau=\frac{b_0}{n2^{n+1}}g\left(w\left(\frac12\right)\right).\end{aligned}\tag{3.4}$$

Similarly,

$$u\left(\frac12\right)\ge\frac{a_0}{n2^{n+1}}f\left(v\left(\frac12\right)\right),\quad w\left(\frac12\right)\ge\frac{c_0}{n2^{n+1}}h\left(u\left(\frac12\right)\right).\tag{3.5}$$

By (H4), there exist positive constants K_1,K_2,K_3 such that

$$f(x)\ge K_1x^p,\quad g(x)\ge K_2x^q,\quad h(x)\ge K_3x^r,\quad\text{for}\quad x\ge0.\tag{3.6}$$

The equations (3.4) and (3.5) together with (3.6) give us

$$\begin{aligned}u\left(\frac12\right)&\ge\frac{a_0}{n2^{n+1}}K_1\left\{\frac{b_0}{n2^{n+1}}K_2\left[\frac{c_0}{n2^{n+1}}K_3u^r\left(\frac12\right)\right]^q\right\}^p\\&=\frac{K_1K_2^pK_3^{pq}}{(n2^{n+1})^{1+p+pq}}(a_0b_0^pc_0^{pq})\left[u\left(\frac12\right)\right]^{pqr}.\end{aligned}$$

Hence

$$u\left(\frac12\right)\ge C_1(a_0b_0^pc_0^{pq})^{\frac{1}{1-pqr}}.\tag{3.7}$$

In a similar manner, we can show that

$$v\left(\frac12\right)\ge C_2(b_0c_0^qa_0^{qr})^{\frac{1}{1-pqr}},\quad w\left(\frac12\right)\ge C_3(c_0a_0^rb_0^{rp})^{\frac{1}{1-pqr}}.\tag{3.8}$$

Now we consider the case $r \geq \frac{1}{2}$. It follows from (3.7) and (3.8) that

$$\begin{aligned}
-u'(r) &= \frac{1}{r^{n-1}} \int_0^r a(s)s^{n-1} f\left(\int_s^1 \frac{1}{\tau^{n-1}}\left(\int_0^\tau b(\xi)\xi^{n-1} g(w(\xi))d\xi\right)d\tau\right) ds \\
&\geq a_0 \int_0^{\frac{1}{2}} s^{n-1} f\left(\int_{\frac{1}{2}}^1 \frac{1}{\tau^{n-1}}\left(\int_0^{\frac{1}{2}} b(\xi)\xi^{n-1} g(w(\xi))d\xi\right)d\tau\right) ds \\
&\geq a_0 \int_0^{\frac{1}{2}} s^{n-1} f\left(b_0 \frac{1}{n2^n} g\left(w\left(\frac{1}{2}\right)\right)\right) ds \\
&\geq \frac{K_1 K_2^p}{n2^n (n2^n)^p} a_0 b_0^p (c_0 a_0^r b_0^{rp})^{\frac{pq}{1-pqr}} \\
&= C_4 (a_0 b_0^p c_0^{pq})^{\frac{1}{1-pqr}};
\end{aligned}$$

and after integrating, we have

$$u(r) \geq C_4 (a_0 b_0^p c_0^{pq})^{\frac{1}{1-pqr}} (1-r), \quad r \geq \frac{1}{2}. \tag{3.9}$$

Similarly,

$$v(r) \geq C_5 (b_0 c_0^q a_0^{qr})^{\frac{1}{1-pqr}} (1-r), \quad w(r) \geq C_6 (c_0 a_0^r b_0^{rp})^{\frac{1}{1-pqr}} (1-r), \quad r \geq \frac{1}{2}. \tag{3.10}$$

Since u, v, w are decreasing in $(0,1)$, this implies that there exist positive constants M_1, M_3, M_5 independent of u, v, w such that the left-side inequalities for u, v, w in Lemma 3.3 hold.

From the equations of u, v, w, we get

$$|u|_0 \leq a_1 f(|v|_0), \quad |v|_0 \leq b_1 g(|w|_0), \quad |w|_0 \leq c_1 h(|u|_0), \tag{3.11}$$

where $|\cdot|_0$ denotes the sup-norm. By (3.9) and (3.10), if $a_0 b_0^p c_0^{pq}$, $b_0 c_0^q a_0^{qr}$ and $c_0 a_0^r b_0^{rp}$ are large, then

$$\begin{aligned}
|u|_0 &\geq C_4 (a_0 b_0^p c_0^{pq})^{\frac{1}{1-pqr}} \gg 1 \quad \text{(i.e. } |u|_0 \text{ is large)}, \\
|v|_0 &\geq C_5 (b_0 c_0^q a_0^{qr})^{\frac{1}{1-pqr}} \gg 1 \quad \text{(i.e. } |v|_0 \text{ is large)}, \\
|w|_0 &\geq C_6 (c_0 a_0^r b_0^{rp})^{\frac{1}{1-pqr}} \gg 1 \quad \text{(i.e. } |w|_0 \text{ is large)}.
\end{aligned}$$

Therefore, by the conditions (3.3) and (3.11) we have that

$$\begin{aligned}
|u|_0 &\leq a_1 f(|v|_0) \leq a_1 C_7 |v|_0^p, \\
|v|_0 &\leq b_1 g(|w|_0) \leq b_1 C_8 |w|_0^q, \\
|w|_0 &\leq c_1 h(|u|_0) \leq c_1 C_9 |u|_0^r.
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
|u|_0 &\leq C_7 a_1 [C_8 b_1 (C_9 c_1 |u|_0^r)^q]^p \leq C_7 \frac{a_1}{a_0} a_0 \left[C_8 \frac{b_1}{b_0} b_0 \left(C_9 \frac{c_1}{c_0} c_0 |u|_0^r\right)^q\right]^p \\
&\leq C_7 L_1 (C_8 L_2)^p (C_9 L_3)^{pq} a_0 b_0^p c_0^{pq} |u|_0^{pqr} = C_{10} (a_0 b_0^p c_0^{pq}) |u|_0^{pqr},
\end{aligned}$$

or

$$|u|_0 \leq C_{11} (a_0 b_0^p c_0^{pq})^{\frac{1}{1-pqr}}.$$

Similarly we have

$$|v|_0 \leq C_{12} (b_0 c_0^q a_0^{qr})^{\frac{1}{1-pqr}}, \quad |w|_0 \leq C_{13} (c_0 a_0^r b_0^{rp})^{\frac{1}{1-pqr}}.$$

Using this and the equation of u', we get

$$-u'(r) \le a_1 f(b_1 g(|w|_0)) \le \frac{a_1}{a_0} a_0 f\bigg(\frac{b_1}{b_0} b_0 g(C_{13}(c_0 a_0^r b_0^{rp})^{\frac{1}{1-pqr}})\bigg) \\ \le L_1 a_0 f(L_2 b_0 g(C_{13}(c_0 a_0^r b_0^{rp})^{\frac{1}{1-pqr}})) \le C_{14}(a_0 b_0^p c_0^{pq})^{\frac{1}{1-pqr}};$$

and after integration, we obtain

$$u(r) \le C_{14}(a_0 b_0^p c_0^{pq})^{\frac{1}{1-pqr}}(1-r), \quad 0 < r < 1.$$

In a similar manner, we can prove the upper estimates for $v(r)$ and $w(r)$. This completes the proof of Lemma 3.3. □

Proof of Theorem 3.1 The existence of a positive solution to (3.2) follows from Theorem 2.1. We prove the uniqueness. Let (u, v, w) and (u_1, v_1, w_1) be two positive solutions of (3.2), and let $\min(a_0 b_0^p c_0^{pq}, b_0 c_0^q a_0^{qr}, c_0 a_0^r b_0^{rp})$ be large enough so that Lemma 3.3 applies. By Lemma 3.3,

$$\frac{M_1}{M_2} u_1 \le u \le \frac{M_2}{M_1} u_1, \quad \frac{M_3}{M_4} v_1 \le v \le \frac{M_4}{M_3} v_1, \quad \frac{M_5}{M_6} w_1 \le w \le \frac{M_6}{M_5} w_1, \quad r \in (0,1).$$

Let $\alpha = \sup\{d_1 > 0 : u(x) \ge d_1 u_1(x) \,\text{in}\,(0,1)\}$, $\beta = \sup\{d_2 > 0 : v(x) \ge d_2 v_1(x) \,\text{in}\,(0,1)\}$ and $\gamma = \sup\{d_3 > 0 : w(x) \ge d_3 w_1(x) \,\text{in}\,(0,1)\}$. Then obviously $\alpha_0 \le \alpha < \infty$, $\beta_0 \le \beta < \infty$, $\gamma_0 \le \gamma < \infty$ and $u \ge \alpha u_1$, $v \ge \beta v_1$, $w \ge \gamma w_1$ in (0.1), where $\alpha_0 = M_1/M_2 > 0$, $\beta_0 = M_3/M_4 > 0$, $\gamma_0 = M_5/M_6 > 0$.

We claim that $\alpha \ge 1$, $\beta \ge 1$ and $\gamma \ge 1$. Without loss of generality, we may assume that $\alpha \le \beta \le \gamma$, then we need only to prove that $\alpha \ge 1$. Suppose to the contrary that $\alpha < 1$. We define

$$G(\tau) = g\bigg(\int_\tau^1 \frac{1}{\xi^{n-1}}\bigg(\int_0^\xi c(\eta)\eta^{n-1} h(u(\eta))d\eta\bigg)d\xi\bigg),$$

and

$$G_1(\tau) = g\bigg(\int_\tau^1 \frac{1}{\xi^{n-1}}\bigg(\int_0^\xi c(\eta)\eta^{n-1} h(u_1(\eta))d\eta\bigg)d\xi\bigg).$$

Since

$$(r^{n-1}u')' = -a(r)r^{n-1} f\bigg(\int_r^1 \frac{1}{s^{n-1}}\bigg(\int_0^s b(\tau)\tau^{n-1} G(\tau)d\tau\bigg)ds\bigg),$$

and

$$(r^{n-1}\alpha u_1')' = -a(r)r^{n-1}\alpha f\left(\int_r^1 \frac{1}{s^{n-1}}\bigg(\int_0^s b(\tau)\tau^{n-1} G_1(\tau)d\tau\bigg)ds\right),$$

then

$$(r^{n-1}(u' - \alpha u_1'))' \le -a(r)r^{n-1}\bigg[f\left(\int_r^1 \frac{1}{s^{n-1}}\bigg(\int_0^s b(\tau)\tau^{n-1} G(\tau)d\tau\bigg)ds\right) \\ - \alpha f\left(\int_r^1 \frac{1}{s^{n-1}}\bigg(\int_0^s b(\tau)\tau^{n-1} G_1(\tau)d\tau\bigg)ds\right)\bigg]. \tag{3.12}$$

Let $p_1 > p_2 > p$, $q_1 > q_2 > q$, $r_1 > r_2 > r$ and $p_1 q_1 r_1 < 1$. We claim that

$$\int_0^\xi c(\eta)\eta^{n-1} h(\alpha u_1(\eta))d\eta \ge \alpha^{r_1} \int_0^\xi c(\eta)\eta^{n-1} h(u_1(\eta))d\eta, \quad \xi \ge 0. \tag{3.13}$$

Since $\alpha \geq \alpha_0 > 0$ and $h(x)/x^{r_2}$ is nonincreasing for large x, we obtain

$$\frac{h(\alpha x)}{\alpha^{r_2} x^{r_2}} \geq \frac{h(x)}{x^{r_2}}.$$

It gives us

$$h(\alpha x) \geq \alpha^{r_2} h(x) \quad \text{for } x \gg 1. \tag{3.14}$$

Let $1/2 < T < 1$. By Lemma 3.3,

$$u_1(\eta) \geq M_1(1-T)(a_0 b_0^p c_0^{pq})^{\frac{1}{1-pqr}} \gg 1, \quad \eta \leq T;$$

and therefore

$$\begin{aligned}&\int_0^\xi c(\eta)\eta^{n-1}\left[h(\alpha u_1(\eta)) - \alpha^{r_1} h(u_1(\eta))\right] d\eta \\ &\quad \geq (\alpha^{r_2} - \alpha^{r_1}) \int_0^\xi c(\eta)\eta^{n-1} h(u_1(\eta)) d\eta \geq 0, \quad \xi \leq T.\end{aligned}$$

For $\xi > T$,

$$\begin{aligned}&\int_0^\xi c(\eta)\eta^{n-1}\left[h(\alpha u_1(\eta)) - \alpha^{r_1} h(u_1(\eta))\right] d\eta \\ &\quad = \int_0^T c(\eta)\eta^{n-1}\left[h(\alpha u_1(\eta)) - \alpha^{r_1} h(u_1(\eta))\right] d\eta \\ &\qquad + \int_T^\xi c(\eta)\eta^{n-1}\left[h(\alpha u_1(\eta)) - \alpha^{r_1} h(u_1(\eta))\right] d\eta \\ &\quad \geq (\alpha^{r_2} - \alpha^{r_1}) \int_0^T c(\eta)\eta^{n-1} h(u_1(\eta)) d\eta - M'(1-T)(1-\alpha)c_1,\end{aligned}$$

where we have used Lemma 3.2 with $H(x) = h(x)$. By (3.6), (3.7) and (3.8),

$$\int_0^T c(\eta)\eta^{n-1} h(u_1(\eta)) d\eta \geq \int_0^{\frac{1}{2}} c(\eta)\eta^{n-1} h(u_1(\eta)) d\eta \geq \frac{c_0 h(u_1(\frac{1}{2}))}{n 2^n} > 0.$$

Since there exists a positive number $k_1 > 0$ such that

$$\alpha^{r_2} - \alpha^{r_1} \geq k_1(1-\alpha), \quad \text{for} \quad 0 < \alpha_0 \leq \alpha < 1,$$

it follows that

$$\int_0^\xi c(\eta)\eta^{n-1}[h(\alpha u_1(\eta)) - \alpha^{r_1} h(u_1(\eta))] d\eta > 0, \quad \xi > T,$$

if T is sufficiently close to 1. This proves the claim (3.13).

Similarly we have

$$\begin{aligned}&\int_0^s b(\tau)\tau^{n-1} g\bigg(\int_\tau^1 \frac{1}{\xi^{n-1}}\bigg(\int_0^\xi c(\eta)\eta^{n-1} h(\alpha u_1) d\eta\bigg) d\xi\bigg) d\tau \\ &\quad \geq \int_0^s b(\tau)\tau^{n-1} g\bigg(\alpha^{r_1} \int_\tau^1 \frac{1}{\xi^{n-1}}\bigg(\int_0^\xi c(\eta)\eta^{n-1} h(u_1) d\eta\bigg) d\xi\bigg) d\tau \\ &\quad \geq \int_0^s b(\tau)\tau^{n-1} \alpha^{q_1 r_1} g\bigg(\int_\tau^1 \frac{1}{\xi^{n-1}}\bigg(\int_0^\xi c(\eta)\eta^{n-1} h(u_1) d\eta\bigg) d\xi\bigg) d\tau.\end{aligned} \tag{3.15}$$

Substituting (3.15) into (3.12) and integrating give us

$$z^{n-1}(u'-\alpha u_1')(z) \le -\int_0^z B(\alpha,r)dr,$$

where

$$B(\alpha,r) = a(r)r^{n-1}\Big[f\left(\alpha^{q_1r_1}\int_r^1 \frac{1}{s^{n-1}}\Big(\int_0^s b(\tau)\tau^{n-1}G_1(\tau)d\tau\Big)ds\right) - \alpha f\left(\int_r^1 \frac{1}{s^{n-1}}\Big(\int_0^s b(\tau)\tau^{n-1}G_1(\tau)d\tau\Big)ds\right)\Big].$$

Using (3.6) and Lemma 3.3, we obtain for $r \le T$,

$$\begin{aligned}
&\int_r^1 \frac{1}{s^{n-1}}\Big(\int_0^s b(\tau)\tau^{n-1}g\Big(\int_\tau^1 \frac{1}{\xi^{n-1}}\Big(\int_0^\xi c(\eta)\eta^{n-1}h(u_1)d\eta\Big)d\xi\Big)d\tau\Big)ds\\
&\ge \int_T^1 \frac{1}{s^{n-1}}\Big(\int_0^T b(\tau)\tau^{n-1}g\Big(\int_T^1 \frac{1}{\xi^{n-1}}\Big(\int_0^T c(\eta)\eta^{n-1}h(u_1)d\eta\Big)d\xi\Big)d\tau\Big)ds\\
&\ge \int_T^1 \frac{1}{s^{n-1}}\Big(\int_0^T b(\tau)\tau^{n-1}g\left(c_0\frac{T^n}{n}(1-T)h(u_1(T))\right)d\tau\Big)ds\\
&\ge \int_T^1 \frac{1}{s^{n-1}}\Big(\int_0^T b(\tau)\tau^{n-1}g\left(\frac{c_0T^n(1-T)K_3}{n}u_1^r(T)\right)d\tau\Big)ds\\
&\ge b_0\frac{T^n(1-T)}{n}g\left(\frac{c_0T^n(1-T)K_3}{n}u_1^r(T)\right)\\
&\ge \frac{T^n(1-T)K_2}{n}\left(\frac{T^n(1-T)K_3}{n}\right)^q b_0c_0^qu_1^{qr}(T)\\
&\ge c_1(T)(b_0c_0^qa_0^{qr})^{\frac{1}{1-pqr}} \gg 1,
\end{aligned}$$

where

$$c_1(T) = \frac{T^{n(q+1)}(1-T)^{(r+1)q+1}K_2K_3^qM_1^{qr}}{n^{q+1}}.$$

Since $f(x)/x^{p_1}$ is nonincreasing for $x \gg 1$,

$$f(\alpha^{q_1r_1}x) \ge \alpha^{p_1q_1r_1}f(x) \quad \text{for} \quad x \gg 1.$$

Note that there exists a positive number $k_2 > 0$ such that

$$\alpha^{p_1q_1r_1} - \alpha \ge k_2(1-\alpha) \quad \text{for} \quad 0 < \alpha_0 \le \alpha < 1,$$

and therefore for $r \le T$,

$$\begin{aligned}
B(\alpha,r) &\ge r^{n-1}(\alpha^{p_1q_1r_1}-\alpha)f\left(\int_r^1 \frac{1}{s^{n-1}}\Big(\int_0^s b(\tau)\tau^{n-1}G_1(\tau)d\tau\Big)ds\right)\\
&\ge r^{n-1}(\alpha^{p_1q_1r_1}-\alpha)K_1[c_1(T)]^p(b_0c_0^qa_0^{qr})^{\frac{p}{1-pqr}}\\
&\ge c_2(T)r^{n-1}(1-\alpha)(b_0c_0^qa_0^{qr})^{\frac{p}{1-pqr}}\\
&> 0,
\end{aligned} \tag{3.16}$$

where $c_2(T) = K_1c_1^p(T)k_2$. This proves that

$$z^{n-1}(u'-\alpha u_1')(z) \le 0, \quad 0 < z \le T.$$

For $z > T$, we have by Lemma 3.2 and (3.16)

$$\begin{aligned}\int_0^z B(\alpha,r)dr &\geq \int_0^{\frac{1}{2}} B(\alpha,r)dr + \int_T^z B(\alpha,r)dr \\ &\geq \frac{c_2(\frac{1}{2})}{n2^n}(1-\alpha)(b_0c_0^q a_0^{qr})^{\frac{p}{1-pqr}} - M'(1-T)(1-\alpha)a_1 \\ &= (1-\alpha)\left[\frac{c_2(\frac{1}{2})}{n2^n}(b_0c_0^q a_0^{qr})^{\frac{p}{1-pqr}} - M'(1-T)a_1\right] \\ &> 0\end{aligned}$$

for large $b_0c_0^q a_0^{qr}$ and T sufficiently close to 1. Hence

$$(u' - \alpha u_1')(z) \leq 0, \quad 0 < z \leq 1.$$

It follows that there exists $\widetilde{\alpha} > \alpha$ such that $u \geq \widetilde{\alpha} u_1$ in $(0,1)$, which is a contradiction! Thus $\alpha \geq 1$ and moreover $\beta \geq 1, \gamma \geq 1$. Hence $u \geq u_1$, $v \geq v_1$, $w \geq w_1$. Similarly, we can prove $u \leq u_1$, $v \leq v_1$, $w \leq w_1$. Consequently, $u = u_1$, $v = v_1$, $w = w_1$ in $(0,1)$, which completes the proof of Theorem 3.1. □

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