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Bifurcation analysis in a delayed diffusive Nicholson's blowflies equation[☆]

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ABSTRACT

The dynamics of a diffusive Nicholson's blowflies equation with a finite delay and Dirichlet boundary condition have been investigated in this paper. The occurrence of steady state bifurcation with the changes of parameter is proved by applying phase plane ideas. The existence of Hopf bifurcation at the positive equilibrium with the changes of specify parameters is obtained, and the phenomenon that the unstable positive equilibrium state without dispersion may become stable with dispersion under certain conditions is found by analyzing the distribution of the eigenvalues. By the theory of normal form and center manifold, an explicit algorithm for determining the direction of the Hopf bifurcation and stability of the bifurcating periodic solutions are derived.

1. Introduction

In order to describe the population dynamics of Nicholson's blowflies, Gurney et al. [1] have proposed the following delay equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -d_m u(t) + \varepsilon u(t-\tau)\mathrm{e}^{-au(t-\tau)}, \tag{1.1}$$

where ε is the maximum per capita daily egg production rate, $1/a$ is the size at which the blowfly population reproduces at its maximum rate, d_m is the per capita daily adult death rate and τ is the generation time. Eq. (1.1) has been extensively studied in the literature, where its results mainly concern the global attractivity of positive equilibrium and oscillatory behaviors of solutions (see [2–7,30,32]). Several studies have also been carried out on Eq. (1.1) with time periodic coefficients (see [8,9]) and on discrete Nicholson's blowflies equation (see [10–15]). After rescaling Eq. (1.1), it takes the form

$$\tilde{u} = au, \qquad \tilde{t} = \frac{t}{\tau}, \qquad \tilde{\tau} = d_m\tau, \qquad \beta = \frac{\varepsilon}{d_m},$$

and by dropping the tildes, then it may be written as

$$\frac{\mathrm{d}u}{\mathrm{d}t}(t) = -\tau u(t) + \beta\tau u(t-1)\mathrm{e}^{-u(t-1)}. \tag{1.2}$$

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To explain interactions among organisms, Yang and So [16] extended Eq. (1.2) to the flowing diffusive form

$$\frac{\partial u}{\partial t}(t,x) = d\triangle u(t,x) - \tau u(t,x) + \beta\tau u(t-1,x)\mathrm{e}^{-u(t-1,x)}. \tag{1.3}$$

Furthermore, some researchers have studied the phenomenon by using an equation in the following form

$$\frac{\mathrm{d}u}{\mathrm{d}t}(t,x) = D_m \Delta u(t,x) - d_m u(t,x) + \varepsilon u(t-\tau,x)\mathrm{e}^{-au(t-\tau,x)}. \tag{1.4}$$

So and Yang [17] investigated global attractivity of the equilibrium of Eq. (1.3) with the Dirichlet boundary condition

$$u(t,0) = u(t,\pi) = 0, \quad \text{for } t \geq 0 \tag{1.5}$$

and they [16] studied the stability and existence of Hopf bifurcation of Eq. (1.3) with Neuman boundary condition. Some numerical and Hopf bifurcation analysis on Eq. (1.3) has been carried out by So, Wu and Yang [18]. Generalized Nicholson's blowflies models with distributed delay in Eqs. (1.3) and (1.4) have been studied extensively (see [19–22]). For the Dirichlet boundary value problem of the diffusive Nicholson's blowflies equation, So and Yang [17] have proved that there is a unique positive steady state solution if and only if $(\beta-1)\tau > \lambda_1$, where λ_1 is the principal eigenvalue of $-\triangle$ with Dirichlet boundary condition.

The purpose of the present paper is to study the bifurcation of Eq. (1.3) with Dirichlet boundary condition (1.5). We prove the existence of positive steady state bifurcation by a direct calculation presented in Robinson [23]. The conclusions are that the problem (1.3) and (1.5) has a unique positive steady state if and only if $\beta > 1 + d/\tau$, and the Eq. (1.3) with Dirichlet boundary condition

$$u(t,0) = u(t,\pi) = \ln\beta, \quad \text{for } t \geq 0 \tag{1.6}$$

has no other positive steady state solution except $u = \ln\beta$. On the other hand, we provide a detailed analysis of Hopf bifurcation for the problems (1.3) and (1.6) by applying the local Hopf bifurcation theory (see [24]). More specifically, we prove that, as β increases, the positive equilibrium $u^* = \ln\beta$ loses its stability and a sequence of Hopf bifurcations occur at u^*. Furthermore, by using the center manifold theory introduced by Lin, So and Wu [25] and normal form method due to Faria [26], we derive an explicit algorithm for determining the stability and direction of the Hopf bifurcations occurring at u^*.

The rest of this paper is organized as follows. In Section 2, existence of positive steady state bifurcation is established. In Section 3, the occurrence of Hopf bifurcation and the phenomenon that the unstable positive equilibrium state without dispersion may become stable with dispersion under certain conditions are found by analyzing the distribution of the eigenvalues. In Section 4, an algorithm for determining the direction and stability of the Hopf bifurcation is derived by using the center manifold due to Lin, So and Wu [25] and normal form method due to Faria [26]. Finally, some numerical analysis is given in order to illustrate the theoretic results found.

2. Positive steady state bifurcation

In the present section, we consider equation

$$\frac{\partial u}{\partial t}(t,x) = d\frac{\partial^2 u}{\partial x^2}(t,x) - \tau u(t,x) + \beta\tau u(t-1,x)\mathrm{e}^{-u(t-1,x)}, \tag{2.1}$$

where $(t,x) \in D = (0,\infty)\times[0,\pi]$, $\beta,\ d,\ \tau > 0$, with Dirichlet boundary condition

$$u(t,0) = u(t,\pi) = 0, \quad \text{for } t \geq 0. \tag{2.2}$$

The steady state $u(x)$ of (2.1) and (2.2) satisfies

$$\begin{aligned} &du_{xx} = \tau u - \tau\beta u\mathrm{e}^{-u}, \\ &u(0) = u(\pi) = 0. \end{aligned} \tag{2.3}$$

Taking $v = u_x$, we can rewrite the equation in (2.3) into a pair of differential equations:

$$\begin{cases} u_x = v, \\ v_x = \dfrac{\tau}{d}(u - \beta u\mathrm{e}^{-u}). \end{cases} \tag{2.4}$$

We can now apply phase plane ideas, treating x as the time variable. It follows from Eq. (2.4) that, on any trajectory,

$$\frac{d}{2}v^2 - \frac{\tau}{2}u^2 - \tau\beta(u\mathrm{e}^{-u} + \mathrm{e}^{-u}) = C,$$

where C is a constant. It is obvious that Eq. (2.4) has two fixed points given by

$$(u,v) = (0,0),\ (\ln\beta, 0).$$

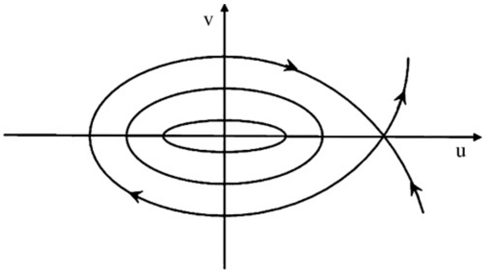

Fig. 2.1. The phase portrait of (2.4).

Clearly, $\ln\beta > 0$ if and only if $\beta > 1$. This relation is assumed throughout this section. The matrix associated with the linearized vector field of Eq. (2.4) is given by

$$\begin{pmatrix} 0 & 1 \\ \frac{\tau}{d}[1-\beta(\mathrm{e}^{-u}-u\mathrm{e}^{-u})] & 0 \end{pmatrix}.$$

The eigenvalues associated with the fixed point (0, 0) are given by $\lambda_{1,2} = \pm \mathrm{i}\sqrt{\frac{\tau(\beta-1)}{d}}$, and the eigenvalues associated with the fixed point $(\ln\beta, 0)$ are given by $\lambda_{1,2} = \pm\sqrt{\frac{\tau\ln\beta}{d}}$. Hence, (0, 0) is a center and $(\ln\beta, 0)$ is a saddle.

Lemma 2.1. *The trajectory starts at $u = \ln\beta$, $v = 0$ which moves around clockwise, can strike the v axis at a value $v = \bar{v} < 0$.*

Proof. Obviously, this lemma holds true if and only if $C^* > -\tau\beta$, where C^* is associated with the trajectory which passes through the point $(\ln\beta, 0)$, and is given by

$$C^* = -\frac{\tau}{2}(\ln\beta)^2 - \tau(1+\ln\beta).$$

Denote $s = \ln\beta$ with $s > 0$. Then $C^* > -\tau\beta$ is equivalent to the inequality

$$s^2 + 2(s+1) - 2\mathrm{e}^s < 0 \tag{2.5}$$

holds. It is easy to verify that (2.5) holds when $s > 0$. The proof is completed. □

Lemma 2.2. *The trajectory starts at $u = 0$, $v = \bar{v}$ which moves around clockwise, can strike the u axis at a value $u = \bar{u}$.*

Proof. Obviously, along this trajectory $u_x = v < 0$, hence, u decreases strictly as x increases. There exists $\delta > 0$ such that $\ln(\beta-\delta) > 0$, then

$$v_x = \frac{\tau}{d}(u - \beta u\mathrm{e}^{-au}) > \frac{\tau}{d}u\left(1 - \frac{\beta}{\beta-\delta}\right) > 0$$

holds when $u < 0$. This implies that v is strictly increasing as a function of x when $u < 0$. Therefore the trajectory can strike the u axis. □

The phase portrait for (2.4) is given in Fig. 2.1. To satisfy the boundary conditions we need the trajectory that starts on the v axis at $x = 0$ and moves back onto the v axis when $x = \pi$. If the trajectory in the right-half plane (resp. left-half plane) moving from v axis strike u axis at a point, denote the point by u_0 (resp. $\bar{u}_0$), the "time" by $t_1(u_0)$ (resp. $t_2(\bar{u}_0)$).

Proposition 2.3. $\lim_{u_0\to 0} t_1(u_0) = \lim_{u_0\to 0} t_2(\bar{u}_0) = \frac{\pi}{2}[\frac{d}{\tau(\beta-1)}]^{\frac{1}{2}}$.

Proof.

$$t_1(u_0) = \sqrt{\frac{d}{2}}\int_0^{u_0}\left[\frac{\tau}{2}u^2 + \tau\beta(u\mathrm{e}^{-u}+\mathrm{e}^{-u}) + C_0\right]^{-\frac{1}{2}}\mathrm{d}u,$$

where

$$C_0 = -\frac{\tau}{2}u_0^2 - \tau\beta(u_0\mathrm{e}^{-u_0}+\mathrm{e}^{-u_0}).$$

Let $u = u_0 \sin x$, $x \in (0, \frac{\pi}{2})$, we obtain

$$t_1(u_0) = \sqrt{\frac{d}{2}} \int_0^{\frac{\pi}{2}} \left[-\frac{\tau}{2} u_0^2 \cos^2 x + \tau\beta(u_0 \sin x \mathrm{e}^{-u_0 \sin x} - u_0 \mathrm{e}^{-u_0}) + \tau\beta(\mathrm{e}^{-u_0 \sin x} - \mathrm{e}^{-u_0})\right]^{-\frac{1}{2}} \cdot u_0 \cos x \mathrm{d}x$$
$$= \sqrt{\frac{d}{2}} \int_0^{\frac{\pi}{2}} \left[-\frac{\tau}{2} + \frac{\tau\beta(u_0 \sin x \mathrm{e}^{-u_0 \sin x} - u_0 \mathrm{e}^{-u_0}) + \tau\beta(\mathrm{e}^{-u_0 \sin x} - \mathrm{e}^{-u_0})}{u_0^2 \cos^2 x}\right]^{-\frac{1}{2}} \mathrm{d}x. \tag{2.6}$$

Then by

$$\lim_{u_0 \to 0} \frac{u_0 \sin x \mathrm{e}^{-u_0 \sin x} - u_0 \mathrm{e}^{-u_0} + \mathrm{e}^{-u_0 \sin x} - \mathrm{e}^{-u_0}}{u_0^2 \cos^2 x} = \lim_{u_0 \to 0} \frac{\mathrm{e}^{-u_0} - \sin^2 x \mathrm{e}^{-u_0 \sin x}}{2\cos^2 x} = \frac{1}{2},$$

it follows that

$$\lim_{u_0 \to 0} t_1(u_0) = \frac{\pi}{2} \left[\frac{d}{\tau(\beta - 1)}\right]^{\frac{1}{2}}.$$

For the case $t_2(\bar{u}_0)$, the proof is similar, and then we omit it. □

Proposition 2.4. *$t_1(u_0)$ and $t_2(\bar{u}_0)$ is strictly increasing as a function of u_0 and $\bar{u}_0$, respectively.*

Proof. Denote

$$\Delta(u_0) = \frac{u_0 \sin x \mathrm{e}^{-u_0 \sin x} - u_0 \mathrm{e}^{-u_0} + \mathrm{e}^{-u_0 \sin x} - \mathrm{e}^{-u_0}}{u_0^2 \cos^2 x}. \tag{2.7}$$

By (2.6), it suffices to prove that $\Delta(u_0)$ is strictly decreasing as a function of u_0. We have

$$\Delta'(u_0) = \frac{\mathrm{e}^{-u_0} - \sin^2 x \mathrm{e}^{-u_0 \sin x}}{u_0 \cos^2 x} - 2\frac{u_0 \sin x \mathrm{e}^{-u_0 \sin x} - u_0 \mathrm{e}^{-u_0} + \mathrm{e}^{-u_0 \sin x} - \mathrm{e}^{-u_0}}{u_0^3 \cos^2 x}.$$

Let

$$A = \sin x,$$

and

$$F(A) = -A^2 \mathrm{e}^{-Au_0} + \mathrm{e}^{-u_0} - 2\left(\frac{A\mathrm{e}^{-Au_0}}{u_0} - \frac{\mathrm{e}^{-u_0}}{u_0}\right) - \frac{2}{u_0^2}(\mathrm{e}^{-Au_0} - \mathrm{e}^{-u_0})$$

with $A \in (0, 1)$, $u_0 > 0$. Then $\Delta'(u_0)$ can be expressed as

$$\Delta'(u_0) = \frac{F(A)}{u_0 \cos^2 x}.$$

Thus, we only need to verify that $F(A) < 0$ for any $u_0 > 0$. For a fixed u_0, clearly,

$$F(1) = 0, \qquad F'(A) = u_0 A^2 \mathrm{e}^{-Au_0} > 0.$$

Therefore, for any $u_0 > 0$, $F(A) < 0$ where $A \in (0, 1)$, which completes the proof. Similarly, we can conclude $t_2(\bar{u}_0)$ is strictly increasing as a function of $\bar{u}_0$. □

Proposition 2.5. $\lim_{u_0 \to \ln\beta} t_1(u_0) = \infty$.

Proof. First, we verify that $\Delta(u_0) \neq \frac{1}{2\beta}$ for any $u_0 \in [0, \ln\beta)$, where $\Delta(u_0)$ is defined by (2.7). We know that $\Delta(u_0)$ is strictly decreasing as a function of u_0 from the proof of Proposition 2.4. By the methods that we have used in the proof of above proposition, we can obtain that $\Delta(u_0)$ is strictly decreasing as a function of x for fixed u_0. Since for $u_0 \in [0, \ln\beta]$,

$$\lim_{x \to \frac{\pi}{2}} \Delta(u_0) = \lim_{x \to \frac{\pi}{2}} \frac{\mathrm{e}^{-u_0}}{2},$$

then we obtain

$$\lim_{x \to \frac{\pi}{2}} \Delta(\ln\beta) = \frac{1}{2\beta}.$$

This implies that $\Delta(u_0) > \frac{1}{2\beta}$ for $u_0 \in [0, \ln\beta)$. Thus, $t_1(u_0) < \infty$ for any $u_0 \in [0, \ln\beta)$.

Now we compute $t_1(\ln\beta)$. Let $\mu = \sin x$, then from (2.6) we derive that

$$t_1(\ln\beta) = \sqrt{\frac{2}{d}}\int_0^1 \left\{ \frac{(\ln\beta)^2}{-\frac{\tau}{2}(1-\mu^2)(\ln\beta)^2 + \tau\beta\ln\beta\mu\left(\frac{1}{\beta}\right)^{\mu} - \tau\ln\beta + \tau\beta\left(\frac{1}{\beta}\right)^{\mu} - \tau} \right\}^{\frac{1}{2}} \mathrm{d}\mu.$$

Hence, from

$$\lim_{\mu\to 1} \frac{-\frac{\tau}{2}(1-\mu^2)(\ln\beta)^2 + \tau\beta\ln\beta\mu\left(\frac{1}{\beta}\right)^{\mu} - \tau\ln\beta + \tau\beta\left(\frac{1}{\beta}\right)^{\mu} - \tau}{(1-\mu)^2(\ln\beta)^2} = \frac{\tau}{2}\ln\beta$$

and $\int_0^1 \frac{1}{1-\mu}\mathrm{d}\mu = \infty$ we have $t_1(\ln\beta) = \infty$. This completes the proof. □

Based on all above analysis, we have the following steady state bifurcation theorem for (2.1) and (2.2).

Theorem 2.6. *If $\beta \in (1, d/\tau + 1]$, then the problem (2.1) and (2.2) has no positive solution; If $\beta \in (d/\tau + 1, +\infty)$, then the problem (2.1) and (2.2) possesses a unique positive steady state.*

Remark 2.7. 1. Suppose that $\beta > 1$ is satisfied. If we treat d or τ as a parameter, we can also obtain the existence of positive steady state bifurcation.

2. The conclusion of Theorem 2.6 is a direct corollary of Proposition 2.3 of So and Yang [17]. But we here provide a new method to prove the conclusions.

Now, we consider Eq. (2.1) with the following boundary value condition

$$u(t,0) = u(t,\pi) = \ln\beta, \quad \text{for } t \ge 0. \tag{2.8}$$

Theorem 2.8. *The problem (2.1) and (2.8) has only one positive steady state solution given by $u = \ln\beta$.*

Proof. In order to satisfy the boundary conditions (2.8) we require that the trajectory starting on the line $u = \ln\beta$ at $x = 0$ should move back onto the $u = \ln\beta$ when $x = \pi$. In fact, from the first integral of (2.4) we have that the trajectory passing through the point $(\ln\beta, 0)$ is given by

$$\frac{d}{2}v^2 - \frac{\tau}{2}u^2 - \beta\tau(u\mathrm{e}^{-u} + \mathrm{e}^{-u}) = -\frac{\tau}{2}(\ln\beta)^2 - \tau(\ln\beta + 1). \tag{2.9}$$

It follows that

$$v^2 = \frac{2}{d}\left[\frac{\tau}{2}u^2 + \beta\tau(u\mathrm{e}^{-u} + \mathrm{e}^{-u}) - \frac{\tau}{2}(\ln\beta)^2 - \tau(\ln\beta + 1)\right], \tag{2.10}$$

and hence, $|v| \to +\infty$ as $u \to +\infty$. For any $u_* \in R$, the trajectory given by (2.10) intersects with the line $u = u_*$ two times at most. Let $(u(x), v(x))$ be the trajectory which starts on the line $u = \ln\beta$ when $x = 0$. There are two cases: $v(0) > 0$ and $v(0) < 0$. In the case of $v(0) > 0$, $u(x)$ is increasing as long as the trajectory stays in the first quadrant. From the uniqueness and the properties of the trajectory given by (2.10), we have that the trajectory of $(u(x), v(x))$ shall always stay in the first quadrant. This shows that the trajectory starting from the line $u = \ln\beta$ with $v(0) > 0$ cannot move back onto the line $u = \ln\beta$. When $v(0) < 0$, it follows that the orbit shall be outside the homoclinic orbit given by (2.10) (see Fig. 2.1), and come back to $u = \ln\beta$, but one part of $u(x)$ will be negative. So there is still no positive steady state solution with $v(0) < 0$. The proof is complete. □

3. Analysis of stability and bifurcation

In this section, we shall carry out the analysis of stability and the existence of Hopf bifurcation of Eq. (2.1). Clearly, $u^* = \ln\beta$ is the unique nontrivial equilibrium of Eq. (2.1) when $\beta \ne 1$, u^* is positive when $\beta > 1$, and u^* is negative when $0 < \beta < 1$. We first transform the fixed point $u = u^*$ of Eq. (2.1) to the origin via the translation $\widehat{u} = u - u^*$ and drop the hats for simplicity of notation, then Eq. (2.1) is transformed into

$$\frac{\partial u}{\partial t}(t,x) = d\triangle u(t,x) - \tau u(t,x) - \tau\ln\beta + \tau u(t-1,x)\mathrm{e}^{-u(t-1,x)} + \tau\ln\beta\mathrm{e}^{-u(t-1,x)}. \tag{3.1}$$

Then we consider Eq. (3.1) with the Dirichlet boundary condition

$$u(t,0) = u(t,\pi) = 0.$$

Denote $X = L^2[0, \pi]$ as the Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and $\mathcal{C} = C([-1, 0], X)$ with the sup norm a Banach space. Then in the abstract space $\mathcal{C}$ this equation is

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t) = d\triangle u(t) - \tau u(t) - \tau \ln \beta + \tau u(t-1)\mathrm{e}^{-\phi(t-1)} + \tau \ln \beta \mathrm{e}^{-u(t-1)}. \tag{3.2}$$

In X, the sequence of eigenvalues of $d\triangle$ is $\{-dk^2\}_{k=1}^{\infty}$, with normalized eigenfunctions $\beta_k(x) = \sqrt{\frac{2}{\pi}} \sin kx$. The set $\{\beta_k\}$ is an orthonormal basis for X. The linearized equation about the equilibrium point zero is

$$\frac{\mathrm{d}}{\mathrm{d}t} \phi(t) = d\triangle \phi(t) - \tau \phi(t) - \tau \ln \beta \phi(t-1) + \tau \phi(t-1), \quad \phi \in \mathcal{C}, \tag{3.3}$$

with characteristic equations

$$\lambda + dk^2 + \tau + (\tau \ln \beta - \tau)\mathrm{e}^{-\lambda} = 0, \quad (k = 1, 2, \ldots). \tag{1k}$$

Let $\lambda_{1,2} = \pm \mathrm{i}\omega$ be solutions of Eq. (1k), then we have

$$\begin{cases} dk^2 + \tau + (\tau \ln \beta - \tau) \cos \omega = 0, \quad k = 1, 2, \ldots, \\ \omega - (\tau \ln \beta - \tau) \sin \omega = 0, \end{cases} \tag{3.4}$$

which leads to

$$\tan \omega = -\frac{\omega}{dk^2 + \tau}, \quad k = 1, 2, \ldots, \tag{2k}$$

$$\beta = \exp\left(1 - \frac{dk^2}{\tau \cos \omega} - \frac{1}{\cos \omega}\right).$$

Let

$$\omega_j^{(k)} \in \left(\left(2j + \frac{1}{2}\right)\pi, (2j+1)\pi\right) \quad \text{and} \quad \omega_{-j}^{(k)} \in \left(\left(2j + \frac{3}{2}\right)\pi, 2(j+1)\pi\right)$$

be the root of Eq. (2k), $k = 1, 2, \ldots; j = 0, 1, 2, \ldots$, and define

$$\beta_j^{(k)} = \exp\left(1 - \frac{dk^2}{\tau \cos \omega_j^{(k)}} - \frac{1}{\cos \omega_j^{(k)}}\right), \quad k = 1, 2, \ldots; j = 0, 1, 2, \ldots,$$

and

$$\beta_{-j}^{(k)} = \exp\left(1 - \frac{dk^2}{\tau \cos \omega_{-j}^{(k)}} - \frac{1}{\cos \omega_{-j}^{(k)}}\right), \quad k = 1, 2, \ldots; j = 0, 1, 2, \ldots.$$

Then we know that $\pm \mathrm{i}\omega_j^{(k)}$ (resp. $\pm\, \mathrm{i}\omega_{-j}^{(k)}$) are the purely imaginary roots of Eq. (1k) with $\beta = \beta_j^{(k)}$ (resp. $\beta = \beta_{-j}^{(k)}$), and Eq. (1k) has no other purely imaginary root. It is not difficult to verify that $\beta_j^{(k+1)} > \beta_j^{(k)}, \beta_{j+1}^{(k)} > \beta_j^{(k)}$ and $\beta_j^{(k)} > e^2$ for $k = 1, 2, \ldots; j = 0, 1, 2, \ldots$. We reorder $\bigcup_{k=1}^{\infty} \{\beta_j^{(k)}\}_{j=0}^{\infty}$ as $\{\beta_0, \beta_1, \beta_2, \ldots\}$, so that $\beta_m \le \beta_{m+1}$, $m \ge 0$. Clearly, $\beta_0 = \beta_0^{(1)}$. Similarly, it is not difficult to verify that $\beta_{-j}^{(k+1)} < \beta_{-j}^{(k)}, \beta_{-(j+1)}^{(k)} < \beta_{-j}^{(k)}$ and $\beta_{-j}^{(k)} < 1$ for $k = 1, 2, \ldots; j = 0, 1, 2, \ldots$. We reorder $\bigcup_{k=1}^{\infty} \{\beta_{-j}^{(k)}\}_{j=0}^{\infty}$ as $\{\beta_{-0}, \beta_{-1}, \beta_{-2}, \ldots\}$, so that $\beta_{-m} \ge \beta_{-(m+1)}$, $m \ge 0$. Clearly, $\beta_{-0} = \beta_{-0}^{(1)}$. Let $\lambda(\beta) = \gamma(\beta) + \mathrm{i}\omega(\beta)$ be the root of Eq. (1k) satisfying $\gamma(\beta_j^{(k)}) = 0$ (resp. $\gamma(\beta_{-j}^{(k)}) = 0$) and $\omega(\beta_j^{(k)}) = \omega_j^{(k)}$ (resp. $\omega(\beta_{-j}^{(k)}) = \omega_{-j}^{(k)}$) when β is close to $\beta_j^{(k)}$ (resp. $\beta_{-j}^{(k)}$). Then we have the following transversality result:

Lemma 3.1. *$\gamma'(\beta_j^{(k)}) > 0$ and $\gamma'(\beta_{-j}^{(k)}) < 0$.*

Proof. Substituting $\lambda(\beta)$ into Eq. (1k) and taking the derivative associated with β, then replacing β by $\beta_j^{(k)}$ yields

$$\begin{aligned} \gamma'(\beta_j^{(k)}) &= \mathrm{Re}\left[\left.\frac{-\tau \mathrm{e}^{-\lambda} \frac{1}{\beta}}{1 - (\tau \ln \beta - \tau)\mathrm{e}^{-\lambda}}\right|_{\beta = \beta_j^{(k)}}\right] \\ &= \frac{\tau}{\beta} \mathrm{Re}\left[\frac{-\cos \omega_j^{(k)} + i \sin \omega_j^{(k)}}{1 - (\tau \ln \beta_j^{(k)} - \tau)\cos \omega_j^{(k)} + (\tau \ln \beta_j^{(k)} - \tau) \sin \omega_j^{(k)}}\right] \end{aligned}$$

$$= \frac{\tau}{\beta} \frac{\tau \ln \beta_j^{(k)} - \tau - \cos \omega_j^{(k)}}{[1 - (\tau \ln \beta_j^{(k)} - \tau) \cos \omega_j^{(k)}]^2 + (\tau \ln \beta_j^{(k)} - \tau)^2 \sin^2 \omega_j^{(k)}}$$

$$= \frac{\tau}{\beta_j^{(k)}} \frac{\frac{\omega_j^{(k)}}{\sin \omega_j^{(k)}} - \cos \omega_j^{(k)}}{[1 - (\tau \ln \beta_j^{(k)} - \tau) \cos \omega_j^{(k)}]^2 + (\tau \ln \beta_j^{(k)} - \tau)^2 \sin^2 \omega_j^{(k)}}.$$

Obviously, $\gamma'(\beta_j^{(k)}) > 0$, as $\sin \omega_j^{(k)} > 0$ and $\cos \omega_j^{(k)} < 0$.

Similarly, one can obtain the second inequality. □

It is easy to verify that $\lambda = 0$ is a root of Eq. (1k) when

$$\beta =: \bar{\beta}_k = e^{-\frac{dk^2}{\tau}}.$$

Let $\lambda = \lambda(\beta)$ be one of the roots of Eq. (1k) satisfying $\lambda(\bar{\beta}_k) = 0$. By substituting $\lambda(\beta)$ into Eq. (1k) and taking the derivative with respect to β yields

$$\left.\frac{d\lambda(\beta)}{d\beta}\right|_{\beta=\bar{\beta}_k} = -\frac{\tau}{\bar{\beta}_k[1 + (dk^2 + \tau)]} < 0.$$

Obviously, $\bar{\beta}_{k+1} < \bar{\beta}_k < 1$ for $k \geq 1$. From $\cos \omega_{-j}^{(1)} > 0$ it follows that

$$\frac{d}{\tau} < \frac{d}{\tau \cos \omega_{-j}^{(1)}} + \frac{1}{\cos \omega_{-j}^{(1)}} - 1.$$

This implies that

$$\bar{\beta}_1 = e^{-\frac{d}{\tau}} > \exp\left(1 - \frac{d}{\tau \cos \omega_{-j}^{(1)}} - \frac{1}{\cos \omega_{-j}^{(1)}}\right) = \beta_{-j}^{(1)}.$$

Denote

$$\omega^* = \omega_0^{(1)}, \beta^* = \beta_0, \text{and } \beta_* = \bar{\beta}_1.$$

From Lemma 3.1 and above analysis, we have the following conclusions on the distribution of the roots of Eq. (1k).

Lemma 3.2. (i) *If $\beta \in (\beta_*, \beta^*)$, then all roots of Eq. (1k)$(k \geq 1)$ have negative real parts.*

(ii) *Eq. (1k) $(k \geq 1)$ have purely imaginary roots if and only if $\beta = \beta_m$ or $\beta = \beta_{-m}$, $m = 0, 1, 2, \ldots$. When $\beta = \beta^*$, all the roots of Eq. (1k), $(k \geq 1)$, except $\pm i\omega^*$, have negative real parts.*

(iii) *If $\beta > \beta^*$, then Eq. (1k) $(k \geq 1)$ have at least a pair of roots with positive real parts; If $\beta < \beta_*$, then Eq. (1k) $(k \geq 1)$ have at least a positive root.*

Proof. First, it is easy to verify that all roots of all the equations (1k) have negative real parts when $\beta = e$, for any $k \geq 1$. From the definition of β_m and β_{-m}, Eq. (1k) $(k \geq 1)$ have purely imaginary roots if and only if $\beta = \beta_m$ or $\beta = \beta_{-m}$. In addition, by the definitions of $\beta^* = \beta_0$ and $\beta_* = \bar{\beta}_1 > \beta_{-0}$, we know that β^* is the smallest value of $\beta > e$ such that Eq. (1k) have roots appearing on the imaginary axis, and β_* is also the largest value of $\beta < e$ such that Eq. (1k) have a root appearing on the imaginary axis. Hence, by the a result of Ruan and Wei [31, Corollary 2.4], we have that all roots of Eq. (1k) have negative real parts for $\beta \in (\beta_*, \beta^*)$, and all the roots of Eq. (1k), $(k \geq 1)$, except $\pm i\omega^*$, have negative real parts when $\beta = \beta^*$. Since $\gamma'(\beta_j^{(k)}) > 0$, Eq. (1k) $(k \geq 1)$ have at least a pair of roots with positive real parts for $\beta \in (\beta^*, \infty)$. Similarly, since $\lambda'(\bar{\beta}_k) < 0$ and $\gamma'(\beta_{-j}^{(k)}) < 0$, Eq. (1k) $(k \geq 1)$ have at least one positive root for $\beta \in (0, \beta_*)$. □

Applying Lemmas 3.1 and 3.2, we have the following result on the dynamics of Eq. (3.1).

Theorem 3.3. *If $\beta \in (\beta_*, \beta^*)$, then the zero solution of Eq. (3.1) is asymptotically stable, and unstable when $\beta > \beta^*$ or $\beta < \beta_*$, as well as Eq. (3.1) undergoes a Hopf bifurcation at the origin when $\beta = \beta^*$.*

Theorem 3.3 shows that the stability of the nontrivial equilibrium of Eq. (2.1) and the existence of Hopf bifurcation as the parameter β varies. Next we discuss the effect of dispersion on the stability.

When the diffusion coefficient $d = 0$, then Eq. (2.1) is reduced to Eq. (1.2), and its characteristic equation is (1k) with $k = 0$ given by

$$\lambda + \tau + (\tau \ln \beta - \tau)e^{-\lambda} = 0. \tag{3.5}$$

Let $\lambda_{1,2} = \pm i\omega$ be solutions of Eq. (3.5), then we have

$$\begin{cases} \tau + (\tau \ln \beta - \tau) \cos \omega = 0, \\ \omega - (\tau \ln \beta - \tau) \sin \omega = 0, \end{cases} \tag{3.6}$$

which leads to

$$\tan \omega = -\frac{\omega}{\tau}, \qquad \beta = \exp\left(1 - \frac{1}{\cos \omega}\right). \tag{3.7}$$

Then, we can obtain the following conclusions.

Lemma 3.4. *There exists a sequence of values of β denoted by*

$$\ldots, \beta_{-1}^{(0)}, \beta_{-0}^{(0)}, \beta_0^{(0)}, \beta_1^{(0)}, \ldots,$$

such that all roots of Eq. (3.5) have negative real parts when $\beta \in (1, \beta_0^{(0)})$, and Eq. (3.5) has at least a root with positive real part when

$$\beta \in (0, 1) \cup (\beta_0^{(0)}, \infty),$$

where

$$\beta_j^{(0)} = \exp\left(1 - \frac{1}{\cos \omega_j}\right), \qquad \beta_{-j}^{(0)} = \exp\left(1 - \frac{1}{\cos \omega_{-j}}\right),$$

and

$$\omega_j \in \left(\left(2j + \frac{1}{2}\right)\pi, (2j+1)\pi\right) \quad \text{and} \quad \omega_{-j} \in \left(\left(2j + \frac{3}{2}\right)\pi, 2(j+1)\pi\right), \quad (j = 0, 1, 2, \ldots)$$

are the root of Eq. (3.7).

Clearly, $\beta_* < 1 < \beta_0^{(0)} < \beta^*$. By combining Theorem 3.3 and Lemma 3.4, we can obtain the following results on the effect of dispersion on the stability.

Theorem 3.5. *When $\beta \in (\beta_*, 1) \cup (\beta_0^{(0)}, \beta^*)$, $u = \ln \beta$ is an asymptotically stable nontrivial equilibrium for Eq. (1.2), and an unstable one for the problem (2.1) and (2.8).*

4. Direction and stability of the Hopf bifurcation

In this section, we shall study the direction of the Hopf bifurcation and stability of the bifurcating periodic solutions by employing the center manifold theorem due to Lin, So and Wu [25] and normal form method due to Faria [26] for partial differential equations with delay. To study the qualitative behavior near the critical point

$$\beta^* = \exp\left(1 - \frac{d}{\tau \cos \omega^*} - \frac{1}{\cos \omega^* w}\right),$$

we change the parameter β by taking $\beta = \beta^* + \alpha$. Then Eq. (1k) has a pair of eigenvalues $\lambda(\alpha)$, $\overline{\lambda(\alpha)}$, $\lambda(\alpha) = \gamma(\alpha) + i\sigma(\alpha)$ of class C^1, with $\lambda(0) = i\omega^*$ and $\gamma'(0) > 0$ by Lemma 3.1.

We use the same notations as in Faria et al. [26]. Let $\Lambda = \{i\omega^*, -i\omega^*\}$, for $u \in \mathcal{C}$, define

$$L(u) = -\tau u(0) - (\tau \ln \beta^* - \tau) u(-1)$$

and

$$\begin{aligned} F(u, \alpha) &= \tau \ln(\beta^* + \alpha) e^{-u(t-1)} + \tau u(t-1) e^{-u(t-1)} - \tau \ln(\beta^* + \alpha) + (\tau \ln \beta^* - \tau) u(-1) \\ &= -\frac{\tau}{\beta^*} \alpha u(-1) + \frac{\tau \ln \beta^*}{2} u^2(-1) - \tau u^2(-1) - \frac{\tau \ln \beta^*}{3!} u^3(-1) + \frac{\tau}{2} u^3(-1) + \mathcal{O}(|u|^3 + |(\alpha, u)|^3). \end{aligned}$$

The Eq. (3.2) is written as

$$\frac{d}{dt} u(t) = d \Delta u(t) + L(u_t) + F(u_t, \alpha). \tag{4.1}$$

From the definition of the associated FDE, (see [26] Definition 4.1), the FDE associated with (3.2) by Λ at the equilibrium point $u = 0$ is given by

$$\dot{x}(t) = -(d+\tau)x(t) - (\tau \ln \beta^* - \tau)x(t-1) + \langle F(x_t\beta_1, \alpha), \beta_1\rangle, \quad x \in C([-1, 0]; \mathbb{R}). \tag{4.2}$$

Defining

$$R(x_t) = -(d+\tau)x(t) - (\tau \ln \beta^* - \tau)x(t-1),$$

Eq. (4.2) is written in the following form

$$\dot{x} = R(x_t) + \langle F(x_t\beta_1, \alpha), \beta_1\rangle. \tag{4.3}$$

From [27–29] and [26], we obtain $\dim \mathscr{P} = \dim P_0 = 2$, and $P_0 = \operatorname{span} \Phi$, where

$$\Phi(\theta) = (\phi_1(\theta), \phi_2(\theta)), \quad \text{with } \phi_1(\theta) = \mathrm{e}^{\mathrm{i}\omega^*\theta},\ \phi_2(\theta) = \mathrm{e}^{-\mathrm{i}\omega^*\theta},$$

$\Psi(0) = \binom{\psi_1(0)}{\psi_2(0)}$, with $\psi_1(0) = \overline{\psi_2(0)} = [1 - \tau(\ln \beta^* - 1)\mathrm{e}^{-\mathrm{i}\omega^*}]^{-1}$, where the bar denotes the complex conjugation.

In $B\mathcal{C} = \mathscr{P} \oplus \mathbf{Ker}\,\pi$ which is decomposed by Λ, then Eq. (4.1) becomes

$$\begin{cases} \dot{z} = Bz + \Psi(0)\langle F(\Phi z\beta_1 + y, \alpha), \beta_1\rangle\ , \\ \dfrac{\mathrm{d}}{\mathrm{d}t}y = A_1 y + (I - \pi)X_0 F(\Phi z\beta_1 + y, \alpha), \quad z \in \mathbb{C}^2,\ y \in \mathscr{Q}^1 \end{cases} \tag{4.4}$$

and in $BC = P \oplus \mathbf{Ker}\,\pi_0$, Eq. (4.2) becomes

$$\begin{cases} \dot{z} = Bz + \Psi(0)\langle F(\Phi z\beta_1 + y, \alpha), \beta_1\rangle, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}y = A_{0,1} y + (I - \pi_0)X_0\langle F((\Phi z + y)\beta_1, \alpha), \beta_1\rangle, \quad z \in \mathbb{C}^2,\ y \in Q^1 \end{cases} \tag{4.5}$$

where $B = \operatorname{diag}(\mathrm{i}\omega^*, -\mathrm{i}\omega^*)$.

The existence of a two-dimensional local center manifold for Eq. (4.1) tangent to $\mathscr{P}$ at $u = 0,\ \alpha = 0$ follows from [25]. Let

$$\dot{z} = Bz + \frac{1}{2}g_2^1(z, 0, \alpha) + \frac{1}{3!}g_3^1(z, 0, \alpha) + \cdots, \quad z \in \mathbb{C}^2, \tag{4.6}$$

be the normal form of Eq. (4.1) on the center manifold [26], and

$$\dot{z} = Bz + \frac{1}{2}g_{0,2}^1(z, 0, \alpha) + \frac{1}{3!}g_{0,3}^1(z, 0, \alpha) + \cdots, \quad z \in \mathbb{C}^2, \tag{4.7}$$

be the normal form of Eq. (4.2) on the center manifold [27].

Theorem 4.1.

$$g_3^1(z, 0, \alpha) = g_{0,3}^1(z, 0, 0) + \begin{pmatrix} cz_1^2 z_2 \\ \bar{c} z_1 z_2^2 \end{pmatrix} + \mathcal{O}(|z|\alpha^2),$$

where

$$c = 3\psi_1(0)(\tau \ln \beta^* - 2\tau)\sum_{k>1} c_k^2 \left[\mathrm{e}^{-\mathrm{i}\omega^*}\frac{2(\tau \ln \beta^* - 2\tau)}{dk^2 + \tau \ln \beta^*} + \mathrm{e}^{-3\mathrm{i}\omega^*}\frac{\tau \ln \beta^* - 2\tau}{2\mathrm{i}\omega^* + dk^2 + \tau + (\tau \ln \beta^* - \tau)\mathrm{e}^{-2\mathrm{i}\omega^*}}\right], \tag{4.8}$$

the "bar" denotes the complex conjugation, and the c_k's are given by the following expression:

$$c_k = \langle \beta_1\beta_k, \beta_1\rangle = \begin{cases} 0, & \text{if } k \text{ even}, \\ -\left(\dfrac{2}{\pi}\right)^{\frac{3}{2}} \dfrac{4}{k(k^2-4)}, & \text{if } k \text{ odd}. \end{cases}$$

Proof. From [26]

$$\bar{f}_3^1(z, 0, \alpha) = \bar{f}_{0,3}^1(z, 0, \alpha) + \frac{3}{2}\Psi(0)\langle D_1 F_2(\Phi z\beta_1, \alpha)\sum_{k>1} h_k(z, \alpha)\beta_k, \beta_1\rangle, \tag{4.9}$$

where $h(z, \alpha) = \sum_{k\geq 1} h_k(z, \alpha)\beta_k$, $h(z, \alpha)$ is the unique solution of

$$(M_2^2 h)(z, \alpha) = f_2^2(z, 0, \alpha),$$

and

$$F_2(u,\alpha)=\tau\ln\beta^* u^2(-1)-\frac{2\tau\alpha}{\beta^*}u(-1)-2\tau u^2(-1).$$

From the definition of F_2 and $f_2^2(z,0,0)$, we have

$$\begin{aligned}
&D_1F_2(v,\alpha)(u)=-\frac{2\tau\alpha}{\beta^*}u(-1)+2(\tau\ln\beta^*-2\tau)u(-1)v(-1),\\
&\langle D_1F_2(\Phi z\beta_1,0)(\psi\beta_k),\beta_1\rangle=\langle 2(\tau\ln\beta^*-2\tau)(\mathrm{e}^{-\mathrm{i}\omega^*}z_1+\mathrm{e}^{\mathrm{i}\omega^*}z_2)\beta_1\psi(-1)\beta_k,\beta_1\rangle\\
&\qquad=2(\tau\ln\beta^*-2\tau)(\mathrm{e}^{-\mathrm{i}\omega^*}z_1+\mathrm{e}^{\mathrm{i}\omega^*}z_2)\psi(-1)\cdot c_k,
\end{aligned}\tag{4.10}$$

$$\begin{aligned}
&f_2^2(z,0,0)=-\Phi\Psi(0)\langle F_2(\Phi z\beta_1,0),\beta_1\rangle\beta_1+X_0F_2(\Phi z\beta_1,0),\\
&\langle f_2^2(z,0,0),\beta_k\rangle=\langle X_0(\tau\ln\beta^*-2\tau)(\mathrm{e}^{-\mathrm{i}\omega^*}z_1+\mathrm{e}^{\mathrm{i}\omega^*}z_2)^2{\beta_1}^2,\beta_k\rangle\\
&\qquad=X_0(\tau\ln\beta^*-2\tau)(\mathrm{e}^{-\mathrm{i}\omega^*}z_1+\mathrm{e}^{\mathrm{i}\omega^*}z_2)^2\cdot c_k,\quad k>1.
\end{aligned}$$

Now we need to compute $h_k(z,0)$, by solving the equation $(M_2^2h)(z,0)=f_2^2(z,0,0)$. The definition of M_2^2 (see [26]) leads to

$$\begin{cases}D_zh_k(z,0)Bz-\dot h_k(z,0)=0,\\ \dot h_k(z,0)(0)+(dk^2+\tau)h_k(z,0)(0)+(\tau\ln\beta^*-\tau)h_k(z,0)(-1)\\ \quad=(\tau\ln\beta^*-2\tau)(\mathrm{e}^{-\mathrm{i}\omega^*}z_1+\mathrm{e}^{\mathrm{i}\omega^*}z_2)^2,\end{cases}\tag{4.11$_k$}$$

where $k>1$ and $\dot h_k(z,0)(0)=\frac{d}{d\theta}h_k(z,0)(\theta)|_{\theta=0.}$ For each $k>1$, it is easy to solve (4.11$_k$) by setting $h_k(z,0)(\theta)=\sum_{|q|=2}h_{q,k}(\theta)z^q$,

$$\begin{aligned}h_k(z,0)(\theta)=c_k\Bigg[&\frac{\tau\ln\beta^*-2\tau}{2\mathrm{i}\omega^*+dk^2+\tau+(\tau\ln\beta^*-\tau)\mathrm{e}^{-2\mathrm{i}\omega^*}}z_1^2+\frac{\tau\ln\beta^*-2\tau}{-2\mathrm{i}\omega^*+dk^2+\tau+(\tau\ln\beta^*-\tau)\mathrm{e}^{2\mathrm{i}\omega^*}}z_2^2\\&+\frac{2(\tau\ln\beta^*-2\tau)}{dk^2+\tau\ln\beta^*}z_1z_2\Bigg].\end{aligned}$$

By using (4.9) and (4.10), we obtain

$$c=3\psi_1(0)(\tau\ln\beta^*-2\tau)\sum_{k>1}c_k^2\left[\mathrm{e}^{-\mathrm{i}\omega^*}\frac{2(\tau\ln\beta^*-2\tau)}{dk^2+\tau\ln\beta^*}+\mathrm{e}^{-3\mathrm{i}\omega^*}\frac{\tau\ln\beta^*-2\tau}{2\mathrm{i}\omega^*+dk^2+\tau+(\tau\ln\beta^*-\tau)\mathrm{e}^{-2\mathrm{i}\omega^*}}\right].\quad\square$$

For Eq. (3.2), the normal form on the center manifold is written in polar coordinates (ρ,ξ) as

$$\begin{cases}\dot\rho=\gamma'(0)\alpha\rho+K\rho^3+\mathcal{O}(\alpha^2\rho+|(\rho,\alpha)|^4),\\ \dot\xi=-\mathrm{i}\omega^*+\mathcal{O}(|(\rho,\alpha)|).\end{cases}\tag{4.12}$$

Denote $K^*=\frac{1}{3!}\mathrm{Re}\,g_{0,3}^1(z,0,0)$, from [26], we have $K=K^*+\frac{1}{3!}\mathrm{Re}\,c$, where c is defined by (4.8). Now we compute K^*, using [27]. Since, for $v\in C([0,1],R)$,

$$R(v)=-dv(0)-\tau v(0)-(\tau\ln\beta^*-\tau)v(-1).$$

Thus

$$R(1)=-d-\tau\ln\beta^*,$$

$$R(\mathrm{e}^{2\mathrm{i}\omega^*\theta})=-d-\tau-(\tau\ln\beta^*-\tau)\mathrm{e}^{-2\mathrm{i}\omega^*}.$$

For $v\in C([0,1],R),\ \alpha\in R$ denote

$$\widetilde F(v,\alpha)=\langle F(v\beta_1,\alpha),\beta_1\rangle.$$

Then

$$\widetilde F(v,\alpha)=-\frac{\tau\alpha}{\beta^*}v(-1)+\frac{4}{3}\left(\frac{2}{\pi}\right)^{\frac32}\left(\frac{\tau\ln\beta^*}{2}-\tau\right)v^2(-1)+\frac{3}{2\pi}\left(\frac{\tau}{2}-\frac{\tau\ln\beta^*}{6}\right)v^3(-1)+\mathcal{O}(|\alpha|^2+|(v,\alpha)|^3).$$

And hence

$$\begin{aligned}&\widetilde F(x_1\mathrm{e}^{\mathrm{i}\omega^*\theta}+x_2\mathrm{e}^{-\mathrm{i}\omega^*\theta}+x_3\cdot 1+x_4\mathrm{e}^{2\mathrm{i}\omega^*\theta},0)\\
&\quad=\frac{4}{3}\left(\frac{2}{\pi}\right)^{\frac32}\left(\frac{\tau\ln\beta^*}{2}-\tau\right)(x_1\mathrm{e}^{-\mathrm{i}\omega^*}+x_2\mathrm{e}^{\mathrm{i}\omega^*}+x_3\cdot1+x_4\mathrm{e}^{-2\mathrm{i}\omega^*})^2\\
&\qquad+\frac{3}{2\pi}\left(\frac{\tau}{2}-\frac{\tau\ln\beta^*}{6}\right)(x_1\mathrm{e}^{-\mathrm{i}\omega^*}+x_2\mathrm{e}^{\mathrm{i}\omega^*}+x_3\cdot1+x_4\mathrm{e}^{-2\mathrm{i}\omega^*})^3+\cdots\\
&\quad:=B_{(2,0,0,0)}x_1^2+B_{(1,1,0,0)}x_1x_2+B_{(1,0,1,0)}x_1x_3+B_{(0,1,0,1)}x_2x_4+B_{(2,1,0,0)}x_1^2x_2+\cdots.\end{aligned}$$

By comparing the corresponding coefficients, we obtain

$$B_{(2,1,0,0)} = \frac{3}{2\pi}\left(\frac{\tau}{2} - \frac{\tau \ln \beta^*}{6}\right) e^{-i\omega^*},$$

$$B_{(1,1,0,0)} = \frac{4}{3}\left(\frac{2}{\pi}\right)^{\frac{3}{2}}\left(\frac{\tau \ln \beta^*}{2} - \tau\right),$$

$$B_{(0,1,0,1)} = B_{(1,0,1,0)} = \frac{4}{3}\left(\frac{2}{\pi}\right)^{\frac{3}{2}}\left(\frac{\tau \ln \beta^*}{2} - \tau\right) e^{-i\omega^*},$$

$$B_{(2,0,0,0)} = \frac{4}{3}\left(\frac{2}{\pi}\right)^{\frac{3}{2}}\left(\frac{\tau \ln \beta^*}{2} - \tau\right) e^{-2i\omega^*}.$$

From [27],

$$\begin{aligned} K^* &= \mathrm{Re}\left[\frac{1}{1 - R(\theta e^{i\omega^*\theta})}(B_{(2,1,0,0)} - \frac{B_{(1,1,0,0)}B_{(1,0,1,0)}}{R(1)} + \frac{B_{(2,0,0,0)}B_{(0,1,0,1)}}{2i\omega^* - R(e^{2i\omega^*\theta})})\right] \\ &= \mathrm{Re}\left[\psi_1(0)(B_{(2,1,0,0)} - \frac{B_{(1,1,0,0)}B_{(1,0,1,0)}}{R(1)} + \frac{B_{(2,0,0,0)}B_{(0,1,0,1)}}{2i\omega^* - R(e^{2i\omega^*\theta})})\right] \\ &= \mathrm{Re}\left\{\psi_1(0)\left[\frac{3}{2\pi}\left(\frac{\tau}{2} - \frac{\tau \ln \beta^*}{6}\right)e^{-i\omega^*} + \frac{\frac{16}{9}\left(\frac{2}{\pi}\right)^3\left(\frac{\tau \ln \beta^*}{2} - \tau\right)^2 e^{-i\omega^*}}{d + \tau \ln \beta^*} + \frac{\frac{16}{9}\left(\frac{2}{\pi}\right)^3\left(\frac{\tau \ln \beta^*}{2} - \tau\right)^2 e^{-3i\omega^*}}{2i\omega^* + d + \tau + (\tau \ln \beta^* - \tau)e^{-2i\omega^*}}\right]\right\}. \end{aligned}$$

Set

$$\begin{aligned} m &= [d + \tau + (\tau \ln \beta^* - \tau)\cos 2\omega^*]^2 + [2\omega^* - (\tau \ln \beta^* - \tau)\sin 2\omega^*]^2, \\ m_k &= [dk^2 + \tau + (\tau \ln \beta^* - \tau)\cos 2\omega^*]^2 + [2\omega^* - (\tau \ln \beta^* - \tau)\sin 2\omega^*]^2, \\ n &= [1 - (\tau \ln \beta^* - \tau)\cos \omega^*]^2 + (\tau \ln \beta^* - \tau)^2 \sin^2 \omega^*, \end{aligned}$$

then

$$\begin{aligned} K^* = {} & \frac{1 - (\tau \ln \beta^* - \tau)\cos \omega^*}{n(d + \tau \ln \beta^*)}\left[\frac{3}{2\pi}(d + \tau \ln \beta^*)\left(\frac{\tau}{2} - \frac{\tau \ln \beta^*}{6}\right) + \frac{16}{9}\left(\frac{2}{\pi}\right)^3\left(\frac{\tau \ln \beta^*}{2} - \tau\right)^2\right]\cos \omega^* \\ & - \frac{(\tau \ln \beta^* - \tau)^2}{n(d + \tau \ln \beta^*)}\left[\frac{3}{2\pi}(d + \tau \ln \beta^*)\left(\frac{\tau}{2} - \frac{\tau \ln \beta^*}{6}\right) + \frac{16}{9}\left(\frac{2}{\pi}\right)^3\left(\frac{\tau \ln \beta^*}{2} - \tau\right)^2\right]\sin^2 \omega^* \\ & + \frac{1}{mn}\frac{16}{9}\left(\frac{2}{\pi}\right)^3\left(\frac{\tau \ln \beta^*}{2} - \tau\right)^2\Big[\{\cos 3\omega^*[d + \tau + (\tau \ln \beta^* - \tau)\cos 2\omega^*] \\ & - [2\omega^* - (\tau \ln \beta^* - \tau)\sin 2\omega^*]\sin 3\omega^*\}[1 - (\tau \ln \beta^* - \tau)\cos \omega^*] \\ & - (\tau \ln \beta^* - \tau)\sin \omega^*\{\cos 3\omega^*[2\omega^* - (\tau \ln \beta^* - \tau)\sin 2\omega^*] \\ & + [d + \tau + (\tau \ln \beta^* \tau - \tau)\cos 2\omega^*]\sin 3\omega^*\}\Big]. \end{aligned}$$

Thus, from this and (4.8) it follows that

$$\begin{aligned} K = {} & K^* + \frac{1}{3!}\mathrm{Re}\, c = K^* + \frac{\tau \ln \beta^* - 2\tau}{2} \\ & \times \sum_{k>1} c_k^2 \Bigg[\frac{2(\tau \ln \beta^* - 2\tau)[1 - (\tau \ln \beta^* - \tau)\cos \omega^*]\cos \omega^*}{n(dk^2 + \tau \ln \beta^*)} - \frac{2(\tau \ln \beta^* - 2\tau)(\tau \ln \beta^* - \tau)\sin^2 \omega^*}{n(dk^2 + \tau \ln \beta^*)} \\ & + \frac{\tau \ln \beta^* - 2\tau}{m_k n}\{\cos 3\omega^*[dk^2 + \tau + (\tau \ln \beta^* - \tau)\cos 2\omega^*][1 - (\tau \ln \beta^* - \tau)\cos \omega^*] \\ & - \sin 3\omega^*[2\omega^* - (\tau \ln \beta^* - \tau)\sin 2\omega^*][1 - (\tau \ln \beta^* - \tau)\cos \omega^*] \\ & - \sin \omega^* \cos 3\omega^*(\tau \ln \beta^* - \tau)[2\omega^* - (\tau \ln \beta^* - \tau)\sin 2\omega^*] \\ & - \sin \omega^* \sin 3\omega^*(\tau \ln \beta^* - \tau)[dk^2 + \tau + (\tau \ln \beta^* - \tau)\cos 2\omega^*]\}\Bigg]. \end{aligned} \tag{4.13}$$

We have shown that all roots of the characteristic equations $(1k)$, except $\pm i\omega^*$, have negative real parts when $\beta = \beta^*$. Hence we have the following result on the properties of the Hopf bifurcation.

Theorem 4.2. *For Eq. (3.2) a generic Hopf bifurcation occurs from $u = 0$, $\beta = \beta^*$. The direction of the bifurcation is $\beta > \beta^*$ (resp. $\beta < \beta^*$) and the bifurcating periodic solutions are stable (resp. unstable) if $K < 0$ (resp. > 0).*

So far, we have derived the formula (4.13) for determining the properties of the Hopf bifurcation occurring at the positive equilibrium when $\beta = \beta^*$ to Eq. (2.1). To illustrate the analytical results found, we will consider some particular cases of Eq. (3.2). We choose the coefficients as follows: $\tau = 2$, $d = 0.7$, then $\omega^* = 2.4124$, $\beta^* = 16.6163$. Furthermore, we obtain $K = -0.0611 < 0$. If we choose $\tau = 2$, $d = 0.35$, then $\omega^* = 2.3551$, $\beta^* = 14.3462$. We obtain $K = -0.0716 < 0$. Therefore, in both those cases, for Eq. (3.2) a generic supercritical Hopf bifurcation occurs from $u = 0$, $\beta = \beta^*$.

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