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Bifurcation from a degenerate simple eigenvalue

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Abstract

It is proved that a symmetry-breaking bifurcation occurs at a simple eigenvalue despite the usual transversality condition fails, and this bifurcation from a degenerate simple eigenvalue result complements the classical one with the transversality condition. The new result is applied to an imperfect pitchfork bifurcation, in which a forward transcritical bifurcation changes to a backward one when the perturbation parameter changes. Several applications in ecological and genetics models are shown.

Keywords: Bifurcation from degenerate simple eigenvalue; Imperfect pitchfork bifurcation; Bifurcation in Banach spaces

1. Introduction

In this paper we revisit the bifurcation problem of the nonlinear equation

$$F(\lambda, u) = 0, \tag{1.1}$$

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where $F \in C^p(\mathbb{R} \times X, Y)$, $p \geqslant 1$, is a nonlinear differentiable mapping, and X, Y are Banach spaces.

If Eq. (1.1) has a trivial solution $u = u_0$ for any $\lambda \in \mathbb{R}$, then a necessary condition for a bifurcation point (λ_0, u_0) is that the linearization of F with respect to u at $\lambda = \lambda_0$ is not invertible. Indeed let $F_u(\lambda_0, u_0)$ be the Fréchet derivative of $F(\lambda, u)$ in u at (λ_0, u_0), then the null space $N(F_u(\lambda_0, u_0))$ contains a non-zero element if (λ_0, u_0) is a bifurcation point. A well-known sufficient condition for bifurcation is that zero is a simple eigenvalue of $F_u(\lambda_0, u_0)$, and the zero eigenvalue moves across $\lambda = \lambda_0$ "transversally". To be more precise, 0 is a simple eigenvalue of $F_u(\lambda_0, u_0)$ if the following assumption is satisfied:

(F1) $\dim N(F_u(\lambda_0, u_0)) = \operatorname{codim} R(F_u(\lambda_0, u_0)) = 1$, and $N(F_u(\lambda_0, u_0)) = \operatorname{span}\{w_0\}$,

where $N(F_u)$ and $R(F_u)$ are the null space and the range of linear operator $F_u = F_u(\lambda_0, u_0)$. Crandall and Rabinowitz [6] prove the following celebrated "bifurcation from a simple eigenvalue" theorem (see [6, Theorem 1.7]):

Theorem 1.1. *Let U be a neighborhood of (λ_0, u_0) in $\mathbb{R} \times X$, and $F(\lambda, u_0) = 0$ for $(\lambda, u_0) \in U$. Assume that the partial derivatives F_u, F_λ and $F_{\lambda u}$ exist and are continuous in U; at (λ_0, u_0), F satisfies* **(F1)** *and*

(F3) $F_{\lambda u}(\lambda_0, u_0)[w_0] \notin R(F_u(\lambda_0, u_0))$.

Let Z be any complement of $\operatorname{span}\{w_0\}$ in X. Then the solution set of $F(\lambda, u) = 0$ near (λ_0, u_0) consists precisely of the curves $u = u_0$ and $\Sigma = \{(\lambda(t), u(t))\colon s \in I \equiv (-\epsilon, \epsilon)\}$, where $\lambda : I \to \mathbb{R}$, $z : I \to Z$ are continuous functions such that $u(t) = u_0 + sw_0 + sz(t)$, $\lambda(0) = \lambda_0$, $z(0) = 0$.

The bifurcation from a simple eigenvalue theorem has been one of the fundamental tools in showing the occurrence of symmetry-breaking spatial patterns in many nonlinear problems. Some novel applications in nonlinear partial differential equation models include the existence of steady periodic water waves [5,30,43], free boundary problem in tumor models and cell growth [14–16], and the existence of nonconstant stationary patterns in spatial ecological models [9,10,20,44,45]. Other applications are also found in nonlinear matrix population models [8], and nonlinear ordinary differential equation population models [21]. In such a bifurcation, a curve of non-trivial solutions emanates from the line of trivial ones. It is also important to determine the direction of the bifurcating curve and the stability of the bifurcating solutions. If in addition, F is C^2 in u, then the bifurcating curve $\Sigma = \{(\lambda(s), u(s))\colon s \in I\}$ in Theorem 1.1 is differentiable. If $\lambda'(0) \neq 0$, then a *transcritical bifurcation* occurs near (λ_0, u_0) (see Fig. 1 left panel); and if $\lambda'(0) = 0$, and $F \in C^3$, $\lambda''(0) \neq 0$, then a *pitchfork bifurcation* occurs near (λ_0, u_0) (see Fig. 1 right panel).

The transversality condition (**F3**) holds in generic situations. But there are important exceptions for which (**F3**) fails. In this paper, we consider a degenerate bifurcation scenario in which (**F1**) is satisfied but (**F3**) is not satisfied. In this case, we prove that, under some higher order transversality conditions on F, the local solution set of (1.1) near the bifurcation point (λ_0, u_0) consists of the line of trivial solutions, and two other solution curves; each of these two curves could be similar to the one in transcritical or pitchfork bifurcation, or in a degenerate case, identical to the curve of trivial solutions. To compare our results with Theorem 1.1, we use some

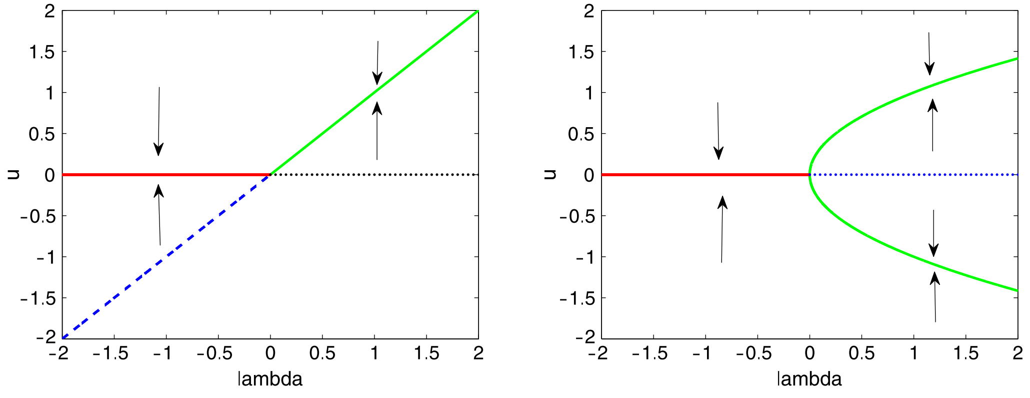

Fig. 1. Bifurcation diagrams. (Left): $F(\lambda, u) = \lambda u - u^2$; (Right): $F(\lambda, u) = \lambda u - u^3$.

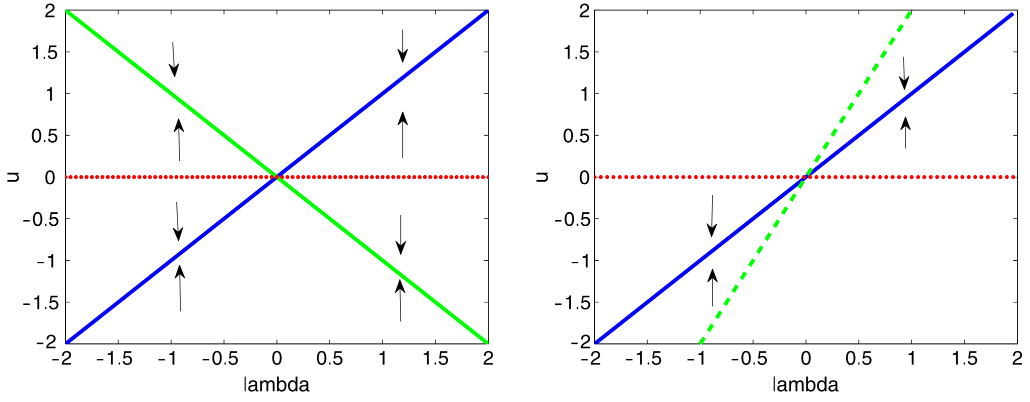

Fig. 2. Bifurcation diagrams. (Left): $F(\lambda, u) = u(\lambda^2 - u^2)$; (Right): $F(\lambda, u) = u(u - \lambda)(u - 2\lambda)$.

normal form mappings $F_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ to illustrate our results. The normal forms of transcritical and pitchfork bifurcations shown in Theorem 1.1 are (always assuming $(\lambda_0, u_0) = (0, 0)$, see Fig. 1)

$$F_1(\lambda, u) = \lambda u - u^2, \quad \text{and} \quad F_2(\lambda, u) = \lambda u - u^3. \tag{1.2}$$

The new bifurcation theorem which we prove here (see Figs. 2 and 3) shows normal forms of

$$F_3(\lambda, u) = u(\lambda - u)(\lambda + au),$$

and

$$F_4(\lambda, u) = u(\lambda - u)(\lambda - au), \quad a > 0(\neq 1); \tag{1.3}$$

and also the degenerate cases

$$F_5(\lambda, u) = \lambda u^2 - u^3, \quad \text{and} \quad F_6(\lambda, u) = \lambda u^2 - u^4. \tag{1.4}$$

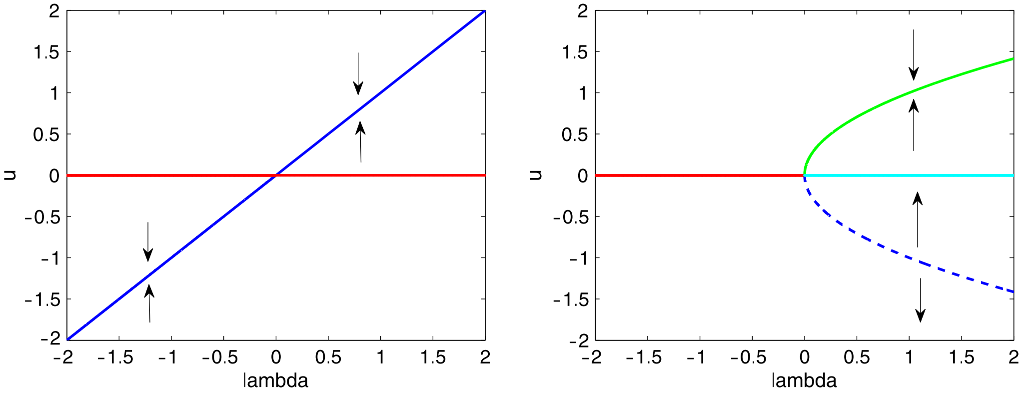

Fig. 3. Bifurcation diagrams. (Left): $F(\lambda, u) = \lambda u^2 - u^3$; (Right): $F(\lambda, u) = \lambda u^2 - u^4$.

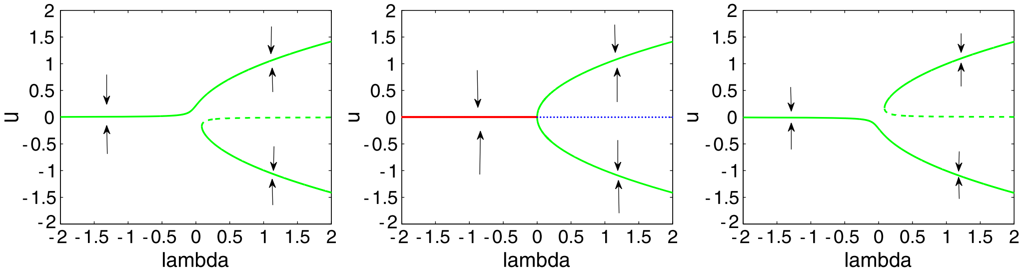

Fig. 4. Imperfect bifurcation not preserving the trivial solutions: bifurcation diagrams of $F(\varepsilon, \lambda, u) = \lambda u - u^3 - \varepsilon = 0$. (Left: $\varepsilon < 0$; Middle: $\varepsilon = 0$; Right: $\varepsilon > 0$.)

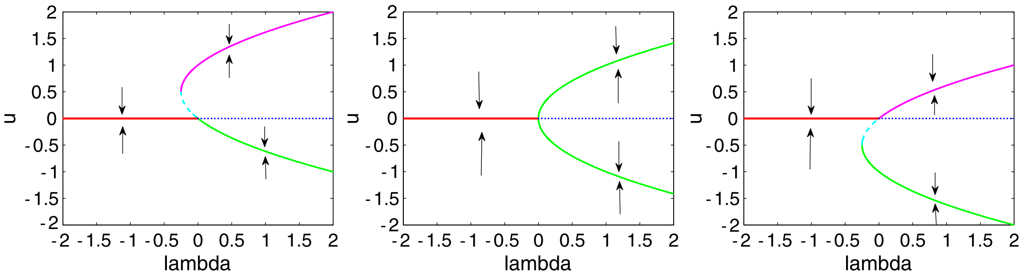

Fig. 5. Imperfect bifurcation preserving the trivial solutions: bifurcation diagrams of $F(\varepsilon, \lambda, u) = \lambda u - u^3 - \varepsilon u^2 = 0$. (Left: $\varepsilon < 0$; Middle: $\varepsilon = 0$; Right: $\varepsilon > 0$.)

The motivation of studying such degenerate bifurcations is to consider some imperfect bifurcations arising from applications. A typical perturbation to a pitchfork bifurcation destroys the original symmetry, then the trivial solutions cannot be preserved (see Fig. 4). But in many application problems, the trivial solutions are preserved under the perturbation, but the perturbed problem may have a different bifurcation structure. Typically a forward (supercritical) bifurcation which produces a stable non-trivial equilibria becomes a backward (subcritical) bifurcation one in which the bifurcating solutions are unstable (see Fig. 5). This can be best demonstrated by the backward bifurcations in epidemics models [11,18], and catastrophic shifts in ecosystems such as deserts, lakes, and corral reefs [34,35]. We show that this phenomenon (as shown in Fig. 5) can be put under the framework of a perturbed bifurcation problem

$$F(\varepsilon, \lambda, u) = 0, \tag{1.5}$$

where ε is a perturbation parameter, and the slight change of ε-value causes variation of the bifurcation diagram in (λ, u)-space. Such imperfect bifurcations have been considered in [23,38], and this study is partially a sequel to these previous ones as we analyze the new bifurcation scenario as in Fig. 5 by applying the new bifurcation from a degenerate simple eigenvalue theorem mentioned above. Some other related recent studies of imperfect bifurcations can be found in [24,32]. The question of whether a perturbation would preserve or destroy the trivial solutions was also considered in [37], but with a different approach.

As remarked in [23], Lyapunov–Schmidt reduction is an important method to reduce an infinite-dimensional problem to a finite-dimensional one, and the theory of singularities of differentiable maps and catastrophe theory are useful in the qualitative studies of such finite-dimensional problems. In particular, the imperfect bifurcation of (Lyapunov–Schmidt) reduced maps has been considered in Golubitsky and Schaeffer [17]. The approach given here (as well as in the one in [23]) directly deals with the original infinite-dimensional problems, and the conditions are based on various partial derivatives of the nonlinear maps on Banach spaces but not derivatives of reduced finite-dimensional maps. This follows the approach in Crandall and Rabinowitz [6,7], which has been widely utilized in applications mentioned above.

In Section 2, we prove the bifurcation from a degenerate simple eigenvalue theorem, and in Section 3 we prove the related stability results. In Section 4 we apply the bifurcation from a degenerate simple eigenvalue theorem to the imperfect bifurcation problem to obtain the precise local bifurcation diagrams near $\varepsilon = \varepsilon_0$. In Section 5, we demonstrate the applications to several imperfect bifurcation problems from mathematical biology.

We use the same labeling of conditions such as (**F1**), (**F2**) on F as in our previous work [38,23], and we use the convention that (**Fi**$'$) stands for the negation of (**Fi**) for i $\in \mathbb{N}$. In the paper, we use $\|\cdot\|$ as the norm of Banach space X, $\langle\cdot,\cdot\rangle$ as the duality pair of a Banach space X and its dual space X^*. For a linear operator L, we use $N(L)$ as the null space of L and $R(L)$ as the range space of L, and we use $L[w]$ to denote the image of w under the linear mapping L. For a multilinear operator L, we use $L[w_1, w_2, \ldots, w_k]$ to denote the image of $(w_1, w_2, \ldots, w_k)$ under L, and when $w_1 = w_2 = \cdots = w_k$, we use $L[w_1]^k$ instead of $L[w_1, w_1, \ldots, w_1]$. For a nonlinear operator F, we use F_u as the partial derivative of F with respect to argument u.

2. Bifurcation from a degenerate simple eigenvalue

We assume that F satisfies (**F1**) at (λ_0, u_0), then we have decompositions of X and Y: $X = N(F_u(\lambda_0, u_0)) \oplus Z$ and $Y = R(F_u(\lambda_0, u_0)) \oplus Y_1$, where Z is a complement of $N(F_u(\lambda_0, u_0))$ in X, and Y_1 is a complement of $R(F_u(\lambda_0, u_0))$. In particular, $F_u(\lambda_0, u_0)|_Z : Z \to R(F_u(\lambda_0, u_0))$ is an isomorphism. Since $R(F_u(\lambda_0, u_0))$ is codimension one, then there exists $l \in Y^*$ such that $R(F_u(\lambda_0, u_0)) = \{v \in Y\colon \langle l, v\rangle = 0\}$.

We comment that in Theorem 1.1, if F is C^2 near (λ_0, u_0), then the curve of non-trivial solutions is differentiable, and one has the formula for the bifurcation direction:

$$\lambda'(0) = -\frac{\langle l, F_{uu}(\lambda_0, u_0)[w_0]^2\rangle}{2\langle l, F_{\lambda u}(\lambda_0, u_0)[w_0]\rangle}. \tag{2.1}$$

If F satisfies

(F4) $F_{uu}(\lambda_0, u_0)[w_0]^2 \notin R(F_u(\lambda_0, u_0))$,

then we have $\lambda'(0) \neq 0$, and a transcritical bifurcation occurs. If F satisfies (**F4**$'$) and $F \in C^3$, then $\lambda'(0) = 0$ and

$$\lambda''(0) = -\frac{\langle l, F_{uuu}(\lambda_0, u_0)[w_0]^3\rangle + 3\langle l, F_{uu}(\lambda_0, u_0)[w_0, v_2]\rangle}{3\langle l, F_{\lambda u}(\lambda_0, u_0)[w_0]\rangle}, \tag{2.2}$$

where v_2 satisfies $F_{uu}(\lambda_0, u_0)[w_0]^2 + F_u(\lambda_0, u_0)[v_2] = 0$. A pitchfork bifurcation typically satisfies $\lambda''(0) \neq 0$.

We recall two important lemmas from our previous work [23]. First is the well-known Lyapunov–Schmidt reduction under the condition (**F1**) which is standard from any textbook (see for example [4,29]).

Lemma 2.1. *Suppose that $F : \mathbb{R} \times X \to Y$ is a C^p ($p \geqslant 1$) mapping such that $F(\lambda_0, u_0) = 0$, and F satisfies* (**F1**) *at (λ_0, u_0). Then the equation $F(\lambda, u) = 0$ for (λ, u) near (λ_0, u_0) can be reduced to*

$$\big\langle l, F\big(\lambda, u_0 + tw_0 + g(\lambda, t)\big)\big\rangle = 0,$$

where $t \in (-\delta, \delta)$, $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$ where δ is a small constant, $l \in Y^*$ such that $\langle l, v\rangle = 0$ if and only if $v \in R(F_u(\lambda_0, u_0))$, and g is a C^p function into Z such that $g(\lambda_0, 0) = 0$ and Z is a complement of $N(F_u(\lambda_0, u_0))$ in X.

Next we recall the following lemma (see [23, Lemma 2.5]) which describes the structure of zero-set of a function defined near a critical point in $\mathbb{R}^2$.

Lemma 2.2. *Suppose that $(x_0, y_0) \in \mathbb{R}^2$ and U is a neighborhood of (x_0, y_0). Assume that $f : U \to \mathbb{R}$ is a C^p function for $p \geqslant 2$, $f(x_0, y_0) = 0$, $\nabla f(x_0, y_0) = 0$, and the Hessian matrix $H = \nabla^2 f(x_0, y_0)$ is non-degenerate. Then*

1. *If H is definite* (*i.e.* $\det(H) > 0$)*, then (x_0, y_0) is the unique zero point of $f(x, y) = 0$ near (x_0, y_0);*
2. *If H is indefinite* (*i.e.* $\det(H) < 0$)*, then there exist two C^{p-1} curves $(x_i(t), y_i(t))$, $i = 1, 2$, $t \in (-\delta, \delta)$, such that the solution set of $f(x, y) = 0$ consists of exactly the two curves near (x_0, y_0), $(x_i(0), y_i(0)) = (x_0, y_0)$. Moreover t can be rescaled and indices can be rearranged so that $(x_1'(0), y_1'(0))$ and $(x_2'(0), y_2'(0))$ are the two linear independent solutions of*

$$f_{xx}(x_0, y_0)\eta^2 + 2f_{xy}(x_0, y_0)\eta\tau + f_{yy}(x_0, y_0)\tau^2 = 0. \tag{2.3}$$

Now we are ready to state our main result on the bifurcation from a degenerate simple eigenvalue.

Theorem 2.3. *Let U be a neighborhood of (λ_0, u_0) in $\mathbb{R} \times X$, and let $F \in C^3(U, Y)$. Assume that $F(\lambda, u_0) = 0$ for $(\lambda, u_0) \in U$. At (λ_0, u_0), F satisfies* (**F1**)*,*

(F3′) $F_{\lambda u}(\lambda_0, u_0)[w_0] \in R(F_u(\lambda_0, u_0))$; *and*
(F4′) $F_{uu}(\lambda_0, u_0)[w_0]^2 \in R(F_u(\lambda_0, u_0))$.

Let $X = N(F_u(\lambda_0, u_0)) \oplus Z$ *be a fixed splitting of* X, *and let* $l \in Y^*$ *such that* $R(F_u(\lambda_0, u_0)) = \{v \in Y \colon \langle l, v\rangle = 0\}$. *Denote by* $v_1 \in Z$ *the unique solution of*

$$F_{\lambda u}(\lambda_0, u_0)[w_0] + F_u(\lambda_0, u_0)[v] = 0, \tag{2.4}$$

and $v_2 \in Z$ *the unique solution of*

$$F_{uu}(\lambda_0, u_0)[w_0]^2 + F_u(\lambda_0, u_0)[v] = 0. \tag{2.5}$$

We assume that the matrix (all derivatives are evaluated at (λ_0, u_0)*)*

$$H = H(\lambda_0, u_0) = \begin{pmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{pmatrix} \tag{2.6}$$

is non-degenerate, i.e., $\det(H) \neq 0$, *where* H_{ij} *is given by*

$$H_{11} = \bigl\langle l, F_{\lambda\lambda u}[w_0] + 2F_{\lambda u}[v_1]\bigr\rangle, \tag{2.7}$$

$$H_{12} = \frac{1}{2}\bigl\langle l, F_{\lambda uu}[w_0]^2 + F_{\lambda u}[v_2] + 2F_{uu}[w_0, v_1]\bigr\rangle, \tag{2.8}$$

$$H_{22} = \frac{1}{3}\bigl\langle l, F_{uuu}[w_0]^3 + 3F_{uu}[w_0, v_2]\bigr\rangle. \tag{2.9}$$

1. *If* H *is definite, i.e.* $\det(H) > 0$, *then the solution set of* $F(\lambda, u) = 0$ *near* $(\lambda, u) = (\lambda_0, u_0)$ *is the line* $\{(\lambda, u_0)\}$.
2. *If* H *is indefinite, i.e.* $\det(H) < 0$, *then the solution set of* $F(\lambda, u) = 0$ *near* $(\lambda, u) = (\lambda_0, u_0)$ *is the union of* C^1 *curves intersecting at* (λ_0, u_0), *including the line of trivial solutions* $\Gamma_0 = \{(\lambda, u_0)\}$ *and two other curves* $\Gamma_i = \{(\lambda_i(s), u_i(s)) \colon |s| < \delta\}$ $(i = 1, 2)$ *for some* $\delta > 0$, *with*

$$\lambda_i(s) = \lambda_0 + \mu_i s + s\theta_i(s), \qquad u_i(s) = u_0 + \eta_i s w_0 + s v_i(s),$$

where (μ_1, η_1) *and* (μ_2, η_2) *are non-zero linear independent solutions of the equation*

$$H_{11}\mu^2 + 2H_{12}\mu\eta + H_{22}\eta^2 = 0, \tag{2.10}$$

$\theta_i(0) = \theta_i'(0) = 0$, $v_i(s) \in Z$, *and* $v_i(0) = v_i'(0) = 0$, $i = 1, 2$.

Proof. We denote the projection from Y into $R(F_u(\lambda_0, u_0))$ by Q. Then the function $g(\lambda, t)$ in Lemma 2.1 is obtained from (see [23])

$$f_1(\lambda, t) \equiv Q \circ F\bigl(\lambda, u_0 + tw_0 + g(\lambda, t)\bigr) = 0. \tag{2.11}$$

Since u_0 is a trivial solution for all λ near λ_0, that is, $F(\lambda, u_0) \equiv 0$, then by Lemma 2.1 we have $g(\lambda, 0) \equiv 0$, hence $g_\lambda(\lambda_0, 0) = 0$ and $g_{\lambda\lambda}(\lambda_0, 0) = 0$. It is easy to calculate that

$$\frac{\partial f_1}{\partial t}(\lambda_0, 0) = Q \circ F_u(\lambda_0, u_0)\big[w_0 + g_t(\lambda_0, 0)\big] = F_u(\lambda_0, u_0)\big[g_t(\lambda_0, 0)\big] = 0,$$

thus $g_t(\lambda_0, 0) = 0$ from $F_u(\lambda_0, u_0)[w_0] = 0$, $F_u(\lambda_0, u_0)|_Z : Z \to R(F_u(\lambda_0, u_0))$ is an isomorphism and $g_t(\lambda_0, 0) \in Z$. Next we calculate the second derivatives of f_1:

$$\begin{aligned}\frac{\partial^2 f_1}{\partial\lambda\partial t}(\lambda_0, 0) &= Q \circ \big(F_{\lambda u}(\lambda_0, u_0)\big[w_0 + g_t(\lambda_0, 0)\big] + F_{uu}(\lambda_0, u_0)\big[w_0 + g_t(\lambda_0, 0), g_\lambda(\lambda_0, 0)\big] \\ &\quad + F_u(\lambda_0, u_0)\big[g_{\lambda t}(\lambda_0, 0)\big]\big) \\ &= F_{\lambda u}(\lambda_0, u_0)[w_0] + F_u(\lambda_0, u_0)\big[g_{\lambda t}(\lambda_0, 0)\big] = 0,\end{aligned}$$

thus $g_{\lambda t}(\lambda_0, 0) = v_1$ from (**F3**$'$), where v_1 is defined as in (2.4); and

$$\begin{aligned}\frac{\partial^2 f_1}{\partial t^2}(\lambda_0, 0) &= Q \circ \big(F_{uu}(\lambda_0, u_0)\big[w_0 + g_t(\lambda_0, 0)\big]^2 + F_u(\lambda_0, u_0)\big[g_{tt}(\lambda_0, 0)\big]\big) \\ &= F_{uu}(\lambda_0, u_0)[w_0]^2 + F_u(\lambda_0, u_0)\big[g_{tt}(\lambda_0, 0)\big] = 0,\end{aligned}$$

thus $g_{tt}(\lambda_0, 0) = v_2$ from (**F4**$'$) where v_2 is defined as in (2.5).

We define the bifurcation function

$$f(\lambda, t) = \big\langle l, F\big(\lambda, u_0 + t w_0 + g(\lambda, t)\big)\big\rangle. \tag{2.12}$$

From the assumptions, f is C^3 in U. Since $g(\lambda, 0) \equiv 0$, then $f(\lambda, 0) \equiv 0$. To prove the statement in Theorem 2.3, we apply Lemma 2.2 to

$$h(\lambda, t) = \begin{cases} \frac{1}{t} f(\lambda, t), & \text{if } t \neq 0, \\ f_t(\lambda, 0), & \text{if } t = 0. \end{cases} \tag{2.13}$$

First we verify that $h(\lambda, t)$ is C^2 at $t = 0$. By the definition,

$$\begin{aligned} h_t(\lambda, 0) &= \lim_{t\to 0} \frac{1}{t}\big(h(\lambda, t) - h(\lambda, 0)\big) = \lim_{t\to 0} \frac{1}{t}\left(\frac{1}{t} f(\lambda, t) - f_t(\lambda, 0)\right) \\ &= \lim_{t\to 0} \frac{1}{t^2}\big(f(\lambda, t) - f(\lambda, 0) - f_t(\lambda, 0)t\big) = \frac{1}{2} f_{tt}(\lambda, 0), \\ h_{\lambda t}(\lambda, 0) &= \lim_{t\to 0} \frac{1}{t}\big(h_\lambda(\lambda, t) - h_\lambda(\lambda, 0)\big) = \lim_{t\to 0} \frac{1}{t}\left(\frac{1}{t} f_\lambda(\lambda, t) - f_{\lambda t}(\lambda, 0)\right) \\ &= \lim_{t\to 0} \frac{1}{t^2}\big(f_\lambda(\lambda, t) - t f_{\lambda t}(\lambda, 0)\big) = \frac{1}{2} f_{\lambda tt}(\lambda, 0), \end{aligned}$$

$$h_\lambda(\lambda, 0) = f_{\lambda t}(\lambda, 0), \qquad h_{\lambda\lambda}(\lambda, 0) = f_{\lambda\lambda t}(\lambda, 0),$$

and

$$\begin{aligned} h_{tt}(\lambda,0) &= \lim_{t\to 0}\frac{1}{t}\big(h_t(\lambda,t)-h_t(\lambda,0)\big) = \lim_{t\to 0}\frac{1}{t}\left(-\frac{1}{t^2}f(\lambda,t)+\frac{1}{t}f_t(\lambda,t)-\frac{1}{2}f_{tt}(\lambda,0)\right)\\ &= \lim_{t\to 0}\frac{1}{t^3}\left[-f(\lambda,t)+tf_t(\lambda,t)-\frac{1}{2}f_{tt}(\lambda,0)t^2\right] = \frac{1}{3}f_{ttt}(\lambda,0). \end{aligned}$$

Hence h_t and h_{tt} exist when $t=0$. Moreover

$$h_{tt}(\lambda,t)-h_{tt}(\lambda,0) = \frac{2}{t^3}\left[f(\lambda,t)-tf_t(\lambda,t)+\frac{t^2}{2}f_{tt}(\lambda,t)-\frac{t^3}{6}f_{ttt}(\lambda,0)\right] = o(t),$$

and

$$\begin{aligned} h_{\lambda\lambda}(\lambda,t)-h_{\lambda\lambda}(\lambda,0) &= \frac{1}{t}f_{\lambda\lambda}(\lambda,t)-f_{\lambda\lambda t}(\lambda,0)\\ &= \frac{1}{t}\big[f_{\lambda\lambda}(\lambda,t)-f_{\lambda\lambda}(\lambda,0)-f_{\lambda\lambda t}(\lambda,0)t\big] = o(t),\\ h_{\lambda t}(\lambda,t)-h_{\lambda t}(\lambda,0) &= -\frac{1}{t^2}f_{\lambda}(\lambda,t)+\frac{1}{t}f_{\lambda t}(\lambda,t)-\frac{1}{2}f_{\lambda\lambda t}(\lambda,0)\\ &= -\frac{1}{t^2}\left[f_{\lambda}(\lambda,t)-tf_{\lambda t}(\lambda,t)+\frac{1}{2}f_{\lambda tt}(\lambda,0)t^2\right] = o(t), \end{aligned}$$

thus $h\in C^2$ at $t=0$ in U.

Next we claim that h defined above satisfies the conditions in Lemma 2.2, that is, $h(\lambda_0,0)=0$, $\nabla h(\lambda_0,0)=(h_\lambda(\lambda_0,0),h_t(\lambda_0,0))=0$, and the Hessian matrix $Hess(h)$ is non-degenerate. Indeed from the information of the partial derivatives of g at $(\lambda_0,0)$, we have

$$\begin{aligned} h(\lambda_0,0) &= f_t(\lambda_0,0) = \big\langle l, F_u(\lambda_0,u_0)\big[w_0+g_t(\lambda_0,0)\big]\big\rangle = 0,\\ h_\lambda(\lambda_0,0) &= f_{\lambda t}(\lambda_0,0)\\ &= \big\langle l, F_{\lambda u}(\lambda_0,u_0)\big[w_0+g_t(\lambda_0,0)\big]+F_{uu}(\lambda_0,u_0)\big[g_\lambda(\lambda_0,0), w_0+g_t(\lambda_0,0)\big]\big\rangle\\ &\quad+\big\langle l, F_u(\lambda_0,u_0)\big[g_{\lambda t}(\lambda_0,0)\big]\big\rangle\\ &= \big\langle l, F_{\lambda u}(\lambda_0,u_0)[w_0]\big\rangle, \end{aligned}$$

and

$$\begin{aligned} h_t(\lambda_0,0) &= \frac{1}{2}f_{tt}(\lambda_0,0)\\ &= \frac{1}{2}\big\langle l, F_{uu}(\lambda_0,u_0)\big[w_0+g_t(\lambda_0,0)\big]^2+F_u(\lambda_0,u_0)\big[g_{tt}(\lambda_0,0)\big]\big\rangle\\ &= \frac{1}{2}\big\langle l, F_{uu}(\lambda_0,u_0)[w_0]^2\big\rangle. \end{aligned}$$

For the Hessian matrix $Hess(h)=\begin{pmatrix} h_{\lambda\lambda} & h_{\lambda t}\\ h_{t\lambda} & h_{tt}\end{pmatrix}$, we evaluate each entry, with the partial derivatives of F being always evaluated at (λ_0,u_0), and the partial derivatives of g being evaluated at $(\lambda_0,0)$. First

$$\begin{aligned}
h_{\lambda\lambda}(\lambda_0,0) &= f_{\lambda\lambda t}(\lambda_0,0)\\
&= \big\langle l, F_{\lambda\lambda u}[w_0+g_t] + 2F_{\lambda uu}[g_\lambda, w_0+g_t] + 2F_{\lambda u}[g_{\lambda t}] + 2F_{uu}[g_{\lambda t}, g_\lambda]\\
&\quad + F_u[g_{\lambda\lambda t}] + F_{uuu}[g_\lambda, g_\lambda, w_0+g_t] + F_{uu}[g_{\lambda\lambda}, w_0+g_t]\big\rangle\\
&= \big\langle l, F_{\lambda\lambda u}[w_0] + 2F_{\lambda u}[v_1]\big\rangle.
\end{aligned}$$

Next we have

$$\begin{aligned}
h_{\lambda t}(\lambda_0,0) &= \frac{1}{2} f_{\lambda tt}(\lambda_0,0)\\
&= \frac{1}{2}\big\langle l, F_{\lambda uu}[w_0+g_t]^2 + F_{\lambda u}[g_{tt}] + 2F_{uu}[g_{\lambda t}, w_0+g_t] + F_u[g_{\lambda tt}]\\
&\quad + F_{uuu}[g_\lambda, w_0+g_t, w_0+g_t] + F_{uu}[g_\lambda, g_{tt}]\big\rangle\\
&= \frac{1}{2}\big\langle l, F_{\lambda uu}[w_0]^2 + F_{\lambda u}[g_{tt}] + 2F_{uu}[g_{\lambda t}, w_0]\big\rangle\\
&= \frac{1}{2}\big\langle l, F_{\lambda uu}[w_0]^2 + F_{\lambda u}[v_2] + 2F_{uu}[v_1, w_0]\big\rangle;
\end{aligned}$$

and finally,

$$\begin{aligned}
h_{tt}(\lambda_0,0) &= \frac{1}{3} f_{ttt}(\lambda_0,0)\\
&= \frac{1}{3}\big\langle l, F_{uuu}[w_0+g_t]^3 + 3F_{uu}[g_{tt}, w_0+g_t] + F_u[g_{ttt}]\big\rangle\\
&= \frac{1}{3}\big\langle l, F_{uuu}[w_0]^3 + 3F_{uu}[w_0, v_2]\big\rangle.
\end{aligned}$$

Therefore from Lemma 2.2, we conclude that the solution set of $h(\lambda,t)=0$ near $(\lambda,t)=(\lambda_0,0)$ is a pair of intersecting curves if the matrix in (2.6) is indefinite, or is a single point if it is definite. Thus the solution set of $F(\lambda,u)=0$ near (λ_0,u_0) is exactly the union of pair of intersecting curves which solve $h(\lambda,t)=0$ and the line of trivial solutions.

For the case of two intersecting curves, we denote the two curves by $(\lambda_i(s),u_i(s)) = (\lambda_i(s), u_0+t_i(s)w_0+g(\lambda_i(s),t_i(s)))$, with $i=1,2$. Then

$$F\big(\lambda_i(s), u_0+t_i(s)w_0+g\big(\lambda_i(s),t_i(s)\big)\big)=0. \tag{2.14}$$

From Lemma 2.2, the vectors $v_i=(\lambda_i'(0), t_i'(0))$ are the solutions of $v^T H v=0$, which are the solutions (μ,η) of (2.10). □

Apparently the more interesting case in Theorem 2.3 is when H_0 is indefinite, *i.e.* $\det(H_0)<0$, thus the following remark is only for that case.

Remark 2.4.

1. If $H_{11} \neq 0$, or equivalently,

$$F_{\lambda\lambda u}(\lambda_0, u_0)[w_0] + 2F_{\lambda u}(\lambda_0, u_0)[v_1] \notin R\big(F_u(\lambda_0, u_0)\big), \tag{2.15}$$

then $\eta_i \neq 0$ for $i = 1, 2$. In that case, both Γ_1 and Γ_2 are transversal to Γ_0. If $\mu_i \neq 0$ also holds for both $i = 1$ and $i = 2$, then (1.1) has exactly three solutions locally for any $\lambda \neq \lambda_0$ (see Fig. 2).

2. If $F_{\lambda u}(\lambda_0, u_0)[w_0] = 0$, then $v_1 = 0$, and similarly if $F_{uu}(\lambda_0, u_0)[w_0]^2 = 0$, then $v_2 = 0$. If $v_1 = v_2 = 0$, then the matrix H_0 in Theorem 2.3 is simplified to

$$H_1 = H_1(\lambda_0, u_0) \equiv \begin{pmatrix} \langle l, F_{\lambda\lambda u}(\lambda_0, u_0)[w_0]\rangle & \frac{1}{2}\langle l, F_{\lambda uu}(\lambda_0, u_0)[w_0]^2\rangle \\ \frac{1}{2}\langle l, F_{\lambda uu}(\lambda_0, u_0)[w_0]^2\rangle & \frac{1}{3}\langle l, F_{uuu}(\lambda_0, u_0)[w_0]^3\rangle \end{pmatrix}, \tag{2.16}$$

and (2.10) becomes

$$\begin{aligned} &\big\langle l, F_{\lambda\lambda u}(\lambda_0, u_0)[w_0]\big\rangle \mu^2 + \big\langle l, F_{\lambda uu}(\lambda_0, u_0)[w_0]^2\big\rangle \mu\eta \\ &\quad + \frac{1}{3}\big\langle l, F_{uuu}(\lambda_0, u_0)[w_0]^3\big\rangle \eta^2 = 0. \end{aligned} \tag{2.17}$$

The local solution set of (1.1) near (λ_0, u_0) described in Theorem 2.3 could be the union of three distinct curves $\Gamma_0 \cup \Gamma_1 \cup \Gamma_2$. But it is possible that one of Γ_1 or Γ_2 is identical to Γ_0, see for example, the mappings defined in (1.4) (Fig. 3). Indeed we have the following corollary:

Corollary 2.5. *Assume the conditions in Theorem* 2.3 *are satisfied, and in addition we assume that*

$$R\big(F_u(\lambda, u_0)\big) \subseteq R\big(F_u(\lambda_0, u_0)\big) \quad \text{for } \lambda \text{ near } \lambda_0, \tag{2.18}$$

then Γ_1 is identical to Γ_0, and Γ_2 is in form of $(\lambda_2(s), u_2(s)) = (\lambda_0 + \mu_2 s + s\theta_2(s), u_0 + sw_0 + sv_2(s))$, *where*

$$\mu_2 = -\frac{H_{22}}{2H_{12}}, \tag{2.19}$$

which determines the bifurcation direction: *if $\mu_2 \neq 0$, then a transcritical bifurcation occurs, and on either side of $\lambda = \lambda_0$,* (1.1) *has exactly two solutions locally*; *and if $\mu_2 = 0$ but a higher order non-degeneracy condition is satisfied, then a pitchfork bifurcation occurs.*

Proof. From (2.13) and (2.18), we have

$$h(\lambda, 0) = f_t(\lambda, 0) = \big\langle l, F_u(\lambda, u_0)\big[w_0 + g_t(\lambda, 0)\big]\big\rangle = 0.$$

Hence one of solution curves of $h(\lambda, t) = 0$ is still given by $t = 0$, while the other solution curve for $h(\lambda, t) = 0$ is non-trivial. Moreover from (2.7) and (2.8),

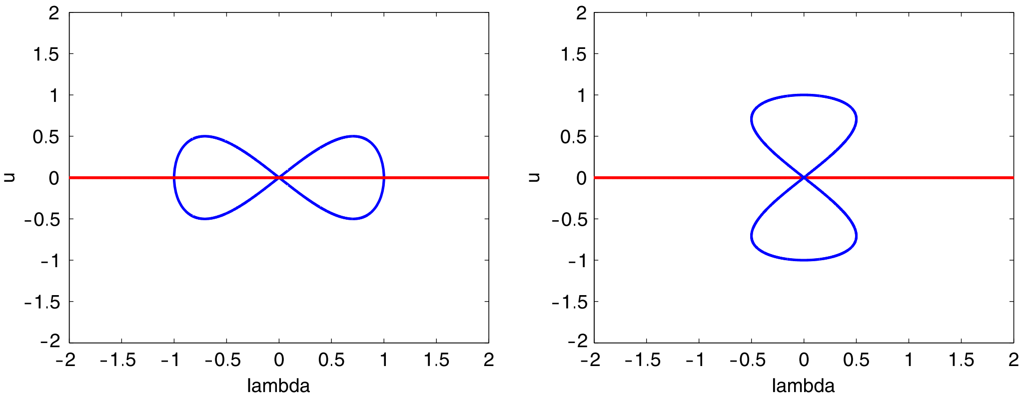

Fig. 6. Global bifurcation diagrams. (Left): $F(\lambda,u)=u(\lambda^2-u^2+u^4)$; (Right): $F(\lambda,u)=u(\lambda^4-\lambda^2+u^2)$.

$$H_{11}=\langle l, F_{\lambda\lambda u}[w_0]+2F_{\lambda u}[v_1]\rangle=0,$$
$$H_{12}=\frac{1}{2}\langle l, F_{\lambda uu}[w_0]^2+2F_{uu}[w_0,v_1]\rangle,$$

thus we can assume that $(\mu_1,\eta_1)=(1,0)$, thus Γ_1 is identical to Γ_0 and $(\mu_2,\eta_2)=(-\frac{H_{22}}{2H_{12}},1)$. □

The condition (2.18) implies that $N(F_u(\lambda,u_0))\neq\{0\}$ for λ near λ_0, since the Fredholm index of $F_u(\lambda,u)$ is locally a constant. This can be satisfied if $w_0\in N(F_u(\lambda,u_0))$ for λ near λ_0 and $R(F_u(\lambda,u_0))=R(F_u(\lambda_0,u_0))$. In this case, (**F3′**) is satisfied for all nearby λ and $v_1=0$. Hence a stronger degeneracy occurs here as each (λ,u_0) is a degenerate point for $F(\lambda,u)=0$. However only at (λ_0,u_0), a bifurcation of non-trivial solutions occurs as (**F4′**) is satisfied at (λ_0,u_0), and it is not satisfied for other (λ,u_0) since $\det(H)=-H_{12}^2\neq 0$. This is demonstrated by the examples in Fig. 3.

We end this section with two more one-dimensional examples to show the global nature of the bifurcation branches Γ_1 and Γ_2 obtained in Theorem 2.3. For the classical bifurcation from simple eigenvalue case, Shi and Wang [41] showed that the connected component of the set of non-trivial solutions of (1.1) containing the curve emanating from (λ_0,u_0) as in Theorem 1.1 is either unbounded or it connects to another (λ_*,u_0) which is another bifurcation point. This result extends the earlier one by Rabinowitz [33] which assumes $X=Y$ and the operators are compact ones. The bifurcation branches Γ_1 and Γ_2 in Theorem 2.3 can be unbounded in $\mathbb{R}\times X$ (see the examples in (1.3) and (1.4)), or they can connect to another bifurcation point (see Fig. 6 left panel), or each of Γ_1 and Γ_2 is bounded (see Fig. 6 right panel). It is easy to verify that the bifurcation at $(\lambda,u)=(0,0)$ in both diagrams of Fig. 6 satisfies the conditions in Theorem 2.3. Note that the latter alternative is not possible for the classical bifurcation from simple eigenvalue case.

3. Stability

In this section, we consider the stability of the bifurcating solutions on Γ_i, $i=0,1,2$ obtained in Theorem 2.3. At the bifurcation point (λ_0,u_0), 0 is an eigenvalue of $F_u(\lambda_0,u_0)$. We are interested in the perturbation of this zero eigenvalue for solution (λ,u) of (1.1) near the bifurcation point. First we recall the following definition of K-simple eigenvalue and a fundamental result due to Crandall and Rabinowitz [7]. Here we denote by $B(X,Y)$ the set of bounded linear maps from X into Y.

Definition 3.1. (See [7, Definition 1.2].) Let $T, K \in B(X,Y)$. We say that $\mu \in \mathbb{R}$ is a K-simple eigenvalue of T, if

$$\dim N(T-\mu K) = \operatorname{codim} R(T-\mu K) = 1, \quad N(T-\mu K) = \operatorname{span}\{w_0\},$$

and

$$K[w_0] \notin R(T-\mu K).$$

Lemma 3.2. *(See [7, Lemma 1.3].) Suppose that $T_0, K \in B(X,Y)$ and μ_0 is a K-simple eigenvalue of T_0. Then there exists $\delta > 0$ such that if $T \in B(X,Y)$ and $\|T - T_0\| < \delta$, then there exists a unique $\mu(T) \in \mathbb{R}$ satisfying $\|\mu(T) - \mu_0\| < \delta$ such that $N(T - \mu(T)K) \neq \emptyset$ and $\mu(T)$ is a K-simple eigenvalue of T. Moreover if $N(T_0 - \mu_0 K) = \operatorname{span}\{w_0\}$ and Z is a complement of $\operatorname{span}\{w_0\}$ in X, then there exists a unique $w(T) \in X$ such that $N(T - \mu(T)K) = \operatorname{span}\{w(T)\}$, $w(T) - w_0 \in Z$ and the map $T \mapsto (\mu(T), w(T))$ is analytic.*

By using Lemma 3.2 in the same way as in [7, Corollary 1.13], we assume that $X \subset Y$, $K = i : X \to Y$ is the inclusion mapping $i(x) = x$ for $x \in X$, and i is continuous. Then we have the following result for the linearized equation for the bifurcating solutions in Theorem 2.3.

Proposition 3.3. *Let X, Y, U, F, Z, λ_0, w_0, v_1 and v_2 be the same as in Theorem 2.3, and let all assumptions in Theorem 2.3 on F be satisfied. In addition we assume that $X \subset Y$, and the inclusion mapping $i : X \to Y$ is continuous. Let $(\lambda_i(s), u_i(s))$ $(i = 1, 2)$ be the solution curves in Theorem 2.3. Then there exist $\varepsilon > 0$, C^2 functions $\gamma : (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \to \mathbb{R}$, $\sigma_i : (-\varepsilon, \varepsilon) \to \mathbb{R}$, $v : (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \to X$, $w_i : (-\varepsilon, \varepsilon) \to X$ such that*

$$F_u(\lambda, u_0)\big[v(\lambda)\big] = \gamma(\lambda) v(\lambda) \quad \text{for } \lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon), \tag{3.1}$$

$$F_u\big(\lambda_i(s), u_i(s)\big)\big[w_i(s)\big] = \sigma_i(s) w_i(s) \quad \text{for } s \in (-\varepsilon, \varepsilon), \tag{3.2}$$

where $\gamma(\lambda_0) = \sigma_i(0) = 0$, $v(\lambda_0) = w_i(0) = w_0$, and $v(\lambda) - w_0 \in X$, $w_i(s) - w_0 \in Z$.

The signs of $\gamma(\lambda)$ and $\sigma_i(s)$ determine the stability of the bifurcating solutions. In [7], the stability of bifurcating solutions obtained in Theorem 1.1 was considered. Here we consider the stability of bifurcating solutions obtained in Theorem 2.3.

Theorem 3.4. *Let the assumptions of Proposition 3.3 hold and let γ, σ_i be the functions provided by Proposition 3.3. In addition, we assume that*

$$w_0 \notin R\big(F_u(\lambda_0, u_0)\big), \quad \text{where } w_0 (\neq 0) \in N\big(F_u(\lambda_0, u_0)\big). \tag{3.3}$$

Then:

1. $\gamma'(\lambda_0) = 0$, *and*

$$\gamma''(\lambda_0) = \frac{H_{11}}{\langle l, w_0 \rangle}. \tag{3.4}$$

If $H_{11} = 0$ *and we assume that* $F \in C^4$ *near* (λ_0, u_0), *then we have* $\gamma''(\lambda_0) = 0$ *and*

$$\gamma'''(\lambda_0) = -\frac{\langle l, F_{\lambda\lambda\lambda u}(\lambda_0, u_0)[w_0] + 3F_{\lambda\lambda u}(\lambda_0, u_0)[v_1] + 3F_{\lambda u}(\lambda_0, u_0)[v_3] \rangle}{\langle l, w_0 \rangle}, \tag{3.5}$$

where $v_3 \in Z$ *is the unique solution of*

$$F_{\lambda\lambda u}(\lambda_0, u_0)[w_0] + 2F_{\lambda u}(\lambda_0, u_0)[v_1] + F_u(\lambda_0, u_0)[v_3] = 0. \tag{3.6}$$

2. $\sigma_i'(0) = 0$ *and*

$$\sigma_i''(0) = \frac{H_{22}\eta_i^2 - H_{11}\mu_i^2}{\langle l, w_0 \rangle}, \tag{3.7}$$

where H_{11}, H_{22}, μ_i, η_i $(i = 1, 2)$ *are defined in Theorem* 2.3.

Proof. Differentiating (3.1) with respect to λ, we obtain that

$$F_{\lambda u}(\lambda, u_0)\big[v(\lambda)\big] + F_u(\lambda, u_0)\big[v'(\lambda)\big] = \gamma'(\lambda)v(\lambda) + \gamma(\lambda)v'(\lambda), \tag{3.8}$$

and evaluating at $\lambda = \lambda_0$, we have

$$F_{\lambda u}(\lambda_0, u_0)[w_0] + F_u(\lambda_0, u_0)\big[v'(\lambda_0)\big] = \gamma'(\lambda_0)w_0.$$

Then the assumption (**F3′**) implies that $\gamma'(\lambda_0) = 0$ and $v'(\lambda_0) = v_1$ from (2.4). We differentiate (3.8) again, and we obtain that

$$\begin{aligned} &F_{\lambda\lambda u}(\lambda, u_0)\big[v(\lambda)\big] + 2F_{\lambda u}(\lambda, u_0)\big[v'(\lambda)\big] + F_u(\lambda, u_0)\big[v''(\lambda)\big] \\ &\quad = \gamma''(\lambda)v(\lambda) + 2\gamma'(\lambda)v'(\lambda) + \gamma(\lambda)v''(\lambda). \end{aligned} \tag{3.9}$$

Setting $\lambda = \lambda_0$ in (3.9), we have

$$F_{\lambda\lambda u}(\lambda_0, u_0)[w_0] + 2F_{\lambda u}(\lambda_0, u_0)[v_1] + F_u(\lambda_0, u_0)\big[v''(\lambda_0)\big] = \gamma''(\lambda_0)w_0, \tag{3.10}$$

and by applying $l \in Y^*$ to (3.10), we obtain (3.4) from (3.3). If F satisfies (2.15), then $H_{11} \neq 0$ and $\gamma''(\lambda_0) \neq 0$. If $H_{11} = 0$, then $\gamma''(\lambda_0) = 0$, and

$$F_{\lambda\lambda u}(\lambda_0, u_0)[w_0] + 2F_{\lambda u}(\lambda_0, u_0)[v_1] + F_u(\lambda_0, u_0)\big[v''(\lambda_0)\big] = 0.$$

We have $v''(\lambda_0) = v_3$, where v_3 is defined by (3.6). We differentiate (3.9) again, and we obtain

$$F_{\lambda\lambda\lambda u}(\lambda,u_0)\big[v(\lambda)\big]+3F_{\lambda\lambda u}(\lambda,u_0)\big[v'(\lambda)\big]+3F_{\lambda u}(\lambda,u_0)\big[v''(\lambda)\big]+F_u(\lambda,u_0)\big[v'''(\lambda)\big]$$
$$=\gamma'''(\lambda)v(\lambda)+3\gamma''(\lambda)v'(\lambda)+3\gamma'(\lambda)v''(\lambda)+\gamma(\lambda)v'''(\lambda).$$

By setting $\lambda=\lambda_0$, we have

$$\gamma'''(\lambda_0)w_0=F_{\lambda\lambda\lambda u}(\lambda_0,u_0)[w_0]+3F_{\lambda\lambda u}(\lambda_0,u_0)[v_1]$$
$$+3F_{\lambda u}(\lambda_0,u_0)[v_3]+F_u(\lambda_0,u_0)\big[v'''(\lambda_0)\big]. \tag{3.11}$$

Thus by applying $l\in Y^*$ to (3.11), we obtain (3.5).

On the other hand, we differentiate $F(\lambda_i(s),u_i(s))=0$ twice to obtain

$$F_{\lambda\lambda}\big[\lambda_i'(s)\big]^2+2F_{\lambda u}\big[u_i'(s)\big]\lambda_i(s)+F_\lambda\lambda_i''(s)$$
$$+F_{uu}\big[u_i'(s)\big]^2+F_u\big[u_i''(s)\big]=0. \tag{3.12}$$

By setting $s=0$ in (3.12), we get

$$u_i''(0)=2\mu_i\eta_iv_1+\eta_i^2v_2. \tag{3.13}$$

Similarly by differentiating (3.2), we obtain

$$F_{\lambda u}\big[w_i(s)\big]\lambda_i'(s)+F_{uu}\big[w_i(s),u_i'(s)\big]+F_u\big[w_i'(s)\big]$$
$$=\sigma_i'(s)w_i(s)+\sigma_i(s)w_i'(s). \tag{3.14}$$

By setting $s=0$ in (3.14), we get

$$\mu_iF_{\lambda u}[w_0]+\eta_iF_{uu}[w_0]^2+F_u\big[w_i'(0)\big]=\sigma_i'(0)w_0. \tag{3.15}$$

Hence by applying l to (3.15), we obtain $\sigma_i'(0)=0$ and

$$w_i'(0)=\mu_iv_1+\eta_iv_2. \tag{3.16}$$

We differentiate (3.14) again, and we have

$$F_{\lambda\lambda u}\big[w_i(s)\big]\big[\lambda_i'(s)\big]^2+2F_{\lambda uu}\big[w_i(s),u_i'(s)\big]\lambda_i'(s)+F_{uuu}\big[w_i(s),u_i'(s),u_i'(s)\big]$$
$$+2F_{\lambda u}\big[w_i'(s)\big]\lambda_i'(s)+F_{\lambda u}\big[w_i(s)\big]\lambda_i''(s)+2F_{uu}\big[w_i'(s),u_i'(s)\big]$$
$$+F_{uu}\big[w_i(s),u_i''(s)\big]+F_u\big[w_i''(s)\big]$$
$$=\sigma_i''(s)w_i(s)+2\sigma_i'(s)w_i'(s)+\sigma_i(s)w_i''(s). \tag{3.17}$$

By setting $s=0$ in (3.17) and using (3.13) and (3.16), we get

$$\sigma_i''(0)w_0=\mu_i^2F_{\lambda\lambda u}[w_0]+2\mu_i\eta_iF_{\lambda uu}[w_0]^2+\eta_i^2F_{uuu}[w_0]^3+2\mu_iF_{\lambda u}[\mu_iv_1+\eta_iv_2]$$
$$+F_{\lambda u}[w_0]\lambda_i''(0)+2\eta_iF_{uu}[\mu_iv_1+\eta_iv_2,w_0]$$
$$+F_{uu}\big[w_0,2\mu_i\eta_iv_1+\eta_i^2v_2\big]+F_u\big[w_i''(0)\big]. \tag{3.18}$$

Thus by applying l to (3.18), we obtain

$$H_{11}\mu_i^2 + 4H_{12}\mu_i\eta_i + 3H_{22}\eta_i^2 = \sigma_i''(0)\langle l, w_0\rangle, \tag{3.19}$$

which implies (3.7) by using (2.10) and (3.3). □

Remark 3.5.

1. If F satisfies (2.15), then we have $\gamma'(0) = 0$ and $\gamma''(0) \neq 0$ from (3.4), thus the trivial solution on Γ_0 in Theorem 2.3 has the same stability near the bifurcation point. Similarly if $H_{22}\eta_i^2 \neq H_{11}\mu_i^2$, we have $\sigma_i'(0) = 0$ and $\sigma_i''(0) \neq 0$ from (3.7), thus the non-trivial solutions on Γ_i $(i = 1, 2)$ in Theorem 2.3 also have the same stability before and after the bifurcation point. This shows that in general, the bifurcation from a degenerate simple eigenvalue does not cause an exchange of stability as in the non-degenerate simple eigenvalue case [7].
2. When Γ_1 and Γ_2 are both distinctive from Γ_0, the stability of non-trivial solutions on Γ_1 and Γ_2 depends on the sign of $H_{11} \cdot H_{22}$. If $H_{11} \cdot H_{22} < 0$, then from Theorems 2.3 and 3.4, $\sigma_1(s) = \sigma_2(s)$ for small s such that $|s| \neq 0$. In this case, the stability of solutions on Γ_1 and Γ_2 are the same, but they are both the opposite of the ones on Γ_0 (see Fig. 2 left panel). But if $H_{11} \cdot H_{22} > 0$, then the solutions on one of Γ_1 or Γ_2 have the same stability as the ones on Γ_0 (see Fig. 2 right panel).
3. From Remark 2.4, if $R(F_u(\lambda, u_0)) \subseteq R(F_u(\lambda_0, u_0))$ for λ near λ_0 and

$$F_{\lambda uu}(\lambda_0, u_0)[w_0]^2 \notin R\big(F_u(\lambda_0, u_0)\big),$$

then one can assume that $(\mu_1, \eta_1) = (1, 0)$ thus Γ_1 is identical to Γ_0, and Γ_2 is distinctive from Γ_0. In this case we have $H_{11} = 0$ so $\gamma''(0) = \sigma_1''(0) = 0$, $\gamma'''(0)$ is given by (3.5), and $\sigma_2''(0) = \frac{H_{22}}{\langle l, w_0\rangle}$ (see Fig. 3). In this case, if $\gamma'''(0) \neq 0$ and $\sigma_2''(0) \neq 0$, then the stability of the non-trivial solutions does not change across the bifurcation point, and all the trivial solutions are always degenerate.

4. Perturbation problem

In this section, we shall consider a nonlinear equation with two parameters ε and λ:

$$F(\varepsilon, \lambda, u) = 0, \tag{4.1}$$

where $F \in C^1(M, Y)$, $M \equiv \mathbb{R} \times \mathbb{R} \times X$, and X, Y are Banach spaces. Here we consider the variation of bifurcation diagrams in (λ, u)-space when the value of an additional parameter ε changes, following the consideration in our previous work [23,38]. To consider the original equation and its linearization together, we define an augmented operator

$$G(\varepsilon, \lambda, u, w) = \begin{pmatrix} F(\varepsilon, \lambda, u) \\ F_u(\varepsilon, \lambda, u)[w] \end{pmatrix}. \tag{4.2}$$

We consider the solutions $(\varepsilon_0, \lambda_0, u_0, w_0)$ of $G(\varepsilon, \lambda, u, w) = 0$. For $(\varepsilon_0, \lambda_0, u_0) \in M$ and $w_0 \in X_1 \equiv \{x \in X\colon \|x\| = 1\}$, by using Hahn–Banach Theorem (see [38, Lemma 7.1]), there exists a closed subspace X_3 of X with codimension 1 such that $X = L(w_0) \oplus X_3$, where $L(w_0) =$

span$\{w_0\}$, and $d(w_0, X_3) = \inf\{\|w - x\|: x \in X_3\} > 0$. Let $X_2 = w_0 + X_3 = \{w_0 + x: x \in X_3\}$. Then X_2 is a closed hyperplane of X with codimension 1. Since X_3 is a closed subspace of X, and X_3 is also a Banach space in the subspace topology. Hence we can regard $M_1 = M \times X_2$ as a Banach space with product topology. Moreover, the tangent space of M_1 is homeomorphic to $M \times X_3$ (see [38] for more on the setting).

Perturbation problem (4.1) in the above framework was first considered in [38] (see Theorems 2.1–2.6) in [38], and some more results of (4.1) were proved in [23, Sections 3 and 4]. All these results show the phenomenon of imperfect bifurcation in which a classical transcritical or pitchfork bifurcation for $\varepsilon = 0$ is perturbed. In the results of [23,38], the trivial solutions in the original transcritical or pitchfork bifurcation is not preserved by the perturbation (see Figs. 1–4 in [23] or Fig. 4 in this paper). In the following new result for the perturbed problem (4.1), the trivial solutions are preserved by the perturbation, but a pitchfork bifurcation is perturbed into a transcritical bifurcation (see Theorem 4.3 below and Fig. 5).

In the following we will still use the conditions (**Fi**) on F defined in previous sections and in [23,38], but we shall understand that the variables are $(\varepsilon_0, \lambda_0, u_0)$ instead of (λ_0, u_0) in all these conditions. In the following theorem, we consider a situation that a classical pitchfork bifurcation occurs and there is a unique degenerate trivial solution when $\varepsilon = \varepsilon_0$, then for a perturbed problem with ε near ε_0, the degenerate trivial solution persists but some new degenerate solutions emerge for $\varepsilon \neq \varepsilon_0$.

Theorem 4.1. *Let $F \in C^3(M, Y)$, and let there be $T_0 = (\varepsilon_0, \lambda_0, u_0, w_0) \in M_1$ such that $G(T_0) = (0, 0)$. Assume that there exists $\delta_0 > 0$ such that*

$$F(\varepsilon, \lambda, u_0) = 0, \qquad F_u(\varepsilon, \lambda_0, u_0)[w_0] = 0, \quad \text{for } |\varepsilon - \varepsilon_0| < \delta_0,\ |\lambda - \lambda_0| < \delta_0. \tag{4.3}$$

Suppose that F satisfies (**F1**), (**F3**), (**F4′**) *at T_0, and*

$$F_{\varepsilon uu}(\varepsilon_0, \lambda_0, u_0)[w_0]^2 + F_{\varepsilon u}(\varepsilon_0, \lambda_0, u_0)[v_2] \notin R\big(F_u(\varepsilon_0, \lambda_0, u_0)\big), \tag{4.4}$$

where $v_2 \in X_3$ is the unique solution of

$$F_{uu}(\varepsilon_0, \lambda_0, u_0)[w_0]^2 + F_u(\varepsilon_0, \lambda_0, u_0)[v] = 0. \tag{4.5}$$

Then the solution set of $G(\varepsilon, \lambda, u, w) = (0, 0)$ near T_0 is the union of C^1 curves intersecting at T_0 including the line of trivial solutions $\Sigma_0 = \{(\varepsilon, \lambda_0, u_0, w_0): |\varepsilon - \varepsilon_0| < \delta_1\}$ and two other curves $\Sigma_i = \{T_{is} = (\varepsilon_i(s), \lambda_i(s), u_i(s), w_i(s)): |s| < \delta_2\}$, for some $\delta_1, \delta_2 > 0$, where $\varepsilon_i(s) = \varepsilon_0 + \mu_i s + s z_{i0}(s)$, $\lambda_i(s) = \lambda_0 + s z_{i1}(s)$, $u_i(s) = u_0 + s\eta_i w_0 + s z_{i2}(s)$, $w_i(s) = w_0 + s\eta_i v_2 + s z_{i3}(s)$, where $z_{i0}(0) = z_{i1}(0) = z_{i2}(0) = z_{i3}(0) = 0$, $i = 1, 2$, $(\mu_1, \eta_1) = (1, 0)$ and

$$(\mu_2, \eta_2) = \left(-\frac{2\langle l, F_{uuu}[w_0]^3 + 3F_{uu}[w_0, v_2]\rangle}{3\langle l, F_{\varepsilon uu}[w_0]^2 + F_{\varepsilon u}[v_2]\rangle}, 1\right). \tag{4.6}$$

Proof. We apply Theorem 2.3 to the equation $G(\varepsilon, \lambda, u, w) = (0, 0)$ with ε as parameter, and we verify all the assumptions in Theorem 2.3. We define the linearized operator $K : \mathbb{R} \times X \times X_3 \to Y \times Y$ of G by

$$K[\tau, v, \psi] = G_{(\lambda,u,w)}(T_0)[\tau, v, \psi]$$
$$= \begin{pmatrix} \tau F_\lambda(\varepsilon_0, \lambda_0, u_0) + F_u(\varepsilon_0, \lambda_0, u_0)[v] \\ \tau F_{\lambda u}(\varepsilon_0, \lambda_0, u_0)[w_0] + F_{uu}(\varepsilon_0, \lambda_0, u_0)[v, w_0] + F_u(\varepsilon_0, \lambda_0, u_0)[\psi] \end{pmatrix}. \tag{4.7}$$

We prove it in several steps, and we recall that $l \in Y^*$ satisfies $N(l) = R(F_u(\varepsilon_0, \lambda_0, u_0))$. In the following, for the simplicity of notations, we denote $U = (\lambda, u, w)$ and $U_0 = (\lambda_0, u_0, w_0)$.

(1) $G = 0$ has a trivial solution $U_0 = (\lambda_0, u_0, w_0)$ for any ε near ε_0. Indeed

$$G(\varepsilon, \lambda_0, u_0, w_0) = \begin{pmatrix} F(\varepsilon, \lambda_0, u_0) \\ F_u(\varepsilon, \lambda_0, u_0)[w_0] \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{4.8}$$

from (4.3).

(2) $\dim N(K) = 1$. Suppose that $(\tau, v, \psi) \in N(K)$ and $(\tau, v, \psi) \neq (0, 0, 0)$. Notice that $F_\lambda(\varepsilon_0, \lambda_0, u_0) = 0$ from (4.3), then the first equation of $K[\tau, v, \psi] = (0, 0)$ is reduced to $F_u(\varepsilon_0, \lambda_0, u_0)[v] = 0$. Hence from (**F1**), we have $v = kw_0$ for $k \in \mathbb{R}$ and

$$\tau F_{\lambda u}(\varepsilon_0, \lambda_0, u_0)[w_0] + F_{uu}(\varepsilon_0, \lambda_0, u_0)[kw_0, w_0] + F_u(\varepsilon_0, \lambda_0, u_0)[\psi] = 0. \tag{4.9}$$

Applying l to (4.9), we obtain $\tau = 0$ from (**F3**) and (**F4′**). Thus $\psi = kv_2$, where v_2 is uniquely determined by (4.5). Therefore $N(K) = \operatorname{span}\{W_0 \equiv (0, w_0, v_2)\}$.

(3) $\operatorname{codim} R(K) = 1$. Let $(h, g) \in R(K)$, and let $(\tau, v, \psi) \in \mathbb{R} \times X \times X_3$ satisfy

$$\tau F_\lambda(\varepsilon_0, \lambda_0, u_0) + F_u(\varepsilon_0, \lambda_0, u_0)[v] = h, \tag{4.10}$$
$$\tau F_{\lambda u}(\varepsilon_0, \lambda_0, u_0)[w_0] + F_{uu}(\varepsilon_0, \lambda_0, u_0)[v, w_0] + F_u(\varepsilon_0, \lambda_0, u_0)[\psi] = g. \tag{4.11}$$

Applying l to (4.10), we get $\langle l, h\rangle = 0$, hence $h \in R(F_u(\varepsilon_0, \lambda_0, u_0))$, and $R(K) \subseteq R(F_u) \times Y$. Conversely, for any $(h, g) \in R(F_u) \times Y$, there exists a unique $v_3 \in X_3$ such that $F_u(\varepsilon_0, \lambda_0, u_0)[v_3] = h$, then v_3 solves (4.10). Substituting $v = v_3$ into (4.11), and applying l, we obtain

$$\tau\big\langle l, F_{\lambda u}[w_0]\big\rangle + \big\langle l, F_{uu}[w_0, v_3]\big\rangle = \langle l, g\rangle. \tag{4.12}$$

Then from (**F3**) there exists a unique τ so that (4.12) holds for any $g \in Y$. With such choice of (τ, v), ψ in (4.11) is uniquely solvable in X_3. Therefore this (τ, v_3, ψ) is a pre-image of (h, g), which implies that $R(K) = R(F_u(\varepsilon_0, \lambda_0, u_0)) \times Y$, and $\operatorname{codim} R(K) = 1$. This proves that G satisfies the condition (**F1**) at (ε_0, U_0).

(4) G satisfies (**F3′**) at (ε_0, U_0), that is $G_{\varepsilon U}(T_0)[W_0] \in R(K)$. From (4.3), we have

$$G_{\varepsilon U}(T_0)[W_0] = \begin{pmatrix} F_{\varepsilon u}(\varepsilon_0, \lambda_0, u_0)[w_0] \\ F_{\varepsilon uu}(\varepsilon_0, \lambda_0, u_0)[w_0]^2 + F_{\varepsilon u}(\varepsilon_0, \lambda_0, u_0)[v_2] \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ F_{\varepsilon uu}(\varepsilon_0, \lambda_0, u_0)[w_0]^2 + F_{\varepsilon u}(\varepsilon_0, \lambda_0, u_0)[v_2] \end{pmatrix},$$

which belongs to $R(K)$ from part (3). Eq. (2.4) now becomes $G_{\varepsilon U}[W_0] + K[\tau, v, \psi] = 0$, which is

$$F_u[v]=0, \tag{4.13}$$

$$F_{\varepsilon uu}[w_0]^2+F_{\varepsilon u}[v_2]+\tau F_{\lambda u}[w_0]+F_{uu}[v,w_0]+F_u[\psi]=0. \tag{4.14}$$

Looking for a solution $(\tau, v, \psi)\in Z_1\equiv\mathbb{R}\times X\times X_3$, first we know that a solution of (4.13) is given by $v=kw_0$ for $k\in\mathbb{R}$, and we choose $k=1$ here. Next we apply l to (4.14) and we obtain

$$\tau=\tau_1\equiv-\frac{\langle l, F_{\varepsilon uu}(\varepsilon_0,\lambda_0,u_0)[w_0]^2+F_{\varepsilon u}(\varepsilon_0,\lambda_0,u_0)[v_2]\rangle}{\langle l, F_{\lambda u}(\varepsilon_0,\lambda_0,u_0)[w_0]\rangle}, \tag{4.15}$$

and subsequently ψ can be uniquely determined with $v=w_0$ and τ_1 given by (4.15). We denote this solution to be $V_1=(\tau_1,w_0,\psi_1)$.

(5) G satisfies (**F4′**) at (ε_0,U_0), that is $G_{UU}(T_0)[W_0]^2\in R(K)$. Notice that

$$G_{UU}(T_0)[W_0]^2=\begin{pmatrix} F_{uu}(\varepsilon_0,\lambda_0,u_0)[w_0]^2\\ F_{uuu}(\varepsilon_0,\lambda_0,u_0)[w_0]^3+2F_{uu}(\varepsilon_0,\lambda_0,u_0)[w_0,v_2]\end{pmatrix}$$

thus (**F4′**) is satisfied for G as F satisfies (**F4′**) at T_0. Eq. (2.5) now becomes $G_{UU}[W_0]^2+K[\tau,v,\psi]=0$, which is

$$F_{uu}[w_0]^2+F_u[v]=0, \tag{4.16}$$

$$F_{uuu}[w_0]^3+2F_{uu}[w_0,v_2]+\tau F_{\lambda u}[w_0]+F_{uu}[v,w_0]+F_u[\psi]=0. \tag{4.17}$$

We look for a solution $(\tau,v,\psi)\in Z_1\equiv\mathbb{R}\times X\times X_3$. Then a solution of (4.16) is given by $v=v_2$ from (**F4′**). We apply l to (4.17) and we obtain

$$\tau=\tau_2\equiv-\frac{\langle l, F_{uuu}(\varepsilon_0,\lambda_0,u_0)[w_0]^3+3F_{uu}(\varepsilon_0,\lambda_0,u_0)[w_0,v_2]\rangle}{\langle l, F_{\lambda u}(\varepsilon_0,\lambda_0,u_0)[w_0]^2\rangle}. \tag{4.18}$$

Then ψ can also be uniquely determined. We denote this solution to be $V_2=(\tau_2,v_2,\psi_2)$.

(6) We prove that $\det(\widetilde{H})<0$, where the matrix $\widetilde{H}$ is given by

$$\widetilde{H}=\widetilde{H}(\varepsilon_0,U_0)=\begin{pmatrix}\widetilde{H}_{11} & \widetilde{H}_{12}\\ \widetilde{H}_{12} & \widetilde{H}_{22}\end{pmatrix} \tag{4.19}$$

and $\widetilde{H}_{ij}$ are given by

$$\widetilde{H}_{11}=\big\langle l_1, G_{\varepsilon\varepsilon U}[W_0]+2G_{\varepsilon U}[V_1]\big\rangle, \tag{4.20}$$

$$\widetilde{H}_{12}=\frac{1}{2}\big\langle l_1, G_{\varepsilon UU}[W_0]^2+G_{\varepsilon U}[V_2]+2G_{UU}[W_0,V_1]\big\rangle, \tag{4.21}$$

$$\widetilde{H}_{22}=\frac{1}{3}\big\langle l_1, G_{UUU}[W_0]^3+3G_{UU}[W_0,V_2]\big\rangle. \tag{4.22}$$

Here $l_1\in(Y\times Y)^*$ is defined by

$$\langle l_1, (y_1, y_2)\rangle = \langle l, y_1\rangle, \tag{4.23}$$

where $l \in Y^*$ is defined previously so that $\langle l, y\rangle = 0$ if and only if $y \in R(F_u(\varepsilon_0, \lambda_0, u_0))$.

From (4.3), (4.20), we have

$$\begin{aligned}\widetilde{H}_{11} &= \langle l, F_{\varepsilon\varepsilon\widetilde{U}}[\widetilde{W}_0] + 2F_{\varepsilon\widetilde{U}}[\widetilde{V}_1]\rangle \\ &= \langle l, F_{\varepsilon\varepsilon u}[w_0] + 2F_{\varepsilon u}[w_0]\rangle = 0.\end{aligned} \tag{4.24}$$

On the other hand, from (4.4), (4.15), (4.21) and (**F4′**), we have

$$\begin{aligned}\widetilde{H}_{12} &= \frac{1}{2}\langle l, F_{\varepsilon\widetilde{U}\widetilde{U}}[\widetilde{W}_0]^2 + F_{\varepsilon\widetilde{U}}[\widetilde{V}_2] + 2F_{\widetilde{U}\widetilde{U}}[\widetilde{W}_0, \widetilde{V}_1]\rangle \\ &= \frac{1}{2}\langle l, F_{\varepsilon uu}[w_0]^2 + \tau_2 F_{\varepsilon\lambda} + F_{\varepsilon u}[v_2] + 2\tau_1 F_{\lambda u}[w_0] + 2F_{uu}[w_0]^2\rangle \\ &= \frac{1}{2}\langle l, F_{\varepsilon uu}[w_0]^2 + F_{\varepsilon u}[v_2] + 2\tau_1 F_{\lambda u}[w_0]\rangle \\ &= -\frac{1}{2}\langle l, F_{\varepsilon uu}[w_0]^2 + F_{\varepsilon u}[v_2]\rangle \neq 0.\end{aligned} \tag{4.25}$$

And from (4.22) and (4.18), we have

$$\begin{aligned}\widetilde{H}_{22} &= \frac{1}{3}\langle l, F_{\widetilde{U}\widetilde{U}\widetilde{U}}[\widetilde{W}_0]^3 + 3F_{\widetilde{U}\widetilde{U}}[\widetilde{W}_0, \widetilde{V}_2]\rangle \\ &= \frac{1}{3}\langle l, F_{uuu}[w_0]^3 + 3\tau_2 F_{\lambda u}[w_0] + 3F_{uu}[w_0, v_2]\rangle \\ &= -\frac{2}{3}\langle l, F_{uuu}[w_0]^3 + 3F_{uu}[w_0, v_2]\rangle\end{aligned} \tag{4.26}$$

where $\widetilde{U} = (\lambda, u)$, $\widetilde{W}_0 = (0, w_0)$, $\widetilde{V}_1 = (\tau_1, w_0)$, and $\widetilde{V}_2 = (\tau_2, v_2)$.

From (4.19), (4.20) and (4.25), we obtain

$$\det(\widetilde{H}) = -\widetilde{H}_{12}^2 = -\frac{1}{4}\langle l, F_{\varepsilon uu}[w_0]^2 + F_{\varepsilon u}[v_2]\rangle^2 < 0. \tag{4.27}$$

Now we can apply Theorem 2.3 to $G(\varepsilon, U) = (0, 0)$. Then the solution set of $G(\varepsilon, U) = (0, 0)$ near T_0 is the union of C^1 curves intersecting at T_0 including the line of trivial solutions $\Sigma_0 = \{(\varepsilon, U_0)\colon |\varepsilon - \varepsilon_0| < \delta_1\}$ and two other curves $\Sigma_i = \{T_{is} = (\varepsilon_i(s), U_i(s))\colon |s| < \delta_2\}$, for some $\delta_1, \delta_2 > 0$, where $\varepsilon_i(s) = \varepsilon_0 + \mu_i s + s z_{i0}(s)$, $U_i(s) = U_0 + s\eta_i W_0 + s Z_i(s)$, where $z_{i0}(0) = Z_i(0) = 0$, $i = 1, 2$, and (μ_1, η_1) and (μ_2, η_2) satisfy

$$2\widetilde{H}_{12}\mu\eta + \widetilde{H}_{22}\eta^2 = 0. \tag{4.28}$$

We can choose $(\mu_1, \eta_1) = (1, 0)$ and $(\mu_2, \eta_2) = (-\widetilde{H}_{22}/(2\widetilde{H}_{12}), 1)$. This completes the proof. □

Following Corollary 2.5, we show the following special case of Theorem 4.1:

Corollary 4.2. *Assume the conditions in Theorem 4.1 are satisfied, and in addition we assume that*

$$R\big(F_u(\varepsilon,\lambda_0,u_0)\big) \subseteq R\big(F_u(\varepsilon_0,\lambda_0,u_0)\big), \quad \text{for } |\varepsilon-\varepsilon_0|<\delta_0. \tag{4.29}$$

Then the solution set of $G(\varepsilon,\lambda,u,w)=(0,0)$ *near* T_0 *consists precisely of the curves* $\Sigma_0=\{(\varepsilon,\lambda_0,u_0,w_0)\colon |\varepsilon-\varepsilon_0|<\delta_1\}$ *and* $\Sigma_2=\{T_{2s}=(\varepsilon_2(s),\lambda_2(s),u_2(s),w_2(s))\colon |s|<\delta_2\}$, *for some* $\delta_1,\delta_2>0$, *where* $\varepsilon_2(s)=\varepsilon_0+\alpha s+sz_{20}(s)$, $\lambda_2(s)=\lambda_0+sz_{21}(s)$, $u_2(s)=u_0+sw_0+sz_{22}(s)$, $w_2(s)=w_0+sv_2+sz_{23}(s)$, $z_{20}(0)=z_{21}(0)=z_{22}(0)=z_{23}(0)=0$, *and*

$$\alpha=-\frac{2\langle l, F_{uuu}(\varepsilon_0,\lambda_0,u_0)[w_0]^3+3F_{uu}(\varepsilon_0,\lambda_0,u_0)[w_0,v_2]\rangle}{3\langle l, F_{\varepsilon uu}(\varepsilon_0,\lambda_0,u_0)[w_0]^2\rangle}. \tag{4.30}$$

Proof. We apply Corollary 2.5 by showing (2.18) holds, that is $R(G_U(\varepsilon,U_0))\subseteq R(K)$. From (4.3) and (4.29), we have

$$\big\langle l_1, G_U(\varepsilon,U_0)[\tau,v,\psi]\big\rangle=\big\langle l,\tau F_\lambda(\varepsilon,\lambda_0,u_0)+F_u(\varepsilon,\lambda_0,u_0)[v]\big\rangle=0.$$

Thus $R(G_U(\varepsilon,U_0))\subseteq R(K)$. And from (4.29), we have $\langle l, F_u(\varepsilon,\lambda_0,u_0)[v_2]\rangle=0$, so we obtain $\langle l, F_{\varepsilon u}(\varepsilon_0,\lambda_0,u_0)[v_2]\rangle=0$. Now we get (4.30), from (4.6). Hence the conclusions follow from Corollary 2.5. □

In Corollary 4.2, we have $\varepsilon_2(0)=\varepsilon_0$, $\varepsilon_2'(0)=\alpha$, $\lambda_2(0)=\lambda_0$, $\lambda_2'(0)=0$, $u_2'(0)=w_0$, and $w_2'(0)=v_2$. To completely determine the turning direction of the curve of degenerate solutions, we calculate $\lambda_2''(0)$. Let $\{T_s=(\varepsilon_2(s),\lambda_2(s),u_2(s),w_2(s))\colon s\in(-\delta,\delta)\}$ be a curve of degenerate solutions which we obtain in Corollary 4.2. Differentiating $G(\varepsilon_2(s),\lambda_2(s),u_2(s),w_2(s))=0$ with respect to s, we obtain (for convenience, we drop all the subscripts 2 in the following)

$$F_\varepsilon\varepsilon'(s)+F_\lambda\lambda'(s)+F_u\big[u'(s)\big]=0, \tag{4.31}$$

$$F_{\varepsilon u}\big[w(s)\big]\varepsilon'(s)+F_{\lambda u}\big[w(s)\big]\lambda'(s)+F_{uu}\big[w(s),u'(s)\big]+F_u\big[w'(s)\big]=0. \tag{4.32}$$

Setting $s=0$ in (4.32), we get exactly (4.5). We differentiate (4.31) and (4.32) again, and we have

$$\begin{aligned}&F_{\varepsilon\varepsilon}\big[\varepsilon'(s)\big]^2+F_\varepsilon\varepsilon''(s)+F_{\lambda\lambda}\big[\lambda'(s)\big]^2+F_\lambda\lambda''(s)+F_{uu}\big[u'(s)\big]^2\\&\quad+F_u\big[u''(s)\big]+2F_{\varepsilon\lambda}\varepsilon'(s)\lambda'(s)+2F_{\varepsilon u}\big[u'(s)\big]\varepsilon'(s)+2F_{\lambda u}\big[u'(s)\big]\lambda'(s)=0,\end{aligned} \tag{4.33}$$

$$\begin{aligned}&F_{\varepsilon\varepsilon u}\big[w(s)\big]\big[\varepsilon'(s)\big]^2+F_{\varepsilon u}\big[w(s)\big]\varepsilon''(s)+F_{\lambda u}\big[w(s)\big]\lambda''(s)\\&\quad+F_{\lambda\lambda u}\big[w(s)\big]\big[\lambda'(s)\big]^2+F_{uuu}\big[u'(s),u'(s),w(s)\big]+F_{uu}\big[w(s),u''(s)\big]\\&\quad+F_u\big[w''(s)\big]+2F_{\varepsilon\lambda u}\big[w(s)\big]\varepsilon'(s)\lambda'(s)+2F_{\varepsilon uu}\big[u'(s),w(s)\big]\varepsilon'(s)\\&\quad+2F_{\lambda uu}\big[u'(s),w(s)\big]\lambda'(s)+2F_{\varepsilon u}\big[w'(s)\big]\varepsilon'(s)+2F_{\lambda u}\big[w'(s)\big]\lambda'(s)\\&\quad+2F_{uu}\big[w'(s),u'(s)\big]=0.\end{aligned} \tag{4.34}$$

Setting $s=0$ in (4.33) and (4.34), we obtain $u''(0)=v_2$ and

$$F_{\lambda u}[w_0]\lambda''(0) + F_{uuu}[w_0]^3 + F_u\big[w''(0)\big] + 2\alpha F_{\varepsilon uu}[w_0]^2 + 2\alpha F_{\varepsilon u}[v_2] + 3F_{uu}[w_0, v_2] = 0$$

and applying l to it, we obtain

$$\begin{aligned}\lambda_2''(0) = \lambda''(0) &= -\frac{\langle l, F_{uuu}[w_0]^3 + 3F_{uu}[w_0, v_2]\rangle + 2\alpha\langle l, F_{\varepsilon uu}[w_0]^2\rangle}{\langle l, F_{\lambda u}[w_0]\rangle} \\ &= \frac{\langle l, F_{uuu}[w_0]^3 + 3F_{uu}[w_0, v_2]\rangle}{3\langle l, F_{\lambda u}[w_0]\rangle},\end{aligned} \tag{4.35}$$

from (4.30). From Theorem 4.1, Corollary 4.2 and above calculations, a one-to-one relation between λ_2 and ε can be established if $\varepsilon_2'(0) = \alpha \neq 0$ by

$$\lambda_2(\varepsilon) - \lambda_0 \approx \frac{1}{2}\lambda_2''(0)\left(\frac{\varepsilon - \varepsilon_0}{\alpha}\right)^2 = \frac{\lambda_2''(0)}{2\alpha^2}(\varepsilon - \varepsilon_0)^2 = \Lambda(\varepsilon - \varepsilon_0)^2, \tag{4.36}$$

where

$$\Lambda = \frac{\lambda_2''(0)}{2\alpha^2} = \frac{3\langle l, F_{\varepsilon uu}[w_0]^2\rangle^2}{8\langle l, F_{\lambda u}[w_0]\rangle \cdot \langle l, F_{uuu}[w_0]^3 + 3F_{uu}[w_0, v_2]\rangle} \tag{4.37}$$

from (4.30) and (4.35).

Now by using the information for the degenerate solutions obtained in Theorem 4.1 and Corollary 4.2, we have the following result for the variation of the bifurcation diagrams of the original equation $F(\varepsilon, \lambda, u) = 0$ in the (λ, u)-space when the parameter ε is perturbed from $\varepsilon = \varepsilon_0$.

Theorem 4.3. *Assume that all the conditions in Theorem 4.1 and Corollary 4.2 are satisfied, and $\{T_{2s}\}$ is defined as in Corollary 4.2. For the purpose of fixing an orientation, we also assume that, in addition to* (**F3**) *and* (4.4)*,*

$$\big\langle l, F_{\lambda u}(\varepsilon_0, \lambda_0, u_0)[w_0]\big\rangle > 0, \tag{4.38}$$

$$\big\langle l, F_{\varepsilon uu}(\varepsilon_0, \lambda_0, u_0)[w_0]^2\big\rangle < 0, \tag{4.39}$$

$$\big\langle l, F_{uuu}(\varepsilon_0, \lambda_0, u_0)[w_0]^3 + 3F_{uu}(\varepsilon_0, \lambda_0, u_0)[w_0, v_2]\big\rangle < 0, \tag{4.40}$$

where v_2 is defined by (4.5)*. Then there exist $\rho, \eta, \delta_3, \delta_4 > 0$ such that for $N = \{(\lambda, u) \in \mathbb{R} \times X\colon |\lambda - \lambda_0| \leqslant \delta_3,\ \|u - u_0\| \leqslant \delta_4\}$, $|\varepsilon - \varepsilon_0| < \rho$, the set $F_\varepsilon = \{(\lambda, u) \in N\colon F(\varepsilon, \lambda, u) = 0\}$ is the union of two curves $\Gamma_0 = \{(\lambda, u_0)\colon |\lambda - \lambda_0| \leqslant \delta_3\}$ and $\Gamma_\varepsilon = \{(\bar\lambda_\varepsilon(t), \bar u_\varepsilon(t))\colon |t| \leqslant \eta\}$ with $\bar\lambda_\varepsilon(0) = \lambda_0$. Moreover,*

- (A) *when $\varepsilon = \varepsilon_0$, $\bar\lambda_{\varepsilon_0}'(0) = 0$, $\bar\lambda_{\varepsilon_0}''(0) > 0$, and $\bar\lambda_{\varepsilon_0}(\pm\eta) = \lambda_0 + \delta_3$;*
- (B) *when $\varepsilon \in (\varepsilon_0 - \rho, \varepsilon_0)$, $\bar\lambda_\varepsilon'(0) < 0$, there exists $t_+ \in (0, \eta)$ such that $\bar\lambda_\varepsilon(t_+) < \lambda_0$, $\bar\lambda_\varepsilon'(t_+) = 0$, $\bar\lambda_\varepsilon''(t_+) > 0$, $\bar\lambda_\varepsilon(\pm\eta) = \lambda_0 + \delta_3$, and $(\bar\lambda_\varepsilon(t_+), \bar u_\varepsilon(t_+))$, (λ_0, u_0) are the only degenerate solutions on Γ_ε;*
- (C) *when $\varepsilon \in (\varepsilon_0, \varepsilon_0 + \rho)$, $\bar\lambda_\varepsilon'(0) > 0$, there exists $t_- \in (-\eta, 0)$ such that $\bar\lambda_\varepsilon(t_-) < \lambda_0$, $\bar\lambda_\varepsilon'(t_-) = 0$, $\bar\lambda_\varepsilon''(t_-) > 0$, $\bar\lambda_\varepsilon(\pm\eta) = \lambda_0 + \delta_3$, and $(\bar\lambda_\varepsilon(t_-), \bar u_\varepsilon(t_-))$, (λ_0, u_0) are the only degenerate solutions on Γ_ε.*

Proof. Apparently from (4.3), $\Gamma_0 = \{(\lambda, u_0)\colon |\lambda - \lambda_0| \leqslant \delta_3\}$ is always a solution curve for any small $\delta_3 > 0$ and ε near ε_0. When $\varepsilon = \varepsilon_0$, the conditions (**F1**) and (**F3**) imply that a bifurcation of non-trivial solutions occurs at (λ_0, u_0) from Theorem 1.1, and a curve of non-trivial solutions $\Gamma_{\varepsilon_0} = \{(\bar{\lambda}_{\varepsilon_0}(t), \bar{u}_{\varepsilon_0}(t))\colon |t| \leqslant \eta\}$ exists. Moreover (2.1), (**F4′**) and (**F6**) imply that $\bar{\lambda}'_{\varepsilon_0}(0) = 0$ and from (2.2), (4.38) and (4.40),

$$\bar{\lambda}''_{\varepsilon_0}(0) = -\frac{\langle l, F_{uuu}[w_0]^3 + 3F_{uu}[w_0, v_2]\rangle}{3\langle l, F_{\lambda u}[w_0]\rangle} > 0. \tag{4.41}$$

Thus when $\varepsilon = \varepsilon_0$, the bifurcation near (λ_0, u_0) is a supercritical pitchfork one, and there is only one degenerate solution on Γ_{ε_0} near (λ_0, u_0). Define $N = \{(\lambda, u) \in \mathbb{R} \times X\colon |\lambda - \lambda_0| \leqslant \delta_3, \|u - u_0\| \leqslant \delta_4\}$, such that F_{ε_0} is the union of Γ_0 and Γ_{ε_0}, and $\bar{\lambda}_{\varepsilon_0}(\pm\eta) = \lambda_0 + \delta_3$, for $s \in [-\eta, \eta]$, $\|\bar{u}_{\varepsilon_0}(t)\| \leqslant \delta_4/2$.

From the conditions are satisfied, a bifurcation of the degenerate solutions of $F = 0$ occurs near $(\varepsilon, \lambda, u) = (\varepsilon_0, \lambda_0, u_0)$ as described as in Theorem 4.1 and Corollary 4.2. Hence there are two curves of degenerate solutions: the trivial one $\Sigma_0 = \{(\varepsilon, \lambda_0, u_0, w_0)\colon |\varepsilon - \varepsilon_0| < \delta_1\}$ and the non-trivial one $\Sigma_2 = \{T_{2s} = (\varepsilon_2(s), \lambda_2(s), u_2(s), w_2(s))\colon |s| < \delta_2\}$. Furthermore from (4.30), (4.39) and (4.40), we have

$$\varepsilon_2'(0) = \alpha = -\frac{2\langle l, F_{uuu}(\varepsilon_0, \lambda_0, u_0)[w_0]^3 + 3F_{uu}(\varepsilon_0, \lambda_0, u_0)[w_0, v_2]\rangle}{3\langle l, F_{\varepsilon uu}(\varepsilon_0, \lambda_0, u_0)[w_0]^2\rangle} < 0,$$

thus there are exactly two degenerate solutions for any $\varepsilon \in (\varepsilon_0 - \rho, \varepsilon_0 + \rho)\backslash\{\varepsilon_0\}$. Similarly $\lambda_2'(0) = 0$, and from (4.35), (4.38) and (4.40), we have

$$\lambda_2''(0) = \frac{\langle l, F_{uuu}(\varepsilon_0, \lambda_0, u_0)[w_0]^3 + 3F_{uu}(\varepsilon_0, \lambda_0, u_0)[w_0, v_2]\rangle}{3\langle l, F_{\lambda u}(\varepsilon_0, \lambda_0, u_0)[w_0]\rangle} < 0,$$

hence $\lambda_2(s) < \lambda_0$ for any $s \neq 0$.

We consider the case of $\varepsilon \in (\varepsilon_0, \varepsilon_0 + \rho)$, and the case of $\varepsilon \in (\varepsilon_0 - \rho, \varepsilon_0)$ is similar. For such ε, (λ_0, u_0) is a degenerate solution of F which satisfies (**F1**). Indeed, by using [22, p. 235, Theorem 5.17], we have $\dim N(F_u(\varepsilon, \lambda_0, u_0)) = 1$ from $\dim N(F_u(\varepsilon, \lambda_0, u_0)) \leqslant \dim N(F_u(\varepsilon_0, \lambda_0, u_0)) = 1$ and $N(F_u(\varepsilon, \lambda_0, u_0)) \neq \emptyset$. Furthermore, since Fredholm index is a constant under a small perturbation (again from [22, p. 235, Theorem 5.17]) and F satisfies (**F1**) at $(\varepsilon_0, \lambda_0, u_0)$, we have $\dim N(F_u(\varepsilon, \lambda_0, u_0)) = \operatorname{codim} R(F_u(\varepsilon, \lambda_0, u_0)) = 1$. Let $l(\varepsilon) \in Y^*$ be the function such that $N(l(\varepsilon)) = R(F_u(\varepsilon, \lambda_0, u_0))$ and $l(\varepsilon_0) = l$. Since $\langle l, F_{\lambda u}(\varepsilon_0, \lambda_0, u_0)[w_0]\rangle > 0$, we have $\langle l(\varepsilon), F_{\lambda u}(\varepsilon, \lambda_0, u_0)[w_0]\rangle > 0$ for $\varepsilon \in (\varepsilon_0, \varepsilon_0 + \rho)$ thus (**F3**) is also satisfied at $(\varepsilon, \lambda_0, u_0)$. Hence all conditions in Theorem 1.1 are satisfied, and a curve of non-trivial solutions $\Gamma_\varepsilon = \{(\bar{\lambda}_\varepsilon(t), \bar{u}_\varepsilon(t))\colon |t| \leqslant \eta\}$ exists such that $(\bar{\lambda}_\varepsilon(0), \bar{u}_\varepsilon(0)) = (\lambda_0, u_0)$, and $\bar{\lambda}'_\varepsilon(0)$ is determined by (2.1). Define $B(\varepsilon) = \langle l(\varepsilon), F_{uu}(\varepsilon, \lambda_0, u_0)[w_0]^2\rangle$, we have

$$\begin{aligned} B'(\varepsilon_0) &= \bigl\langle l'(\varepsilon_0), F_{uu}[w_0]^2\bigr\rangle + \bigl\langle l, F_{\varepsilon uu}[w_0]^2\bigr\rangle \\ &= -\bigl\langle l'(\varepsilon_0), F_u[v_2]\bigr\rangle + \bigl\langle l, F_{\varepsilon uu}[w_0]^2\bigr\rangle \\ &= \bigl\langle l, F_{\varepsilon uu}[w_0]^2\bigr\rangle < 0. \end{aligned} \tag{4.42}$$

In (4.42), we obtain $\langle l'(\varepsilon_0), F_u[v_2]\rangle = -\langle l, F_{\varepsilon u}[v_2]\rangle = 0$ by differentiating the equation $\langle l(\varepsilon), F_u(\varepsilon, \lambda_0, u_0)[v_2]\rangle = 0$. Thus $B(\varepsilon) < 0$ for $\varepsilon \in (\varepsilon_0, \varepsilon_0 + \rho)$, and from (2.1), we have

$$\bar{\lambda}'_\varepsilon(0) = -\frac{\langle l(\varepsilon), F_{uu}(\varepsilon, \lambda_0, u_0)[w_0]^2\rangle}{2\langle l(\varepsilon), F_{\lambda u}(\varepsilon, \lambda_0, u_0)[w_0]\rangle} > 0.$$

On the other hand, since $\bar{\lambda}_{\varepsilon_0}(\pm\eta) = \lambda_0 + \delta_3$, and for $s \in [-\eta, \eta]$, $\|\bar{u}_{\varepsilon_0}(t)\| \leqslant \delta_4/2$. Then by choosing ε close enough to ε_0, we may assume that $\bar{\lambda}_\varepsilon(\pm\eta) = \lambda_0 + \delta_3$, and for $s \in [-\eta, \eta]$, $\|\bar{u}_\varepsilon(t)\| \leqslant \delta_4$. Hence there exists $t_- \in (-\eta, 0)$ such that $(\bar{\lambda}_-, \bar{u}_-) = (\bar{\lambda}(t_-), \bar{u}(t_-))$ is the only degenerate solution on the curve Γ_ε other than (λ_0, u_0).

We verify that $(\bar{\lambda}_-, \bar{u}_-)$ is a degenerate solution which satisfies the condition of Saddle-node Bifurcation Theorem (see [7, Theorem 3.2]). Again the condition (**F1**) is satisfied at $(\bar{\lambda}_-, \bar{u}_-)$ since $(\bar{\lambda}_-, \bar{u}_-)$ is a perturbation of (λ_0, u_0). The only other condition is (**F2**): $F_\lambda(\varepsilon, \bar{\lambda}_-, \bar{u}_-) \notin R(F_u(\varepsilon, \bar{\lambda}_-, \bar{u}_-))$. To prove that, recall that $\Sigma_2 = \{T_{2s} = (\varepsilon_2(s), \lambda_2(s), u_2(s), w_2(s))\colon |s| < \delta_2\}$ is the curve of non-trivial degenerate solutions of $F = 0$. Since we assume $\varepsilon \in (\varepsilon_0, \varepsilon_0 + \rho)$, then $(\bar{\lambda}_-, \bar{u}_-) = (\lambda_2(s_-), u_2(s_-))$ for some $s_- \in (-\delta_2, 0)$. Define $A(s) = \langle l(s), F_\lambda(\varepsilon_2(s), \lambda_2(s), u_2(s))\rangle$, where $l(s) \in Y^*$ satisfying $N(l(s)) = R(F_u(\varepsilon_2(s), \lambda_2(s), u_2(s)))$. Then $A'(0) = \langle l, F_{\lambda u}(\varepsilon_0, \lambda_0, u_0)[w_0]\rangle > 0$ and $A(0) = 0$, so $A(s) < 0$ for $s < 0$ which implies (**F2**) holds at $(\bar{\lambda}_-, \bar{u}_-)$. We also define $B(s) = \langle l(s), F_{uu}(\varepsilon_2(s), \lambda_2(s), u_2(s))[w_2(s)]^2\rangle$, we have

$$\begin{aligned}
B'(0) &= \big\langle l'(0), F_{uu}[w_0]^2\big\rangle + \alpha\big\langle l, F_{\varepsilon uu}[w_0]^2\big\rangle + \big\langle l, F_{uuu}[w_0]^3 + 2F_{uu}[v_2, w_0]\big\rangle \\
&= \big\langle l'(0), F_{uu}[w_0]^2\big\rangle - \frac{2}{3}\big\langle l, F_{uuu}[w_0]^3 + 3F_{uu}[w_0, v_2]\big\rangle + \big\langle l, F_{uuu}[w_0]^3 + 2F_{uu}[v_2, w_0]\big\rangle \\
&= \frac{1}{3}\big\langle l, F_{uuu}[w_0]^3 + 3F_{uu}[w_0, v_2]\big\rangle < 0, \qquad (4.43)
\end{aligned}$$

since $\varepsilon'_2(0) = \alpha$, $\lambda'_2(0) = 0$, $u'_2(0) = w_0$, $w'_2(0) = v_2$ and (4.30). In (4.43), we obtain $\langle l'(0), F_{uu}[w_0]^2\rangle = \langle l, F_{uu}[v_2, w_0]\rangle$ by differentiating $\langle l(s), F_u(\varepsilon_2(s), \lambda_2(s), u_2(s))[w_0]\rangle = 0$ twice and using (4.35). In particular $B(s_-) > 0$ from $B(0) = 0$. Now from [7, Theorem 3.2], near (λ_-, u_-), the solutions of $F(\varepsilon, \cdot) = 0$ form a curve, which is indeed identical to Γ_ε defined earlier. Hence $\bar{\lambda}'(t_-) = 0$, $\bar{\lambda}''(t_-) > 0$ and $\bar{\lambda}(\pm\eta) = \lambda_0 + \delta_1$. Here

$$\bar{\lambda}''(t_-) = -\frac{\langle l(s_-), F_{uu}(\varepsilon, \lambda_-, u_-)[w_-, w_-]\rangle}{\langle l(s_-), F_\lambda(\varepsilon, \lambda_-, u_-)\rangle} = -\frac{B(s_-)}{A(s_-)} > 0. \qquad \square \qquad (4.44)$$

The results in this section provide an abstract framework for the bifurcations shown in Fig. 5. That is, the bifurcation diagram changes from a "forward" transcritical bifurcation when $\varepsilon > \varepsilon_0$, to a pitchfork bifurcation when $\varepsilon = \varepsilon_0$, and to a "backward" transcritical bifurcation when $\varepsilon < \varepsilon_0$. Here the forward and backward refer to the portion of the solution curve with $t > 0$, which often represents the positive solutions.

5. Examples

Reaction–diffusion models have been used to described various spatiotemporal phenomena in spatial ecology, population genetics. In the ecological models in form

$$\Delta u+\lambda u f(x,u)=0,\quad x\in\Omega,\qquad u(x)=0,\quad x\in\partial\Omega,\tag{5.1}$$

it was usually assumed that the growth rate per capita $f(x,u)$ is decreasing in u due to the crowding effect, and this represents the typical logistic growth. However it has been increasingly recognized that for many species, the function $f(x,u)$ is increasing for small u because the population is too small to support a growth, and is decreasing for large u again due to the competition for the limited resource. The latter growth pattern is termed as Allee effect in ecological studies, see for examples, Cantrell and Cosner [3], Shi and Shivaji [40], Stephens and Sutherland [42]. If $f(x,0)<0$, then the growth pattern $f(x,u)$ is of a strong Allee effect; and if $f(x,0)\geqslant 0$, then it is of a weak Allee effect [40].

Example 5.1. In this example, we demonstrate the transition of the bifurcation diagrams of (5.1) when f is changed from logistic type to weak Allee effect type. To be more specific, we consider the following semilinear elliptic equation

$$\begin{cases}\Delta u+\lambda\big[a(x)u-b(x)u^3-\varepsilon c(x)u^2\big]=0, & x\in\Omega,\\ u=0, & x\in\partial\Omega,\end{cases}\tag{5.2}$$

where λ is a positive parameter, ε is a real parameter, Ω is a bounded region with smooth boundary in $\mathbb{R}^n$ for $n\geqslant 1$, $a(\cdot),b(\cdot),c(\cdot)\in C^\alpha(\overline{\Omega})$, and there exists an open subset $\Omega_0\subseteq\Omega$ such that $a(x)>0$ for any $x\in\Omega_0$. It is well-known that the eigenvalue problem

$$\begin{cases}\Delta\phi+\lambda a(x)\phi=0 & \text{in } \Omega,\\ \phi=0 & \text{on } \partial\Omega,\end{cases}\tag{5.3}$$

has a principal eigenvalue $\lambda_1>0$ such that the corresponding eigenfunction $\phi_1(x)>0$ in Ω (see [2]). Moreover λ_1 can be expressed as

$$\lambda_1=\inf\left\{\int_\Omega\left|\nabla\phi(x)\right|^2dx:\ \int_\Omega a(x)\phi^2(x)\,dx=1,\ \phi\in H_0^1(\Omega)\right\}.$$

This apparently implies that $\int_\Omega a(x)\phi_1^2(x)\,dx>0$, and here we also assume that

$$\int_\Omega b(x)\phi_1^4(x)\,dx>0,\quad \text{and}\quad \int_\Omega c(x)\phi_1^3(x)\,dx>0.\tag{5.4}$$

When $\varepsilon=0$, it is well-known that (5.2) has a unique positive solution and a unique negative solution when $\lambda>\lambda_1$, and a pitchfork bifurcation occurs at $(\lambda_1,0)$ (see [40]). Here we demonstrate our new abstract theory by analyzing the bifurcation problem with ε small.

Define a nonlinear mapping $F:\mathbb{R}\times\mathbb{R}\times C_0^{2,\alpha}(\overline{\Omega})\to C^\alpha(\overline{\Omega})$, by

$$F(\varepsilon,\lambda,u)=\Delta u+\lambda\big[a(x)u-b(x)u^3-\varepsilon c(x)u^2\big].\tag{5.5}$$

Then $u=0$ is a trivial solution for any λ and ε, and $\lambda=\lambda_1$ is a bifurcation point for positive and negative solutions of (5.2) for any fixed ε. It is easy to verify that $N(F_u(0,\lambda_1,0))=\text{span}\{\phi_1\}$, $R(F_u(0,\lambda_1,0))=\{v\in C^\alpha(\overline{\Omega}):\ \int_\Omega v\phi_1\,dx=0\}$; $F_{\lambda u}(0,\lambda_1,0)[\phi_1]=a(x)\phi_1\notin R(F_u(0,\lambda_1,0))$,

$F_{uu}(0,\lambda_1,0)[\phi_1]^2 = 0 \in R(F_u(0,\lambda_1,0))$, thus (**F1**), (**F3**) and (**F4′**) are satisfied and $v_2 = 0$ at $(\varepsilon,\lambda,u) = (0,\lambda_1,0)$. The condition (4.3) is also satisfied as $F_u(\varepsilon,\lambda_1,0) = F_u(0,\lambda_1,0)$ hence $R(F_u(\varepsilon,\lambda_1,0)) = R(F_u(0,\lambda_1,0))$. Furthermore we can compute that

$$\langle l, F_{\lambda u}(0,\lambda_1,0)[\phi_1]\rangle = \int_\Omega a(x)\phi_1^2(x)\,dx > 0,$$

$$\langle l, F_{uuu}(0,\lambda_1,0)[\phi_1]^3\rangle = -6\lambda_1 \int_\Omega b(x)\phi_1^4(x)\,dx < 0,$$

$$\langle l, F_{\varepsilon uu}(0,\lambda_1,0)[\phi_1]^2\rangle = -2\lambda_1 \int_\Omega c(x)\phi_1^3(x)\,dx < 0,$$

hence (4.38), (4.39) and (4.40) are also satisfied. From Theorem 4.3, there exists $\delta > 0$ such that all the degenerate solutions of (5.2) near $T_0 = (0,\lambda_1,0,\phi_1)$ consist precisely of the curves $(\varepsilon,\lambda_1,0,\phi_1)$ and $T_2 = (\varepsilon_2(s),\lambda_2(s),u_2(s),w_2(s))$, where $\varepsilon_2(s) = \alpha s + s z_{20}(s)$, $\lambda_2(s) = \lambda_1 + s z_{21}(s)$, $u_2(s) = u_0 + s\phi_1 + s z_{22}(s)$, $w_2(s) = \phi_1 + s z_{23}(s)$, for $s \in (-\delta,\delta)$, where

$$\alpha = -\frac{2\int_\Omega b(x)\phi_1^4\,dx}{\int_\Omega c(x)\phi_1^3\,dx},$$

and the bifurcation diagrams near $\varepsilon = 0$ are exactly as in the ones in Fig. 5.

Restricting to the positive solutions of (5.2), then near the bifurcation point $(\lambda,u) = (\lambda_1,0)$, the bifurcation of positive solutions is a forward one when $\varepsilon > 0$, and it is a backward one when $\varepsilon < 0$. Indeed by combining with a well-known uniqueness result, we can obtain the following global bifurcation result of (5.2).

Theorem 5.1. *Suppose that $a(\cdot), b(\cdot), c(\cdot) \in C^\alpha(\overline{\Omega})$, and there exists an open subset $\Omega_0 \subseteq \Omega$ such that $a(x) > 0$ for any $x \in \Omega_0$. Let λ_1 be the principal eigenvalue of* (5.3)*, and the corresponding eigenfunction ϕ_1 satisfies* (5.4).

1. *There exists $\delta > 0$ such that for ε satisfying $\varepsilon \in (-\delta,\delta)$, all non-trivial solutions of* (5.2) *near $(\lambda_1,0)$ lie on a smooth curve $\Gamma_\varepsilon = \{(\lambda_\varepsilon(t), u_\varepsilon(t,\cdot))\colon |t| < \eta\}$, such that $\lambda_\varepsilon(0) = \lambda_1$, $u_\varepsilon(0,\cdot) = 0$, $u_\varepsilon(t,x) > 0$ when $t \in (0,\eta)$ and $x \in \Omega$, and $u_\varepsilon(t,x) < 0$ when $t \in (-\eta,0)$ and $x \in \Omega$. Moreover if $\varepsilon \in (0,\delta)$, then there exists $t_- \in (-\eta,0)$ such that $\lambda'_\varepsilon(t_-) = 0$ and $\lambda''_\varepsilon(t_-) > 0$, and $\lambda'_\varepsilon(t)(t - t_-) > 0$ for $t \in (-\eta,\eta)\backslash\{t_-\}$; if $\varepsilon = 0$, then $\lambda'_0(0) = 0$, $\lambda''_0(0) > 0$, and $\lambda'_0(t)t > 0$ for $t \in (-\eta,\eta)\backslash\{0\}$; and if $\varepsilon \in (-\delta,0)$, then there exists $t_+ \in (-\eta,0)$ such that $\lambda'_\varepsilon(t_+) = 0$ and $\lambda''_\varepsilon(t_+) > 0$, and $\lambda'_\varepsilon(t)(t - t_+) > 0$ for $t \in (-\eta,\eta)\backslash\{t_+\}$ (see Fig.* 5 *for an illustration*);
2. *If in addition, we assume that $a(x) \equiv a > 0$, $b(x) \equiv b > 0$ and $c(x) \equiv c > 0$, then all the positive and negative solutions of* (5.2) *lie on a smooth curve $\Gamma_\varepsilon = \{(\lambda_\varepsilon(t), u_\varepsilon(t,\cdot))\colon t \in \mathbb{R}\}$, such that $\lambda_\varepsilon(0) = \lambda_1$, $u_\varepsilon(0,\cdot) = 0$, $u_\varepsilon(t,x) > 0$ when $t > 0$ and $x \in \Omega$, and $u_\varepsilon(t,x) < 0$ when $t < 0$ and $x \in \Omega$. Moreover if $\varepsilon \in (0,\delta)$, then there exists $t_- < 0$ such that $\lambda'_\varepsilon(t_-) = 0$ and $\lambda''_\varepsilon(t_-) > 0$, and $\lambda'_\varepsilon(t)(t - t_-) > 0$ for $t \in \mathbb{R}\backslash\{t_-\}$; if $\varepsilon = 0$, then $\lambda'_0(0) = 0$, $\lambda''_0(0) > 0$, and $\lambda'_0(t)t > 0$ for $t \in \mathbb{R}\backslash\{0\}$; and if $\varepsilon \in (-\delta,0)$, then there exists $t_+ > 0$ such that $\lambda'_\varepsilon(t_+) = 0$ and $\lambda''_\varepsilon(t_+) > 0$, and $\lambda'_\varepsilon(t)(t - t_+) > 0$ for $t \in (-\eta,\eta)\backslash\{t_+\}$.*

Proof. The local bifurcation diagrams follow from the calculation above and Theorem 4.3. For the global bifurcation diagrams, we assume that $a(x) \equiv a > 0$, $b(x) \equiv b > 0$ and $c(x) \equiv c > 0$. We first prove the case of $\varepsilon = 0$. From [39, Theorem 2.3] (see also [31]), (5.2) has no positive solution if $\lambda \leqslant \lambda_1$, and has exactly one positive solution v_λ if $\lambda > \lambda_1$. Moreover, all v_λ's lie on a smooth curve, v_λ is stable and v_λ is increasing with respect to λ. Hence by letting $\lambda_0(t) = \lambda_1 + (\lambda - \lambda_1)t^2$, $u_0(t,\cdot) = v_{\lambda_0(t)}(\cdot)$ for $t > 0$, and $u_0(t,\cdot) = -v_{\lambda_0(-t)}(\cdot)$ for $t < 0$, we prove the global bifurcation diagram for $\varepsilon = 0$ as the positive (hence also negative) solution of (5.2) is unique for $\lambda > \lambda_1$.

For the case of $\varepsilon \in (-\delta, 0)$, it is easy to see that the negative solution branch satisfies the conditions in [39, Theorem 2.3], hence the negative solution is unique for any $\lambda > \lambda_1$. For the positive solutions, the local bifurcation near $(\lambda_1, 0)$ and the saddle-node bifurcation point $(\lambda_\varepsilon(t_+), u_\varepsilon(t_+))$ have been shown above. From the global bifurcation theorem (see [33,41]), the local curve of positive solutions from $(\lambda_1, 0)$ is unbounded, and indeed it can be extended to $\lambda = \infty$ as all positive solutions are bounded by $\sqrt{a/b}$. Thus there exists at least a positive solution for any $\lambda > \lambda_\varepsilon(t_+)$ and they are on the global branch emanating from $(\lambda_1, 0)$. To prove the uniqueness of positive solution for $\lambda > \lambda_1 + \delta_0$, we first use [1, Theorem A] to conclude the uniqueness when $\lambda > M$ for some large $M > 0$. For $\lambda \in [\lambda_1 + \delta_0, M]$, the unique positive solution v_λ of (5.2) with $\varepsilon = 0$ is stable, hence a small perturbation for ε near 0 is still stable, uniformly for $\lambda \in [\lambda_1 + \delta_0, M]$. Hence the positive solution for $\lambda \in [\lambda_1 + \delta_0, M]$ is also unique. The case of $\varepsilon \in (0, \delta)$ is similar and we omit the details. □

We extract the global bifurcation for the positive solutions in the following corollary:

Corollary 5.2. *Let $a, b, c > 0$, and let $\lambda_1(a)$ be the principal eigenvalue of (5.3) with $a(x) \equiv a$. Then for the boundary value problem*

$$\begin{cases} \Delta u + \lambda\big(au - bu^3 - \varepsilon cu^2\big) = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{5.6}$$

there exists $\delta > 0$ such that

1. *When $\varepsilon \in [0, \delta)$, (5.6) has no positive solution when $\lambda \leqslant \lambda_1(a)$, it has a unique positive solution $\bar{u}_\lambda$ when $\lambda > \lambda_1(a)$, all $\bar{u}_\lambda$ are on a smooth curve $\Gamma_\varepsilon = \{(\lambda, \bar{u}_\lambda)\colon \lambda > \lambda_1(a)\}$, and $\bar{u}_\lambda$ is stable and it is increasing in λ;*
2. *When $\varepsilon \in (-\delta, 0)$, there exists $\lambda_* \in (0, \lambda_1(a))$ such that (5.6) has no positive solution when $\lambda \leqslant \lambda_*$, it has a unique positive solution $\bar{u}_\lambda$ when $\lambda \geqslant \lambda_1(a)$ and $\lambda = \lambda_*$, and it has exactly two positive solutions $\bar{u}_\lambda$ and $\underline{u}_\lambda$ such that $\bar{u}_\lambda > \underline{u}_\lambda$ when $\lambda_* < \lambda < \lambda_1(a)$; all $\bar{u}_\lambda$ are on a smooth curve $\overline{\Gamma}_\varepsilon = \{(\lambda, \bar{u}_\lambda)\colon \lambda \geqslant \lambda_*\}$, and $\bar{u}_\lambda$ is stable and it is increasing in λ; all $\underline{u}_\lambda$ are on a smooth curve $\underline{\Gamma}_\varepsilon = \{(\lambda, \underline{u}_\lambda)\colon \lambda_* < \lambda < \lambda_1(a)\}$, $\underline{u}_\lambda$ is unstable, $\lim_{\lambda \to \lambda_*^+} \underline{u}_\lambda = \bar{u}_\lambda$, and $\lim_{\lambda \to \lambda_1^-(a)} \underline{u}_\lambda = 0$.*

From (4.37), we can have an estimate of λ_*:

$$\lambda_* \approx \lambda_1(a) + \Lambda\varepsilon^2, \tag{5.7}$$

where

$$\Lambda = -\frac{\lambda_1(a)[\int_\Omega c(x)\phi_1^3(x)\,dx]^2}{4\int_\Omega a(x)\phi_1^2(x)\,dx \cdot \int_\Omega b(x)\phi_1^4(x)\,dx}. \tag{5.8}$$

Example 5.2. Here we revisit an equation from the study of gene frequency under the combined influence of selection and migration (in equilibrium form):

$$\begin{cases} \Delta u + \mu s(x)u(1-u)\big[hu + (1-h)(1-u)\big] = 0, & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{5.9}$$

where $s \in L^\infty(\Omega)$, $0 \leqslant h \leqslant 1$, and $0 \leqslant u(x) \leqslant 1$. This model was introduced by Fisher [12], and it was considered by Fleming [13], Henry [19, Chapter 10], see also more recent work in [25–28]. Here we only consider the case that $\int_\Omega s(x)\,dx < 0$ and $s(x) > 0$ on a subset Ω_0 with positive measure. Clearly $u = 0$ and $u = 1$ are constant solutions to (5.9), and from [19, Lemma 10.1.2, Lemma 10.1.3], [36], $\mu_0(h) = \inf\{\int_\Omega |\nabla\phi|^2\,dx\colon (1-h)\int_\Omega s(x)\phi^2\,dx = 1\}$ is a bifurcation point from $u = 0$. To cast Eq. (5.9) into the framework in Section 4, we define

$$\lambda = \mu(1-h), \qquad \varepsilon = 3 - \frac{1}{1-h}. \tag{5.10}$$

Then (5.9) becomes

$$\begin{cases} \Delta u + \lambda s(x)\big[u - \varepsilon u^2 + (-1+\varepsilon)u^3\big] = 0, & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{5.11}$$

We consider the bifurcation near $\varepsilon = \varepsilon_0 = 0$.

Define a nonlinear mapping $F : \mathbb{R} \times \mathbb{R} \times C_N^{2,\alpha}(\overline{\Omega}) \to C^\alpha(\overline{\Omega})$, by

$$F(\varepsilon, \lambda, u) = \Delta u + \lambda s(x)\big[u - \varepsilon u^2 + (-1+\varepsilon)u^3\big], \tag{5.12}$$

where $C_N^{2,\alpha}(\overline{\Omega}) = \{u \in C^{2,\alpha}(\overline{\Omega})\colon \frac{\partial u}{\partial n} = 0\}$. Let λ_0 be the positive principal eigenvalue of

$$\begin{cases} \Delta\phi + \lambda s(x)\phi = 0, & x \in \Omega, \\ \dfrac{\partial\phi}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{5.13}$$

and ϕ_1 is the corresponding positive eigenfunction. Then $\lambda = \lambda_0 = \frac{1}{3}\mu_0$ is a bifurcation point for positive and negative solutions of (5.11) for any fixed ε. One can verify that $N(F_u(0,\lambda_0,0)) = \operatorname{span}\{\phi_1\}$, $R(F_u(0,\lambda_0,0)) = \{v \in C^\alpha(\overline{\Omega})\colon \int_\Omega v\phi_1\,dx = 0\}$; and $F_{\lambda u}(0,\lambda_0,0)[\phi_1] = s(x)\phi_1 \notin R(F_u(0,\lambda_0,0))$, $F_{uu}(0,\lambda_0,0)[\phi_1]^2 = 0 \in R(F_u(0,\lambda_0,0))$, thus (**F1**), (**F3**) and (**F4′**) are satisfied and $v_2 = 0$ at $(\varepsilon,\lambda,u) = (0,\lambda_0,0)$. The condition (4.3) is also satisfied as $F_u(\varepsilon,\lambda_0,0) = F_u(0,\lambda_0,0)$ hence $R(F_u(\varepsilon,\lambda_0,0)) = R(F_u(0,\lambda_0,0))$. Furthermore we can compute that

$$\langle l, F_{\lambda u}(0,\lambda_0,0)[\phi_1]\rangle = \int_\Omega s(x)\phi_1^2(x)\,dx = \frac{1}{\lambda_0}\int_\Omega |\nabla\phi_1|^2\,dx > 0,$$

$$\langle l, F_{uuu}(0,\lambda_0,0)[\phi_1]^3\rangle = -6\lambda_0\int_\Omega s(x)\phi_1^4(x)\,dx = -18\int_\Omega |\nabla\phi_1|^2\phi_1^2\,dx < 0,$$

$$\langle l, F_{\varepsilon uu}(0,\lambda_0,0)[\phi_1]^2\rangle = -2\lambda_0\int_\Omega s(x)\phi_1^3(x)\,dx = -4\int_\Omega |\nabla\phi_1|^2\phi_1\,dx < 0,$$

hence (4.38), (4.39) and (4.40) are also satisfied. Then all conditions in Theorem 4.3 are satisfied, and similar to Example 5.1, the bifurcation diagrams in (λ, u)-space near $\varepsilon = 0$ are exactly as in the ones in Fig. 5. In particular, the bifurcation is supercritical when $\varepsilon \in (0,\delta)$ (or $h \in (2/3 - \delta_1, 2/3)$), and it is subcritical when $\varepsilon \in (-\delta, 0)$ (or $h \in (2/3, 2/3 + \delta_1)$).

A similar global bifurcation picture as the ones in Example 5.1 can be obtained for (5.9). As proved in [19], the positive solution is unique for any $\mu > \mu_0$ for $h \in [1/3, 2/3]$. For $h \in (2/3, 2/3 + \delta_1)$, there exists a $\mu_* < \mu_0$ such that (5.9) has no positive solution when $\mu < \mu_*$, has exactly one positive solution when $\mu > \mu_0$ or $\mu = \mu_*$, and has exactly two positive solutions when $\mu_* < \mu < \mu_0$. This is based on the perturbation results proved above and the uniqueness of the positive solution for large μ. The latter uniqueness result can be proved by using the method outlined in [19, p. 319, Exercise 3] and topological degree argument as in [1].

From (4.37), we can have the following estimate of the saddle-node bifurcation point $\mu_* = \mu_*(h)$ ($h \in (2/3, 2/3 + \delta_1)$):

$$\mu_*(h) = \frac{\lambda_0}{1-h}\left[1 - \frac{(\int_\Omega |\nabla\phi_1|^2\phi_1\,dx)^2}{3\int_\Omega |\nabla\phi_1|^2\phi_1^2\,dx \cdot \int_\Omega |\nabla\phi_1|^2\,dx}\left(\frac{2-3h}{1-h}\right)^2\right]. \tag{5.14}$$

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