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Exact multiplicity of solutions to a diffusive logistic equation with harvesting ☆

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ABSTRACT

An Ambrosetti–Prodi type exact multiplicity result is proved for a diffusive logistic equation with harvesting. We show that a modified diffusive logistic mapping has exactly either zero, or one, or two pre-images depending on the harvesting rate. It implies that the original diffusive logistic equation with harvesting has at most two positive steady state solutions.

1. Introduction

In this paper, we revisit the steady state equation of a diffusive logistic model with harvesting

$$\begin{cases} \Delta u + au - u^2 - cg(x) = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where a, c are positive constants, Ω is a smooth bounded region with $\partial\Omega$ of class $C^{2,\alpha}$ in $\mathbf{R}^n$ for $n \geqslant 1$, and $g \in C^\alpha(\overline{\Omega})$ for $0 < \alpha < 1$. Eq. (1.1) arises from the studies of population biology of one species which disperses in a habitat Ω with hostile boundary $\partial\Omega$, and the population is subject to a harvesting rate $c \cdot g(x)$. We refer to [5] for more explanation of the mathematical model.

We denote by λ_k the kth eigenvalue of

$$\begin{cases} \Delta\phi + \lambda\phi = 0, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega. \end{cases} \tag{1.2}$$

In particular, $\lambda_1 > 0$ is the principal eigenvalue with a positive eigenfunction ϕ_1. In [5], we assume that $g(x) \geqslant 0$. Under this condition, the main results of [5] can be summarized as follows:

1. If $a \leqslant \lambda_1$, then for any $c > 0$, (1.1) has no non-negative solution.
2. If $a > \lambda_1$, there exists $c_2 > 0$ such that when $0 < c \leqslant c_2$, (1.1) has a positive solution u_1, and u_1 is the maximal solution which is also stable; and when $c > c_2$, there is no non-negative solution.
3. There exists a $\delta > 0$, such that if $\lambda_1 < a < \lambda_1 + \delta$ for some $\delta > 0$, (1.1) has exactly two positive solutions u_1 and u_2 when $c \in (0, c_2)$, has exactly one positive solution when $c = c_2$, and has no non-negative solution when $c > c_2$.

Our main result about the diffusive logistic equation with harvesting in this paper is

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Theorem 1.1. *For any $c \in \mathbf{R}, g \in C^{\alpha}(\overline{\Omega})$ and $g(x) \geqslant (\not\equiv)0$ for $x \in \overline{\Omega}$, if $\lambda_1 < a < \lambda_2$, then (1.1) has either zero, or one, or two positive solutions. Moreover, there exists $c_2 > 0$ such that (1.1) has at least one positive solution u_1 and at most two positive solutions when $c \in (0, c_2)$, has exactly one positive solution when $c = c_2$, and has no non-negative solution when $c > c_2$.*

Compared to the previous exact multiplicity result in Oruganti et al. [5], the main improvement is that the parameter a here is not restricted to a small interval $(\lambda_1, \lambda_1 + \delta)$, but the interval (λ_1, λ_2). On the other hand, we have less precise information on the exact multiplicity of positive solutions for a given $g(x)$, but we give an upper bound of number of positive solutions. The exact multiplicity result in [5] was proved by a perturbation argument based on implicit function theorem. Here we take a quite different approach following the classical paper of Ambrosetti and Prodi [1]. We rewrite (1.1) into

$$\begin{cases} \Delta u + au - u^2 = g(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1.3}$$

We have dropped the constant c in the equation since c can be arbitrary constant in our result. The left hand side of the equation in (1.3) defines a differentiable mapping $F : u \mapsto \Delta u + au - u^2$ for $u \in C_0^{2,\alpha}(\overline{\Omega})$. Hence the question of exact multiplicity of solutions to (1.1) becomes the number of pre-images for a given $g \in C^{\alpha}(\overline{\Omega})$.

Although the mapping F is well-defined, the behavior of the nonlinearity $au - u^2$ makes it impossible to prove a global inversion theorem as in Ambrosetti and Prodi [1]. Instead we consider a modified problem as follows: fix $a \in (\lambda_1, \lambda_2)$; let $j(u)$ be defined as

$$j(u) = \begin{cases} -2, & u \in [0, a], \\ \text{properly defined}, & u \in [-\varepsilon, 0) \cup (a, a + \varepsilon], \\ 0, & u \in (-\infty, -\varepsilon) \cup (a + \varepsilon, +\infty). \end{cases} \tag{1.4}$$

Here $j(u)$ is properly defined for $u \in [-\varepsilon, 0) \cup (a, a + \varepsilon]$ so that $j(u)$ is continuous. Next we define $\tilde{f}(u)$ to be the unique function satisfying $\tilde{f}''(u) = j(u), \tilde{f}(0) = 0$ and $\tilde{f}'(0) = a$. Notice that $\tilde{f}(u) \equiv f(u)$ for $u \in [0, a]$, and $\tilde{f}''(u) \leqslant 0$ for all $u \in \mathbf{R}$. Moreover we can choose a small enough ε and the value of $j(u)$ in $[-\varepsilon, 0) \cup (a, a + \varepsilon]$ appropriately so that $\tilde{f}(u)$ satisfies

$$\lim_{u \to -\infty} \frac{\tilde{f}(u)}{u} = \lim_{u \to -\infty} \tilde{f}'(u) = l_- \in (\lambda_1, \lambda_2), \lim_{u \to +\infty} \frac{\tilde{f}(u)}{u} = \lim_{u \to +\infty} \tilde{f}'(u) = l_+ < 0. \tag{1.5}$$

Then we have the following theorem of global inversion with singularity:

Theorem 1.2. *Let $\Omega \subset \mathbf{R}^n$ be a bounded open subset of class $C^{2,\alpha}$, and let $\tilde{f}$ and $j = \tilde{f}''$ satisfy (1.4) and (1.5). Consider the boundary-value problem:*

$$\begin{cases} \Delta u + \tilde{f}(u) = g(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.6}$$

where $g \in C^{0,\alpha}(\overline{\Omega})$. Then there exists a closed connected C^1-manifold M of codimension 1 in $C^{0,\alpha}(\overline{\Omega})$, such that $C^{0,\alpha}(\overline{\Omega}) \setminus M$ consists exactly of two connected components A_1 and A_2 with the following properties:

(a) *if $g \in A_1$, then the problem (1.6) has no solution;*
(b) *if $g \in A_2$, then the problem (1.6) has exactly two solutions; and*
(c) *if $g \in M$, then the problem (1.6) has a unique solution.*

Moreover if $g(x) \geqslant (\not\equiv)0$ for $x \in \overline{\Omega}$, then there exists $c_2 > 0$ such that

$$\begin{cases} \Delta u + \tilde{f}(u) = c \cdot g(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.7}$$

has exactly two solutions when $c \in (0, c_2)$ with at least one of them being positive, has exactly one solution which is positive when $c = c_2$, and has no solution when $c > c_2$.

Theorem 1.1 is a corollary of Theorem 1.2, since any positive solution u of (1.1) with $g \geqslant 0$ satisfies $0 \leqslant u(x) \leqslant a$, hence u also satisfies (1.6). In general it is not known whether both solutions when $c \in (0, c_2)$ are positive. This question is closely related to whether the anti-maximum principle holds for the parameter a and the function $g(x)$. Two positive cases are (i) when $\lambda_1 < a < \lambda_1 + \delta$ for some $\delta = \delta(g) > 0$ [5]; and (ii) $n = 1, \Omega = (-1, 1)$ and $g(x)$ satisfies an integral constraint [7].

The structure of a differential mapping between infinite dimensional spaces have been studied by many people since the pioneer work of Ambrosetti and Prodi [1], see the survey article [3] and the references therein. It would also be interesting to know more about the structure of the separating manifold M in Theorem 1.2, in which we show that each ray $\{cg : c > 0\}$ intersects M exactly once for $g \geqslant (\not\equiv)0$. By using the arguments in [6] (see also [8,4]), we can show that the opposite ray $\{cg : c < 0\}$ stays entirely in A_2. However for sign-changing g, no such information is known.

For the remaining part of the paper, we prove Theorem 1.2. In Section 2, we recall the abstract framework of differentiable mapping; and we prove the main result and related lemmas in Section 3. Our proofs mostly follow the lines in Refs. [1,2], but we use a different eigenvalue problem as in Shi and Wang [9] while the ones in Refs. [1,2] cannot be used in our situation.

2. Preliminaries

In order to obtain the exact multiplicity of solutions to (1.6), we recall some preliminaries about the inversion of differentiable mappings between Banach spaces as preparation for the main results of the paper. Since all definitions and theorems are from [1] and [2], and we omit the proofs.

Definition 2.1. A mapping $\Phi : X \to Y$ (X and Y are topological spaces) is said to be *proper* if for every compact set $K \subset Y$, the set $\Phi^{-1}(K)$ is compact in X.

Let us remark that if Φ is proper then it maps closed sets into closed sets.

Definition 2.2. Let X and Y be topological spaces. A continuous mapping $\Phi : \ X \to Y$ is said to be *locally invertible* at $u_0 \in X$, if there exists a neighborhood U of u_0 and a neighborhood V of $y_0 = \Phi(u_0)$ such that Φ induces a homeomorphism between U and V.

We set $N(y) = \sharp\Phi^{-1}(\{y\})$ (cardinal number of $\Phi^{-1}(\{y\})$).

Proposition 2.3. *Let X and Y be metrizable topological spaces, and let $\Phi : \ X \to Y$ be a proper, continuous mapping which is locally invertible at every point. Then the function $y \mapsto N(y)$ is finite and locally constant.*

As a corollary of this proposition, we obtain that if Y is connected then $N(y)$ is constant. For our purpose it is fundamental to study the set of the points at which the mapping is not invertible.

Definition 2.4. Let $\Phi : \ X \to Y$ be a continuous mapping ($X, \ Y$ topological spaces). We say that $u \in X$ is a *singular point* if Φ is not locally invertible at u; $y \in Y$ is said to be a *critical point* if $y = \Phi(u)$, for some singular point $u \in X$.

Proposition 2.5. *Let X and Y be a metrizable topological spaces and $\Phi : X \to Y$ a continuous proper mapping. We denote the singular set (consist of all singular points) by W. Then $N(y)$ is constant on every connected component of $Y \setminus \Phi(W)$.*

Now we introduce some notions that we shall use in the study of the singular and critical set of a differentiable mapping.

Definition 2.6. Let X be a Banach space. A set $M \subset X$ is said to be a C^k-manifold of codimension 1, if for every point $u_0 \in M$ there exists a neighborhood U of u_0 and a C^k-functional $\Gamma : \ U \to R$ such that

(a) $\Gamma'(u_0) \neq 0$;
(b) $M \cap U = \{u : \ u \in U, \Gamma(u) = 0\}$.

Then we have an important result:

Proposition 2.7. *Let M be a closed connected C^k-manifold ($k \geqslant 1$) of codimension 1 in the Banach space X. Then $X \setminus M$ has at most two components.*

Here is the local structure theorem near an "ordinary singular point" of Ambrosetti and Prodi [1]:

Theorem 2.8. *Let X and Y be Banach spaces, let Λ be an open subset of X, and let $\phi : \ \Lambda \to Y$ be a mapping of class C^k with $k \geqslant 2$. Assume that $u_0 \in \Lambda$ is such that:*

(I) $\phi'(u_0)$ has a kernel of dimension 1 and an image of codimension 1.
(II) If $v_0 \in X$ is a non-zero vector such that $\phi'(u_0)v_0 = 0$ and γ_0 is a functional on Y such that $Im(\phi'(u_0)) = \{z : \langle z, \gamma_0\rangle = 0\}$, then the linear functional

$$z \mapsto \langle \phi''(u_0)[z, v_0], \gamma_0\rangle$$

is not identically zero.
Then the singular set W of ϕ is, in a neighborhood of u_0, a C^{k-1}-manifold of codimension 1. If the condition (II) *is replaced by (II*)*

$$\langle \phi''(u_0)[v_0, v_0], \gamma_0\rangle \neq 0,$$

then we can find an open neighborhood U of u_0 such that $\phi(W \cap U)$ is a C^{k-1}-manifold of codimension 1.

Definition 2.9. We say that $u \in M$ is an *ordinary singular point* if (I) and (II*) hold.

If u_0 is an ordinary singular point, then we can compute locally the number of the solutions of the equation $\phi(u) = y$.

Theorem 2.10. *Let $\phi : \Lambda \to Y$ be a mapping of class C^k with $k \geqslant 2$, and let $u_0 \in \Lambda$ be an ordinary singular point. Then denoted by s a vector which is transversal to $\phi(W)$ in $y_0 = \phi(u_0)$, there exists a neighborhood U of u_0 and an $\varepsilon \in R$ such that (a) $\forall y \in (y_0, y_0 + \varepsilon s]$ the equation $\phi(u) = y$ has two solutions in U. (b) $\forall y \in [y_0 - \varepsilon s, y_0)$ the equation $\phi(u) = y$ has no solution in U.*

3. Proof of the main results

3.1. Proof of Theorem 1.2

We consider the mapping $\phi : C_0^{2,\alpha}(\overline{\Omega}) \to C^{0,\alpha}(\overline{\Omega})$ defined by

$$\phi(u) = \Delta u + \widetilde{f}(u).$$

From the assumptions on $\widetilde{f}$ it follows that ϕ is of class C^2. The Fréchet derivative of ϕ evaluated on $u \in C^{2,\alpha}(\overline{\Omega})$ is given by

$$\phi'(u) : v \mapsto \Delta v + \widetilde{f}'(u)v.$$

To complete the proof of Theorem 1.2, we state here some lemmas, which we shall prove in the following subsections.

Lemma 3.1. *The mapping ϕ is proper.*

Lemma 3.2. *The singular set W of ϕ is non-empty, closed and connected; each point in W is an ordinary singular point.*

Lemma 3.3. *If $g \in \phi(W)$, then (1.6) has a unique solution.*

Assuming these three lemmas, now we complete the proof of the Theorem 1.2. Since all the points of W are ordinary singular points, then by Theorem 2.8, $\phi(W)$ is a manifold of codimension 1. By Lemma 3.1, ϕ is proper; and since W is closed and connected, $\phi(W)$ is also closed and connected. From Proposition 2.7 we can say that $C^{0,\alpha}(\overline{\Omega}) \setminus \phi(W)$ has at most 2 connected components. Moreover since ϕ is proper, then by Proposition 2.5, the number of the solutions of $\phi(u) = g$ is constant on each connected component of $C^{0,\alpha}(\overline{\Omega}) \setminus \phi(W)$. To compute such number, we first observe that, for every neighborhood U of $u_0 \in W$, there exists a neighborhood V of $g_0 = \phi(u_0)$ such that $\phi^{-1}(V) \subset U$. Otherwise, there should exists a neighborhood U^* of u_0 and a sequence u_n such that $u_n \notin U^*$ and $\lim_{n\to\infty}\phi(u_n) = g_0$. Since ϕ is proper, we might extract a subsequence converging to a point u^* such that $u_n \notin U^*$ and $\phi(u^*) = g_0$, $u^* \neq u_0$. This would be against Lemma 3.3.

On the other hand, since u_0 is an ordinary singular point, by Theorem 2.10, we can compute locally the number of solutions to the equation $\phi(u) = g$ when g lies on a segment which is transversal to $\phi(W)$ in g_0. The number of solutions is 2 or 0 according to g lying on which side of $\phi(W)$. Hence part (a) and (b) of the theorem hold. Finally if $g \in \phi(W)$, by Lemma 3.3, the solution of $\phi(u) = g$ is unique.

Finally if $g \geqslant 0$ and $g \not\equiv 0$, then the existence of $c_2 > 0$ is shown in Oruganti et al. [5], and other statements of the exact multiplicity results follow from Oruganti et al. [5]. That completes the proof of the theorem.

3.2. Proof of Lemma 3.1

First we prepare several lemmas. The first one is part of Theorem 0.5 in Ambrosetti and Prodi [2].

Proposition 3.4. *Let $\Omega \subset \mathbf{R}^n$ be a bounded subset of class $C^{2,\alpha}(\overline{\Omega})$, and let v be the solution of the boundary-value problem*

$$\begin{cases} \Delta v = h, & x \in \Omega, \\ v = 0, & x \in \partial\Omega, \end{cases} \tag{3.1}$$

where h is a bounded function. Then for every fixed α $(0 < \alpha < 1)$ the following estimate holds

$$\|v\|_{1,\alpha} \leqslant k_\alpha \|h\|_{L_\infty}$$

where k_α is a suitable constant.

Now we consider the following eigenvalue problem:

$$\begin{cases} \Delta v + \widetilde{f}'(u)v = -\mu v, & x \in \Omega, \\ v = 0, & x \in \partial\Omega. \end{cases} \tag{3.2}$$

It is well-known that (3.2) has a sequence of eigenvalues $\mu_i = \mu_i(\widetilde{f}'(u))$ satisfies $\mu_1 < \mu_2 \leqslant \cdots$, and $\mu_i \to \infty$ as $i \to \infty$. From [9] (page 3689), for $i = 1, 2, \cdots$,

$$\mu_i(\widetilde{f}'(u)) = Min_i Max_i \frac{\int_\Omega \left(|\nabla v|^2 - \widetilde{f}'(u)v^2\right)dx}{\int_\Omega v^2 dx} \tag{3.3}$$

where Max_i is taken over all $v(\neq 0) \in T_i$, and Min_i is over all i-dimentional subspaces T_i of $H_0^1(\Omega)$. The following result on the monotonicity of the eigenvalues is well-known:

Lemma 3.5 [9] Lemma 2.1. *Suppose that $W_1, W_2 \in L^\infty(\Omega)$ satisfy $W_2(x) \geqslant W_1(x)$ a.e., then $\mu_i(W_2) \leqslant \mu_i(W_1)$, here $\mu_i(W_i)$ are the eigenvalues of (3.2) with $\widetilde{f}'(u)$ replaced by W_i. If in addition the Lebesgue measure $m(\{W_2 > W_1\}) > 0$, then $\mu_i(W_2) < \mu_i(W_1)$.*

The following lemma is a key to the global inversion result:

Lemma 3.6. *Let u_n be a sequence in $C_0^{2,\alpha}(\overline{\Omega})$ and $\phi(u_n) = \Delta u_n + \widetilde{f}(u_n) = g_n$. If the sequence g_n is bounded in $C^{0,\alpha}(\overline{\Omega})$, then the sequence u_n is bounded in $C^{0,\alpha}(\overline{\Omega})$.*

Proof 1. Suppose that the opposite holds: $\lim_{n\to+\infty}\|u_n\|_{0,\alpha} = +\infty$. We set $z_n = u_n\|u_n\|_{0,\alpha}^{-1}$; we have $z_n \in C_0^{2,\alpha}(\overline{\Omega})$ and $\|z_n\|_{0,\alpha}$=1.

We introduce the real function h defined as follows:

$$h(t) = \begin{cases} \frac{\widetilde{f}(t)}{t}, & \text{for } t\neq 0, \\ \widetilde{f}'(0), & \text{for } t = 0. \end{cases}$$

In virtue of the hypothesis on $\widetilde{f}$, h is of class C^1 and is bounded.

From the equation $\Delta u_n + \widetilde{f}(u_n) = g_n$, dividing by $\|u_n\|_{0,\alpha}$, we get

$$\Delta z_n + h(u_n)z_n = \frac{g_n}{\|u_n\|_{0,\alpha}}. \tag{3.4}$$

The sequence $g_n\|u_n\|_{0,\alpha}^{-1} - h(u_n)z_n$ is bounded in $L^\infty(\Omega)$. By Proposition 3.4, we have that $\|z_n\|_{1,\alpha}$ is bounded. Therefore we can extract a subsequence converging in $C^1(\overline{\Omega})$ to a function z^*, and since $\|z_n\|_{0,\alpha} = 1$, then $\|z^*\|_{0,\alpha} = 1$ from the continuity of the norm. In particular $z^*\neq 0$.

We write (3.4) in distribution sense:

$$-\int_\Omega \sum_i \frac{\partial z_n}{\partial x_i}\frac{\partial \omega}{\partial x_i}dx + \int_\Omega h(u_n)z_n\omega dx = \int_\Omega \frac{g_n}{\|u_n\|_{0,\alpha}}\omega dx \tag{3.5}$$

for every $\omega \in C_0^\infty(\Omega)$. We observe that for the points $x \in \Omega$ where we have $z^*(x) < 0$, $\lim_{n\to+\infty}u_n(x) = -\infty$ and hence $\lim_{n\to+\infty}h(u_n(x)) = l_-$, while for the points where we have $z^*(x) > 0$, it results in $\lim_{n\to+\infty}h(u_n(x)) = l_+$. Thus if we set

$$a(x) = \begin{cases} l_-, & \text{if } z^*(x) < 0, \\ l_+, & \text{if } z^*(x) > 0, \\ f\prime(0), & \text{if } z^*(x) = 0, \end{cases}$$

we have for any $x \in \Omega$, $\lim_{n\to+\infty}h(u_n(x))z_n(x) = a(x)z^*(x)$. Taking the limit, from (3.5) we obtain (by Lebesgue's dominant convergent theorem)

$$-\int_\Omega \sum_i \frac{\partial z^*}{\partial x_i}\frac{\partial \omega}{\partial x_i}dx + \int_\Omega az^*\omega dx = 0, \quad \forall\omega \in C_0^\infty(\Omega). \tag{3.6}$$

This relation shows that there exists $k \geqslant 1$ such that $\mu_k(a(x))) = 0$ is an eigenvalue of the problem

$$\begin{cases} \Delta v + a(x)v = -\mu v, & x \in \Omega, \\ v = 0, & x \in \partial\Omega. \end{cases}$$

We claim that $\mu_1(a(x)) = 0$. In fact, since $a(x) \leqslant l_-$, by Lemma 3.5, $\mu_k(a(x)) \geqslant \mu_k(l_-) = \lambda_k - l_-$. In particular $\mu_2(a(x)) \geqslant \mu_2(l_-) = \lambda_2 - l_- > 0$. We must have $\mu_1(a(x)) = 0$. This prove z^* is of one sign in all of Ω. Now, if we suppose $z^*(x) > 0$ in Ω, then the following equation is fulfilled

$$\Delta z^* + l_+z^* = 0, z^*|_{\partial\Omega} = 0,$$

which is a contradiction, since l_+ is not an eigenvalue for $-\Delta$. On the other hand, if we have $z^*(x) < 0$ in Ω, then we have

$$\Delta z^* + l_-z^* = 0, z^*|_{\partial\Omega} = 0.$$

Also this relation cannot be true since l_- is not an eigenvalue for $-\Delta$. This contradiction shows that u_n must be bounded in $C^{0,\alpha}(\overline{\Omega})$.

3.2.1. Proof of Lemma 3.1

Let $\{u_n\}$ be a sequence in $C_0^{2,\alpha}(\overline{\Omega})$ such that $\phi(u_n) = \Delta u_n + \widetilde{f}(u_n) = g_n$ is convergent in $C^{0,\alpha}(\overline{\Omega})$. By Lemma 3.6 we know that $\{u_n\}$ is bounded in $C^{0,\alpha}(\overline{\Omega})$, and then $\Delta u_n = g_n - \widetilde{f}(u_n)$ is a bounded sequence in $C^{0,\alpha}(\overline{\Omega})$. But since, under our hypothesis for Ω, the operator Δ is an isomorphism of $C_0^{2,\alpha}(\overline{\Omega})$ onto $C^{0,\alpha}(\overline{\Omega})$, then we can say that $\{u_n\}$ is a bounded sequence in $C_0^{2,\alpha}(\overline{\Omega})$. Hence we can extract a subsequence from $\{u_n\}$ converging in $C^{0,\alpha}(\overline{\Omega})$, then the equation itself shows that this subsequence converges in $C_0^{2,\alpha}(\overline{\Omega})$, which proves Lemma 3.1.

3.3. Proof of Lemmas 3.2 and 3.3

3.3.1. Proof of Lemma 3.2

First we prove that each point u_0 of the singular manifold W is an ordinary singular point, namely the hypothesis (I) and (II*) of Theorem 2.8 are fulfilled. We consider the eigenvalue problem:

$$\begin{cases} \Delta v + \widetilde{f}'(u_0)v = -\mu v, & x \in \Omega, \\ v = 0, & x \in \partial\Omega. \end{cases} \tag{3.7}$$

Note that u is singular if and only if $\mu_i(u) = 0$ for some i. From the proof of Lemma 3.1, we know $\mu_1(u) = 0$. Then it is a simple eigenvalue: the kernel of $\phi'(u_0)$ is associated with a non-zero vector $v_0 \in C_0^{2,\alpha}(\overline{\Omega})$. It is known that $Im\phi'(u_0)$ consists of the elements $g \in C^{0,\alpha}(\overline{\Omega})$, for which $\int_\Omega g(x)v_0(x)dx = 0$. Then the hypothesis (I) of Theorem 2.8 is satisfied. The functional γ_0 which is associated with $Im\phi'(u_0)$ is

$$z \to \int_\Omega z(x)v_0(x)\,dx.$$

Now we compute ϕ''. Since the second differential of the linear term vanishes, we have

$$(\phi''(u_0)[v,w])(x) = \widetilde{f}''(u_0(x))v(x)w(x).$$

Then the condition (II*) of Theorem 2.8 becomes

$$\int_\Omega \widetilde{f}''(u_0)v_0^3 dx \neq 0.$$

This relation is satisfied since $\widetilde{f}''(t) \leqslant 0$ for any $t \in \mathbf{R}$, and $\widetilde{f}''(t) \not\equiv 0$, and v_0 is of the same sign on the whole Ω, since it is the first eigenfunction of (3.7).

Next, we show that W is non-empty, closed, and connected. We show W has a cartesian representation on a linear subspace of $C_0^{2,\alpha}(\overline{\Omega})$ of codimension 1. Namely, let $s \in C_0^{2,\alpha}(\overline{\Omega})$ with $s(x) > 0$ for all $x \in \Omega$ and let Z be any linear subspace of $C_0^{2,\alpha}(\overline{\Omega})$ of codimension 1, such that $s \notin Z$. Every element $u \in C_0^{2,\alpha}(\overline{\Omega})$ can be represented in a unique way in the form $u = z + \tau s, \tau \in \mathbf{R}$, and $z \in Z$. We consider the eigenvalue problem:

$$\begin{cases} \Delta v + \widetilde{f}'(z+\tau s)v = -\mu v, & x \in \Omega, \\ v = 0, & x \in \partial\Omega, \end{cases}$$

where z is a fixed element of Z, and $\tau \in \mathbf{R}$. Also, $\mu_i(f\prime(z+\tau s))$ is a monotone function of $\tau \in \mathbf{R}$, thus there is only one τ such that $\mu_1(\widetilde{f}'(z+\tau s)) = 0$ for fixed $z \in Z$. Then we have proved that every straight line $\tau \to z + \tau s$ meets the manifold W at a unique point: it is easy to show that this point depends continuously on Z.

3.3.2. Proof of Lemma 3.3

Let $u_0 \in W, \phi(u_0) = g_0$, we assume that equation $\phi(u) = g_0$ has another solution $\widetilde{u}$. We set

$$\omega(x) = \begin{cases} \frac{\widetilde{f}(\widetilde{u}(x)) - \widetilde{f}(u_0(x))}{\widetilde{u}(x) - u_0(x)}, & \text{if } \widetilde{u}(x) \neq u_0(x), \\ \widetilde{f}'(u_0(x)), & \text{if } \widetilde{u}(x) = u_0(x). \end{cases}$$

Then $\widetilde{u} - u_0$ is a nontrivial solution of the problem

$$\begin{cases} \Delta v + \omega v = -\mu v, & x \in \Omega, \\ v = 0, & x \in \partial\Omega, \end{cases}$$

with $\mu = 0$. On the other hand, since $\widetilde{f}''(t) \leqslant 0$, it follows that $\omega(x) \leqslant (\not\equiv)\widetilde{f}'(u_0(x))$ for $x \in \Omega$. By the hypothesis we have $u_0 \in W$, thus also the problem

$$\begin{cases} \Delta v + \widetilde{f}'(u_0)v = -\mu v, & x \in \Omega, \\ v = 0, & x \in \partial\Omega, \end{cases}$$

has $\mu = 0$ as the first eigenvalue. Then $\mu_1(w) > \mu_1(\widetilde{f}'(u_0)) = 0$ from Lemma 3.5, thus $\mu_1(\omega) > 0$ for all $i \geqslant 1$. That is a contradiction. Hence u_0 is the unique solution of $\phi(u) = g_0$.

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