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Existence, uniqueness and stability of positive solutions to sublinear elliptic systems

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The existence, stability and uniqueness of positive solutions to a semilinear elliptic system with sublinear nonlinearities are proved. It is shown that the precise global bifurcation diagram of the positive solutions is a monotone curve with different asymptotical behaviour according to the form of the nonlinearities. Equations with Hölder continuous nonlinearities are also considered.

1. Introduction

Reaction–diffusion systems are used to model many chemical and biological phenomena in the natural world [25, 26], and systems of coupled partial differential equations are also used in other physical models such as nonlinear Schrödinger systems in multi-component Bose–Einstein condensates and nonlinear optics [19, 23]. The steady-state solutions or standing-wave solutions of such systems of nonlinear partial differential equations satisfy a nonlinear elliptic system with more than one equation. Much effort has been devoted to the existence of solutions of such systems (see, for example, [4, 8, 10, 11, 13, 15–17, 24, 30, 33, 34]), but it is usually difficult to determine whether or not the solution is unique.

We consider the positive solutions of a semilinear elliptic system of the form

$$\left.\begin{aligned} \Delta u + \lambda f(u,v) &= 0, \quad x \in \Omega, \\ \Delta v + \lambda g(u,v) &= 0, \quad x \in \Omega, \\ u(x) = v(x) &= 0, \quad x \in \partial\Omega, \end{aligned}\right\} \tag{1.1}$$

where $\lambda > 0$ and Ω is a bounded smooth domain. Here f and g are real-valued functions defined on $\mathbb{R}^2_+ := (0,\infty) \times (0,\infty)$ which satisfy

(A1) $f, g \in C^{\alpha}(\overline{\mathbb{R}^2_+}) \cap C^1(\mathbb{R}^2_+)$ for $\alpha \in (0,1)$,

(A2) (cooperativeness) define the Jacobian of the vector field (f, g) as

$$J(u,v) = \begin{pmatrix} \dfrac{\partial f}{\partial u}(u,v) & \dfrac{\partial f}{\partial v}(u,v) \\ \dfrac{\partial g}{\partial u}(u,v) & \dfrac{\partial g}{\partial v}(u,v) \end{pmatrix} \equiv \begin{pmatrix} f_u(u,v) & f_v(u,v) \\ g_u(u,v) & g_v(u,v) \end{pmatrix}. \tag{1.2}$$

Then $f_v(u,v) \geqslant 0$ and $g_u(u,v) \geqslant 0$ for $(u,v) \in \mathbb{R}^2_+$.

We consider the existence, uniqueness and stability of positive solutions to (1.1). From assumption (A1), we look for positive solutions $(u,v) \in C_0^{2,\alpha}(\bar{\Omega})$. The stability of a solution is determined by the following eigenvalue problem:

$$\left.\begin{aligned} \Delta\xi + \lambda f_u(u,v)\xi + \lambda f_v(u,v)\eta &= -\mu\xi, \quad x \in \Omega, \\ \Delta\eta + \lambda g_u(u,v)\xi + \lambda g_v(u,v)\eta &= -\mu\eta, \quad x \in \Omega, \\ \xi(x) = \eta(x) &= 0, \qquad x \in \partial\Omega, \end{aligned}\right\} \tag{1.3}$$

From the maximum principle of cooperative elliptic systems (see lemma 2.1 and [40]), equation (1.3) has a real principal eigenvalue $\mu_1(u,v)$, which has the smallest real part among all spectrum points. A solution (u,v) is *stable* if $\mu_1(u,v) > 0$ and it is *unstable* otherwise.

For a scalar semilinear elliptic equation

$$\Delta u + \lambda f(u) = 0, \quad x \in \Omega, \qquad u(x) = 0, \quad x \in \partial\Omega, \tag{1.4}$$

the stability of a solution can be defined in a similar fashion. If the nonlinear function is sublinear, that is, if $f(u)/u$ is a non-increasing function, then it is well known that a positive solution of (1.4) is stable and one can prove that the solution must be unique with various methods. Brezis and Kamin [2] presented several different proofs for the uniqueness (see also [20, 27, 28, 35, 36]).

Here we use the stability defined above to prove that under some general sublinear conditions similar to the one for scalar equations, the positive solution of (1.1) is stable. Then one can use bifurcation theory to prove the existence and uniqueness of the positive solution. We also obtain the precise global bifurcation diagrams of the system in (λ, u, v) space under these conditions. In all cases that we consider, the bifurcation diagram is a single monotone solution curve (see §§ 3 and 4 for more precise statements). The notation of sublinearity and superlinearity of the nonlinear vector field $(f(u,v), g(u,v))$ or of those in higher dimensions were considered in [39]. But our definition is quite different and our purpose is to prove uniqueness under a sublinear assumption. Our definition of sublinear nonlinearity is similar to the one in [28] for the scalar case.

Dalmasso [11, 12] obtained an existence and uniqueness result for a more special sublinear system and it was extended by Shi and Shivaji [37]. The uniqueness of positive solutions for large λ was proved in [15–18]. For superlinear-type systems, Clément *et al.* [8] obtained *a priori* estimates of positive solutions and used topological methods to prove the existence of solutions. Rellich–Pohozaev-type identities can be established for systems with variational structure [24], which is very useful for the non-existence result, among others. We note that many of these results are for the special Hamiltonian system case,

$$\left.\begin{aligned} \Delta u + \lambda f(v) = 0, \quad & x \in \Omega, \\ \Delta v + \lambda g(u) = 0, \quad & x \in \Omega, \\ u(x) = v(x) = 0, \quad & x \in \partial\Omega, \end{aligned}\right\} \tag{1.5}$$

while we consider equations in a more general form.

For (1.1) satisfying (A2) with Ω being a finite ball or the whole space, it is known that a positive solution is radially symmetric [3, 41]. Hence, the system can be converted into a system of ordinary differential equations. Several authors have taken that approach for the existence of the solutions [30, 33, 34] and much success has been achieved for Lane–Emden systems. Using the scaling invariant in the Lane–Emden system, the uniqueness of the radial positive solution has been shown in [10, 11, 22] for a system with three equations. Korman [21] obtained a uniqueness and exact multiplicity result for the one-dimensional case. A more general approach, using shooting method and linearized equations for the radial case, has been taken by the present authors [6, 7].

We recall the maximum principle and prove the main stability result in § 2. In § 3 we use the stability result and bifurcation theory to prove the existence and uniqueness of solution under several different assumptions on the differentiable nonlinearities; in § 4 we consider a similar question for nonlinearities that are merely Hölder continuous, and we use the monotonicity method for the existence and a method in [2, 11] for the uniqueness.

2. The maximum principle and stability

Let (u, v) be a solution of (1.1). The stability of (u, v) is determined by the linearized equation (1.3), which can be written as

$$\boldsymbol{L}\boldsymbol{u} = J\boldsymbol{u} + \mu\boldsymbol{u}, \tag{2.1}$$

where

$$\boldsymbol{u} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \quad \boldsymbol{L}\boldsymbol{u} = \begin{pmatrix} -\Delta\xi \\ -\Delta\eta \end{pmatrix} \quad \text{and} \quad J = \lambda \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}. \tag{2.2}$$

If we assume that (f, g) is cooperative (satisfying (A2)), then system (2.1), (2.2) is a linear elliptic system of cooperative type. If we also assume that $f_v(u(x), v(x)) \not\equiv 0$ and $g_u(u(x), v(x)) \not\equiv 0$, then J is irreducible, and the maximum principles hold for irreducible cooperative systems. We now recall some known results.

LEMMA 2.1. *Suppose that Ω is a bounded, open, connected subset of $\mathbb{R}^n$ satisfying a uniform exterior cone condition, and $u, v \in C^0(\bar{\Omega})$, with L and J as given in (2.2). Suppose that the entries of J are in $L^q(\Omega)$ with $q \geqslant pn/(p-n)$ and that (f, g) satisfies (A2), $f_v(u(x), v(x)) \not\equiv 0$ and $g_u(u(x), v(x)) \not\equiv 0$. Let $X \equiv [W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)]^2$ and $Y \equiv [L^p(\Omega)]^2$, where $p > n$. Then we have the following.*

(i) $\mu_1 = \inf\{\mathrm{Re}(\mu) \colon \mu \in \mathrm{spt}(L-J)\}$ *is a real eigenvalue of* $L-J$, *where* $\mathrm{spt}(L-J)$ *is the spectrum of* $L-J$.

(ii) *For $\mu = \mu_1$, there exists a unique (up to a constant multiple) eigenfunction $\boldsymbol{u}_1 \in X$, and $\boldsymbol{u}_1 > 0$ in Ω.*

(iii) *For $\mu < \mu_1$, the equation $L\boldsymbol{u} = J\boldsymbol{u} + \mu\boldsymbol{u} + \boldsymbol{f}$ is uniquely solvable for any $\boldsymbol{f} \in Y$, and the solution $\boldsymbol{u}(\in X) > 0$ as long as $\boldsymbol{f} \geqslant 0$.*

(iv) *(Maximum principle.) For $\mu \leqslant \mu_1$, suppose that $\boldsymbol{u} \in [W^{2,p}(\Omega)]^2$ satisfies $L\boldsymbol{u} \geqslant J\boldsymbol{u} + \mu\boldsymbol{u}$ in Ω, and $\boldsymbol{u} \geqslant 0$ on $\partial\Omega$. Then $\boldsymbol{u} \geqslant 0$ in Ω.*

(v) *If there exists a $\boldsymbol{u} \in [W^{2,p}(\Omega)]^2$ that satisfies $L\boldsymbol{u} \geqslant J\boldsymbol{u}$ in Ω, $\boldsymbol{u} \geqslant 0$ on $\partial\Omega$ and either $\boldsymbol{u} \not\equiv 0$ on $\partial\Omega$ or $L\boldsymbol{u} \not\equiv J\boldsymbol{u}$ in Ω, then $\mu_1 > 0$.*

For a more general result and proofs see proposition 3.1, theorem 1.1 and remark 1.4 of [40]. Moreover, from a standard compactness argument, the eigenvalues $\{\mu_i\}$ of $L - J$ are countable, and $|\mu_i - \mu_1| \to \infty$ as $i \to \infty$. We note that μ_i is not necessarily real valued.

We prove the following basic result on the stability of a positive solution.

THEOREM 2.2. *Suppose that f, g satisfies (A1) and (A2) and that (u, v) is a positive solution of (1.1). We assume that there exists $q > n$ such that*

$$\text{each entry of } J(u, v) \text{ is in } L^q(\Omega). \tag{2.3}$$

If f and g also satisfy the condition

(A3) *for any $(u, v) \in \mathbb{R}^2_+$,*

$$f(u, v) > f_v(u, v)v + g_v(u, v)u, \qquad g(u, v) > g_u(u, v)u + f_u(u, v)v, \tag{2.4}$$

then (u, v) is stable.

Proof. We choose $p > n$ such that $q \geqslant pn/(p - n)$. Then lemma 2.1 can be applied to $L - J$ defined in (2.1) and (2.2). Let (u, v) be a positive solution of (1.1), and let (μ_1, ξ, η) be the corresponding principal eigenpair such that $\xi > 0$ and $\eta > 0$ in Ω. Multiplying the equation for u in (1.1) by η, multiplying the equation for η in (1.3) by u, integrating over Ω and subtracting, we obtain

$$\lambda \int_\Omega f\eta \,\mathrm{d}x = \lambda \int_\Omega g_u u\xi \,\mathrm{d}x + \lambda \int_\Omega g_v u\eta \,\mathrm{d}x + \mu_1 \int_\Omega u\eta \,\mathrm{d}x. \tag{2.5}$$

Similarly, from the equations for v and ξ, we find

$$\lambda \int_\Omega g\xi \,\mathrm{d}x = \lambda \int_\Omega f_u v\xi \,\mathrm{d}x + \lambda \int_\Omega f_v v\eta \,\mathrm{d}x + \mu_1 \int_\Omega v\xi \,\mathrm{d}x. \tag{2.6}$$

Suppose that $\mu_1 \leqslant 0$. Then we have

$$\lambda \int_\Omega f\eta \,\mathrm{d}x \leqslant \lambda \int_\Omega g_u u\xi \,\mathrm{d}x + \lambda \int_\Omega g_v u\eta \,\mathrm{d}x \tag{2.7}$$

and

$$\lambda \int_\Omega g\xi \,\mathrm{d}x \leqslant \lambda \int_\Omega f_u v\xi \,\mathrm{d}x + \lambda \int_\Omega f_v v\eta \,\mathrm{d}x. \tag{2.8}$$

If (A3) is satisfied, then, adding (2.7) and (2.8), we obtain

$$\int_{\Omega} f\eta \,\mathrm{d}x + \int_{\Omega} g\xi \,\mathrm{d}x \leqslant \int_{\Omega} (g_v u + f_v v)\eta \,\mathrm{d}x + \int_{\Omega} (f_u v + g_u u)\xi \,\mathrm{d}x, \tag{2.9}$$

which is in contradiction with (A3) and $\xi > 0$, $\eta > 0$ in Ω. Hence, $\mu_1 > 0$ if (A3) is satisfied. □

The maximum principle can also be used to construct solutions of the elliptic system (1.1) which satisfy the cooperative condition (A2). Let $X = [C_0^{2,\alpha}(\bar{\Omega})]^2$. A pair of functions $(\bar{u}, \bar{v}) \in X$ is an *upper solution* for (1.1) if

$$\left.\begin{aligned} \Delta\bar{u} + \lambda f(\bar{u}, \bar{v}) \leqslant 0, \quad & x \in \Omega, \\ \Delta\bar{v} + \lambda g(\bar{u}, \bar{v}) \leqslant 0, \quad & x \in \Omega, \\ \bar{u}(x) \geqslant 0, \quad \bar{v}(x) \geqslant 0, \quad & x \in \partial\Omega, \end{aligned}\right\} \tag{2.10}$$

and $(\underline{u}, \underline{v}) \in X$ is a *lower solution* for (1.1) if

$$\left.\begin{aligned} \Delta\underline{u} + \lambda f(\underline{u}, \underline{v}) \geqslant 0, \quad & x \in \Omega, \\ \Delta\underline{v} + \lambda g(\underline{u}, \underline{v}) \geqslant 0, \quad & x \in \Omega, \\ \underline{u}(x) \leqslant 0, \quad \underline{v}(x) \leqslant 0, \quad & x \in \partial\Omega. \end{aligned}\right\} \tag{2.11}$$

The following existence result based on the monotone iteration method is well known [29, 32].

THEOREM 2.3. *Suppose that f and g satisfy (A1) and (A2), and suppose that $(\bar{u}, \bar{v})$ and $(\underline{u}, \underline{v})$ are pairs of upper and lower solutions which satisfy $\bar{u}(x) \geqslant \underline{u}(x)$ and $\bar{v}(x) \geqslant \underline{v}(x)$ for $x \in \Omega$. Define $X_1 \subset X$ by*

$$X_1 = \{(u, v) \in X : \underline{u}(x) \leqslant u(x) \leqslant \bar{u}(x),\ \underline{v}(x) \leqslant v(x) \leqslant \bar{v}(x)\}.$$

Then (1.1) possesses a minimal solution $(u_{\mathrm{m}}, v_{\mathrm{m}})$ and a maximal solution $(u_{\mathrm{M}}, v_{\mathrm{M}})$ in X_1; that is, for any solution $(u_a, v_a) \in X_1$, then

$$u_{\mathrm{m}} \leqslant u_a \leqslant u_{\mathrm{M}} \quad \textit{and} \quad v_{\mathrm{m}} \leqslant v_a \leqslant v_{\mathrm{M}}.$$

3. Existence and uniqueness: smooth case

In this section we always assume that f and g are smooth, i.e. they satisfy

(A1′) $f, g \in C^1(\overline{\mathbb{R}_+^2})$.

We note that if (A1′) is satisfied, then $J(u, v) \in C^0(\bar{\Omega})$ and (2.3) is automatically true. Let (λ_1, φ_1) be the principal eigenpair of

$$-\Delta\varphi = \lambda\varphi, \quad x \in \Omega, \qquad \varphi(x) = 0, \quad x \in \partial\Omega, \tag{3.1}$$

such that $\varphi_1(x) > 0$ in Ω and $\|\varphi_1\|_\infty = 1$.

First we recall an existence and uniqueness result proved in [37].

THEOREM 3.1. *Consider*

$$\left.\begin{aligned} \Delta u + \lambda f_1(v) = 0, &\quad x \in \Omega, \\ \Delta v + \lambda g_1(u) = 0, &\quad x \in \Omega, \\ u(x) = v(x) = 0, &\quad x \in \partial\Omega. \end{aligned}\right\} \tag{3.2}$$

Suppose that f_1 *and* g_1 *satisfy, for any* $u \geqslant 0$ *and* $v \geqslant 0$,

$$f_1'(v) > 0, \qquad g_1'(u) > 0, \tag{3.3}$$

$$\frac{\mathrm{d}}{\mathrm{d}v}\left(\frac{f_1(v)}{v}\right) < 0, \qquad \frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{g_1(u)}{u}\right) < 0 \tag{3.4}$$

and

$$\lim_{v\to\infty} \frac{f_1(v)}{v} = \lim_{u\to\infty} \frac{g_1(u)}{u} = 0. \tag{3.5}$$

Then

(i) *if at least one of* $f_1(0)$ *and* $g_1(0)$ *is positive, then (3.2) has a unique positive solution* $(u(\lambda), v(\lambda))$ *for all* $\lambda > 0$,

(ii) *if* $f_1(0) = g_1(0) = 0$, *and* $f_1'(0) > 0$ *and* $g_1'(0) > 0$, *then, for some* $\lambda_* = \lambda_1/\sqrt{f_1'(0)g_1'(0)} > 0$, *(3.2) has no positive solution when* $\lambda \leqslant \lambda_*$, *and (3.2) has a unique positive solution* $(u(\lambda), v(\lambda))$ *for* $\lambda > \lambda_*$.

Moreover, $\{(\lambda, u(\lambda), v(\lambda)) \colon \lambda > \lambda_*\}$ *(in the first case, we assume that* $\lambda_* = 0$*) is a smooth curve, so* $u(\lambda)$ *and* $v(\lambda)$ *are strictly increasing in* λ, *and* $(u(\lambda), v(\lambda)) \to (0, 0)$ *as* $\lambda \to \lambda_*^+$.

Theorem 3.1 is identical to theorem 6.1 of [37] and we omit the proof here. Note that if f_1 and g_1 satisfy (3.4), then it is necessary that $f_1(0) \geqslant 0$ and $g_1(0) \geqslant 0$. Hence, f_1 and g_1 are positive for $u, v > 0$ here. If $f_1(0) = 0$, then we must have $f_1'(0) > 0$ since $f_1'(0) > f_1(v)/v$ for $v > 0$. Some examples of smooth sublinear functions satisfying conditions (3.3)–(3.5) are

$$\begin{aligned} f(u) &= \ln(u+1) + k, \\ f(u) &= 1 - \mathrm{e}^{-u} + k, \\ f(u) &= (1+u)^p - 1 + k, \quad 0 < p < 1, \end{aligned}$$

and

$$f(u) = \frac{u}{(m+u)} + k, \quad k \geqslant 0.$$

In the following we consider the case when f and g depend on both u and v. In all results in this section we assume that f and g are additionally separated in the form

$$f(u, v) = f_1(v) + f_2(u) \quad \text{and} \quad g(u, v) = g_1(u) + g_2(v). \tag{3.6}$$

All of our results generalize theorem 3.1, but the structure and bifurcation of the solution sets are different. First we have the following.

THEOREM 3.2. *Consider*

$$\left.\begin{aligned} \Delta u + \lambda[f_1(v) + f_2(u)] &= 0, \quad x \in \Omega, \\ \Delta v + \lambda[g_1(u) + g_2(v)] &= 0, \quad x \in \Omega, \\ u(x) = v(x) &= 0, \quad x \in \partial\Omega. \end{aligned}\right\} \tag{3.7}$$

Suppose that $f_1(v)$ and $g_1(u)$ satisfy (3.3)–(3.5), and suppose that $f_2(u)$ and $g_2(v)$ satisfy, for any $u \geqslant 0$ and $v \geqslant 0$,

$$f_2(u) \geqslant 0, \qquad f_2'(u) \leqslant 0, \qquad g_2(v) \geqslant 0, \qquad g_2'(v) \leqslant 0 \tag{3.8}$$

and

$$f_1(0) + f_2(0) > 0 \quad \text{or} \quad g_1(0) + g_2(0) > 0. \tag{3.9}$$

Then (3.7) has a unique positive solution $(u(\lambda), v(\lambda))$ for all $\lambda > 0$. Moreover, $\{(\lambda, u(\lambda), v(\lambda))\colon \lambda > 0\}$ is a smooth curve so that $u(\lambda)$ and $v(\lambda)$ are strictly increasing in λ, and $(u(\lambda), v(\lambda)) \to (0,0)$ as $\lambda \to 0^+$.

Proof. Our proof follows that of theorem 6.1 in [37]. First we extend f_i and g_i to be defined on $\mathbb{R}$ for $u, v < 0$ in the following way: if $f_i(0) = 0$ or $g_i(0) = 0$, then we define $f_i(x)$ or $g_i(x) \equiv 0$ for $x < 0$; if $f_i(0) > 0$ or $g_i(0) > 0$, then we define $f_i(x)$ or $g_i(x) \equiv 0$ for $x < -\theta$ for some $\theta > 0$ and we define the function properly so it is continuous on $\mathbb{R}$. From the assumptions, $f(u,v) = f_1(v) + f_2(u)$ and $g(u,v) = g_1(u) + g_2(v)$ satisfy (A3). Hence, from theorem 2.2, any positive solution of (3.7) is stable. We define

$$F(\lambda, u, v) = \begin{pmatrix} \Delta u + \lambda[f_1(v) + f_2(u)] \\ \Delta v + \lambda[g_1(u) + g_2(v)] \end{pmatrix}, \tag{3.10}$$

where $\lambda \in \mathbb{R}$ and $u, v \in C_0^{2,\alpha}(\bar{\Omega})$. Here, since f_i and g_i are C^1, $F\colon \mathbb{R} \times X \to Y$ is continuously differentiable, where $X = [C_0^{2,\alpha}(\bar{\Omega})]^2$ and $Y = [C^{\alpha}(\bar{\Omega})]^2$.

Apparently $(\lambda, u, v) = (0,0,0)$ is a solution of (3.7). We apply the implicit function theorem at $(\lambda, u, v) = (0,0,0)$. Note that the Fréchet derivative of F is given here by

$$F_{(u,v)}(\lambda, u, v) \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \Delta\phi + \lambda[f_1'(v)\psi + f_2'(u)\phi] \\ \Delta\psi + \lambda[g_1'(u)\phi + g_2'(v)\psi] \end{pmatrix}, \tag{3.11}$$

Thus, $F_{(u,v)}(0,0,0)(\phi,\psi)^{\mathrm{T}} = (\Delta\phi, \Delta\psi)^{\mathrm{T}}$, and it is an isomorphism from X to Y. The implicit function theorem implies that $F(\lambda, u, v) = 0$ has a unique solution $(\lambda, u(\lambda), v(\lambda))$ for $\lambda \in (0, \delta)$ for some small $\delta > 0$, and that $(u'(0), v'(0))$ is the unique solution of

$$\begin{gathered} \Delta\phi + f_1(0) + f_2(0) = 0, \quad \Delta\psi + g_1(0) + g_2(0) = 0, \quad x \in \Omega, \\ \phi(x) = \psi(x) = 0, \quad x \in \partial\Omega. \end{gathered} \tag{3.12}$$

Then $(u'(0), v'(0)) = ([f_1(0) + f_2(0)]e, [g_1(0) + g_2(0)]e)$, where e is the unique positive solution of

$$\Delta e + 1 = 0, \quad x \in \Omega, \qquad e(x) = 0, \quad x \in \partial\Omega. \tag{3.13}$$

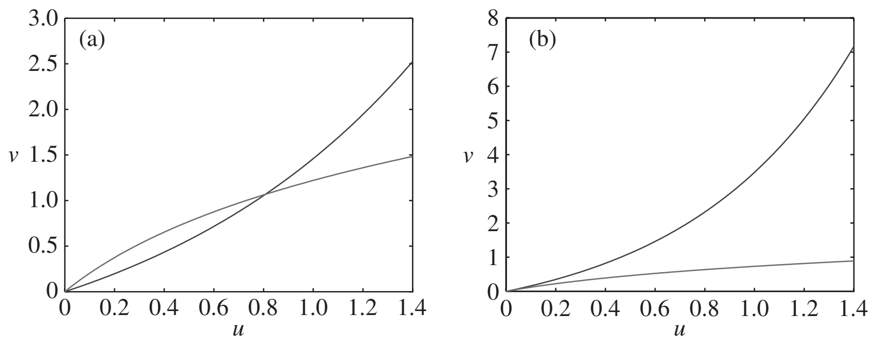

Figure 1. Possible graphs of $f_1(v) - au = 0$ and $g_1(u) - bv = 0$ in (3.15). Here $f_1(v) = \ln(v+1)$ and $g_1(u) = \ln(2u+1)$. (a) $a = b = 0.9$, unique intersection; (b) $a = b = 1.5$, no intersection.

If $f_1(0)+f_2(0) > 0$ and $g_1(0)+g_2(0) > 0$, then $(u(\lambda), v(\lambda))$ is positive for $\lambda \in (0,\delta)$. If $f_1(0)+f_2(0) > 0$ and $g_1(0)+g_2(0) = 0$, then $u(\lambda) > 0$ for $\lambda \in (0,\delta)$ and $g_2(0) = 0$. Thus, $g_2(v) = 0$ for all $v \in \mathbb{R}$ by (3.8) and the extension of g_2 to $\mathbb{R}$. However,

$$\Delta v(\lambda) = -\lambda[g_1(u(\lambda)) + g_2(v(\lambda))] = -\lambda g_1(u(\lambda)) < 0.$$

Hence, $v(\lambda) > 0$ as well. A similar conclusion holds when $f_1(0) + f_2(0) = 0$ and $g_1(0)+g_2(0) > 0$. Therefore, (3.7) has a positive solution $(u(\lambda), v(\lambda))$ for $\lambda \in (0,\delta)$.

Now we can follow the proof of theorem 6.1 in [37] to obtain the remaining part of the proof. In particular, we can show that the solution $(\lambda, u(\lambda), v(\lambda))$ is strictly increasing in λ as $(\partial u(\lambda)/\partial\lambda, \partial v(\lambda)/\partial\lambda)$ satisfies the following equation:

$$F_{(u,v)}(\lambda, u, v)\begin{pmatrix} \dfrac{\partial u(\lambda)}{\partial\lambda} \\ \dfrac{\partial v(\lambda)}{\partial\lambda} \end{pmatrix} = -\begin{pmatrix} f_1(v) + f_2(u) \\ g_1(u) + g_2(v) \end{pmatrix}. \tag{3.14}$$

Then $(\partial u(\lambda)/\partial\lambda, \partial v(\lambda)/\partial\lambda) > 0$ from the maximum principle (lemma 2.1(iii)) and the fact that $\mu_1((u(\lambda), v(\lambda))) > 0$ from theorem 2.2. □

Next we consider the case when $f(u,v)$ and $g(u,v)$ are not necessarily positive for all $(u,v) \in \mathbb{R}_+^2$. Consider

$$\left.\begin{aligned} \Delta u + \lambda[f_1(v) - au] &= 0, \quad x \in \Omega, \\ \Delta v + \lambda[g_1(u) - bv] &= 0, \quad x \in \Omega, \\ u(x) = v(x) &= 0, \quad x \in \partial\Omega, \end{aligned}\right\} \tag{3.15}$$

where $a > 0$ and $b > 0$.

First we observe the following geometric properties of the curves $f_1(v) - au = 0$ and $g_1(u) - bv = 0$ (see figure 1).

Lemma 3.3. *Suppose that $f_1(v)$ and $g_1(u)$ satisfy (3.3)–(3.5) and that $a, b > 0$.*

(i) *If $f_1(0) = g_1(0) = 0$ and $ab \geqslant f_1'(0)g_1'(0)$, then the curves $f_1(v) - au = 0$ and $g_1(u) - bv = 0$ have no intersection points in the first quadrant. Furthermore, for any $(u,v) \in \mathbb{R}_+^2$, $f_1(v) - au \leqslant 0$ or $g_1(u) - bv \leqslant 0$.*

(ii) *If* $f_1(0) = g_1(0) = 0$ *and* $ab < f_1'(0)g_1'(0)$, *then the curves* $f_1(v) - au = 0$ *and* $g_1(u) - bv = 0$ *have a unique intersection point* (u_*, v_*) *in the first quadrant.*

(iii) *If at least one of* $f_1(0)$ *and* $g_1(0)$ *is positive, then the curves* $f_1(v) - au = 0$ *and* $g_1(u) - bv = 0$ *have a unique intersection point* (u_*, v_*) *in the first quadrant.*

Proof. First we assume that $f_1(0) = g_1(0) = 0$. If (u_*, v_*) is an intersection point of $f_1(v) - au = 0$ and $g_1(u) - bv = 0$, then, from (3.4),

$$f_1'(0) \geqslant \frac{f_1(v_*)}{v_*} = \frac{au_*}{v_*}, \qquad g_1'(0) \geqslant \frac{g_1(u_*)}{u_*} = \frac{bv_*}{u_*}.$$

Thus, by multiplying the two inequalities, we obtain $f_1'(0)g_1'(0) \geqslant ab$. If $ab = f_1'(0)g_1'(0)$, then we must have $(u_*, v_*) = (0, 0)$ from the strict decreasing property in (3.4). Hence, when $ab \geqslant f_1'(0)g_1'(0)$, $f_1(v) - au = 0$ and $g_1(u) - bv = 0$ have no positive intersection points.

If $ab < f_1'(0)g_1'(0)$, then near $(u, v) = (0, 0)$, the curve $f_1(v) - au = 0$ is below $g_1(u) - bv = 0$, and hence the existence of an intersection is clear from (3.5). Suppose that (u_*, v_*) and (u_{**}, v_{**}) are two distinct intersection points. Without loss of generality we assume that $u_{**} > u_*$; then, from the monotonicity of f_1 or g_1, it is necessary that $v_{**} > v_*$. But we have

$$\frac{f_1(v_*)}{av_*} > \frac{f_1(v_{**})}{av_{**}} = \frac{bu_{**}}{g_1(u_{**})} > \frac{bu_*}{g_1(u_*)} = \frac{f_1(v_*)}{av_*}, \tag{3.16}$$

which is a contradiction. Hence, the intersection point (u_*, v_*) is unique. The case that at least one of $f_1(0)$ and $g_1(0)$ is positive is similar and we therefore omit the details. □

The following result classifies the structure of the solution set of (3.15) under different conditions on f_1, g_1 at $u, v = 0$ and the parameters a, b.

THEOREM 3.4. *Suppose that* $f_1(v)$ *and* $g_1(u)$ *satisfy (3.3)–(3.5) and that* $a = b > 0$.

(i) *If* $f_1(0) = g_1(0) = 0$, $f_1'(0) > 0$, $g_1'(0) > 0$ *and* $a^2 \geqslant f_1'(0)g_1'(0)$, *then (3.15) has no positive solution for any* $\lambda > 0$.

(ii) *If* $f_1(0) = g_1(0) = 0$, $f_1'(0) > 0$, $g_1'(0) > 0$ *and* $a^2 < f_1'(0)g_1'(0)$, *then, for some*

$$\lambda_* = \frac{\lambda_1}{\sqrt{f_1'(0)g_1'(0)} - a} \geqslant \frac{2\lambda_1}{f_1'(0) + g_1'(0) - 2a}, \tag{3.17}$$

equation (3.15) has no positive solution when $\lambda \leqslant \lambda_*$ *and has a unique positive solution* $(u(\lambda), v(\lambda))$ *for* $\lambda > \lambda_*$.

(iii) *If at least one of* $f_1(0)$ *and* $g_1(0)$ *is positive, then (3.15) has a unique positive solution* $(u(\lambda), v(\lambda))$ *for all* $\lambda > 0$.

Moreover, in the last two cases, $\{(\lambda, u(\lambda), v(\lambda)) : \lambda > \lambda_*\}$ *(in the second case, we assume* $\lambda_* = 0$*) is a smooth curve,* $(u(\lambda), v(\lambda)) \to (0, 0)$ *as* $\lambda \to \lambda_*^+$.

Proof. First we prove that if $(u(x), v(x))$ is a positive solution of (3.15), $u(x_1) = \max_{x\in\bar{\Omega}} u(x)$ and $v(x_2) = \max_{x\in\bar{\Omega}} v(x)$, then, for $i = 1, 2$,

$$f_1(v(x_i)) - au(x_i) > 0 \quad \text{and} \quad g_1(u(x_i)) - av(x_i) > 0. \tag{3.18}$$

In fact, from the maximum principle and the monotonicity of f_1 and g_1,

$$0 \geqslant \Delta u(x_1) = \lambda[au(x_1) - f_1(v(x_1))] \geqslant \lambda[au(x_2) - f_1(v(x_2))]. \tag{3.19}$$

Similarly, we also have $g_1(u(x_i)) - av(x_i) \geqslant 0$, and the strict inequalities hold because of the strong maximum principle.

If $f_1(0) = g_1(0) = 0$, $f_1'(0) > 0$, $g_1'(0) > 0$ and $a^2 \geqslant f_1'(0)g_1'(0)$, then, from lemma 3.3, for any $(u, v) \in \mathbb{R}_+^2$ either $f_1(v) - au \leqslant 0$ or $g_1(u) - av \leqslant 0$. Hence, (3.15) has no positive solution from (3.18). In other cases, from (3.18), we must have $0 < u(x) < u_*$ and $0 < v(x) < v_*$ for $x \in \Omega$ if $(u(x), v(x))$ is a positive solution of (3.15).

Now we assume that $f_1(0) = g_1(0) = 0$, $f_1'(0) > 0$, $g_1'(0) > 0$ and $a^2 < f_1'(0)g_1'(0)$. We claim that, when

$$\lambda < \frac{2\lambda_1}{f_1'(0) + g_1'(0) - 2a}, \tag{3.20}$$

(3.15) has no positive solution. First of all, the denominator in (3.20) is positive, since

$$f_1'(0) + g_1'(0) \geqslant 2\sqrt{f_1'(0)g_1'(0)} > 2a.$$

Now, from integration of the equations, we obtain

$$\int_\Omega (|\nabla u|^2 + |\nabla v|^2 + a\lambda u^2 + a\lambda v^2)\,\mathrm{d}x = \int_\Omega \lambda(f_1(v)u + g_1(u)v)\,\mathrm{d}x. \tag{3.21}$$

From the left-hand side of (3.21), we see that

$$\begin{aligned}\int_\Omega (|\nabla u|^2 + |\nabla v|^2 + a\lambda u^2 + a\lambda v^2)\,\mathrm{d}x &\geqslant \int_\Omega (\lambda_1 u^2 + \lambda_1 v^2 + a\lambda u^2 + a\lambda v^2)\,\mathrm{d}x \\ &\geqslant (\lambda_1 + a\lambda)\int_\Omega (u^2 + v^2)\,\mathrm{d}x, \end{aligned} \tag{3.22}$$

while the right-hand side of (3.21) can be estimated as

$$\begin{aligned}\int_\Omega \lambda(f_1(v)u + g_1(u)v)\,\mathrm{d}x &\leqslant \lambda\int_\Omega f_1'(0)uv\,\mathrm{d}x + \lambda\int_\Omega g_1'(0)uv\,\mathrm{d}x \\ &\leqslant \tfrac{1}{2}\lambda(f_1'(0) + g_1'(0))\int_\Omega (u^2 + v^2)\,\mathrm{d}x. \end{aligned} \tag{3.23}$$

Now (3.22) and (3.23) imply that

$$\lambda \geqslant \frac{2\lambda_1}{f_1'(0) + g_1'(0) - 2a} \tag{3.24}$$

is necessary for the existence of a positive solution (u, v).

Next, similarly to the proof of theorem 3.2, we define

$$F(\lambda, u, v) = \begin{pmatrix} \Delta u + \lambda[f_1(v) - au] \\ \Delta v + \lambda[g_1(u) - av] \end{pmatrix}, \tag{3.25}$$

where $\lambda \in \mathbb{R}$ and $u, v \in C_0^{2,\alpha}(\bar{\Omega})$. Then $(\lambda, 0, 0)$ is a trivial solution of (3.15) for any $\lambda > 0$. The linearization of F at $(\lambda, 0, 0)$ is

$$F_{(u,v)}(\lambda, 0, 0) \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \Delta\phi + \lambda[f_1'(0)\psi - a\phi] \\ \Delta\psi + \lambda[g_1'(0)\phi - a\psi] \end{pmatrix}. \tag{3.26}$$

Since f_1, g_1 satisfy (3.3), then, from lemma 2.1, $F_{(u,v)}(\lambda, 0, 0)$ has a principal eigenvalue $\mu_1(\lambda)$, which is the only eigenvalue with positive eigenfunction. Then one can verify that the principal eigenpair of $F_{(u,v)}(\lambda, 0, 0)$ is $(\mu_1(\lambda), \varphi_1, c\varphi_1)$, where $\mu = \mu_1(\lambda)$ and c satisfy

$$(\mu + \lambda a + \lambda_1)^2 = f_1'(0)g_1'(0)\lambda^2, \qquad c = \frac{\mu + \lambda a + \lambda_1}{\lambda f_1'(0)}. \tag{3.27}$$

In particular, when $\mu_1(\lambda) = 0$, we can show that the corresponding $\lambda = \lambda_*$ is the positive root of

$$[f_1'(0)g_1'(0) - a^2]\lambda^2 - 2a\lambda_1\lambda - \lambda_1^2 = 0, \tag{3.28}$$

and $c = c_*$ is the positive root of

$$f_1'(0)c^2 - g_1'(0) = 0. \tag{3.29}$$

Hence, when $\lambda = \lambda_*$, $F_{(u,v)}(\lambda_*, 0, 0)$ is not invertible and $\lambda = \lambda_*$ is a potential bifurcation point. More precisely, the null space $N(F_{(u,v)}(\lambda_*, 0, 0)) = \operatorname{span}\{(\varphi_1, c_*\varphi_1)\}$, the range space

$$R(F_{(u,v)}(\lambda_*, 0, 0)) = \left\{(\phi, \psi) \in Y : \int_\Omega [c_*\phi + \psi]\varphi_1 \, \mathrm{d}x = 0\right\}.$$

Next, note that $(c_*\varphi_1, \varphi_1)$ is the principal eigenvector of the conjugate operator $F^*_{(u,v)}(\lambda_*, 0, 0)$. Finally, we can verify that

$$F_{\lambda(u,v)}(\lambda_*, 0, 0)[\varphi_1, c_*\varphi_1]^{\mathrm{T}} \notin R(F_{(u,v)}(\lambda_*, 0, 0)).$$

If this is not true, then

$$F_{\lambda(u,v)}(\lambda_*, 0, 0) \begin{pmatrix} \varphi_1 \\ c_*\varphi_1 \end{pmatrix} = \begin{pmatrix} (c_* f_1'(0) - a)\varphi_1 \\ (g_1'(0) - ac_*)\varphi_1 \end{pmatrix} \in R(F_{(u,v)}(\lambda_*, 0, 0)),$$

and the definition of $R(F_{(u,v)}(\lambda_*, 0, 0))$ implies that

$$f_1'(0)c_*^2 - 2ac_* + g_1'(0) = 0. \tag{3.30}$$

But, (3.29) and (3.30) would, taken together, imply that $c_* = g_1'(0)/a = a/f_1'(0)$, which is in contradiction with $a^2 < f_1'(0)g_1'(0)$. Hence, $F_{\lambda(u,v)}(\lambda_*, 0, 0)[\varphi_1, c_*\varphi_1]^{\mathrm{T}} \notin R(F_{(u,v)}(\lambda_*, 0, 0))$.

Now we are in a position to apply a bifurcation from a simple eigenvalue theorem of Crandall and Rabinowitz [9]. The non-trivial solutions of $F(\lambda, u, v) = (0, 0)$ near $(\lambda_*, 0, 0)$ are in the form of $(\lambda(s), u(s), v(s))$ for $s \in (-\delta, \delta)$, where $u(s) = s\varphi_1 + o(s)$ and $v(s) = c_* s\varphi_1 + o(s)$. In particular, the solution $(u(s), v(s))$ is positive when $s \in (0, \delta)$ from the positivity of φ_1. Moreover, one can use a global bifurcation theorem of Rabinowitz [31] (see also [38]) to conclude that an unbounded continuum Σ of positive solutions of $F(\lambda, u, v) = (0, 0)$ emanates from $(\lambda_*, 0, 0)$. Since (3.15) has no positive solution when

$$\lambda < \frac{2\lambda_1}{f_1'(0) + g_1'(0) - 2a},$$

and all solutions (u, v) are bounded by (u_*, v_*), Σ must be unbounded in the positive λ direction. This proves that (3.15) has at least one positive solution for each $\lambda > \lambda_*$.

Since (A3) is satisfied for $f(u, v) = f_1(v) - au$ and $g(u, v) = g_1(u) - av$, any positive solution (u, v) of (3.15) is stable from theorem 2.2. Hence, the implicit function theorem can be applied to any positive solution (λ, u, v) so that it belongs to a smooth curve of positive solutions $\Gamma = \{(\lambda, u(\lambda), v(\lambda))\}$. Let

$$\lambda_{**} = \inf\{\lambda\colon (\lambda, u(\lambda), v(\lambda)) \in \Gamma\}.$$

Then

$$\lambda_{**} \geqslant \frac{2\lambda_1}{f_1'(0) + g_1'(0) - 2a} > 0.$$

From the boundedness of $(\lambda, u(\lambda), v(\lambda)) \in \Gamma$, there exists a sequence (λ^n) such that $\lambda^n > \lambda_{**}$, $\lambda^n \to \lambda_{**}$, and $(u(\lambda^n), v(\lambda^n)) \to (u_{**}, v_{**})$ in $[C^2(\bar{\Omega})]^2$ as $n \to \infty$. Hence, (u_{**}, v_{**}) satisfies $F(\lambda_{**}, u_{**}, v_{**}) = 0$ and $u_{**}(x) \geqslant 0$, $v_{**}(x) \geqslant 0$. From the maximum principle, either $u_{**}(x) > 0$ and $v_{**}(x) > 0$, or $u_{**}(x) \equiv 0$ and $v_{**}(x) \equiv 0$ for $x \in \Omega$. But if (u_{**}, v_{**}) is positive, then it is stable, so one can apply implicit function theorem to extend Γ, which contradicts with the definition of λ_{**}. Thus $(u_{**}, v_{**}) = (0, 0)$, and we must have $\lambda_{**} = \lambda_*$ since it is the only bifurcation point for the line of trivial solutions. Since we can use the same argument for each smooth curve of positive solutions, there is only one such curve, which is Σ from the global bifurcation theorem, and $\Sigma = \{(\lambda, u(\lambda), v(\lambda))\colon \lambda > \lambda_*\}$. This also implies that (3.15) has no positive solution when $\lambda \leqslant \lambda_*$. Here we note that the argument that stability implies uniqueness described above was used in [5].

The case where at least one of $f_1(0)$ and $g_1(0)$ is positive is similar, and the existence of a curve of positive solutions can be proved via the implicit function theorem at $(\lambda, u, v) = (0, 0, 0)$ in a similar way to the proof of theorem 3.2. □

We note that the condition $a = b$ in theorem 3.4 is needed for the stability condition (A3). When $a \neq b$, the bifurcation arguments can still be applied if $ab < f_1'(0)g_1'(0)$, but it is not known whether the stability result in theorem 2.2 holds.

Our last result is about sublinear nonlinearities with positive linear terms. Consider

$$\left.\begin{aligned} \Delta u+\lambda[f_1(v)+au]&=0, \quad x\in\Omega,\\ \Delta v+\lambda[g_1(u)+bv]&=0, \quad x\in\Omega,\\ u(x)=v(x)&=0, \quad x\in\partial\Omega, \end{aligned}\right\} \tag{3.31}$$

where $a,b>0$.

THEOREM 3.5. *Suppose that $f_1(v)$ and $g_1(u)$ satisfy (3.3)–(3.5) and $a=b>0$. Let $\lambda^*=\lambda_1/a$, and we define λ_* as follows:*

(i) *if $f_1(0)=g_1(0)=0$, $f_1'(0)>0$ and $g_1'(0)>0$, then*

$$\lambda_*=\frac{\lambda_1}{\sqrt{f_1'(0)g_1'(0)}+a}>0; \tag{3.32}$$

(ii) *if at least one of $f_1(0)$ and $g_1(0)$ is positive, then $\lambda_*=0$.*

Then (3.31) has no positive solution when $\lambda\leqslant\lambda_*$ or $\lambda\geqslant\lambda^*$, and (3.31) has a unique positive solution $(u(\lambda),v(\lambda))$ for $\lambda\in(\lambda_*,\lambda^*)$. Moreover,

$$\{(\lambda,u(\lambda),v(\lambda))\colon \lambda\in(\lambda_*,\lambda^*)\}$$

is a smooth curve so that $u(\lambda)$ and $v(\lambda)$ are strictly increasing in λ, $(u(\lambda),v(\lambda))\to(0,0)$ as $\lambda\to\lambda_*^+$ and $\|u(\lambda)\|_\infty+\|v(\lambda)\|_\infty\to\infty$ as $\lambda\to(\lambda^*)^-$.

Proof. First we assume that $f_1(0)=g_1(0)=0$, $f_1'(0)>0$ and $g_1'(0)>0$. We multiply the equation of u in (3.31) by φ_1 and integrate on Ω, obtaining

$$\lambda_1\int_\Omega u\varphi_1\,\mathrm{d}x=\lambda\int_\Omega[f_1(v)\varphi_1+au\varphi_1]\,\mathrm{d}x>\lambda a\int_\Omega u\varphi_1\,\mathrm{d}x. \tag{3.33}$$

Hence, $\lambda<\lambda_1/a$. On the other hand, from (3.33), we have

$$(\lambda_1-\lambda a)\int_\Omega u\varphi_1\,\mathrm{d}x=\lambda\int_\Omega f_1(v)\varphi_1\leqslant\lambda f_1'(0)\int_\Omega v\varphi_1\,\mathrm{d}x. \tag{3.34}$$

Similarly,

$$(\lambda_1-\lambda a)\int_\Omega v\varphi_1\,\mathrm{d}x=\lambda\int_\Omega g_1(u)\varphi_1\leqslant\lambda g_1'(0)\int_\Omega u\varphi_1\,\mathrm{d}x. \tag{3.35}$$

Multiplying (3.34) and (3.35), we obtain that

$$(\lambda_1-\lambda a)^2\leqslant\lambda^2f_1'(0)g_1'(0). \tag{3.36}$$

Hence, if (λ,u,v) is a positive solution of (3.31), then λ must satisfy (3.36) and

$$\lambda<\frac{\lambda_1}{a}. \tag{3.37}$$

Define

$$h(\lambda)=(\lambda_1-\lambda a)^2-\lambda^2f_1'(0)g_1'(0). \tag{3.38}$$

Then there exists a unique $\lambda_* \in (0, \lambda_1/a)$ (defined as in (3.32)) such that $h(\lambda) > 0$ for $0 \leqslant \lambda < \lambda_*$ and $h(\lambda) < 0$ when $\lambda_* < \lambda < \lambda_1/a$. Hence, (3.31) can have positive solutions only when $\lambda \in (\lambda_*, \lambda^*)$.

A local bifurcation analysis using bifurcation from a simple eigenvalue theorem similar to the one in the proof of theorem 3.4 can be carried out at $\lambda = \lambda_*$, and we omit the details. The positive solution $(\lambda, u(\lambda), v(\lambda))$ is strictly increasing in λ as $(\partial u(\lambda)/\partial\lambda, \partial v(\lambda)/\partial\lambda)$ satisfies

$$F_{(u,v)}(\lambda, u, v) \begin{pmatrix} \dfrac{\partial u(\lambda)}{\partial\lambda} \\ \dfrac{\partial v(\lambda)}{\partial\lambda} \end{pmatrix} = -\begin{pmatrix} f_1(v) + au \\ g_1(u) + av \end{pmatrix}, \tag{3.39}$$

and hence $(\partial u(\lambda)/\partial\lambda, \partial v(\lambda)/\partial\lambda) > 0$ from the same arguments in the proof of theorem 3.2. The proof of existence of solutions when at least one of $f_1(0)$ and $g_1(0)$ is positive is also similar.

Note that (A3) is satisfied for $f(u, v) = f_1(v) + au$ and $g(u, v) = g_1(u) + av$, so any positive solution (u, v) of (3.31) is stable from theorem 2.2. Hence, in all cases, we can show as before that any positive solution is on a smooth curve, that there is only one such solution curve Σ, and that the left endpoint of Σ is the bifurcation point (depending on the case studied) from the trivial solutions. It only remains to prove that the Σ can be extended to λ^*.

Let $\lambda^{**} = \sup\{\lambda\colon (\lambda, u(\lambda), v(\lambda)) \in \Sigma\}$. From (3.37), we know that Σ can be extended at most to λ^*. Hence, $\lambda^{**} \leqslant \lambda^*$. As $\lambda \to (\lambda^{**})^-$, we must have

$$M_\lambda = \|u(\lambda)\|_\infty + \|v(\lambda)\|_\infty \to \infty,$$

otherwise a limiting process will yield a positive solution at λ^{**}; then Σ can be extended further beyond $\lambda = \lambda^{**}$, which contradicts with the definition of λ^{**}. Now we define $U(\lambda) = u(\lambda)/M_\lambda$ and $V(\lambda) = v(\lambda)/M_\lambda$. Then $(U(\lambda), V(\lambda))$ satisfies

$$\left.\begin{aligned} \Delta U + \lambda\left[\frac{f_1(v)}{M_\lambda} + aU\right] &= 0, \quad x \in \Omega, \\ \Delta V + \lambda\left[\frac{g_1(u)}{M_\lambda} + aV\right] &= 0, \quad x \in \Omega, \\ U(x) = V(x) &= 0, \quad x \in \partial\Omega. \end{aligned}\right\} \tag{3.40}$$

From the boundedness of $(U(\lambda), V(\lambda))$ we can obtain a sequence (λ^n) such that $\lambda^n < \lambda^{**}$, $\lambda^n \to \lambda^{**}$ and $(U(\lambda^n), V(\lambda^n)) \to (U^{**}, V^{**})$ in $[C^2(\bar{\Omega})]^2$ as $n \to \infty$. From (3.5), $(\lambda^{**}, U^{**}, V^{**})$ satisfies

$$\left.\begin{aligned} \Delta U^{**} + \lambda^{**} a U^{**} &= 0, \quad x \in \Omega, \\ \Delta V^{**} + \lambda^{**} a V^{**} &= 0, \quad x \in \Omega, \\ U^{**}(x) = V^{**}(x) &= 0, \quad x \in \partial\Omega, \end{aligned}\right\} \tag{3.41}$$

$\|U^{**}\|_\infty + \|V^{**}\|_\infty = 1$, and $U^{**} \geqslant 0$, $V^{**} \geqslant 0$. Therefore, we must have $\lambda^{**} = \lambda_1/a$, $U^{**} = s\varphi_1$ and $V^{**} = (1-s)\varphi_1$ for some $s \in [0, 1]$. This proves that $\lambda^{**} = \lambda^*$, which completes the proof. □

4. Existence and uniqueness: the Hölder continuous case

In this section we always assume that f and g satisfy (A1) and (A2), so f and g may not be differentiable. Then the nonlinear operator $F(\lambda, u, v)$ defined in § 3 is not necessarily differentiable, and hence the implicit function theorem and bifurcation theorems cannot be used easily. We shall prove results similar to theorems 3.4 and 3.5 by using different methods.

THEOREM 4.1. *Consider (3.15). Suppose that* $f_1(0) = g_1(0) = 0$, $f_1(v)$ *and* $g_1(u)$ *satisfy (3.5) for any* $u > 0$ *and any* $v > 0$,

$$f_1'(v) > 0, \qquad g_1'(u) > 0, \tag{4.1}$$

and there exists $0 < p, q < 1$ *such that, for any* $u > 0$ *and any* $v > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}v}\left(\frac{f_1(v)}{v^q}\right) \leqslant 0, \qquad \frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{g_1(u)}{u^p}\right) \leqslant 0. \tag{4.2}$$

Then, for $\lambda \in (0, \infty)$, *(3.15) has a unique positive solution* $(u(\lambda), v(\lambda))$. *Moreover,*

$$\{(\lambda, u(\lambda), v(\lambda)) \colon \lambda > 0\}$$

is a continuous curve, so $(u(\lambda), v(\lambda)) \to (0, 0)$ *as* $\lambda \to 0^+$.

Proof. First we show that the curves $-au + f_1(v) = 0$ and $-bv + g_1(u) = 0$ intersect at a unique point in $\mathbb{R}^2_+$. From (3.5), for large u, $v = g_1(u)/b$ is below the curve $u = f_1(v)/a$. Suppose that the two curves do not intersect. Then, for any $u > 0$, we have

$$f_1^{-1}(au) > \frac{g_1(u)}{b}. \tag{4.3}$$

We note that (4.2) implies that, for $\tau \in (0, 1)$,

$$f_1(\tau v) \geqslant \tau^q f_1(v) \quad \text{and} \quad g_1(\tau u) \geqslant \tau^p g_1(u). \tag{4.4}$$

For $u > 0$ small, $g_1(u) < 1$ and $u < 1$, and then (4.3) and (4.4) imply that

$$au > f_1\left(\frac{g_1(u)}{b}\right) \geqslant [g_1(u)]^q f_1\left(\frac{1}{b}\right) \geqslant u^{pq}[g_1(1)]^q f_1\left(\frac{1}{b}\right), \tag{4.5}$$

which contradicts $pq < 1$. Hence, the two curves must intersect. Suppose that (u_*, v_*) is an intersection point, and that (u_{**}, v_{**}) is another. Without loss of generality we assume $u_{**} > u_*$; then, from the monotonicity of f_1 or g_1, it is necessary that $v_{**} > v_*$. From

$$a = \frac{f_1(v_*)}{u_*} = \frac{f_1(v_*)}{v_*^q}\frac{v_*^q}{u_*} \geqslant \frac{f_1(v_{**})}{v_{**}^q}\frac{v_*^q}{u_*} = a\left(\frac{v_*}{v_{**}}\right)^q \frac{u_{**}}{u_*}.$$

Thus, we have

$$\left(\frac{v_{**}}{v_*}\right)^q \geqslant \frac{u_{**}}{u_*}, \tag{4.6}$$

and similarly

$$\left(\frac{u_{**}}{u_*}\right)^p \geqslant \frac{v_{**}}{v_*}. \tag{4.7}$$

But (4.6) and (4.7) yield

$$\left(\frac{u_{**}}{u_*}\right)^{pq} \geqslant \frac{u_{**}}{u_*},$$

which contradicts $u_{**} > u_*$ and $pq < 1$. Therefore, the curves $-au + f_1(v) = 0$ and $-bv + g_1(u) = 0$ intersect at only one point.

Now we use the monotonicity method to prove the existence of a solution. Let (u_*, v_*) be the unique intersection point of $-au + f_1(v) = 0$ and $-bv + g_1(u) = 0$. Then, from the proof of theorem 3.4, any positive solution of (3.15) satisfies $0 < u(x) < u_*$ and $0 < v(x) < v_*$. Let $(\bar{u}, \bar{v}) = (u_*, v_*)$. Then it is clear that $(\bar{u}, \bar{v})$ is an upper solution of (3.15).

Let

$$L_f = \frac{f_1(v_*)}{v_*^q}, \qquad L_g = \frac{g_1(u_*)}{u_*^p}. \tag{4.8}$$

Since $0 \leqslant u \leqslant u_*$ and $0 \leqslant v \leqslant v_*$, in this range $f_1(v) \geqslant L_f v^q$ and $g_1(u) \geqslant L_g u^p$. We construct a lower solution in the form of $(\underline{u}, \underline{v}) = (\varepsilon_1\varphi_1, \varepsilon_2\varphi_1)$, where $\varepsilon_1, \varepsilon_2$ will be specified later. Recall that φ_1 is the positive principal eigenfunction with $\|\varphi_1\|_\infty = 1$. Now

$$\begin{aligned}\Delta(\varepsilon_1\varphi_1) + \lambda[-a\varepsilon_1\varphi_1 + f_1(\varepsilon_2\varphi_1)] &\geqslant -\varepsilon_1\lambda_1\varphi_1 - \lambda a\varepsilon_1\varphi_1 + \lambda L_f\varepsilon_2^q\varphi_1^q \\ &= \varepsilon_1\varphi_1^q[-(\lambda a + \lambda_1)\varphi_1^{1-q} + \lambda L_f\varepsilon_2^q\varepsilon_1^{-1}] \\ &\geqslant \varepsilon_1\varphi_1^q[-(\lambda a + \lambda_1) + \lambda L_f\varepsilon_2^q\varepsilon_1^{-1}].\end{aligned}$$

Similarly,

$$\Delta(\varepsilon_2\varphi_1) + \lambda[-b\varepsilon_2\varphi_1 + g_1(\varepsilon_1\varphi_1)] \geqslant \varepsilon_2\varphi_1^p[-(\lambda b + \lambda_1) + \lambda L_g\varepsilon_1^p\varepsilon_2^{-1}].$$

Hence, if we choose

$$\left.\begin{aligned} 0 < \varepsilon_1 \leqslant \left(\frac{\lambda L_g}{\lambda b + \lambda_1}\right)^{q/(1-pq)}\left(\frac{\lambda L_f}{\lambda a + \lambda_1}\right)^{1/(1-pq)}, \\ \left(\frac{\varepsilon_1(\lambda a + \lambda_1)}{\lambda L_f}\right)^{1/q} \leqslant \varepsilon_2 \leqslant \frac{\lambda L_g}{\lambda b + \lambda_1}\varepsilon_1^p, \end{aligned}\right\} \tag{4.9}$$

then $(\underline{u}, \underline{v}) = (\varepsilon_1\varphi_1, \varepsilon_2\varphi_1)$ is a lower solution of (3.15). We can choose smaller ε_1 and ε_2 but still satisfy (4.9), so that $(\underline{u}, \underline{v}) \leqslant (\bar{u}, \bar{v}) = (u_*, v_*)$. Then from theorem 2.3, there exists a positive solution (u, v) of (3.15) between the lower and upper solutions.

Next we prove the uniqueness of the solution for any $\lambda > 0$. Here we follow an argument of [2, 11]. We fix $\lambda > 0$. We define $G_c(x, y)$ as the Green function of the linear elliptic operator $-\Delta + cI$ with zero boundary condition where $c \geqslant 0$. Then it is well known that $G_c(x, y) > 0$ for $x, y \in \Omega$. Let (u_j, v_j), $j = 1, 2$, be two positive solutions of (3.15). Define

$$S = \{s \in (0, 1] \colon u_1 - tu_2 \geqslant 0,\ v_1 - tv_2 \geqslant 0 \text{ for } t \in [0, s]\}.$$

Since $u_j, v_j \in C^{2,\alpha}(\bar{\Omega})$ with $\alpha = \min\{p, q\}$, there exists small $\varepsilon > 0$ such that $\varepsilon \in S$, and thus $S \neq \emptyset$. Let $\theta = \sup S$ and assume that $\theta < 1$. Then $u_1 - \theta u_2 \geqslant 0$ and $v_1 - \theta v_2 \geqslant 0$ in $\bar{\Omega}$. Then, from (4.1) and (4.4), we have

$$\begin{aligned} u_1(x) &= \lambda \int_\Omega G_{\lambda a}(x,y) f_1(v_1(y))\,\mathrm{d}y \\ &\geqslant \lambda \int_\Omega G_{\lambda a}(x,y) f_1(\theta v_2(y))\,\mathrm{d}y \\ &\geqslant \lambda\theta^q \int_\Omega G_{\lambda a}(x,y) f_1(v_2(y))\,\mathrm{d}y \\ &= \theta^q u_2(x). \end{aligned} \tag{4.10}$$

Similarly, we have $v_1(x) \geqslant \theta^p v_2(x)$. From these relations, we obtain

$$\begin{aligned} (-\Delta + \lambda a)(u_1 - \theta u_2) &= \lambda[f_1(v_1) - \theta f_1(v_2)] \\ &\geqslant \lambda[f_1(\theta^p v_2) - \theta f_1(v_2)] \\ &\geqslant \lambda(\theta^{pq} - \theta) f_1(v_2) \end{aligned} \tag{4.11}$$

and, similarly,

$$(-\Delta + \lambda b)(v_1 - \theta v_2) \geqslant \lambda(\theta^{pq} - \theta) g_1(u_2). \tag{4.12}$$

Since $f_1(v_2(x)) > 0$ and $g_1(u_2(x)) > 0$ for $x \in \Omega$, $\theta^{pq} - \theta > 0$, from the strong maximum principle, (4.11) and (4.12), we have

$$u_1(x) - \theta u_2(x) > 0, \quad v_1(x) - \theta v_2(x) > 0 \quad \text{for } x \in \Omega, \tag{4.13}$$

and the Hopf lemma implies that the normal derivatives on the boundary satisfy

$$\frac{u_1(x) - \theta u_2(x)}{\partial\nu} < 0, \quad \frac{v_1(x) - \theta v_2(x)}{\partial\nu} < 0 \quad \text{for } x \in \partial\Omega. \tag{4.14}$$

This would allow that $u_1(x) - (\theta+\varepsilon)u_2(x) \geqslant 0$ and $v_1(x) - (\theta+\varepsilon)v_2(x) \geqslant 0$ for a small $\varepsilon > 0$, which contradicts the definition of θ. Hence, we must have $\theta = 1$, $u_1 \geqslant u_2$ and $v_1 \geqslant v_2$. On the other hand, substituting u_1, v_1 with u_2, v_2, we have $u_1 \leqslant u_2$ and $v_1 \leqslant v_2$. Therefore, $u_1 \equiv u_2$ and $v_1 \equiv v_2$, and this proves the uniqueness of the solution.

Finally, we prove that the set of solutions $\{(\lambda, u(\lambda),\ v(\lambda))\colon \lambda > 0\}$ is a continuous curve. Indeed, for any $\lambda_a > 0$, choose a sequence $\lambda^n \to \lambda_a$. Then, from the uniqueness of solutions and the form of the upper/lower solution, one can see that $(u(\lambda^n), v(\lambda^n)) \to (u(\lambda_a), v(\lambda_a))$ as $n \to \infty$. Hence, the set of solutions is a continuous curve. □

THEOREM 4.2. *Consider (3.31) with $a, b > 0$. Suppose that $f_1(0) = g_1(0) = 0$, $f_1(v)$ and $g_1(u)$ satisfy (3.5) and, for any $u > 0$ and $v > 0$, (4.1) and (4.2) hold. Then (3.31) has a unique positive solution $(u(\lambda), v(\lambda))$ for $\lambda \in (0, \lambda^*)$, where $\lambda^* = \min\{\lambda_1/a, \lambda_1/b\}$. Moreover, $\{(\lambda, u(\lambda), v(\lambda))\colon \lambda \in (0, \lambda^*)\}$ is a continuous curve, so $u(\lambda)$ and $v(\lambda)$ are strictly increasing in λ, $(u(\lambda), v(\lambda)) \to (0, 0)$ as $\lambda \to 0^+$, and*

$$\|u(\lambda)\|_\infty + \|v(\lambda)\|_\infty \to \infty \quad \textit{as } \lambda \to (\lambda^*)^-.$$

Proof. The proof is similar to that of theorem 4.1, so we only point out the difference. For the existence part we still use $(\underline{u}, \underline{v}) = (\varepsilon_1\varphi_1, \varepsilon_2\varphi_1)$ as a lower solution of (3.31), but now $\varepsilon_1, \varepsilon_2 \in (0, 1)$ satisfy

$$\left.\begin{aligned} 0 < \varepsilon_1 \leqslant \left(\frac{\lambda D_g}{\lambda_1 - \lambda b}\right)^{q/(1-pq)} \left(\frac{\lambda D_f}{\lambda_1 - \lambda a}\right)^{1/(1-pq)}&, \\ \left(\frac{\varepsilon_1(\lambda_1 - \lambda a)}{\lambda D_f}\right)^{1/q} \leqslant \varepsilon_2 \leqslant \frac{\lambda D_g}{\lambda_1 - \lambda b}\varepsilon_1^p&, \end{aligned}\right\} \tag{4.15}$$

where

$$D_f = \Big(\inf_{x\in\Omega} \varphi_1^{q-1}(x)\Big) f_1(1), \qquad D_g = \Big(\inf_{x\in\Omega} \varphi_1^{p-1}(x)\Big) g_1(1),$$

and here we assume that $\lambda < \min\{\lambda_1/a, \lambda_1/b\}$. Note that, from the proof of theorem 3.5, we can only have the solution for $\lambda < \min\{\lambda_1/a, \lambda_1/b\}$.

For any $c < \lambda_1$ we define e_c to be the unique positive solution of

$$\Delta e + ce + 1 = 0, \quad x \in \Omega, \qquad e(x) = 0, \quad x \in \partial\Omega. \tag{4.16}$$

For the upper solution we choose $(\bar{u}, \bar{v}) = (M_1 e_{\lambda a}, M_2 e_{\lambda b})$, where $M_1 \geqslant 1$ and $M_2 \geqslant 1$. Then

$$\begin{aligned} \Delta(M_1 e_{\lambda a}) + \lambda[aM_1 e_{\lambda a} + f_1(M_2 e_{\lambda b})] &= -M_1 + \lambda f_1(M_2 e_{\lambda b}) \\ &\leqslant -M_1 + \lambda M_2^q f_1(e_{\lambda b}) \\ &\leqslant -M_1 + \lambda M_2^q C_f, \end{aligned}$$

where $C_f = \max_{x\in\bar{\Omega}} f_1(e_{\lambda b}(x))$. Similarly, if $C_g = \max_{x\in\bar{\Omega}} g_1(e_{\lambda a}(x))$, then

$$\Delta(M_2 e_{\lambda b}) + \lambda[bM_2 e_{\lambda b} + g_1(M_1 e_{\lambda a})] \leqslant -M_2 + \lambda M_1^p C_g.$$

Thus, if we choose M_1 and M_2 by

$$M_1 \geqslant (\lambda^{1+q} C_g^q C_f)^{1/(1-pq)} \quad \text{and} \quad \left(\frac{M_1}{\lambda C_f}\right)^{1/q} \geqslant M_2 \geqslant \lambda M_1^p C_g, \tag{4.17}$$

then $(\bar{u}, \bar{v}) = (M_1 e_{\lambda a}, M_2 e_{\lambda b})$ is an upper solution of (3.31). By choosing M_1 and M_2 larger (but still satisfying (4.17)) so that $(\underline{u}, \underline{v}) \leqslant (\bar{u}, \bar{v})$, we have a pair of upper and lower solutions needed in theorem 2.3. Hence, the existence of a positive solution is proved.

For the uniqueness proof we only change the operator $-\Delta + \lambda c$ to $-\Delta - \lambda c$ for $c = a, b$. The elliptic operator is still positive and the maximum principle still holds, since $\lambda < \min\{\lambda_1/a, \lambda_1/b\}$. The other parts of the proof are same as those in the proof of theorem 3.5. □

We remark that the stability defined in § 1 can still be established for f and g in theorems 4.1 and 4.2, although f_v and g_u become ∞ near $\partial\Omega$. By using remark 3.1 of [1] we can extend the spectral theory to the case when the Jacobian has a singularity of the form u^{p-1} or v^{q-1} with $0 < p, q < 1$.

EXAMPLE 4.3. We can apply the existence and uniqueness results of theorems 4.1 and 4.2 to

$$\left.\begin{aligned} \Delta u + \lambda[\pm au + v^q] = 0, &\quad x \in \Omega, \\ \Delta v + \lambda[\pm bv + u^p] = 0, &\quad x \in \Omega, \\ u(x) = v(x) = 0, &\quad x \in \partial\Omega, \end{aligned}\right\} \tag{4.18}$$

where $0 < p, q < 1$. When $p, q > 1$ and pq are subcritical, (4.18) (with $-$ sign) was considered in [14]. Here we prove the existence and uniqueness of the positive solution for the sublinear case.

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